The inverse along a product and its applications

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Abstract

In this paper, we study the recently defined notion of the inverse along an element. An existence criterion for the inverse along a product is given in a ring. As applications, we present the equivalent conditions for the existence and expressions of the inverse along a matrix.

Keywords:
Von Neumann regularity, Inverse along an element, Green's relations, Matrices over a ring
2010 MSC: 15A09, 16E50

1. Introduction

In this paper, R is an associative ring with unity 1. An element $a \in R$ is (von Neumann) regular if there exists $x \in R$ such that $axa = a$. Such x, an inner inverse of a, is denoted by a^-. We call b an outer inverse of a provided that $bab = b$. If b is both an inner and an outer inverse of a, then it is a reflexive inverse of a, and is denoted by a^+.

Given a semigroup S, S^1 denotes the monoid generated by S. Following Green [1], Green's preorders and relations in a semigroup are defined by

$a \leq_{\mathcal{L}} b \Leftrightarrow S^1a \subset S^1b \Leftrightarrow$ there exists $x \in S^1$ such that $a = xb$.

$a \leq_{\mathcal{R}} b \Leftrightarrow aS^1 \subset bS^1 \Leftrightarrow$ there exists $x \in S^1$ such that $a = bx$.

$a \leq_{\mathcal{H}} b \Leftrightarrow a \leq_{\mathcal{L}} b$ and $a \leq_{\mathcal{R}} b$.

$a\mathcal{L}b \Leftrightarrow S^1a = S^1b \Leftrightarrow$ there exist $x, y \in S^1$ such that $a = xb$ and $b = ya$.

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$a\mathcal{R}b \Leftrightarrow aS^1 = bS^1 \Leftrightarrow$ there exist $x, y \in S^1$ such that $a = bx$ and $b = ay$.
$a\mathcal{H}b \Leftrightarrow a\mathcal{L}b$ and $a\mathcal{R}b$.

Recently, Mary [4] introduced the notion of the inverse along an element that is based on Green's relation in a semigroup S. Given $a, d \in S$, an element $a \in S$ is invertible along d [4] if there exists b such that $dab = d = bad$ and $b \leq_{\mathcal{H}} d$. The element b above is unique if it exists, and is denoted by $a^{\|d}$. Recall that $a^{\|d}$ exists implies that d is regular. Later, Mary and Patrício [5] proved that a is invertible along d if and only if $d\mathcal{H}dad$, which gave a new existence criterion for the inverse along an element. Further, given a regular element d, they [5, 6] characterized the existence of $a^{\|d}$ by means of a unit and d^- in a ring. Moreover, the representation of $a^{\|d}$ is given. As applications, they [6] derived the equivalent conditions for the existence and the formula of the inverse along a regular lower triangular matrix. More results on the inverse along an element can be found in mathematical literature [3, 9].

Motivated by papers [5, 6], we investigate the inverse along a product pmq (m is regular) in a ring, extending the results in [5, 6]. As applications, the inverse along a regular matrix $\begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix}$ is given under some conditions.

2. The inverse along a product *pmq*

In this section, we begin with some lemmas which play important roles in the sequel.

Lemma 2.1. *Given $a, b \in R$, then $1 + ab$ is invertible if and only if $1 + ba$ is invertible. Moreover, $(1 + ba)^{-1} = 1 - b(1 + ab)^{-1}a$.*

Lemma 2.1 is known as the Jacobson's Lemma (see e.g. [2]).

Lemma 2.2. ([8, Theorem 1]) *Let R be a ring and e an idempotent in R. Then $exe + 1 - e$ is invertible in R if and only if exe is invertible in eRe.*

The next theorem, a main result of this paper, gives an existence criterion of the inverse along a product pmq in a ring.

Theorem 2.3. *Let $p, a, q, m \in R$ with m regular. If $m \leq_{\mathcal{L}} pm$ and $m \leq_{\mathcal{R}} mq$, then the following conditions are equivalent*

(i) *a is invertible along pmq.*

(ii) *$u = mqap + 1 - mm^-$ is invertible.*

(iii) $v = qapm + 1 - m^{-}m$ *is invertible.*

In this case,
$$a^{\| pmq} = pu^{-1}mq = pmv^{-1}q.$$

PROOF. It follows from Lemma 2.1 that (ii)$\Leftrightarrow$(iii). Next, it is sufficient to prove (i)$\Leftrightarrow$(ii).

(i)$\Rightarrow$(ii) Suppose that a is invertible along pmq. From $m \leq_{\mathcal{L}} pm$ and $m \leq_{\mathcal{R}} mq$, then there exist p' and q' such that $p'pm = m = mqq'$. In view of [5, Theorem 2.2], we know that a is invertible along pmq if and only if $pmq\mathcal{H}pmqapmq$. There are $x, y \in R$ such that

$$pmq = xpmqapmq = pmqapmqy. \tag{1}$$

Multiplying the above equation (1) by p' on the left yields

$$mq = mqapmqy.$$

Multiplying the above equation (1) by q' on the right yields

$$pm = xpmqapm.$$

Hence,

$$mqapmm^{-}(mqyq'm^{-}mm^{-}) = mm^{-} = (mm^{-}p'xpmm^{-})mqapmm^{-}.$$

The equalities above show that $mqapmm^{-}$ is invertible in $mm^{-}Rmm^{-}$. By Lemma 2.2, $mqapmm^{-} + 1 - mm^{-}$ is invertible in R. Again, Lemma 2.1 ensures that $u = mqap + 1 - mm^{-}$ is invertible.

(ii)$\Rightarrow$(i) Suppose that u, therefore v are invertible. From $um = mv = mqapm$, it follows that $pmq = pu^{-1}mqapmq = pmqapmv^{-1}q$ and $pu^{-1}mq = pmv^{-1}q$. Pose $b = pu^{-1}mq = pmv^{-1}q$, then $b \leq_{\mathcal{H}} pmq$ since $pu^{-1}mq = pu^{-1}p'pmq = pmqq'v^{-1}q$.

Hence, a is invertible along pmq. Moreover,

$$a^{\| pmq} = pu^{-1}mq = pmv^{-1}q.$$

The proof is completed. $\square$

If p is left invertible and q is right invertible, then $m\mathcal{L}pm$ and $m\mathcal{R}mq$. As a special result of Theorem 2.3, we have the following corollary.

Corollary 2.4. *Let $p, a, q, m \in R$ with m regular. If p is left invertible and q is right invertible, then the following conditions are equivalent*

(i) *a is invertible along pmq.*

(ii) *$u = mqap + 1 - mm^-$ is invertible.*

(iii) *$v = qapm + 1 - m^-m$ is invertible.*

In this case,

$$a^{\| pmq} = pu^{-1}mq = pmv^{-1}q.$$

Taking $p = q = 1$, we get

Corollary 2.5. ([5, Theorem 3.2] and [6, Theorem 1.3]) *Let m be a regular element of a ring R. Then the following are equivalent*

(i) *a is invertible along m.*

(ii) *$u = ma + 1 - mm^-$ is invertible.*

(iii) *$v = am + 1 - m^-m$ is invertible.*

In this case,

$$a^{\| m} = u^{-1}m = mv^{-1}.$$

3. Applications to the inverse along a matrix

Mary, Patrício [6] gave some equivalent conditions for the existence of the inverse along a regular lower triangular matrix $\begin{bmatrix} d_1 & 0 \\ d_2 & d_4 \end{bmatrix}$ over a Dedekind-finite ring. It would be interesting to find the related existence criteria and formula of the inverse along a regular matrix $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix}$, in the general case.

By $R_{2\times 2}$ we denote the ring of 2×2 matrices over R. Let $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & 0 \end{bmatrix} \in R_{2\times 2}$ and $D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d_2 & 0 \\ d_1 & d_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =: PMQ$. Given a lower triangular matrix $M = \begin{bmatrix} d_2 & 0 \\ d_1 & d_3 \end{bmatrix}$ with d_2 and d_3 regular, Patrício and Puystjens [7] proved that M is regular if and only if $w = (1 - d_3 d_3^+) d_1 (1 - d_2^+ d_2)$ is regular. In this case,

$$MM^- = \begin{bmatrix} d_2 d_2^+ & 0 \\ (1 - ww^-)(1 - d_3 d_3^+) d_1 d_2^+ & d_3 d_3^+ + ww^-(1 - d_3 d_3^+) \end{bmatrix}.$$

Next, we consider the inverse along a regular matrix, whose $(2,2)$ entry is zero.

Theorem 3.1. *Let* $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & 0 \end{bmatrix} \in R_{2\times 2}$ *with* d_2 *and* d_3 *regular. If* $c^{\parallel d_2}$ *exists, then* $A^{\parallel D}$ *exists if and only if* $\xi = \beta - \alpha c^{\parallel d_2} a$ *is invertible.*

In this case, $A^{\parallel D} = \begin{bmatrix} \xi^{-1}(d_1 - \alpha c^{\parallel d_2}) & \xi^{-1} d_3 \\ c^{\parallel d_2}[1 - a\xi^{-1}(d_1 - \alpha c^{\parallel d_2})] & -c^{\parallel d_2} a \xi^{-1} d_3 \end{bmatrix}$, *where*

$$\begin{aligned} \alpha &= d_1 c + d_3 d - (1 - ww^-)(1 - d_3 d_3^+) d_1 d_2^+, \\ \beta &= d_1 a + d_3 b + (1 - ww^-)(1 - d_3 d_3^+), \\ \xi &= \beta - \alpha c^{\parallel d_2} a. \end{aligned}$$

Proof. We have $MAP = \begin{bmatrix} d_2 c & d_2 a \\ d_1 c + d_3 d & d_1 a + d_3 b \end{bmatrix}$. Hence,

$$U = MAP + I - MM^- = \begin{bmatrix} u & d_2 a \\ \alpha & \beta \end{bmatrix}, \text{ where}$$

$$\begin{aligned} u &= d_2 c + 1 - d_2 d_2^+, \\ \alpha &= d_1 c + d_3 d - (1 - ww^-)(1 - d_3 d_3^+) d_1 d_2^+, \\ \beta &= d_1 a + d_3 b + (1 - ww^-)(1 - d_3 d_3^+). \end{aligned}$$

Since $c^{\parallel d_2}$ exists, it follows that $u = d_2 c + 1 - d_2 d_2^+$ is invertible and $c^{\parallel d_2} = u^{-1} d_2$. Using Schur complements we get the factorization

$$U = \begin{bmatrix} 1 & 0 \\ \alpha u^{-1} & 1 \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & \xi \end{bmatrix} \begin{bmatrix} 1 & c^{\parallel d_2} a \\ 0 & 1 \end{bmatrix},$$

where $\xi = \beta - \alpha c^{\parallel d_2} a$. Hence, U is invertible if and only if ξ is invertible.

Note that $U^{-1} = \begin{bmatrix} 1 & -c^{\parallel d_2} a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u^{-1} & 0 \\ 0 & \xi^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\alpha u^{-1} & 1 \end{bmatrix}$. Then

$$A^{\parallel D} = PU^{-1}M = \begin{bmatrix} \xi^{-1}(d_1 - \alpha c^{\parallel d_2}) & \xi^{-1} d_3 \\ c^{\parallel d_2}[1 - a\xi^{-1}(d_1 - \alpha c^{\parallel d_2})] & -c^{\parallel d_2} a \xi^{-1} d_3 \end{bmatrix}.$$

The proof is completed. $\square$

Remark 3.2. In the above Theorem, if c is not invertible along d_2, $A^{\|D}$ may exist. Next, we give an example to illustrate it.

Take $D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and A to be the 2×2 identity matrix over any field. Since 0 is not invertible along 1 then the $(1, 2)$ entry of A is not invertible the $(2, 1)$ entry of D, and yet A is invertible along D since they are both invertible.

Now, suppose that d_4 in the matrix D is regular and set $e = 1 - d_4 d_4^+$, $f = 1 - d_4^+ d_4$ and $s = d_1 - d_3 d_4^+ d_2$. We have the following decomposition

$$D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} = \begin{bmatrix} 1 & d_3 d_4^+ \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s & d_3 f \\ ed_2 & d_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d_4^+ d_2 & 1 \end{bmatrix} =: PMQ.$$

We next discuss the inverse of $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ along a regular matrix D, under certain conditions.

Theorem 3.3. *Let* $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2\times 2}$ *with* d_4 *regular. With the notations above, if* $d_3 f = 0$ *and* $a^{\|s}$ *exists, then* $A^{\|D}$ *exists if and only if* $\xi = \beta - \alpha a^{\|s}(ad_3 d_4^+ + c)$ *is invertible.*

In this case, $A^{\|D} = \begin{bmatrix} x_1 d_2 + x_2 s & x_1 d_4 \\ \xi^{-1}(d_2 - \alpha a^{\|s}) & \xi^{-1} d_4 \end{bmatrix}$, *where*

$$\begin{aligned}
u &= sa + 1 - ss^+, \\
t &= ed_2(1 - s^+ s), \\
\alpha &= d_2 a + d_4 b - (1 - tt^-)ed_2 s^+, \\
\beta &= (d_2 a + d_4 b)d_3 d_4^+ + d_2 c + d_4 d + (1 - tt^-)e, \\
\xi &= \beta - \alpha a^{\|s}(ad_3 d_4^+ + c), \\
x_1 &= [(1 - a^{\|s} a)d_3 d_4^+ - a^{\|s} c]\xi^{-1}, \\
x_2 &= u^{-1} - x_1 \alpha u^{-1}.
\end{aligned}$$

Proof. If $d_3 f = 0$, then $M = \begin{bmatrix} s & 0 \\ ed_2 & d_4 \end{bmatrix}$. Note that the regularity of D is equivalent to the regularity of M. Hence, it follows from [7, Theorem 1] that

$$I - MM^- = \begin{bmatrix} 1 - ss^+ & 0 \\ -(1 - tt^-)ed_2 s^+ & (1 - tt^-)e \end{bmatrix},$$

where $t = ed_2(1 - s^+s)$.

Note that $MQAP = \begin{bmatrix} sa & s(ad_3d_4^+ + c) \\ d_2a + d_4b & (d_2a + d_4b)d_3d_4^+ + d_2c + d_4d \end{bmatrix}$. We have

$$U = MQAP + I - MM^- = \begin{bmatrix} u & s(ad_3d_4^+ + c) \\ \alpha & \beta \end{bmatrix},$$

where

$$\begin{aligned} u &= sa + 1 - ss^+, \\ \alpha &= d_2a + d_4b - (1 - tt^-)ed_2s^+, \\ \beta &= (d_2a + d_4b)d_3d_4^+ + d_2c + d_4d + (1 - tt^-)e. \end{aligned}$$

In this case,

$$U^{-1} = \begin{bmatrix} 1 & -a^{\|s}(ad_3d_4^+ + c) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u^{-1} & 0 \\ 0 & \xi^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\alpha u^{-1} & 1 \end{bmatrix},$$

where $\xi = \beta - \alpha a^{\|s}(ad_3d_4^+ + c)$.

By calculations, $A^{\|D} = PU^{-1}MQ = \begin{bmatrix} x_1d_2 + x_2s & x_1d_4 \\ \xi^{-1}(d_2 - \alpha a^{\|s}) & \xi^{-1}d_4 \end{bmatrix}$, where

$$\begin{aligned} x_1 &= [(1 - a^{\|s}a)d_3d_4^+ - a^{\|s}c]\xi^{-1}, \\ x_2 &= u^{-1} - x_1\alpha u^{-1}. \end{aligned}$$

The proof is completed. □

Remark 3.4. In Theorem 3.3, $A^{\|D}$ may exist and yet $d_3f \neq 0$ and $a^{\|s}$ exists.

Indeed, suppose that $R = \mathbb{Z}/6\mathbb{Z}$ and let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \in R_{2\times 2}$. Note that A is invertible along D, using Corollary 2.5. From $d_4^+ = 2$, we have $f = 1 - d_4^+d_4 = 3$ and $d_3f = 3 \neq 0$. Note that $sa + 1 - ss^- = 5$ is invertible, from which a is invertible along s.

In Theorem 3.3, if d_4 is invertible, then $e = f = 0$. Hence, we have the following corollary.

Corollary 3.5. *Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2\times 2}$ with d_4 invertible. If $a^{\|s}$ exists, then $A^{\|D}$ exists if and only if $\xi = \beta - \alpha a^{\|s}(ad_3d_4^{-1} + c)$ is invertible.*

In this case, $A^{\|D} = \begin{bmatrix} x_1 d_2 + x_2 s & x_1 d_4 \\ \xi^{-1}(d_2 - \alpha a^{\|s}) & \xi^{-1} d_4 \end{bmatrix}$, *where*

$$\begin{aligned}
s &= d_1 - d_3 d_4^{-1} d_2, \\
u &= sa + 1 - ss^+, \\
\alpha &= d_2 a + d_4 b, \\
\beta &= \alpha d_3 d_4^{-1} + d_2 c + d_4 d, \\
\xi &= \beta - \alpha a^{\|s}(a d_3 d_4^{-1} + c), \\
x_1 &= [(1 - a^{\|s} a) d_3 d_4^{-1} - a^{\|s} c]\xi^{-1}, \\
x_2 &= u^{-1} - x_1 \alpha u^{-1}.
\end{aligned}$$

In Theorem 3.3, take $d_3 = 0$, then $s = d_1$. We can get the formula and equivalence for the existence of the inverse along a regular lower triangular matrix obtained in [6].

Corollary 3.6. ([6, Theorem 3.1]) *Let* $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & 0 \\ d_2 & d_4 \end{bmatrix} \in R_{2\times 2}$ *with* d_4 *regular. With the notations above, if* $a^{\|d_1}$ *exists, then* $A^{\|D}$ *exists if and only if* $\xi = \beta - \alpha a^{\|d_1} c$ *is invertible.*

In this case, $A^{\|D} = \begin{bmatrix} a^{\|d_1} & -a^{\|d_1} c \xi^{-1} d_4 \\ \xi^{-1}(d_2 - \alpha a^{\|d_1}) & \xi^{-1} d_4 \end{bmatrix}$, *where*

$$\begin{aligned}
u &= d_1 a + 1 - d_1 d_1^+, \\
t &= e d_2 (1 - d_1^+ d_1), \\
\alpha &= d_2 a + d_4 b - (1 - tt^-) e d_2 d_1^+, \\
\beta &= d_2 c + d_4 d + (1 - tt^-) e, \\
\xi &= \beta - \alpha a^{\|d_1} c.
\end{aligned}$$

By taking $ed_2 = 0$ in Theorem 3.3, we get the following corollary.

Corollary 3.7. *Let* $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2\times 2}$ *with* d_4 *regular. With the notations above, if* $ed_2 = d_3 f = 0$ *and* $a^{\|s}$ *exists, then* $A^{\|D}$ *exists if and only if* $\xi = \beta - \alpha a^{\|s}(a d_3 d_4^+ + c)$ *is invertible.*

In this case, $A^{\|D} = \begin{bmatrix} x_1d_2 + x_2s & x_1d_4 \\ \xi^{-1}(d_2 - \alpha a^{\|s}) & \xi^{-1}d_4 \end{bmatrix}$, *where*

$$\begin{aligned} u &= sa + 1 - ss^+, \\ \alpha &= d_2a + d_4b, \\ \beta &= \alpha d_3d_4^+ + d_2c + d_4d + e, \\ \xi &= \beta - \alpha a^{\|s}(ad_3d_4^+ + c), \\ x_1 &= [(1 - a^{\|s}a)d_3d_4^+ - a^{\|s}c]\xi^{-1}, \\ x_2 &= u^{-1} - x_1\alpha u^{-1}. \end{aligned}$$

Question 3.8. Given a regular matrix D, can we give further equivalent conditions such that $A^{\|D}$ exists without additional conditions ?

ACKNOWLEDGMENTS

The authors are highly grateful to the referee for valuable comments which led to improvements of the paper. In particular, Remarks 3.2 and 3.4 were suggested to the authors by the referee. The first author is grateful to China Scholarship Council for supporting him to purse his further study in University of Minho, Portugal. Pedro Patrício and Yulin Zhang were financed by the Research Centre of Mathematics of the University of Minho with the Portuguese Funds from the "Fundação para a Ciência e a Tecnologia", through the Project PEst-OE/MAT/UI0013/2014. Jianlong Chen and Huihui Zhu were supported by the National Natural Science Foundation of China (No. 11201063 and No. 11371089), the Specialized Research Fund for the Doctoral Program of Higher Education (No. 20120092110020), the Natural Science Foundation of Jiangsu Province (No. BK20141327), the Foundation of Graduate Innovation Program of Jiangsu Province(No. CXLX13-072), the Scientific Research Foundation of Graduate School of Southeast University and the Fundamental Research Funds for the Central Universities (No. 22420135011).

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