Recurrence relations for Clifford algebra-valued orthogonal polynomials

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Abstract

The theory of orthogonal polynomials of one real or complex variable is well established as well as its generalization for the multidimensional case. Hypercomplex function theory (or Clifford analysis) provides an alternative approach to deal with higher dimensions. In this context, we study systems of orthogonal polynomials of a hypercomplex variable with values in a Clifford algebra and prove some of their properties.

Key words: Clifford analysis; generalized Appell polynomials; recurrence relations; special functions

1 Introduction

Hypercomplex function theory (or Clifford analysis) generalize to higher dimensions the theory of holomorphic functions of one complex variable by using Clifford algebras. In this framework the analogue of holomorphic functions is obtained as null-solutions to a generalized Cauchy-Riemann system and are called monogenic (or hyperholomorphic) functions. During the last decade, several authors considered the problem of constructing Clifford algebra-valued orthogonal polynomials (cf. [3–5, 13, 14]). In particular, the paper [14] constructs orthogonal bases of polynomials in the space of square integrable monogenic functions in the unit ball of $\mathbb{R}^{n+1}$. The resulting polynomial systems are obtained via the algebraic approach of Gelfand-Tsetlin bases and generalize the results already obtained in [13] for the case $n = 2$. The construction process relies on building blocks that are non-monogenic Clifford algebra-valued polynomials. By using the matrix approach followed in [7] we establish recurrence relations for those building blocks and we show that they can be obtained as well from the Appell sequence constructed in [10, 15] by a simple shift of their coefficients.

2 Basic notions

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ with a non-commutative product according to the multiplication rules

$$e_k e_l + e_l e_k = -2\delta_{kl}, \quad k, l = 1, \ldots, n,$$

where δ_{kl} is the Kronecker symbol. The associative 2^n-dimensional Clifford algebra $\mathcal{C}\ell_{0,n}$ over $\mathbb{R}$ is the set of numbers of the form

$$a = \sum_A a_A e_A, \quad a_A \in \mathbb{R},$$

with $A \subseteq \{1, \cdots, n\}$, $e_A = e_{l_1} e_{l_2} \cdots e_{l_r}$, where $1 \leq l_1 < \cdots < l_r \leq n$ and $e_\emptyset =: e_0 =: 1$. The conjugate of a is given by $\bar{a} = \sum_A a_A \bar{e}_A$, where $\bar{e}_A = \bar{e}_{l_r} \bar{e}_{l_{r-1}} \cdots \bar{e}_{l_1}$, with $\bar{e}_j = -e_j$, $j = l_1, \ldots, l_r$.

Let $\mathbb{R}^{n+1}$ be embedded in $\mathcal{C}\ell_{0,n}$ by identifying $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ with

$$x = x_0 + \underline{x} \in \mathcal{A}_n := \operatorname{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset \mathcal{C}\ell_{0,n},$$

where $\underline{x} = x_1 e_1 + \ldots + x_n e_n$ is called a vector and the elements of $\mathcal{A}_n$ are the paravectors.

The generalized Cauchy-Riemann operator and its conjugate are given, respectively, by

$$\overline{\partial} := \frac{1}{2}(\partial_0 + \partial_{\underline{x}}) \quad \text{and} \quad \partial := \frac{1}{2}(\partial_0 - \partial_{\underline{x}}),$$

where

$$\partial_0 := \frac{\partial}{\partial x_0} \quad \text{and} \quad \partial_{\underline{x}} := e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}.$$

The analogue of the class of holomorphic functions is now formed by $\mathcal{C}^1$-functions f that satisfy the equation $\overline{\partial} f = 0$ (resp. $f \overline{\partial} = 0$) and they are called (left) *monogenic* (resp. right monogenic) (also called hyperholomorphic or Clifford holomorphic). For more details, see [2, 11].

Let f be a monogenic function that is hypercomplex-differentiable in some domain $\Omega \subset \mathbb{R}^{n+1}$ in the sense of [12]. Then f is real-differentiable and its (hypercomplex) derivative is given by $f' = \partial f$ in Ω. Moreover, let f be in the Hilbert module of monogenic square integrable Clifford algebra-valued functions with the following Clifford algebra-valued inner product

$$(f, g)_{\mathcal{C}\ell_{0,n}} = \int_{B^{n+1}} \bar{f}\, g \, d\lambda^{n+1}, \tag{1}$$

where λ^{n+1} is the Lebesgue measure and B^{n+1} is the unit ball in $\mathbb{R}^{n+1}$.

3 Clifford algebra-valued orthogonal polynomials

The importance of homogeneous monogenic polynomials and their applications is already visible in [2]. However, in its sequel [9] the authors devoted a entire chapter to the construction of monogenic homogeneous polynomials that are orthogonal with respect to the inner product (1).

We follow that construction considering different normalization constants in order to obtain some simplification and better accordance between formulae. More explicitly, for each $k \in \mathbb{N}_0$ and for arbitrarily fixed $\mathcal{C}\ell_{0,n}$-valued monogenic polynomials $P_j(\underline{x})$ $(j = 0, \dots, k)$, we apply the Cauchy-Kovalevskaya extension to the polynomials $c_{k,j}(n)\binom{k}{j}\underline{x}^{k-j}\, P_j(\underline{x})$. The normalized constants $c_{k,j}(n)$ are defined by

$$c_{k,j}(n) = \begin{cases} \dfrac{(k-j)!!(n+2j-2)!!}{(n+k+j-1)!!}, & \text{if } k,j \text{ have different parities} \\ c_{k-1,j}(n), & \text{if } k,j \text{ have the same parity} \end{cases}, \tag{2}$$

for $k \geq 1$, $j = 0, \dots, k$ and $c_{0,0}(n) = 1$.

For each j $(j = 0, \dots, k)$, the resulting homogeneous monogenic polynomial

$$\widetilde{X}^{(k)}_{n+1,j}(x) := X^{(k-j)}_{n+1,j}(x)\, P_j(\underline{x}), \quad x \in \mathcal{A}_n \tag{3}$$

is a product of a non-monogenic (in general) homogeneous polynomial $X^{(k-j)}_{n+1,j}$ of degree $k-j$, by the original fixed monogenic polynomial P_j. The polynomials $X^{(k-j)}_{n+1,j}$ $(j = 0, \dots, k)$ can be represented by

$$X^{(k-j)}_{n+1,j}(x) = F^{(k-j)}_{n+1,j}(x) + \frac{j+1}{n+2j} F^{(k-j-1)}_{n+1,j-1}(x)\, \underline{x}, \tag{4}$$

where

$$F^{(k-j)}_{n+1,j}(x) = \frac{(j+1)_{k-j}}{(n-1+2j)_{k-j}} |x|^{k-j} C^{\frac{n-1}{2}+j}_{k-j}\left(\frac{x_0}{|x|}\right),$$

with $F^{(-1)}_{n+1,k+1} \equiv 0$, $x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}$, $|.|$ is the usual Euclidean norm in $\mathbb{R}^{n+1}$, $(\mu)_m = \mu(\mu+1)(\mu+2)\dots(\mu+m-1)$, and C^{ν}_m is the Gegenbauer polynomial of degree m and parameter $\nu \neq 0$.

These polynomials were used as building blocks of orthogonal bases for the space of homogeneous monogenic polynomials of degree k (cf. [14]). Indeed, based on the algebraic concept of Gelfand-Tsetlin bases, the construction is recursive and starts with an orthogonal basis in $\mathbb{R}^2$ with values in the Clifford algebra $\mathcal{C}\ell_{0,2}$. The resulting basis is formed by the polynomials

$$f_{k,\mu} = X^{(k-k_n)}_{n+1,k_n} X^{(k_n-k_{n-1})}_{n,k_{n-1}} \cdots X^{(k_3-k_2)}_{3,k_2} \zeta^{k_2},$$

where $\zeta := x_1 - x_2\, e_1 e_2$ and μ is an arbitrary sequence of integers $(k_{n+1}, k_n, \dots, k_3, k_2)$ such that $k = k_{n+1} \geq k_n \geq \dots \geq k_3 \geq k_2 \geq 0$.

4 The structural Appell sequence and recurrence relations

The classical concept of Appell sequences [1] was generalized in [10,15] to the hypercomplex case as sequences of homogeneous monogenic polynomials $(\mathcal{F}_k)_{k\geq 0}$ of exact degree k such that $\partial \mathcal{F}_k = k\,\mathcal{F}_{k-1},\ k=1,2,\ldots$. In those papers, the authors constructed the generalized Appell sequence $(\mathcal{P}_k^n)_{k\in\mathbb{N}_0}$, where the polynomials are given by

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} \binom{k}{s} c_{s,0}(n)\, x_0^{k-s}\, \underline{x}^s,$$

with $c_{s,0}(n)$ $(s=0,\ldots,k)$ the coefficients (2).

It is worth to notice that $\mathcal{P}_k^n \equiv X_{n+1,0}^{(k)}$ for all $k\in\mathbb{N}_0$. For $j=1,\ldots,k$ the sequence of non-monogenic polynomials $\left(X_{n+1,j}{}^{(k-j)}\right)_{k\in\mathbb{N}}$ can be generated from the monogenic Appell sequence $(\mathcal{P}_k^n)_{k\in\mathbb{N}}$ by a simply shift of their coefficients. Indeed (see [8]),

Theorem 4.1 *For all $k\in\mathbb{N}_0$ and each fixed j $(j=0,\ldots,k)$, it holds*

$$X_{n+1,j}^{(k-j)}(x) = \binom{k}{j}\mathcal{P}_{k-j}^{n+2j}(x), \quad x\in\mathcal{A}_n.$$

As a consequence of this result, the polynomials $X_{n+1,j}^{(k-j)}$ $(j=0,\ldots,k)$, explicitly given by (4), admit the simple representation

$$X_{n+1,j}^{(k-j)}(x) = \binom{k}{j}\sum_{s=0}^{k-j}\binom{k-j}{s} c_{s,0}(n+2j)\, x_0^{k-j-s}\,\underline{x}^s, \quad x\in\mathcal{A}_n.$$

From the general theory of orthogonal polynomials it is well-known that orthogonal polynomials satisfy a three-term recurrence. It is then natural to ask if we have a similar result for the orthogonal monogenic sequence $\left(\widetilde{X}_{n+1,j}^{(k)} : j=0,\ldots,k\right)_{k\in\mathbb{N}_0}$. The matrix approach followed in [6] was advantageously used to obtain matrix recurrences for the sequence $\left(X_{n+1,j}{}^{(k-j)}\right)_{k\in\mathbb{N}}$ for each $j=0,\ldots,k$ (see [7]). Combining this results with Theorem 4.1, a three-term type recurrence can be derived (see [8]):

Theorem 4.2 *For all $k\in\mathbb{N}_0$ and each fixed j $(j=0,\ldots,k)$, the monogenic polynomials $\widetilde{X}_{n+1,j}^{(k-j)}(x),\ x\in\mathcal{A}_n$ satisfy the three-term type recurrence*

$$\begin{aligned}(n+k+1+j)(k+2-j)\widetilde{X}_{n+1,j}^{(k+2)} - \left[(n+2k+2)x_0+\underline{x}\right](k+2)\widetilde{X}_{n+1,j}^{(k+1)}&\\ +(k+2)(k+1)\,|x|^2\,\widetilde{X}_{n+1,j}^{(k)} &= 0,\end{aligned}$$

$$\widetilde{X}_{n+1,j}^{(j)} = P_j(\underline{x}),\ \ \widetilde{X}_{n+1,j}^{(j+1)} = (j+1)(x_0+\frac{1}{n+2j}\underline{x})P_j(\underline{x}).$$

Acknowledgements

This work was supported by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications of the University of Aveiro, the Research Centre of Mathematics of the University of Minho and the Portuguese Foundation for Science and Technology ("FCT - Fundação para a Ciência e a Tecnologia"), within projects PEst-OE/MAT/UI4106/2014 and PEstOE/MAT/UI0013/2014.

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