Model spaces and Toeplitz kernels in reflexive Hardy spaces

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Abstract

This paper considers model spaces in an H_p setting. The existence of unbounded functions and the characterisation of maximal functions in a model space are studied, and decomposition results for Toeplitz kernels, in terms of model spaces, are established.

1 Introduction and notation

In the theory of complex functions and linear operators, there has been a significant body of work attempting to understand the structure and properties of kernels of Toeplitz operators, or Toeplitz kernels, and to describe them (or at least determine their dimension) explicitly for some concrete classes of symbols (see, for example, [1, 2, 7, 11, 18, 19]).

Linked with this is the theory of *model spaces*, which have generated an enormous interest; they provide the natural setting for truncated Toeplitz operators and are relevant in connection with the study of a variety of topics such as the Schrödinger operator, classical extremal problems, and Hankel operators (see for instance [10] and references therein).

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Model spaces constitute a particular type of Toeplitz kernel whose properties are in general more fully understood. Indeed, denoting by $\mathbb{D}$ the unit disk, Beurling's theorem characterises the nontrivial subspaces of $H^2(\mathbb{D})$ which are invariant under the (unilateral) shift S as consisting of the $H^2(\mathbb{D})$ multiples of some inner function θ, i.e., as being of the form $\theta H^2(\mathbb{D})$. The so-called model spaces K_θ are the nontrivial invariant subspaces for the backward shift S^*; they are the orthogonal complements in $H^2(\mathbb{D})$ of the shift-invariant subspaces $\theta H^2(\mathbb{D})$.

An equivalent definition, which is better suited to the context of the Hardy spaces H^p with $p \in (1, \infty), p \neq 2$, in which the Hilbert space structure is absent, is to say that K_θ is the kernel of the Toeplitz operator whose symbol is $\overline{\theta}$, the complex conjugate of the inner function θ, assumed to be nonconstant. This approach to model spaces in $H^p(\mathbb{D})$, or in $H_p^+ := H^p(\mathbb{C}^+)$ which will be our main setting (here $\mathbb{C}^+$ denotes the upper half-plane), provides a simple operator theory point of view, as well as a functional analytic description of S^*-invariant subspaces which is almost as simple as Beurling's description of S-invariant subspaces: K_θ consists of the H_p^- multiples of θ which belong to H_p^+ (using the notation $H_p^\pm$ for $H^p(\mathbb{C}^\pm)$).

This paper's results take further some ideas introduced in [2], looking at model spaces and Toeplitz operators in a general H_p context $(1 < p < \infty)$, rather than simply H_2, and working on the upper half-plane rather than the disk. One advantage of this choice is that some formulae are simpler in the half-plane context, although they can generally be translated to analogous results on the disk; some questions, however, are meaningful only in a half-plane context.

The themes considered in this work include near invariance (a property of all Toeplitz kernels, and model spaces in particular), the dependence of a Toeplitz kernel on the symbol of the corresponding Toeplitz operator and the H_p space where it is defined, some associated factorisation and decomposition results, and the existence of a *maximal function* in every Toeplitz kernel that uniquely defines the latter. The results also generalise some properties of model spaces to general Toeplitz kernels and show that we can use model spaces to "quantify" (in a loose sense of the word), for infinite-dimensional kernels, some properties relating the dimensions of finite-dimensional kernels.

More precisely, the structure of this paper is as follows. The first two sections are of an auxiliary nature. In Section 2 we present some results on Toeplitz kernels and near invariance in an H_p^+ context, very much in the spirit of [2]. In Section 3 we turn our attention to model spaces, regarded as Toeplitz kernels of a particular kind, and present their basic properties and some factorisation and decomposition results. The main results of the paper are contained in the next three sections. Section 4 addresses the question when model spaces consist entirely of bounded functions, i.e., form subspaces of H_∞^+; the answer for the half-plane turns out to be significantly more interesting than in the disk case and provides an example where results on the disk do not carry over to the upper half-plane and vice-versa. Then in Section 5 we are mainly concerned with characterising maximal functions in a model space, i.e., those which are contained in no smaller Toeplitz kernel. Finally, in Section 6 we establish decomposition results relating two Toeplitz kernels determined by symbols that differ only by an inner factor.

We take $1 < p < \infty$ and H_p^+, H_p^- to be the Hardy spaces of the upper and lower half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$ respectively. We write L_p to denote $L^p(\mathbb{R})$.
The class of invertible elements in $H_\infty^\pm$ is denoted by $\mathcal{G}H_\infty^\pm$. Similarly for $\mathcal{G}L_\infty$.
We write $P^+ : L_p \to H_p^+$ for the projection with kernel H_p^-.

For $g \in L_\infty(\mathbb{R})$ and $1 < p < \infty$, the *Toeplitz operator* $T_g : H_p^+ \to H_p^+$ is defined by

$$T_g f_+ = P^+(g f_+), \qquad (f_+ \in H_p^+).$$

We shall require the functions

$$\lambda_\pm(\xi) = \xi \pm i \qquad \text{and} \qquad r(\xi) = \frac{\xi - i}{\xi + i}\ , \tag{1.1}$$

and write S for the operator T_r on H_p^+ of multiplication by r, with S^* the operator $T_{\overline{r}}$.

2 Near invariance and T-kernels

Definition 2.1. *[2] Let $\mathcal{E}$ be a proper closed subspace of H_p^+ and η a complex-valued function defined almost everywhere on $\mathbb{R}$. We say that $\mathcal{E}$ is* nearly η-invariant *if and only if, for every $f_+ \in \mathcal{E}$ such that $\eta f_+ \in H_p^+$, we have $\eta f_+ \in \mathcal{E}$; that is*

$$\eta\mathcal{E} \cap H_p^+ \subset \mathcal{E}. \tag{2.1}$$

If $\mathcal{E}$ is nearly η - invariant with $\eta \in L_\infty$, then we also say that $\mathcal{E}$ is nearly T_η-invariant*.*

We abbreviate "nearly η - invariant" to "n. η-invariant".

We denote by $\mathcal{N}_p$ the set of all complex-valued functions η, defined a.e. on $\mathbb{R}$, such that every kernel of a Toeplitz operator (abbreviated to *T-kernel*) in H_p^+ is n. η-invariant, i.e., such that for all $g \in L_\infty$ we have

$$\eta \ker T_g \cap H_p^+ \subset \ker T_g. \tag{2.2}$$

It is shown in [2] that $\mathcal{N}_p \supset \widetilde{\mathcal{N}}_p$, where

$$\widetilde{\mathcal{N}}_p := \{\eta : L_p \cap \eta H_p^- \subset H_p^-\},$$

and that many well-known classes of functions are contained in $\widetilde{\mathcal{N}}_p$, amongst them $\mathcal{L}^-_{\infty,m} := \lambda_-^m H_\infty^-$ for all $m \in \mathbb{Z}$, the set of all rational functions with poles belonging to $\mathbb{C}^+ \cup \mathbb{R} \cup \{\infty\}$, and H_p^- for all $p \in (1, \infty)$.

On the other hand, if we extend the notation for T-kernels, defining

$$\ker T_g := \{\varphi_+ \in H_p^+ : g\varphi_+ \in H_p^-\} \tag{2.3}$$

for all complex-valued g defined a.e. on $\mathbb{R}$, it is clear that we also have

$$\eta \ker T_g \cap H_p^+ \subset \ker T_{\eta^{-1}g} \tag{2.4}$$

if $\eta^{\pm 1}$ are defined a.e. on $\mathbb{R}$ (whether or not they belong to $\mathcal{N}_p$). We have moreover:

Proposition 2.2. *If $\eta \in \widetilde{\mathcal{N}}_p$, then $\ker T_{\eta^{-1}g} \subset \ker T_g$ for all $g \in L_\infty$.*

Proof. Let $\varphi_+ \in H_p^+$ and $\eta^{-1} g\varphi_+ = \varphi_- \in H_p^-$. Then $g\varphi_+ = \eta\varphi_- \in L_p \cap \eta H_p^- \subset H_p^-$, so that $\varphi_+ \in \ker T_g$. □

Taking (2.4) into account we have thus:

Corollary 2.3. *If $\eta \in \widetilde{\mathcal{N}}_p$, $g \in L_\infty$, then*

$$\eta \ker T_g \cap H_p^+ \subset \ker T_{\eta^{-1}g} \subset \ker T_g. \tag{2.5}$$

The inclusions in (2.4) and in Proposition 2.2 may be strict or not.
Regarding the first inclusion, it is easy to see that if O_+ is outer in H_∞^+ then

$$O_+^{-1} \ker T_g \cap H_p^+ = \ker T_{O_+ g}, \tag{2.6}$$

and in particular

$$h_+ \in \mathcal{G}H_\infty^+ \Rightarrow h_+ \ker T_g = \ker T_{h_+^{-1} g}. \tag{2.7}$$

On the other hand, for any non-constant inner function θ, if $\ker T_g \neq \{0\}$ then

$$\theta \ker T_{\theta g} \subsetneq \ker T_g \tag{2.8}$$

since either $\ker T_{\theta g} = \{0\}$ and (2.8) is obvious, or $\ker T_{\theta g} \neq \{0\}$ and (2.8) follows from (2.4) and the proposition below.

Proposition 2.4. *If $\tilde{g}$ is a complex-valued function defined a.e. on $\mathbb{R}$, $\ker T_{\tilde{g}} \neq \{0\}$ and θ is a non-constant inner function, then $\theta \ker T_{\tilde{g}}$ is not a n. $\bar{\theta}$-invariant subspace of H_p^+.*

Proof. For $\mathcal{E} = \theta \ker T_{\tilde{g}}$, we have $\bar{\theta}\mathcal{E} = \ker T_{\tilde{g}} \subset H_p^+$. But if $\ker T_{\tilde{g}} \subset \mathcal{E}$, then for any $\varphi_+ \in \ker T_{\tilde{g}}$ we would have $\varphi_+ = \theta\psi_+$ with $\psi_+ \in \ker T_{\tilde{g}}$ and, repeating this reasoning, φ_+ would be divisible in H_p^+ by arbitrarily large powers of θ, implying that $\varphi_+ = 0$. $\square$

We remark however that $\theta \ker T_{\tilde{g}}$ is a n. S^*-invariant subspace of H_p^+ if $\theta(i) \neq 0$. Indeed if $r^{-1}\theta\varphi_+ \in H_p^+$, with $\varphi_+ \in \ker T_{\tilde{g}}$, then we must have $\varphi_+(i) = 0$, so that $r^{-1}\varphi_+ \in H_p^+$, and $\tilde{g}r^{-1}\varphi_+ = r^{-1}\varphi_-$ with $\varphi_- \in H_p^-$, implying that $r^{-1}\varphi_+ \in \ker T_{\tilde{g}}$ and $r^{-1}\theta\varphi_+ \in \theta \ker T_{\tilde{g}}$.

Regarding the inclusion in Proposition 2.2, we have the following two results.

Proposition 2.5. *If $\eta^{\pm 1} \in \widetilde{\mathcal{N}}_p$ and g, $\eta g \in L_\infty$, then $\ker T_{\eta^{-1} g} = \ker T_g$.*

Proof. From Corollary 2.3 we have, on the one hand, $\ker T_{\eta^{-1}g} \subset \ker T_g$ and, on the other hand, $\ker T_g = \ker T_{\eta(\eta^{-1}g)} \subset \ker T_{\eta^{-1}g}$. $\square$

In particular, if O_- is outer in H_∞^- then

$$\ker T_{O_- g} = \ker T_g \tag{2.9}$$

and

$$h_- \in \mathcal{G}H_\infty^- \Rightarrow \ker T_{h_- g} = \ker T_g. \tag{2.10}$$

Proposition 2.6. *If $\eta = \bar{\theta}\tilde{\eta}$, where $\theta \in H_\infty^+$ is a non-constant inner function and $\tilde{\eta} \in \widetilde{\mathcal{N}}_p$, then*

$$\ker T_{\eta^{-1}g} \subsetneq \ker T_g$$

if $\ker T_g \neq \{0\}$.

Proof. If $\ker T_{\eta^{-1}g} = \{0\}$, the inclusion is obviously strict. If $\ker T_{\eta^{-1}g} \neq \{0\}$ then, by an analogue of Theorem 2.2 in [2] and Proposition 2.2 above,

$$\ker T_{\eta^{-1}g} = \ker T_{\theta\tilde{\eta}^{-1}g} \subsetneq \ker T_{\tilde{\eta}^{-1}g} \subset \ker T_g.$$

$\square$

Note that studying T-kernels is closely related to studying sets of the form $\eta \ker T_g \cap H_p^+$ since we can write, for the kernel of any operator T_G in H_p^+,

$$\ker T_G = h_+(\bar{\theta}_1 \ker T_{\bar{\theta}_2} \cap H_p^+) \tag{2.11}$$

where $h_+ \in \mathcal{G}H_\infty^+$ and θ_1, θ_2 are inner functions, which may be chosen to be Blaschke products ([7], Theorem 1).

3 Model spaces in H_p^+

Definition 3.1. *If θ is an inner function, then $K_\theta^p := H_p^+ \cap \theta H_p^-$, for $p \in (1, \infty)$.*

We omit the superscript p in K_θ^p unless it is required for clarity.

This definition makes it clear that K_θ is a T-kernel, since $K_\theta = \ker T_{\bar{\theta}}$. Model spaces are thus n. η-invariant for all $\eta \in \widetilde{\mathcal{N}}_p$; in the case of $\eta \in H_\infty^-$, model spaces are moreover T_η-invariant. A particular case is that of $S^* = T_{r^{-1}}$, where r is given by (1.1), in which case the converse is true ([9]) and we can say that $K \subset H_p^+$ is a model space if and only if K is $S^* - invariant$.

Given $p \in (1, \infty)$, to each inner function θ we can associate a bounded projection $P_\theta : L_p \to K_\theta$ defined by

$$P_\theta = \theta P^- \overline{\theta} P^+. \tag{3.1}$$

Its restriction to H_p^+ is also a projection onto K_θ, which we denote in the same way. We have $K_\theta = P_\theta H_p^+ = P_\theta L_p$ and $H_p^+ = K_\theta \oplus \theta H_p^+$ (for $p = 2$ this is an orthogonal decomposition).

We also have $K_\theta = P^+(\theta H_p^-)$ and

$$K_\theta = \theta\,\overline{K_\theta}. \tag{3.2}$$

Given any non-constant inner function θ, we have $K_\theta \neq \{0\}$. An approach to this result, which gives more information on the structure of model spaces, uses the following factorisation result.

Theorem 3.2. *Given any non-constant inner function θ, we may choose $a \in \mathbb{R}$ and inner functions θ_1, θ_2 where θ_1 is non-constant, analytic in a neighbourhood of a and $\theta_1(a) = 1$, such that $\theta = \theta_1\theta_2$.*

Proof. If θ has an elementary Blaschke factor b, then the result is clear, taking $a = 0$ and $\theta_1 = b/b(0)$. So we may assume that θ is a singular inner function.
If the measure μ determining θ is an atom concentrated at ∞, then we may take a to be any finite point, and the result is clear.
Otherwise, let I be any open interval such that $\mu(\mathbb{R} \setminus I) > 0$, and choose $a \in I$. Define a decomposition of μ into positive singular measures by setting $\mu = \mu_1 + \mu_2$, where $\mu_1(A) = \mu(A \setminus I)$ and $\mu_2(A) = \mu(A \cap I)$. These determine inner functions θ_1 and θ_2 with the required properties, and by multiplying them by unimodular constants, if necessary, we may also assume that $\theta_1(a) = 1$. □

It is easy to see that, if θ_1 be a non-constant inner function, analytic in a neighbourhood of a point $a \in \mathbb{R}$, with $\theta_1(a) = 1$, and $\Lambda_{\theta_1,a}$ is the function

$$\Lambda_{\theta_1,a}(\xi) = \frac{\theta_1(\xi) - 1}{\xi - a}, \quad \xi \in \mathbb{R}, \tag{3.3}$$

then $\Lambda_{\theta_1,a} \in K_{\theta_1}$. If, in addition, θ_1 is a singular inner function then $\Lambda_{\theta_1^\mu,a} \in K_{\theta_1}$ for all $\mu \in (0, 1]$.
So, if θ is a Blaschke product, then $\dfrac{1}{\xi - \overline{z_+}} \in K_\theta$ for every zero z_+ of θ. If θ is a singular inner function, we can write $\theta = \theta_1\theta_2$ as in Theorem 3.2 and

$$\theta_2\Lambda_{\theta_1^\mu,a} \in K_\theta \quad \text{for all } \mu \in (0, 1].$$

Otherwise, $\theta = \alpha B\mathcal{S}$ where $\alpha \in \mathbb{C}$, $|\alpha| = 1$, B is a Blaschke product and $\mathcal{S}$ is a singular inner function, and it is easy to see that $K_\theta \supset K_{\mathcal{S}}$.

In any case, we explicitly see that K_θ is infinite-dimensional unless θ is a finite Blaschke product. In the latter case, we can write

$$\theta = h_- r^n h_+, \quad \text{with } h_\pm \in \mathcal{G}H_\infty^\pm,\ n \in \mathbb{N} \tag{3.4}$$

and K_θ is an n-dimensional linear space described by

$$K_\theta = h_+ \text{ span}\left\{\lambda_+^{-1} r^j : j = 0, 1, \ldots, n-1\right\} = h_+ K_{r^n} \tag{3.5}$$

(recall that $\lambda_\pm(\xi) = \xi \pm i$).
Thus, in the case where θ is a rational inner function, it is clear from (3.5) that $K_\theta \subset \lambda_+^{-1} H_\infty^+ \subset H_\infty^+$. The question whether $K_\theta \subset H_\infty^+$ in other cases is fairly delicate and will be dealt with later in this paper.

To have a better understanding of infinite-dimensional model spaces K_θ, it will be useful to characterise some dense subsets. While K_θ may not be itself contained in H_∞^+, there are nevertheless dense subsets of K_θ contained in $\lambda_+^{-1} H_\infty^+$. Indeed, for each $w \in \mathbb{C}^+$, let

$$k_w(\xi) = \frac{i}{2\pi} \frac{1}{\xi - \overline{w}}, \quad \xi \in \mathbb{R}, \tag{3.6}$$

and, given an inner function θ, let k_w^θ be defined for each $w \in \mathbb{C}^+$ by

$$k_w^\theta(\xi) = \frac{i}{2\pi} \frac{1 - \overline{\theta(w)}\theta(\xi)}{\xi - \overline{w}} = P_\theta k_w(\xi). \tag{3.7}$$

These are the reproducing kernel functions for K_θ^2, but they play the same role in K_θ^p for each $p \in (1, \infty)$, namely

$$\int_{-\infty}^{+\infty} f(x) \overline{k_w^\theta(x)}\, dx = f(w) \quad \text{for all } f \in K_\theta^p. \tag{3.8}$$

Let also f_k^θ be the functions defined, for each $k \in \mathbb{Z}_0^+$, by

$$f_k^\theta = \frac{r^k}{\lambda_+} - \frac{a_0 + a_1\lambda_+ + \ldots + a_k \lambda_+^k}{\lambda_+^{k+1}} \theta,$$

where $a_j = (\lambda_-^j \bar{\theta})^{(j)}_{(-i)}/j!$, $j = 0, 1, ..., k-1$. As in the case of reproducing kernel functions, these are easily recognisable functions of K_θ^p, providing the following density result. We have $k_w^\theta, f_k^\theta \in K_\theta^p \cap \lambda_+^{-1} H_\infty^+$ for all $w \in \mathbb{C}^+, k \in \mathbb{Z}_0^+$ and $p \in (1, \infty)$, and

$$K_\theta^p = \text{clos}_{H_p^+} \text{span}\{k_w^\theta : w \in \mathbb{C}^+\} = \text{clos}_{H_p^+} \text{span}\{f_k^\theta : k \in \mathbb{Z}_0^+\}.$$

Definition 3.3. *For inner functions θ_1 and θ_2, we write $\theta_2 \preceq \theta_1$ if and only if θ_2 divides θ_1, in the sense that $\theta_1 = \theta_2\theta_3$ for some inner function θ_3.*
We also write $\theta_2 \prec \theta_1$ if $\theta_1 = \theta_2\theta_3$ for some non-constant *inner function θ_3.*

The results in the next theorem may be considered as generally known; see, for instance, [17].

Theorem 3.4. *Let θ_1, θ_2 and θ_3 be inner functions. We have, for $p \in (1, \infty)$:*

(i) *$\theta_2 \preceq \theta_1$ if and only if $K^p_{\theta_2} \subset K^p_{\theta_1}$;*

(ii) *$\theta_2 \prec \theta_1$ if and only if $K^p_{\theta_2} \subsetneq K^p_{\theta_1}$;*

(iii) *$\theta_2\theta_3 \preceq \theta_1$ if and only if $\theta_3 K^p_{\theta_2} \subset K^p_{\theta_1}$;*

(iv) *$\theta_1 \preceq \theta_3 \implies \theta_1 K_{\theta_2} \subset K_{\theta_3\theta_2}$, where the inclusion is strict if θ_1 is not constant.*

An alternative short proof of (i)–(iii) is provided in Section 5 using the characterisation of maximal functions in a model space instead of the H^+_p–H^+_q duality.
For any inner functions θ_1, θ_2 we have

$$K_{\theta_1} \subset K_{\theta_1\theta_2}, \quad \theta_1 K_{\theta_2} \subset K_{\theta_1\theta_2}, \tag{3.9}$$

and the two subspaces at the left-hand side of these inclusions provide a direct sum decomposition

$$K_{\theta_1\theta_2} = K_{\theta_1} \oplus \theta_1 K_{\theta_2}. \tag{3.10}$$

For $p = 2$, (3.10) yields an orthogonal decomposition of $K_{\theta_1\theta_2}$. We also have the following.

Theorem 3.5. *Let θ, θ_1 be inner functions and let $n, m \in \mathbb{N}$ with $n \geq m$. Then if*

$$\theta^n \preceq r^m \theta_1 \tag{3.11}$$

then

$$K_{\theta_1} = K_{\theta^s} \oplus \theta^s K_{\theta_1/\theta^s} \tag{3.12}$$

for any $s \in \mathbb{Z}^+_0$, $s \leq n - m$.

Proof. If (3.11) holds, let $r^m\theta_1 = \theta^n\widetilde{\theta}$ with $\widetilde{\theta}$ inner. Then

$$\theta_1 = \theta^{n-m}(r^{-m}\widetilde{\theta}\theta^m) \tag{3.13}$$

and, since $\theta_1 \in H_\infty^+$, we must have $\theta(i) = 0$ or $\widetilde{\theta}(i) = 0$. In any case, $r^{-m}\widetilde{\theta}\theta^m$ is an inner function and so (3.13) implies that $\theta^{n-m} \preceq \theta_1$. Now (3.12) follows from Theorem 3.4 and (3.10). □

Inner functions and model spaces can be related by an equivalence relation as follows.

Definition 3.6. *If θ_1 and θ_2 are inner functions, we say that $\theta_1 \sim \theta_2$ if and only if there are functions $h_\pm \in \mathcal{G}H_\infty^\pm$ such that*

$$\theta_1 = h_-\theta_2 h_+. \tag{3.14}$$

It is easy to see that we have $\theta_1 = h_-\theta_2 h_+$ and $\theta_1 = \tilde{h}_-\theta_2\tilde{h}_+$ with $h_\pm \in \mathcal{G}H_\infty^\pm$, $\tilde{h}_\pm \in \mathcal{G}H_\infty^\pm$, if and only if $\frac{h_-}{\tilde{h}_-} = \frac{\tilde{h}_+}{h_+} = c \in \mathbb{C}\setminus\{0\}$, and we can choose $h_\pm$ in (3.14) such that $\|h_-\|_\infty = \|h_+\|_\infty = 1$.
Moreover, if (3.14) holds for given θ_1, θ_2, then $h_-\overline{h}_+ = h_+^{-1}(\overline{h_-})^{-1}$; since the left-hand side represents a function in H_∞^- and the right-hand side represents a function in H_∞^+, both are constant and we have

$$\overline{h_+} = h_-^{-1}c, \quad \overline{h_-} = h_+^{-1}c^{-1}, \quad \text{with } c \in \mathbb{C}\setminus\{0\}. \tag{3.15}$$

Definition 3.7. *If θ_1 and θ_2 are inner functions, we say that $K_{\theta_1} \sim K_{\theta_2}$ if and only if*

$$K_{\theta_1} = h_+K_{\theta_2} \quad \text{with } h_+ \in \mathcal{G}H_\infty^+. \tag{3.16}$$

It is clear that

$$\theta_1 \sim \theta_2 \Rightarrow K_{\theta_1} \sim K_{\theta_2} \tag{3.17}$$

since, by (2.7), (2.10) and (3.15), if (3.14) holds then

$$K_{\theta_1} = \ker T_{\bar{\theta}_1} = \ker T_{h_-^{-1}\bar{\theta}_2 h_+^{-1}} = h_+ \ker T_{\bar{\theta}_2} = h_+K_{\theta_2}.$$

If θ is a finite Blaschke product, then $K_\theta \sim K_{\tilde{\theta}}$ if and only if $\tilde{\theta}$ is also a finite Blaschke product of the same degree. However, model spaces associated with infinite Blaschke products may be equivalent, in the sense of Definition 3.7, to model spaces associated to singular inner functions. In particular, for any singular inner function θ there exists an infinite Blaschke product B such that

$$K_\theta \sim K_B. \tag{3.18}$$

In fact the function

$$B = \frac{\theta - a}{1 - \bar{a}\theta} \tag{3.19}$$

is a Blaschke product for all a with $|a| < 1$ outside a set of measure zero [3, 8]. Thus any inner function θ can be factorised as

$$\theta = h_- B h_+ \tag{3.20}$$

where B is a Blaschke product and $h_\pm \in \mathcal{G}H_\infty^\pm$ with

$$h_- = 1 + a\bar{B}, \quad h_+ = \frac{1}{1 + \bar{a}B}\,. \tag{3.21}$$

It follows from (3.20) that $\theta \sim B$ and

$$K_\theta = h_+ K_B. \tag{3.22}$$

If $K_{\theta_1} \sim K_{\theta_2}$ then the two model spaces are isomorphic (although not usually isometric in the case $p \neq 2$) and share several properties, namely that they are either both contained in H_∞^+ or they are not.
The projections associated with K_{θ_1} and K_{θ_2} are related as follows.

Theorem 3.8. *If $K_{\theta_1} \sim K_{\theta_2}$ and $h_+ \in \mathcal{G}H_\infty^+$ is such that* (3.16) *holds, then*

$$\widetilde{P}_{\theta_1} := h_+ P_{\theta_2} h_+^{-1} P^+ \tag{3.23}$$

is a projection from H_p^+ (or L_p) onto K_{θ_1} such that

$$\widetilde{P}_{\theta_1}|_{K_{\theta_1}} = P_{\theta_1}. \tag{3.24}$$

Proof. $\widetilde{P}_{\theta_1}$ is obviously a projection and, for any $\varphi_+ \in H_p^+$, $\widetilde{P}_{\theta_1}\varphi_+ \in h_+ K_{\theta_2} = K_{\theta_1}$. Moreover, if $\varphi_+ \in K_{\theta_1}$ then $h_+^{-1}\varphi_+ \in K_{\theta_2}$, $P_{\theta_2} h_+^{-1}\varphi_+ = h_+^{-1}\varphi_+$, and we have $\widetilde{P}_{\theta_1}\varphi_+ = h_+ P_{\theta_2} h_+^{-1}\varphi_+ = \varphi_+$. $\square$

4 Model spaces contained in H_∞^+

Let

$$K_\theta^\infty := H_\infty^+ \cap \theta H_\infty^- \tag{4.1}$$

for an inner function θ. Since $\theta \in K_\theta^\infty$, we can extend the inclusion $\theta_1 K_{\theta_2} \subset K_{\theta_1\theta_2}$ in (3.9) as follows.

Proposition 4.1. *For any inner functions θ_1, θ_2 we have*

$$K_{\theta_1}^{\infty} K_{\theta_2} \subset K_{\theta_1\theta_2}. \tag{4.2}$$

Proof. Let $f_1^+ \in K_{\theta_1}^{\infty}$, $f_2^+ \in K_{\theta_2}$. Then $f_1^+ f_2^+ \in K_{\theta_1\theta_2}$ because $f_1^+ f_2^+ \in H_p^+$ and

$$\overline{\theta_1}\,\overline{\theta_2} f_1^+ f_2^+ = (\overline{\theta_1} f_1^+)(\overline{\theta_2} f_2^+) \in H_p^-.$$

□

Using the fact that model spaces are T-kernels and the n. η-invariance of T-kernels for all $\eta \in \overline{K}_\theta = \bar{\theta} K_\theta$ ([2]), we also have

$$K_{\theta_1} K_{\theta_2} \cap H_p^+ \subset K_{\theta_1\theta_2}, \tag{4.3}$$

since $K_{\theta_1} K_{\theta_2} = \overline{K}_{\theta_1}(\theta_1 K_{\theta_2})$ and $\theta_1 K_{\theta_2} \subset K_{\theta_1\theta_2}$ by (3.9).
From (4.3) we have $K_{\theta_1} K_{\theta_2} \subset K_{\theta_1\theta_2}$ if either K_{θ_1} or K_{θ_2} is contained in H_∞^+, as happens when θ_1 or θ_2 are finite Blaschke products. The question whether there are infinite-dimensional model spaces satisfying this boundedness condition has different answers depending on whether the setting is the disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ or the upper-half plane.

4.1 The case of the disk

This is the easier case, and the following result holds, which we include for completeness.

Theorem 4.2. *Let $\theta \in H_\infty(\mathbb{D})$ be inner; then, for any $p \in (1, \infty)$, the model space $K_\theta = H_p(\mathbb{D}) \cap \theta \bar{z} \overline{H_p(\mathbb{D})}$ is a subspace of $H_\infty(\mathbb{D})$ if and only if θ is a rational function.*

Proof. If θ is rational then we have $\theta = h_- z^n h_+$ with $h_+, \overline{h_-} \in \mathcal{G}H_\infty(\mathbb{D})$, $\overline{h_\pm} = h_\mp^{-1}$ and n equal to the number of zeros of θ, taking their multiplicity into account. By (2.7) and (2.10), $K_\theta = \ker T_{\bar\theta} = h_+ \ker T_{\overline{z^n}}$ and it follows that $K_\theta \subset H_\infty(\mathbb{D})$.
Conversely, note that the reproducing kernel functions k_w^θ, with

$$k_w^\theta(z) := \frac{1 - \overline{\theta(w)}\theta(z)}{1 - \overline{w} z}, \quad w \in \mathbb{D},$$

lie in K_θ, for any $p \in (1, \infty)$. Indeed, their $H_p(\mathbb{D})$ norm is bounded by a constant times $(1 - |w|)^{-1+1/p}$, as can be seen by estimating the norm of $1/(1 - \overline{w} z)$ directly – it is enough to consider real positive w and do a direct

calculation. This can be achieved quite simply using an isometry with the H_p space of the half-plane as in [16, Prop. 2.15].
However, if θ is not a finite Blaschke product, then for each $\varepsilon > 0$ we can find a point $w \in \mathbb{D}$ with $|w| > 1 - \varepsilon$ and $|\theta(w)| < 1/2$. Thus, taking $z = w/|w|$ we have $\|k_w^\theta\|_\infty \geq 1/(2(1 - |w|))$, that is

$$\sup_{f \in K_p^\theta} \frac{\|f\|_\infty}{\|f\|_p} = \infty. \tag{4.4}$$

If every function in K_θ is bounded then we have a natural embedding $J : K_\theta \to H_\infty(\mathbb{D})$. But the closed graph theorem now implies that J is a bounded operator, contradicting (4.4). $\square$

4.2 The case of the (upper) half-plane

As in the setting of H_p spaces of the disk, if θ is a rational inner function then $K_\theta^p \subset H_\infty^+$, for all $p \in (1, \infty)$. Now, however, we may have $K_\theta^p \subset H_\infty^+$ for some classes of irrational inner functions θ, as well as model spaces which are not contained in H_∞^+.
Indeed, Dyakonov [4] (see also [5, 6]) gave the following necessary and sufficient conditions for $K_\theta^p \subset H_\infty^+$ (note that they do not depend on p).

$$\begin{array}{ll} 1. & \theta' \in H_\infty^+; \\ 2. & \inf\{|\theta(z)| : 0 < \operatorname{Im} z < \epsilon\} > 0 \quad \text{for some} \quad \epsilon > 0. \end{array} \tag{4.5}$$

In particular, if for $\lambda \in \mathbb{R}^+$, e_λ denotes the singular inner function

$$e_\lambda(\xi) = e^{i\lambda\xi}, \quad \xi \in \mathbb{R}, \tag{4.6}$$

then for any $p \in (1, \infty)$ the (Paley–Wiener type) model space $K_{e_\lambda}^p$ consists of entire functions and is contained in H_∞^+.
However, if θ possesses a sequence of zeroes tending to the real axis, or if θ has a singular inner factor other than e_λ for some $\lambda > 0$, then the model space K_θ^p contains unbounded functions. This follows from the well-known fact that, for a singular inner function determined by a measure ν, the non-tangential boundary limits are 0 almost everywhere with respect to ν (see for example [3, Chap. 1]).

The following result gives an alternative, and occasionally more usable, necessary and sufficient condition for the inclusion into H_∞^+.

Theorem 4.3. $K_\theta^p \subset H_\infty^+$ *if and only if*

$$\sup_{w \in \mathbb{C}^+} \frac{1 - |\theta(w)|^2}{\operatorname{Im} w} < \infty.$$

Proof. Note that, by Dyakonov's result, it is sufficient to discuss the case $p = 2$. By the closed graph theorem a necessary and sufficient condition for K_θ^2 to embed into H_∞^+ is that, for all $f \in K_\theta^2$, we have $f \in H_\infty^+$ and there is a constant $C > 0$ such that

$$\|f\|_\infty \leq C\|f\|_2 \tag{4.7}$$

for all $f \in K_\theta^2$.
Since for $f \in K_\theta^2$ we have $\sup_{w\in\mathbb{C}^+} |f(w)| = \sup_{w\in\mathbb{C}^+} |\langle f, k_w^\theta\rangle|$, condition (4.7) is equivalent to the condition that the L_2 norms of the k_w^θ are uniformly bounded, independently of w. For $p = 2$ we have

$$\|k_w^\theta\|_2^2 = \langle k_w^\theta, k_w^\theta\rangle = |k_w^\theta(w)|$$

and the result follows from (3.7).

□

The following refinement of (4.5) is an immediate consequence of Theorem 4.3 and (4.5) itself.

Corollary 4.4. *We have* $K_\theta^p \subset H_\infty^+$ *if and only if*

$$\lim_{\varepsilon \to 0^+} \inf\{|\theta(z)| : 0 < \operatorname{Im} z < \varepsilon\} = 1. \tag{4.8}$$

Dyakonov's condition that $\inf\{|\theta(z)| : 0 < \operatorname{Im} z < \epsilon\} > 0$ for some $\epsilon > 0$ has appeared elsewhere in the literature, being applied to realization theory [15] and finite-time controllability [14]. (The context is the right half-plane but it is easy to transcribe the results for the upper half-plane.) In particular, for a Blaschke product with zeroes $\lambda_n = x_n + iy_n$, $n \geq 1$, the condition is shown in [15] to be equivalent to the property that $\inf y_n > 0$ and

$$\sup_{x\in\mathbb{R}} \sum_{n=1}^\infty \frac{y_n}{y_n^2 + (x - x_n)^2} < \infty,$$

which in turn can be expressed as a Carleson measure condition on the measure $\mu := \sum_{n=1}^\infty y_n \delta_{\lambda_n}$, tested on reproducing kernels k_λ lying on a horizontal line.

A more general question, to which we do not know a complete answer except in the case $p = 2$, is to ask when a T-kernel contains only bounded functions.

5 Maximal and minimal functions in model spaces

It was shown in [2] that for every $\varphi_+ \in H_p^+ \setminus \{0\}$ there exists a T-kernel containing φ_+, denoted by $\mathcal{K}_{\min}(\varphi_+)$, such that for any $g \in L_\infty$ we have

$$\varphi_+ \in \ker T_g \Rightarrow \mathcal{K}_{\min}(\varphi_+) \subset \ker T_g \tag{5.1}$$

and, if $\varphi_+ = I_+ O_+$ is an inner-outer factorisation of φ_+,

$$\mathcal{K}_{\min}(\varphi_+) = \ker T_{\overline{I}_+ \overline{O}_+ / O_+}. \tag{5.2}$$

$\mathcal{K}_{\min}(\varphi_+)$ is called the minimal kernel for φ_+. It can be shown moreover that a nontrivial, proper, n. S^*-invariant subspace $\mathcal{E}$ of H_p^+ $(1 < p < \infty)$ is a T-kernel if and only if there exists $\varphi_+ \in H_p^+$ such that $\mathcal{E} = \mathcal{K}_{\min}(\varphi_+)$, i. e., such that $f_+ \in \mathcal{E}$ if and only if $f_+ \in H_p^+$ and $\overline{I}_+ \frac{\overline{O}_+}{O_+} f_+ \in H_p^-$, where $\varphi_+ = I_+ O_+$ is an inner-outer factorisation of φ_+ ([2]).

Definition 5.1. *If $K = \mathcal{K}_{\min}(\varphi_+)$, we say that φ_+ is a* maximal function *for K.*

Being T-kernels, model spaces are minimal kernels for some of their elements. Given a model space K_θ, it is thus natural to try to characterise the maximal functions for K_θ.

We start by remarking that, writing $\theta = \theta_1 \theta_2$ as in Theorem 3.2 and defining $\Lambda_{\theta_1,a}$ as in (3.3), we have (for any $p \in (1, \infty)$)

$$\theta_2 \Lambda_{\theta_1,a} \in K_{\theta_1 \theta_2} = K_\theta. \tag{5.3}$$

Since $\Lambda_{\theta_1,a}$ is outer, it follows from (5.2) that

$$\mathcal{K}_{\min}(\theta_2 \Lambda_{\theta_1,a}) = \ker T_{\overline{\theta_1 \theta_2}} = K_\theta. \tag{5.4}$$

Depending on the inner function θ associated with the model space, other maximal functions can be defined for K_θ, which may also be useful. The following theorems describe, in different ways, the maximal functions of a given model space K_θ.

Theorem 5.2. *$K_\theta = \mathcal{K}_{\min}(\varphi_+)$ if and only if $\varphi_+ \in H_p^+$ and $\varphi_+ = \theta \varphi_-$ with φ_- outer in H_p^-.*

Proof. If $K_\theta = \mathcal{K}_{\min}(\varphi_+)$, then $\varphi_+ \in K_\theta$, so that $\varphi_+ \in H_p^+$ and $\overline{\theta}\varphi_+ = \varphi_-$ with $\varphi_- \in H_p^-$. If φ_- is not outer in H_p^-, then $\varphi_- = I_- O_-$ where I_- is a non-constant inner function in H_∞^- and O_- is outer in H_p^-. Thus $\varphi_+ \in \ker T_{\overline{I_- \theta}} \subsetneq \ker T_{\bar{\theta}} = K_\theta$, which contradicts the assumption.
Conversely, if $\varphi_+ \in H_p^+$ and $\varphi_+ = \theta\varphi_-$ with φ_- outer in H_p^-, then $\varphi_+ \in K_\theta$. Moreover, for any $g \in L_\infty$, if $\varphi_+ \in \ker T_g$ then $g\theta\varphi_- = \eta_- \in H_p^-$, so that

$$g = \bar{\theta}\frac{\eta_-}{\varphi_-}$$

where φ_- is outer in H_p^-. Thus, for any $\psi_+ \in H_p^+$ such that $\bar{\theta}\psi_+ = \psi_- \in H_p^-$, i.e., for any $\psi_+ \in K_\theta$, we have

$$g\psi_+ = \bar{\theta}\frac{\eta_-}{\varphi_-}\psi_+ = \frac{\eta_-\psi_-}{\varphi_-} \in H_p^-$$

because the right-hand side represents a function which is in L_p and in the Smirnov class $\overline{\mathcal{N}_+}$. It follows that $\psi_+ \in \ker T_g$. Thus $K_\theta \subset \ker T_g$ and we have $K_\theta = \mathcal{K}_{\min}(\varphi_+)$. □

Remark 5.3. *The result of Theorem 5.2 provides an alternative proof to some properties in Theorem 3.4 that were proved using the $L_p - L_q$ duality. Consider, for instance, Theorem 3.4 (i) and assume that $K_{\theta_2}^p \subset K_{\theta_1}^p$. Let $\varphi_{\theta_2}^+$ be a maximal function for $K_{\theta_2}^p$, so that by Theorem 5.2 we have $\varphi_{\theta_2}^+ = \theta_2 O_{2-}$ where O_{2-} is outer in H_p^-. Since $K_{\theta_2}^p \subset K_{\theta_1}^p$, then $\theta_2 O_{2-} = \theta_1\psi_-$ with $\psi_- \in H_p^-$; if $\psi_- = I_- O_-$ is an inner-outer factorisation (in H_p^-) then it follows that $\bar{\theta}_2 I_- O_- = \bar{\theta}_1 O_{2-}$ and, by the uniqueness of inner-outer factorisations, we conclude that $\bar{\theta}_2 I_- = \lambda\bar{\theta}_1$ ($\lambda \in \mathbb{C}$), whence $\theta_2 \preceq \theta_1$. The same reasoning can be applied to prove (iii) in Theorem 3.4.*

As a consequence of Theorem 5.2 we also have:

Theorem 5.4. *If $\mathcal{K}_{\min}(\varphi_+)$ is a model space K_{θ_1}, then $\mathcal{K}_{\min}(\theta\varphi_+)$ is also a model space and we have*

$$\mathcal{K}_{\min}(\theta\varphi_+) = K_\theta \oplus \theta\mathcal{K}_{\min}(\varphi_+) = K_{\theta\theta_1}. \tag{5.5}$$

Proof. If $\mathcal{K}_{\min}(\varphi_+) = K_{\theta_1}$, where θ_1 is an inner function, then by Theorem 5.2 we have $\varphi_+ = \theta_1\varphi_-$ with φ_- outer in H_p^-. Therefore $\theta\varphi_+ = \theta\theta_1\varphi_-$ and, using Theorem 5.2 again, $\mathcal{K}_{\min}(\theta\varphi_+) = K_{\theta\theta_1}$. Since $K_{\theta\theta_1} = K_\theta \oplus \theta K_{\theta_1}$ by (3.10), we conclude that (5.5) holds. □

We have the following relation for maximal functions in model spaces that are equivalent in the sense of Definition 3.7.

Theorem 5.5. *Let θ_1, θ_2 be inner functions and let $K_{\theta_1} \sim K_{\theta_2}$. If (3.16) holds, then φ_+ is a maximal function for K_{θ_1} if and only if $\varphi_+ = h_+\psi_+$, where ψ_+ is a maximal function for K_{θ_2}.*

Proof. Let ψ_+ be a maximal function for K_{θ_2} and let $\psi_+ = I_+O_+$ be its inner-outer factorisation. Thus

$$K_{\theta_2} = \mathcal{K}_{\min}(\psi_+) = \ker T_{\overline{I}_+\overline{O}_+/O_+}$$

by (5.2). On the other hand, if $\varphi_+ = h_+\psi_+$ then

$$\mathcal{K}_{\min}(\varphi_+) = \mathcal{K}_{\min}(h_+\psi_+) = \ker T_{\overline{I}_+ \frac{\bar{h}_+\overline{O}_+}{h_+O_+}} = h_+ \ker T_{\overline{I}_+ \frac{\overline{O}_+}{O_+}} = h_+K_{\theta_2}$$

by (2.7) and (2.10). Now it follows from (3.16) that $\mathcal{K}_{\min}(\varphi_+) = K_{\theta_1}$. Conversely, if φ_+ is a maximal function for K_{θ_1} then, from the first part of the proof,

$$\mathcal{K}_{\min}(h_+^{-1}\varphi_+) = h_+^{-1}K_{\theta_1} = K_{\theta_2}$$

and thus $h_+^{-1}\varphi_+$ is a maximal function for K_{θ_2}. $\square$

If B is a Blaschke product vanishing at $z_0^+ \in \mathbb{C}^+$, we have from (5.2)

$$K_B = \mathcal{K}_{\min}\left(\frac{B}{\xi - z_0^+}\right). \tag{5.6}$$

Thus it follows from Theorem 5.5 and (3.17) that if θ is any non-constant inner function which can be factorised as in (3.20), a maximal function for K_θ will be

$$\varphi_+^\theta = h_+\varphi_+^B, \quad \text{with} \quad \varphi_+^B = \frac{B}{\lambda_{z_0^+}}\ , \tag{5.7}$$

where

$$\lambda_{z_0^+}(\xi) := \xi - z_0^+ \tag{5.8}$$

and we assume that $B(z_0^+) = 0$.

Note that φ_+^θ and φ_+^B in (5.7), as well as the maximal functions in (5.4), do not depend on p and belong to $\lambda_+^{-1}H_\infty^+$ (whether or not K_θ^p is contained in

H_∞^+).

We can also see that, given any inner function θ_1, from (3.20) and (5.7) we have

$$\theta_1 = h_- \lambda_{z_0^+} \varphi_+^{\theta_1} \tag{5.9}$$

and that the decomposition $K_{\theta\theta_1} = K_{\theta_1} \oplus \theta_1 K_\theta$ (where θ is an inner function) can also be written in terms of a maximal function for K_{θ_1} as

$$K_{\theta\theta_1} = K_{\theta_1} \oplus h_- \lambda_{z_0^+} \varphi_+^{\theta_1} K_\theta \,. \tag{5.10}$$

Another property relating model spaces with minimal kernels is the following.

Theorem 5.6. *Let $\varphi_1^+, \varphi_2^+, \ldots, \varphi_n^+$ be such that $\mathcal{K}_{\min}\left(\varphi_j^+\right) = K_{\theta_j}$ for each $j = 1, 2, \ldots, n$, where θ_j is an inner function. Then there is a minimal kernel K containing $\{\varphi_j^+ : j = 1, 2, \ldots, n\}$, and for $\theta = \mathrm{LCM}(\theta_1, \theta_2, \ldots, \theta_n)$ we have*

$$K = K_\theta = \mathrm{clos}_{H_p^+}(K_{\theta_1} + \cdots + K_{\theta_n}) = K_{\theta_j} \oplus \theta_j K_{\theta\overline{\theta_j}}$$

for each j.

Proof. $\mathrm{clos}_{H_p^+}(K_{\theta_1} + \cdots + K_{\theta_n})$ is a closed subspace of H_p^+, invariant for $S^* = T_{r^{-1}}$, so it is a model space $K_{\widetilde{\theta}}$. Now $K_{\widetilde{\theta}}$ is a T-kernel, and $K_{\widetilde{\theta}} \supset \{\varphi_1^+, \varphi_2^+, \ldots, \varphi_n^+\}$. Since every T-kernel containing $\{\varphi_1^+, \varphi_2^+, \ldots, \varphi_n^+\}$ must be closed, and contain each K_{θ_j}, it also contains $K_{\widetilde{\theta}}$, so that the latter is the minimal kernel K.
Since $K_{\widetilde{\theta}} \supset K_{\theta_j}$, we have $\theta_j \preceq \widetilde{\theta}$, for every j, by Theorem 3.4 and, since $\theta = \mathrm{LCM}(\theta_1, \ldots, \theta_n)$, we have $\theta \preceq \widetilde{\theta}$. On the other hand, $K_{\widetilde{\theta}} \subset K_\theta$, since $K_{\widetilde{\theta}} \subset H_p^+$ and $\overline{\theta} K_{\widetilde{\theta}} \subset H_p^-$; therefore, $\widetilde{\theta} \preceq \theta$. It follows that $\widetilde{\theta} = \theta$. □

As a motivation for the next definition, we remark now that if $\varphi_+ = I_+ O_+$ is the inner-outer factorisation of a maximal function for $\ker T_g$, so that $\ker T_g = \ker T_{\overline{I}_+ \overline{O}_+ / O_+}$, it may happen that

$$\overline{O}_+ / O_+ = \overline{I}_{1+} \overline{O}_{1+} / O_{1+}, \tag{5.11}$$

where I_{1+} is a non-constant inner function and O_{1+} is an outer function in H_p^+ (take for instance $O_+(\xi) = \frac{1}{(\xi+i)^2}$). In that case, we have

$$\ker T_g = \ker T_{\overline{I}_+ \frac{\overline{O}_+}{O_+}} = \ker T_{\overline{I_+ I_{1+}} \frac{\overline{O}_{1+}}{O_{1+}}}\,, \tag{5.12}$$

where $I_+ \prec I_+I_{1+}$. This cannot happen, however, when $\ker T_{\overline{O}_+/O_+} = \text{span}\{O_+\}$, which is equivalent to saying that O_+^2 is rigid in $H_{p/2}^+$ ([2]). In fact, (5.12) would imply that $I_+I_{1+}O_{1+} \in \ker T_{\overline{I}_+ \frac{\overline{O}_+}{O_+}}$ and thus $I_{1+}O_{1+} \in \ker T_{\overline{O}_+/O_+} = \text{span}\{O_+\}$, which is impossible for non-constant θ_1.

Definition 5.7. *If $g \in L_\infty$, we say that O_+ is a* minimal function *for* $\ker T_g$ *if and only if for some inner function I_+ we have* $\ker T_g = \mathcal{K}_{\min}(I_+O_+)$ *and* $\mathcal{K}_{\min}(O_+) = \text{span}\{O_+\}$.

In H_2^+, every non-trivial T-kernel has a minimal function ([18],[19]). The following theorem shows that this property also holds for model spaces in H_p^+; whether the same is true in general for T-kernels in H_p^+ is an open question, to the authors' knowledge.

Theorem 5.8. *For any $p \in (1,\infty)$ and any inner function θ, there exists a minimal function O_+ in K_θ.*

Proof. With the notation of (3.20) and (5.7), it is enough to consider $O_+ = \frac{h_+}{\lambda_{\overline{z}_0^+}}$ and $I_+ = B\frac{\lambda_{\overline{z}_0^+}}{\lambda_{z_0^+}}$. □

6 On the relations between $\ker T_g$ and $\ker T_{\theta g}$

If θ is a non-constant inner function, $g \in L_\infty$ and $\ker T_g \neq \{0\}$, we have $\ker T_{\theta g} \subsetneq \ker T_g$. We may then ask how much "smaller" $\ker T_{\theta g}$ is, with respect to $\ker T_g$, and in particular when is it non-trivial.

Definition 6.1. *Let $g \in L_\infty$ and θ be an inner function. If* $\ker T_g \neq \{0\}$ *and* $\ker T_{\theta g} = \{0\}$*, we say that θ* annihilates $\ker T_g$.

It is clear that a necessary and sufficient condition for $\ker T_g$ not to be annihilated by θ is that there exists φ_+ such that

$$\theta\varphi_+ \in \ker T_g, \quad \varphi_+ \in H_p^+ \setminus \{0\}, \tag{6.1}$$

and in this case $\varphi_+ \in \ker T_{\theta g}$.

If θ is a finite Blaschke product we have the following result from [1], taking into account that in this case $\theta \sim r^k$, where k is the number of zeroes of θ.

Theorem 6.2. *If $g \in L_\infty$ and θ is a finite Blaschke product, then*

$$\dim\ker T_g < \infty \Leftrightarrow \dim\ker T_{\theta g} < \infty. \tag{6.2}$$

We have $\dim\ker T_g < \infty$ if and only if there exists $k_0 \in \mathbb{Z}$ such that $\ker T_{r^{k_0}g} = \{0\}$ and, in this case, $\dim\ker T_g \leq \max\{0, k_0\}$. Moreover, if $\dim\ker T_g < \infty$, we have

$$\dim\ker T_{\theta g} = \max\{0, \dim\ker T_g - k\} \tag{6.3}$$

where k is the number of zeroes of θ counting their multiplicity.

Thus, in particular, if $\dim\ker T_g = d < \infty$ and θ is a finite Blaschke product such that $\dim K_\theta \leq d$, then

$$\dim\ker T_{\theta g} = \dim\ker T_g - \dim K_\theta\,.$$

If θ is not a finite Blaschke product and $\dim\ker T_g < \infty$, then $\ker T_{\theta g} = \{0\}$, since $\theta\varphi_+ \in \ker T_g$ implies that $\theta_1\varphi_+ \in \ker T_g$ for all inner function θ_1 such that $\theta_1 \prec \theta$. On the contrary, if $\ker T_g$ is infinite-dimensional then $\ker T_{\theta g}$ may or may not be finite-dimensional, and in particular it may be $\{0\}$. It is clear that θ annihilates $\ker T_g$ if $\bar{g} \in H^+_\infty$ is an inner function and $\theta \succ \bar{g}$, but that may also happen when no such relation holds between θ and $\bar{g}$, as in the example that follows.

Example 6.3. *Let $g(\xi) = e^{i/\xi}$, $\theta(\xi) = e^{i\xi}$. For $p = 2$, we have*

$$f_+ \in \ker T_{\theta g} \Leftrightarrow f_+ \in H^+_2\,,\ e^{i\xi}e^{i/\xi}f_+ = f_- \in H^-_2\,. \tag{6.4}$$

Using the isometry from H^+_2 onto H^-_2 defined by $f \mapsto \tilde{f}$ with $\tilde{f}(\xi) = \frac{1}{\xi}f(\frac{1}{\xi})$, we obtain from (6.4):

$$e^{i\xi}e^{i/\xi}f_+ = f_- \Leftrightarrow e^{-i\xi}e^{-i/\xi}\tilde{f}_- = \tilde{f}_+ \tag{6.5}$$

($\tilde{f}_\pm \in H^\mp_2$). Since, by Coburn's Lemma, we have $\ker T_{\theta g} = 0$ or $\ker T_{\overline{\theta g}} = 0$, it follows from (6.5) that $f_+ = 0$. Therefore, in this case, $\ker T_g$ is infinite-dimensional and $\ker T_{\theta g} = \{0\}$.

Condition (6.1) implies a certain "lower bound" for T-kernels not to be annihilated by an inner function θ. We have the following.

Theorem 6.4. *Let $g \in L_\infty$ and $\theta \in H^+_\infty$ be an inner function. Suppose that $\ker T_{\theta g} \neq \{0\}$, and let φ_+ be a maximal function for $\ker T_{\theta g}$. Then, for any $z_0 \in \mathbb{C}^+$ and any $h_- \in \mathcal{G}H^-_\infty$,*

$$\ker T_g \supset (h_-\lambda_{z_0}\varphi_+K_\theta \cap H^+_p) \oplus \ker T_{\theta g} \tag{6.6}$$

where $\lambda_{z_0}(\xi) = \xi - z_0$.

Proof. We have $K_\theta = \theta\,\overline{K_\theta}$ with $\overline{K_\theta} \subset \widetilde{\mathcal{N}}_p$ and we also have $h_-, \lambda_{z_0} \in \widetilde{\mathcal{N}}_p$. Thus if $\varphi_+ \in \ker T_{\theta g}$, which is equivalent to $\theta\varphi_+ \in \ker T_g$, it follows that $h_-\lambda_{z_0}\bar{k}_+\theta\varphi_+ \in \ker T_g$ for all $k_+ \in K_\theta$ such that the left-hand side of this relation represents a function in H_p^+. Thus $(h_-\lambda_{z_0}\varphi_+ K_\theta \cap H_p^+) \subset \ker T_g$. Clearly, we also have $\ker T_{\theta g} \subset \ker T_g$. Moreover, as we show next,

$$h_-\lambda_{z_0}\varphi_+ K_\theta \cap \ker T_{\theta g} = \{0\}. \tag{6.7}$$

To prove this, we start by remarking that $\ker T_{\theta g} = \ker T_{h_-^{-1}\theta g}$. Now assume that $\mathcal{K}_{\min}(\varphi_+) = \ker T_{\theta g}$ and $\varphi_+ = I_+O_+$ is an inner-outer factorisation; let moreover $\psi_+ = h_-\lambda_{z_0}\varphi_+ k_+$, with $k_+ \in K_\theta$, be a function in H_p^+. Then

$$\psi_+ \in \ker T_{\theta g} \Leftrightarrow \psi_+ \in \ker T_{\overline{I_+O_+}/O_+} = \ker T_{h_-^{-1}\overline{I_+O_+}/O_+}$$

$$\Leftrightarrow \lambda_{z_0} k_+ \overline{O_+} = \psi_- \in H_p^-.$$

Therefore we have $k_+ = \frac{\psi_-}{\overline{O_+}\lambda_{z_0}} \in \overline{\mathcal{N}}_+ \cap L_p = H_p^-$ and, since $k_+ \in H_p^+$, it follows that $k_+ = 0$. Thus

$$(h_-\lambda_{z_0}\varphi_+ K_\theta \cap H_p^+) \cap \ker T_{\theta g} = h_-\lambda_{z_0}\varphi_+ K_\theta \cap \ker T_{\theta g} = \{0\}.$$

□

Remark 6.5. *Let $h_- = 1$, $z_0^+ = i$ (so that $\lambda_{z_0^+} = \lambda_-$) and let $f = \lambda_-\varphi_+$, $\mathcal{K} = \operatorname{span}\{P_\theta(\lambda_+^{-1} r^k) : k \in \mathbb{Z}_0^+\}$. The previous result implies that whenever $\ker T_{\theta g} \neq \{0\}$ we must have*

$$\ker T_g \supset f\mathcal{K} \oplus \ker T_{\theta g}, \tag{6.8}$$

where $f \neq 0$ and $\mathcal{K}$ is dense in K_θ.

Moreover, with the same assumptions as in Theorem 6.4:

Corollary 6.6. *If $h_-\lambda_{z_0}\varphi_+ K_\theta \subset H_p^+$ then, for $f = h_-\lambda_{z_0}\varphi_+$, we have $\ker T_g \supset fK_\theta \oplus \ker T_{\theta g}$.*

In particular, if $\theta g = \bar{\theta}_1$, then $\ker T_{\theta g}$ is a model space K_{θ_1}, and $\ker T_g = K_{\theta\theta_1}$. Choosing for K_{θ_1} a maximal function $\varphi_+^{\theta_1}$ such that $\theta_1 = h_-\lambda_{z_0}\varphi_+^{\theta_1}$ as in (5.9), we see from (5.10) that the inclusion in Corollary 6.6 becomes an equality in this case.
Another case in which the inclusions of Theorem 6.4 and Corollary 6.6 can also be replaced by equalities is the one that we study below.

We start by remarking that, in the case of an infinite-dimensional $\ker T_g$, it follows from Theorem 6.2 that, if θ is a finite Blaschke product, then $\ker T_{\theta g}$ is an infinite-dimensional proper subspace of $\ker T_g$. Thus it is not possible to relate their dimensions as in Theorem 6.2 for finite-dimensional T-kernels. We can, however, present an alternative relation which not only generalises Theorem 6.2 but moreover sheds new light on the meaning of (6.3) when $k < \dim\ker T_g < \infty$.

Let $r_z(\xi) := \frac{\xi - z}{\xi - \bar{z}}$ and let

$$B = B_1 \cdot B_2 \cdots B_n$$

with $B_j = r_{z_j}^{k_j}$, $j = 1, 2, \dots n$, and $k_j \in \mathbb{N}$, $z_j \in \mathbb{C}^+$ for each $j = 1, 2, \dots n$. Let moreover

$$k = \sum_{j=1}^{n} k_j.$$

With this notation, we have the following.

Theorem 6.7. *Let* $g \in L_\infty$. *If* $\dim\ker T_g \le k$, *then* $\ker T_{Bg} = \{0\}$; *if* $\dim\ker T_g > k$, *then*

$$\ker T_g = \ker T_{Bg} \oplus \lambda_{z_1}\varphi_+ K_B \tag{6.9}$$

where

$$\lambda_{z_1}(\xi) := \xi - z_1 \tag{6.10}$$

and φ_+ *is a maximal function for* $\ker T_{Bg}$, *i.e.*,

$$\mathcal{K}_{\min}(\varphi_+) = \ker T_{Bg}. \tag{6.11}$$

Proof. If $\dim\ker T_g > k$, then $\ker T_{Bg} \neq \{0\}$ by Theorem 6.2; let φ_+ be a maximal function for $\ker T_{Bg}$. Since, for any inner function $\theta \in H_\infty^+$, $\mathcal{K}_{\min}(\varphi_+) = \ker T_{\theta g}$ implies that $\tilde{\theta}\varphi_+ \notin \ker T_{\theta g}$ whenever $\tilde{\theta}$ is a non-constant inner function, we have that

$$\widetilde{B}\varphi_+ \in \ker T_g \setminus \ker T_{Bg} \quad \text{if } \widetilde{B} \preceq B,\ \widetilde{B} \notin \mathbb{C}. \tag{6.12}$$

Let us define, for $g \in L_\infty$,

$$(\ker T_g)_- := g \ker T_g \subset H_p^-. \tag{6.13}$$

It is easy to see that $(\ker T_g)_-$ is nearly α_+-invariant for all $\alpha_+ \in H_\infty^+$, in the sense that

$$\alpha_+(\ker T_g)_- \cap H_p^- \subset (\ker T_g)_-. \tag{6.14}$$

Let moreover

$$\varphi_- = gB\varphi_+. \tag{6.15}$$

It is clear that φ_- cannot have a non-constant inner factor (in H_∞^-), i.e., φ_- is an outer function in H_p^-; otherwise there would be some non-constant inner function $\theta \in H_\infty^+$ such that $\varphi_- = \bar{\theta}\tilde{\varphi}_-$ with $\tilde{\varphi}_- \in H_p^-$, and it would follow from (6.15) that $\varphi_+ \in \ker T_{\theta Bg} \subsetneq \ker T_{Bg}$, contradicting (6.11). Therefore,

$$\varphi_-(\bar{z}_j) \neq 0, \quad \text{for all } j = 1, 2, \ldots, n. \tag{6.16}$$

From (6.15), (6.16) and (6.12) we also see that not only

$$\varphi_- \in (\ker T_g)_- \setminus \overline{B}H_p^- \tag{6.17}$$

but also

$$\bar{\beta}\varphi_- \in (\ker T_g)_- \setminus \overline{B}H_p^- \quad \text{if } \beta \prec B, \tag{6.18}$$

where β is an inner function.
Let now ψ_- be any element of $(\ker T_g)_-$. We have

$$\psi_- - \frac{\psi_-(\bar{z}_1)}{\varphi_-(\bar{z}_1)}\varphi_- = r_{z_1}^{-1}\tilde{\psi}_{1-} \in (\ker T_g)_- \tag{6.19}$$

where, by (6.14), $\tilde{\psi}_{1-} \in (\ker T_g)_-$. Repeating the same reasoning k_1 times, we get (for some constants $a_0, a_1, \ldots, a_{k_1-1}$),

$$\begin{aligned} \psi_- &= (a_0 + a_1 r_{z_1}^{-1} + \ldots + a_{k_1-1} r_{z_1}^{-(k_1-1)})\varphi_- + \overline{B}_1\psi_{1-} \\ &= p_{z_1}^-\varphi_- + \overline{B}_1\psi_{1-} \end{aligned} \tag{6.20}$$

where $p_{z_1}^-\varphi_- \in (\ker T_g)_- \setminus \overline{B}H_p^-$ by (6.18), $\overline{B}_1\psi_{1-} \in (\ker T_g)_-$ and $\psi_{1-} \in (\ker T_g)_-$ by (6.14).
Analogously, for some constants $b_0, b_1, \ldots, b_{k_2-1}$, we have

$$\begin{aligned} \psi_- &= (b_0 + b_1 r_{z_2}^{-1} + \ldots + b_{k_2-1} r_{z_2}^{-(k_2-1)})\varphi_- + \overline{B}_2\psi_{2-} \\ &= p_{z_2}^-\varphi_- + \overline{B}_2\psi_{2-} \end{aligned} \tag{6.21}$$

and substituting in (6.20) we obtain

$$\psi_- = (p_{z_1}^- + \overline{B}_1 p_{z_2}^-)\varphi_- + \overline{B}_1\overline{B}_2\psi_{2-} \tag{6.22}$$

with

$$(p_{z_1}^- + \overline{B}_1 p_{z_2}^-)\varphi_- \in (\ker T_g)_- \setminus \overline{B}H_p^- \tag{6.23}$$

$$\overline{B}_1\overline{B}_2\psi_{2-} \in (\ker T_g)_-. \tag{6.24}$$

Assuming, for simplicity, that $n = 2$, (6.24) is equivalent to

$$\overline{B}\psi_{2-} \in (\ker T_g)_- \cap \overline{B}H_p^-. \tag{6.25}$$

Since

$$(p_{z_1}^- + \overline{B}_1 p_{z_2}^-)\varphi_- \in \lambda_{z_1}\varphi_-\overline{K}_B = \lambda_{z_1}\varphi_-\overline{B}K_B$$

and

$$(\ker T_g)_- \cap \overline{B}H_p^- = \overline{B}(\ker T_{Bg})_-,$$

it follows from (6.22), (6.23) and (6.25) that

$$(\ker T_g)_- = \overline{B}(\ker T_{Bg})_- \oplus \lambda_{z_1}\varphi_-\overline{B}K_B.$$

Therefore

$$g^{-1}(\ker T_g)_- = B^{-1}g^{-1}(\ker T_{Bg})_- \oplus (B^{-1}g^{-1}\varphi_-)\lambda_{z_1}K_B$$

$$\Leftrightarrow \ker T_g = \ker T_{Bg} \oplus \varphi_+\lambda_{z_1}K_B.$$

□

Remark 6.8. *It is not difficult to see, using the n. η-invariance of T-kernels for $\eta \in H_\infty^-$, that the decomposition* (6.9) *still holds if we replace $\lambda_{z_1}\varphi_+K_B$ by $h_-\lambda_{z_1}\varphi_+K_B$, for any $h_- \in H_\infty^-$ such that the latter is contained in H_p^+, as happens in* (5.10) *for model spaces. For $p = 2$, we may ask whether, by choosing appropriate functions φ_+ and h_- as in* (5.10)*, we can make the direct sum in* (6.9) *orthogonal.*

Theorem 5.4 implies that if φ_+ is a maximal function for a model space $K_{\theta_1} = \ker T_{\overline{\theta}_1}$, then $\theta\varphi_+$ is a maximal function for the model space $K_{\theta\theta_1} = \ker T_{\overline{\theta\theta_1}}$ (where θ is any inner function). As a consequence of Theorem 6.7 we can now generalise this result, when θ is a finite Blaschke product, to any T-kernel.

Theorem 6.9. *Let B be a finite Blaschke product and let $g \in L_\infty$. If φ_+ is a maximal function for $\ker T_g$, then $B\varphi_+$ is a maximal function for $\ker T_{\overline{B}g}$.*

Proof. Assume that B is a (non-constant) finite Blaschke product and let z_1 be one of its zeroes. Assume moreover that φ_+ is such that

$$\mathcal{K}_{\min}(\varphi_+) = \ker T_g$$

and let $\varphi_+ = I_+ O_+$ be an inner-outer factorisation. Then, by (5.2),

$$\ker T_g = \ker T_{\overline{I}_+ \overline{O}_+ / O_+} \quad \text{and} \quad \mathcal{K}_{\min}(B\varphi_+) = \ker T_{\overline{B}\,\overline{I}_+ \overline{O}_+ / O_+}.$$

So, from Theorem 6.7,

$$\begin{aligned} \mathcal{K}_{\min}(B\varphi_+) &= \ker T_{\overline{I}_+ \overline{O}_+ / O_+} \oplus \lambda_{z_1} \varphi_+ K_B \\ &= \ker T_g \oplus \lambda_{z_1} \varphi_+ K_B = \ker T_{\overline{B} g}. \end{aligned}$$

□

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