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EQUIDISTRIBUTION FOR HIGHER-RANK ABELIAN ACTIONS ON HEISENBERG NILMANIFOLDS

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ABSTRACT. We prove quantitative equidistribution results for actions of Abelian subgroups of the $(2g+1)$-dimensional Heisenberg group acting on compact $(2g+1)$-dimensional homogeneous nilmanifolds. The results are based on the study of the C^∞-cohomology of the action of such groups, on tame estimates of the associated cohomological equations and on a renormalization method initially applied by Forni to surface flows and by Forni and the second author to other parabolic flows. As an application we obtain bounds for finite Theta sums defined by real quadratic forms in g variables, generalizing the classical results of Hardy and Littlewood [25, 26] and the optimal result of Fiedler, Jurkat, and Körner [17] to higher dimension.

CONTENTS

1. INTRODUCTION

In the analysis of the time evolution of a dynamical system many problems reduce to the study of the *cohomological equation*; in the case, for example, of a smooth vector field X on a connected compact manifold M this means finding a function u on M that is a solution of the equation

$$Xu = f, \tag{1.1}$$

where f is a given function on M.

For a detailed discussion of the cohomological equation for flows and transformations in ergodic theory the reader may consult [28].

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In the 2006 paper [14], Forni and the second author used renormalization techniques coupled with the study of the cohomological equation to derive the equidistribution speed of nilflows on Heisenberg three-manifolds. This approach had initially been used by Forni for the study of flows on translation surfaces and subsequently by Forni and the second author [13] for the study of horocycle flows, where precise asymptotics of the equidistribution of these flows were obtained (see also [3]). Renormalization fails for homogeneous flows on higher-step nilmanifolds as, in general, the automorphism group of the underlying nilpotent group is rather poor, lacking semi-simple elements. In a recent paper [16] Forni and the second author developed a novel "rescaling technique" to overcome this difficulty in higher-step nilmanifold; as a consequence they obtained non-trivial estimates on Weyl sums, estimates which have recently been improved independently by Wooley [50].

The present paper moves in a different direction: the study of higher-rank Abelian actions, a theme of research that has been the subject of several investigations, primarily by A. Katok and co-authors (e.g., [29, 35, 11, 34, 11, 33]). In fact, homogeneous actions of Abelian subgroups of higher-dimensional Heisenberg groups provide a setting where renormalization methods may still be applied, yielding precise quantitative estimates of the rate of equidistribution of the orbits once an in-depth analysis of the cohomological equations is carried out. Thus, an important part of this work is devoted to the study of the full cohomology of the actions of these groups; our attention has been focused on obtaining tame estimates for the solutions of cohomological equations with minimal loss of smoothness, a result that has its own interest in view of future applications to the study of some perturbations of these actions.

An immediate consequence of the quantitative estimates of the rate of equidistribution are bounds on exponential sums for quadratic forms in terms of certain diophantine properties of the form. To our knowledge these bounds, which generalise the classical results of Hardy and Littlewood [25, 26] and the optimal result of Fiedler, Jurkat, and Körner [17], are new.

Cohomology in Heisenberg manifolds. In this article we study the cohomology of the action of an abelian subgroup P of the $(2g+1)$-dimensional Heisenberg group H^g on the algebra of smooth functions on a homogeneous manifold H^g/Γ. The linearity of the problem and the fact that the unitary dual of H^g is classical knowledge make the use of harmonic analysis particularly suitable to our goal, as it was the case in the works of L. Flaminio and G. Forni [13, 14, 15]. As a consequence, our results on the cohomology of P also apply to more general H^g-modules, those for which the action of the center of H^g has a spectral gap.

Before stating our results, let us fix some notation.

Let G be a connected Lie group of Lie algebra $\mathfrak{g}$, and let $\mathsf{M} = \mathsf{G}/\Gamma$ be a compact homogeneous space of G. Then G acts by left translations on $C^\infty(\mathsf{M})$ via

$$(h.f)(m) = f(h^{-1}m), \quad h \in \mathsf{G}, \quad f \in C^\infty(\mathsf{M}). \tag{1.2}$$

Let F be a closed G-invariant subspace of $C^\infty(\mathsf{M})$. The space F is a tame graded Fréchet space [24, Def. II.1.3.2] topologized by the family of increasing Sobolev norms $\|\cdot\|_s$, defining L^2 Sobolev spaces $W^s(\mathsf{M})$.

For any connected Lie subgroup $\mathsf{P} < \mathsf{G}$ with Lie algebra $\mathfrak{p}$, the action by translations of P on G/Γ turns F into a $\mathfrak{p}$-module. Therefore we may consider the Chevalley-Eilenberg cochain complex $A^*(\mathfrak{p}, F) := \Lambda^*\mathfrak{p}' \otimes F$ of F-valued alternating forms on $\mathfrak{p}$, endowed with the usual differential "d". By *cohomology of the* $\mathfrak{p}$*-module* F we simply mean the Lie-algebra cohomology $H^*(\mathfrak{p}, F)$ of this cochain complex. When $F = C^\infty(\mathsf{M})$ we also refer to this cohomology as *the cohomology of the action of* P *on* M.

A natural question that arises when we consider a Lie group or Lie algebra cohomology with values in a topological module is whether the *reduced* cohomology coincides with the ordinary cohomology; that is, whether the spaces $B^*(\mathfrak{p}, F)$ of coboundaries are closed in the spaces $Z^*(\mathfrak{p}, F)$ of cocycles. Following A. Katok [27], we give the following definition.

DEFINITION 1.1. The $\mathfrak{p}$-module F is *cohomologically C^∞-stable in degree k* if the space $B^k(\mathfrak{p}, F)$ of F-valued coboundaries of degree k is closed in the C^∞ topology.

Let $Z_k(\mathfrak{p}, F)$ denote the space of closed currents of dimension k, that is, the space of all continuous linear functionals on $A^k(\mathfrak{p}, M)$ vanishing on $B^k(\mathfrak{p}, F)$. By the Hahn-Banach Theorem, $B^k(\mathfrak{p}, F)$ is a closed subspace of $A^k(\mathfrak{p}, F)$ if and only if it is equal to the intersection of the kernels of all $D \in Z_k(\mathfrak{p}, F)$.

We recall that a tame linear map $\phi : F_1 \to F_2$ between tame graded Fréchet spaces satisfies a tame estimate of degree r with base b if, denoting by $\|\cdot\|_s$ the norms defining the grading, we have $\|\phi(f)\|_s \le C\|f\|_{s+r}$ for all $s \ge b$ and $f \in F_1$; the constant C may depends on s.

The tame grading of F implies that $A^*(\mathfrak{p}, F)$ is a tame graded Fréchet cochain complex and that the differentials are tame maps of degree 1. Thus, besides C^∞-stability, another question that arises naturally is whether, for a given a coboundary ω, there exists a primitive Ω whose norm is tamely estimated by the norm of ω.

DEFINITION 1.2. We say that the $\mathfrak{p}$-module F is *tamely cohomologically C^∞-stable in degree $k \ge 1$* if there exists a tame map $\mathrm{d}_{-1} : B^k(\mathfrak{p}, F) \to A^{k-1}(\mathfrak{p}, F)$ assigning to every coboundary $\omega \in B^k(\mathfrak{p}, F)$ a primitive of ω.

A related question, which is fundamental in perturbation theory, is whether the chochain complex $A^k(\mathfrak{p}, F)$ has a tame splitting [24] (see [30, 12]). Recall that a graded Fréchet space F_1 is a tame direct summand of a graded Fréchet space F_2 if there are tame maps $L : F_1 \to F_2$ and $M : F_2 \to F_1$ such that $M \circ L$ is the identity map of F_1 [24, Def. II.1.3.1]. In this situation we also say that the short exact sequence $0 \to F_1 \to F_2 \to F_2/L(F_1) \to 0$ splits tamely.

DEFINITION 1.3. We say that the $\mathfrak{p}$-module F has *tame splitting* in degree k if the space $B^k(\mathfrak{p}, F)$ is a tame direct summand of $A^k(\mathfrak{p}, F)$.

Let H^g be the Heisenberg group of dimension $2g+1$. Any compact homogeneous space $\mathsf{M} = \mathsf{H}^g/\Gamma$ is a circle bundle $p : \mathsf{M} \to \mathsf{H}^g/(\Gamma\,\mathsf{Z}(\mathsf{H}^g))$ over the $2g$-dimensional torus $\mathsf{T}^{2g} = \mathsf{H}^g/(\Gamma\,\mathsf{Z}(\mathsf{H}^g))$, with fibers given by the orbits of the center $\mathsf{Z}(\mathsf{H}^g)$ of H^g. The space of C^∞ functions on M splits as a direct sum of the H^g-invariant subspace $p^*(C^\infty(\mathsf{T}^{2g}))$ and the H^g-invariant subspace $F_0 = C_0^\infty(\mathsf{M})$ formed by the smooth functions on M having zero average on the fibers of the fibration p. The following theorem is a particular case of Theorem 3.16 below.

DEFINITION 1.4. A connected Abelian subgroup of H^g without central elements will be called *an isotropic subgroup of* H^g. A *Legendrian* subgroup of H^g is an isotropic subgroup of H^g of maximal dimension g.

THEOREM 1.5. *Let* P *be a d-dimensional isotropic subgroup of* H^g *with Lie algebra* $\mathfrak{p}$*. The* $\mathfrak{p}$*-module* F_0 *is tamely cohomologically* C^∞*-stable in all degrees. In fact, for all* $k = 1,\dots,d$ *there are linear maps*

$$\mathrm{d}_{-1} : B^k(\mathfrak{p}, F_0) \to A^{k-1}(\mathfrak{p}, F_0)$$

associating to each $\omega \in B^k(\mathfrak{p}, F_0)$ *a primitive of* ω *and satisfying tame estimates of degree* $(k+1)/2+\varepsilon$ *for any* $\varepsilon > 0$.

We have $H^k(\mathfrak{p}, F_0) = 0$ *for* $k < d$*; in degree* d*, we have that* $H^d(\mathfrak{p}, F_0)$ *is infinite-dimensional if* $d < g$ *or one-dimensional if* $d = g$ *(that is, if* $\mathfrak{p}$ *is a Legendrian subspace) in each irreducible* $\mathfrak{p}$*-sub-module of* F_0.

The $\mathfrak{p}$*-module* F_0 *has* tame splitting *in all degrees: for* $k = 0,\dots,d$ *and for any* $\varepsilon > 0$ *there exist a constant* C *and linear maps*

$$M^k : A^k(\mathfrak{p}, F_0) \to B^k(\mathfrak{p}, F_0)$$

such that the restriction of M^k *to* $B^k(\mathfrak{p}, F_0)$ *is the identity map, and the following estimates hold:*

$$\|M^k\omega\|_s \le C\|\omega\|_{s+w}, \quad \forall\ \omega \in A^k(\mathfrak{p}, F_0),$$

where $w = (k+3)/2+\varepsilon$*, if* $k < d$ *and* $w = d/2+\varepsilon$ *if* $k = d$.

Let $\mathsf{P} < \mathsf{H}^g$ be a subgroup as in the theorem above and let $\bar{\mathsf{P}}$ be the group obtained by projecting P on $\mathsf{H}^g/\mathsf{Z}(\mathsf{H}^g) \approx \mathbb{R}^{2g}$. As before we set $\mathsf{T}^{2g} = \mathsf{H}^g/(\Gamma\,\mathsf{Z}(\mathsf{H}^g))$. The P-module $p^*(C^\infty(\mathsf{T}^{2g}))$ is naturally isomorphic to the $\bar{\mathsf{P}}$-module $C^\infty(\mathsf{T}^{2g})$. It should be considered as folklore that the cohomology of the action of a subgroup $\bar{\mathsf{P}}$ on a torus depends on the Diophantine properties of $\bar{\mathsf{P}}$, considered as a vector space. The Diophantine condition $\bar{\mathfrak{p}} \in \mathrm{DC}_\tau(\bar\Gamma)$ mentioned in the theorem below will be made precise in Section 3.1.

THEOREM 1.6. *Let* P *be an isotropic subgroup of* H^g*, let* $\mathsf{M} := \mathsf{H}^g/\Gamma$ *be a compact homogeneous space, and let* $F := C^\infty(\mathsf{M})$*. Let* $\bar{\mathsf{P}}$ *be the projection of* P *into* $\mathsf{H}^g/\mathsf{Z}(\mathsf{H}^g) \approx \mathbb{R}^{2g}$*, let* $\bar{\mathfrak{p}}$ *be its Lie algebra, and let* $\bar\Gamma = \Gamma/(\Gamma\cap\mathsf{Z}(\mathsf{H}^g)) \approx \mathbb{Z}^{2g}$*. Then the action of* P *on* M *is tamely cohomologically* C^∞*-stable and has a tame splitting in all degrees if and only if* $\bar{\mathfrak{p}} \in DC_\tau(\bar\Gamma)$ *for some* $\tau > 0$*. In this case we have*

$$H^k(\mathfrak{p}, F) = \Lambda^k\mathfrak{p} \ \text{if}\ k < \dim\mathfrak{p}, \quad H^k(\mathfrak{p}, F) = \Lambda^k\mathfrak{p} \oplus H^k(\mathfrak{p}, F_0) \ \text{if}\ k = \dim\mathfrak{p}.$$

Equidistribution of isotropic subgroups on Heisenberg manifolds. We denote by $\mathsf{M} = \mathsf{H}^g/\Gamma$ the standard Heisenberg nilmanifold (see Section 2 for details on the definitions and notations). Let $(X_1,\dots,X_g,\Xi_1,\dots,\Xi_g,T)$ be a fixed rational basis of $\mathfrak{h}^g = \mathrm{Lie}(\mathsf{H}^g)$ satisfying the canonical commutation relations. Then the symplectic group $\mathrm{Sp}_{2g}(\mathbb{R})$ acts on H^g by automorphisms[1]. For $1 \le d \le g$, let P^d be the subgroup generated by $(X_1,\dots,X_d)$ and, for any $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$, set $X_i^\alpha := \alpha^{-1}(X_i)$, $1 \le i \le d$. We define a parametrization of the subgroup $\alpha^{-1}(\mathsf{P}^d)$ according to

$$\mathsf{P}_x^{d,\alpha} := \exp(x_1 X_1^\alpha + \cdots + x_d X_d^\alpha), \quad x = (x_1,\dots,x_d) \in \mathbb{R}^d.$$

Given a Jordan region $U \subset \mathbb{R}^d$ and a point $m \in \mathsf{M}$, we define a d-dimensional $\mathfrak{p}$-current $\mathscr{P}_U^{d,\alpha} m$ by

$$\left\langle \mathscr{P}_U^{d,\alpha} m, \omega \right\rangle := \int_U f(\mathsf{P}_x^{d,\alpha} m)\,\mathrm{d}x \tag{1.3}$$

for any degree d $\mathfrak{p}$-form $\omega = f\,\mathrm{d}X_1^\alpha \wedge \cdots \wedge \mathrm{d}X_d^\alpha$, with $f \in C_0^\infty(\mathsf{M})$ (here $C_0^\infty(\mathsf{M})$ denotes the space of smooth functions with zero average along the fibers of the central fibration of M).

It is well-known that the Diophantine properties of a real number may be formulated in terms of the speed of excursion, into the cusp of the modular surface, of a geodesic ray having that number as limit point on the boundary of hyperbolic space. This observation allows us to define the Diophantine properties of the subgroup $\mathsf{P}^{d,\alpha}$ in terms of bounds on the *height* of the projection, in the Siegel modular variety $\Sigma_g = \mathsf{K}_g \backslash \mathrm{Sp}_{2g}(\mathbb{R})/\mathrm{Sp}_{2g}(\mathbb{Z})$, of the orbit of α under the action of some one-parameter semi-group of the Cartan subgroup of $\mathrm{Sp}_{2g}(\mathbb{R})$ (here K_g denotes the maximal compact subgroup of $\mathrm{Sp}_{2g}(\mathbb{R})$). We refer to Section 4.4 for the definition of height function.

Let $\{\exp t\widehat{\delta}(d)\}_{t\in\mathbb{R}}$ be the Cartan subgroup of $\mathrm{Sp}_{2g}(\mathbb{R})$ defined by the formula $\exp(t\widehat{\delta}(d))X_i = e^t X_i$, for $i = 1,\dots,d$ and $\exp(t\widehat{\delta}(d))X_i = X_i$, for $i = d+1,\dots,g$. Roughly, the Definition 4.10 states that $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$ satisfies a $\widehat{\delta}(d)$-Diophantine condition of type σ if the height of the projection of $\exp(-t\widehat{\delta}(d))\alpha$ in the Siegel modular variety Σ_g is bounded by $e^{2td(1-\sigma)}$; if, for any $\varepsilon > 0$, the height considered above is bounded by $e^{2td\varepsilon}$, then we say that $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$ satisfies a $\widehat{\delta}(d)$-Roth condition; finally we say that α is of bounded type if the height of $\exp(-\widehat{\delta})\alpha$ stays bounded as $\widehat{\delta}$ ranges in a positive cone $\mathfrak{a}^+$ in the Cartan algebra of diagonal symplectic matrices (see Definition 4.10).

As the height function is defined on the Siegel modular variety Σ_g, the Diophantine properties of α depend only on its class $[\alpha]$ in the quotient space $\mathfrak{M}_g = \mathrm{Sp}_{2g}(\mathbb{R})/\mathrm{Sp}_{2g}(\mathbb{Z})$.

[1] By acting on the left on the components of elements of $\mathfrak{h}^g$ in the given basis.

The definitions above agree with the usual definitions in the $g = 1$ case. Several authors (Lagarias [36], Dani [10], Kleinbock and Margulis [32], Chevallier [7]) proposed, in different contexts, various generalizations of the $g = 1$ case. We postpone to Remark 4.11 the discussion of these generalizations.

We may now state our main equidistribution result.

THEOREM 1.7. *Let $\mathsf{P}^d < \mathsf{H}^g$ be an isotropic subgroup of dimension $d \le g$. Set $Q(T) = [0,T]^d$. For any $s > \frac{1}{4}d(d+11)+g+1/2$ and any $\varepsilon > 0$ there exists a constant $C = C(\mathsf{P},\alpha,s,g,\varepsilon) > 0$ such that, for all $T \gg 1$ and all test $\mathfrak{p}$-forms $\omega \in \Lambda^d\mathfrak{p}\otimes W^s_0(M)$,*

- *there exists a full measure set $\Omega_g(w_d) \subset \mathfrak{M}_g$ such that if $[\alpha] \in \Omega_g(w_d)$ then*
$$\left|\left\langle \mathscr{P}^{g,\alpha}_{Q(T)} m, \omega\right\rangle\right| \le C(\log T)^{d+1/(2g+2)+\varepsilon}\, T^{d/2}\, \|\omega\|_s;$$
- *if $[\alpha] \in \mathfrak{M}_g$ satisfies a $\widehat{\delta}(d)$-Diophantine condition of exponent $\sigma > 0$ then*
$$\left|\left\langle \mathscr{P}^{d,\alpha}_{Q(T)} m, \omega\right\rangle\right| \le C\, T^{d(1-\sigma'/2)}\, \|\omega\|_s;$$
for all $\sigma' < \sigma$;
- *if $[\alpha] \in \mathfrak{M}_g$ satisfies a $\widehat{\delta}(d)$-Roth condition, then*
$$\left|\left\langle \mathscr{P}^{d,\alpha}_{Q(T)} m, \omega\right\rangle\right| \le C\, T^{d/2+\varepsilon}\, \|\omega\|_s;$$
- *if $[\alpha] \in \mathfrak{M}_g$ is of bounded type, then*
$$\left|\left\langle \mathscr{P}^{d,\alpha}_{Q(T)} m, \omega\right\rangle\right| \le C\, T^{d/2}\, \|\omega\|_s.$$

The exponent of the logarithmic factor in the first case is certainly not optimal. When $d = 1$, a more precise result is stated in Proposition 5.9, which coincides with the optimal classical result for $d = g = 1$ (Fiedler, Jurkat, and Körner [17]).

The method of proof is, to our knowledge, the first generalization of the methods of renormalization of Forni [20] and of Flaminio and Forni [14, 15] to actions of higher dimensional Lie groups. A different direction is the one taken by Flaminio and Forni in [16], where equidistribution of nilflows on higher step nilmanifolds requires a subtler rescaling technique, due to the lack of a renormalization flow.

A drawback of the inductive scheme that we adopted is that we are limited to consider averages on cubes $Q(T)$ (the generalization to pluri-rectangles is however feasible, but more cumbersome to state). For more general regions, growing by homotheties, we can obtain weak estimates where the power $T^{d/2}$ is replaced by T^{d-1}. However, N. Shah's ideas [45] suggest that equidistributions estimates as strong as those stated above are valid for general regions with smooth boundary.

Application to higher-dimensional Theta sums. In their fundamental 1914 paper [25], Hardy and Littlewood introduced a renormalization formula to study the exponential sums $\sum_{n=0}^N e(n^2x/2+\xi n)$, usually called *finite theta sums*, where

$N \in \mathbb{N}$ and $e(t) := \exp(2\pi i t)$. Their algorithm provided optimal bounds for these sums when x is of bounded type.

Since then, Hardy and Littlewood's renormalization method has been applied or improved by several authors obtaining finer estimates on finite theta sums (Berry and Goldberg [4], Coutsias and Kazarinoff [8], Fedotov and Klopp [18]). Optimal estimates have been obtained by Fiedler, Jurkat, and Körner [17]. Differently from the previously quoted authors, who relied heavily on the continued fractions properties of the real number x, Fiedler, Jurkat, and Körner's method was based on an approximation of x by rational numbers with denominators bounded by $4N$.

In this paper we consider the g-dimensional generalization, the finite theta sums

$$\sum_{n \in \mathbb{Z}^g \cap [0,N]^g} e(\mathcal{Q}[n] + \ell(n)), \tag{1.4}$$

where $\mathcal{Q}[x] := x^\top \mathcal{Q} x$ is the quadratic form defined by a symmetric $g \times g$ real matrix $\mathcal{Q}$, and $\ell(x) := \ell^\top x$ is the linear form defined by a vector $\ell \in \mathbb{R}^g$. In the spirit of Flaminio and Forni [14], our method consists of reducing the sum (1.4) to a Birkhoff sum along an orbit (depending on ℓ) of some Legendrian subgroup (depending on $\mathcal{Q}$) in the standard $(2g+1)$-dimensional Heisenberg nilmanifold.

The occurrence of Heisenberg nilmanifolds is not a surprise; in fact the connection between the Heisenberg group and the theta series is well known and very much exploited [1, 2, 48, 14, 42, 43].

The application to g-dimensional finite theta sums (1.4) is the following corollary of Theorem 5.11.

COROLLARY 1.8. *Let $\mathcal{Q}[x] = x^\top \mathcal{Q} x$ be the quadratic form defined by the symmetric $g \times g$ real matrix $\mathcal{Q}$, let $\alpha = \begin{pmatrix} I & 0 \\ \mathcal{Q} & I \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{R})$, and let $\ell(x) = \ell^\top x$ be the linear form defined by $\ell \in \mathbb{R}^g$. Set*

$$\Theta(\mathcal{Q}, \ell; N) := N^{-g/2} \sum_{n \in \mathbb{Z}^g \cap [0,N]^g} e(\mathcal{Q}[n] + \ell(n)).$$

- *There exists a full measure set $\Omega_g \subset \mathfrak{M}_g$ such that if $[\alpha] \in \Omega_g$ and $\varepsilon > 0$ then*
$$\Theta(\mathcal{Q}, \ell; N) = \mathcal{O}\left((\log N)^{g+1/(2g+2)+\varepsilon}\right).$$
- *If $[\alpha] \in \mathfrak{M}_g$ satisfies a $\widehat{\delta}(g)$-Roth condition, then for any $\varepsilon > 0$*
$$\Theta(\mathcal{Q}, \ell; N) = \mathcal{O}\left(N^{\varepsilon}\right).$$
- *If $[\alpha] \in \mathfrak{M}_g$ is of bounded type, then*
$$\Theta(\mathcal{Q}, \ell; N) = \mathcal{O}(1).$$

The Diophantine conditions in terms of the symmetric matrix $\mathcal{Q}$ are discussed in Remark 4.11.

As we mentioned above, dynamical methods have already been used to study the sums $\Theta(\mathcal{Q}, \ell; N)$. Götze and Gordin [21], generalizing [38], show that some smoothings of $\Theta(\mathcal{Q}, \ell; N)$ have a limit distribution. See also Marklof [39, 40].

Geometrical methods, similar to ours, to estimate finite theta sums are also used by Griffin and Marklof [23] and Cellarosi and Marklof [9]. They focus on the distributions of these sums as $\mathcal{Q}$ and ℓ are uniformly distributed in the $g = 1$ case. As they are only interested in theta sums, they may consider a single irreducible representation ρ of the Heisenberg group and a single intertwining operator between ρ and $L^2(\mathsf{M})$. The other more technical difference is that as $\mathcal{Q}$ and ℓ vary, it is more convenient to generalize the ergodic sums (1.3) to the case when ω is transverse current.

Estimates of theta sums are also crucial in the paper of Götze and Margulis [22], which focuses on the finer aspects of the "quantitative Oppenheim conjecture". There, it is a matter of estimating the error terms when counting the number of integer lattice points of given size for which an indefinite irrational quadratic form takes values in a given interval. This is clearly a subtler problem than the one considered here.

Article organization. In Section 2, we introduce the necessary background on the Heisenberg and symplectic groups. In Section 3 we prove the results about the cohomology of isotropic subgroups of the Heisenberg groups. Section 4 deals with the relation between Diophantine properties and dynamics on the Siegel modular variety. Finally, in Section 5 we prove the main equidistribution result and the applications to finite theta sums.

Applications to the rigidity problem of higher-rank Abelian actions on Heisenberg nilmanifolds, as a consequence of the tame estimates for these actions, will be the subject of further works.

2. HEISENBERG GROUP AND SIEGEL SYMPLECTIC GEOMETRY

2.1. The Heisenberg group and the Schrödinger representation.

The Heisenberg group and Lie algebra. Let ω denote the *canonical symplectic form* on $\mathbb{R}^{2g} \approx \mathbb{R}^g \times \mathbb{R}^g$, i.e., the non-degenerate alternate bilinear form $\omega((x,\xi),(x',\xi')) = \xi \cdot x' - \xi' \cdot x$, where we use the notations $(x,\xi) \in \mathbb{R}^g \times \mathbb{R}^g$ and $\xi \cdot x := \xi_1 x_1 + \cdots + \xi_g x_g$. The *Heisenberg group* over $\mathbb{R}^g$ (or the *real* $(2g+1)$-*dimensional Heisenberg group*) is the set $\mathsf{H}^g = \mathbb{R}^g \times \mathbb{R}^g \times \mathbb{R}$ equipped with the product law

$$(x,\xi,t)\cdot(x',\xi',t') = (x+x',\xi+\xi',t+t'+\tfrac{1}{2}\omega((x,\xi),(x',\xi'))) . \tag{2.1}$$

It is a central extension of $\mathbb{R}^{2g}$ by $\mathbb{R}$, as we have an exact sequence

$$0 \to \mathsf{Z}(\mathsf{H}^g) \to \mathsf{H}^g \to \mathbb{R}^{2g} \to 0 ,$$

with $\mathsf{Z}(\mathsf{H}^g) = \{(0,0,t)\} \approx \mathbb{R}$.

The Lie algebra of H^g is the vector space $\mathfrak{h}^g = \mathbb{R}^g \times \mathbb{R}^g \times \mathbb{R}$ equipped with the commutator

$$\big[\,(q,p,t),(q',p',t')\,\big] = (0,0,p\cdot q' - p'\cdot q).$$

Let $T = (0,0,1) \in Z(\mathfrak{h}^g)$. If (X_i) is a basis of $\mathbb{R}^g$, and (Ξ_i) the symplectic dual basis, we obtain a basis (X_i, Ξ_j, T) of $\mathfrak{h}^g$ satisfying the *canonical commutation relations*:

$$[X_i, X_j] = 0, \quad [\Xi_i, \Xi_j] = 0, \quad [\Xi_i, X_j] = \delta_{ij} T, \qquad 1 \le i, j \le g. \tag{2.2}$$

A basis (X_i, Ξ_j, T) of $\mathfrak{h}^g$ satisfying the relations (2.2) will be called a *Heisenberg basis of* $\mathfrak{h}^g$. The Heisenberg basis (X_i^0, Ξ_j^0, T) where X_i^0 and Ξ_j^0 are the standard bases of $\mathbb{R}^g$, will be called the *standard Heisenberg basis*.

Given a Lagrangian subspace $\mathfrak{l} \subset \mathbb{R}^g \times \mathbb{R}^g$, there exists a Heisenberg basis (X_i, Ξ_j, T) such that (X_i) spans $\mathfrak{l}$; in this case the span $\mathfrak{l}' = \langle \Xi_j \rangle$ is also Lagrangian and we say that the basis (X_i, Ξ_j, T) is *adapted to the splitting* $\mathfrak{l} \times \mathfrak{l}' \times Z(\mathfrak{h}^g)$ of $\mathfrak{h}^g$.

Standard lattices and quotients. The set $\Gamma := \mathbb{Z}^g \times \mathbb{Z}^g \times \frac{1}{2}\mathbb{Z}$ is a discrete and co-compact subgroup of the Heisenberg group H^g, which we shall call the *standard lattice* of H^g. The quotient

$$\mathsf{M} := \mathsf{H}^g / \Gamma$$

is a smooth manifold that will be called the *standard Heisenberg nilmanifold*. The natural projection map

$$p \colon \mathsf{M} \to \mathsf{H}^g / (\Gamma \mathsf{Z}(\mathsf{H}^g)) \approx (\mathsf{H}^g / Z(\mathsf{H}^g)) / (\Gamma / \Gamma \cap \mathsf{Z}(\mathsf{H}^g)) \tag{2.3}$$

maps M onto a $2g$-dimensional torus $\mathbb{T}^{2g} := \mathbb{R}^{2g}/\mathbb{Z}^{2g}$. All lattices of H^g were described by Tolimieri in [48]. Henceforth we will limit ourselves to consider only a standard Heisenberg nilmanifold, our results extending trivially to the general case. Observe that $\exp T$ is the element of $\mathsf{Z}(\mathsf{H}^g)$ generating $\Gamma \cap \mathsf{Z}(\mathsf{H}^g)$.

Unitary H^g-modules and Schrödinger representation. The *Schrödinger representation* is a unitary representation of $\rho : \mathsf{H}^g \to U(L^2(\mathbb{R}^g, \mathrm{d}y))$ of the Heisenberg group into the group of unitary operators on $L^2(\mathbb{R}^g, \mathrm{d}y)$; it is explicitly given by

$$(\rho(x,\xi,t)\varphi)(y) = e^{it - i\xi\cdot y - \frac{1}{2} i\xi\cdot x}\varphi(y+x), \quad (\varphi \in L^2(\mathbb{R}^g), \ (x,\xi,t) \in \mathsf{H}^g)$$

(see [19]). Composing the Schrödinger representation with the automorphism $(x,\xi,t) \mapsto (|h|^{1/2}x, \epsilon|h|^{1/2}\xi, ht)$ of H^g, where $h \neq 0$ and $\epsilon = \mathrm{sign}(h) = \pm 1$, we obtain the *Schrödinger representation with parameters h*: for all $\varphi \in L^2(\mathbb{R}^g, \mathrm{d}y)$

$$(\rho_h(x,\xi,t)\varphi)(y) = e^{iht - i\epsilon|h|^{1/2}\xi\cdot y - \frac{1}{2} ih\xi\cdot x}\varphi(y + |h|^{1/2}x). \tag{2.4}$$

According to the Stone-von Neumann theorem [37], the unitary irreducible representations $\pi : \mathsf{H}^g \to U(\mathscr{H})$ of the Heisenberg group on a Hilbert space $\mathscr{H}$ are

- either trivial on the center; then they are equivalent to a one-dimensional representation of the quotient group $\mathsf{Z}(\mathsf{H}^g)\backslash\mathsf{H}^g$, i.e., equivalent to a character of $\mathbb{R}^{2g}$,
- or infinite dimensional and unitarily equivalent to a Schrödinger representation with some parameter $h \neq 0$.

Infinitesimal Schrödinger representation. The space of smooth vectors of the Schrödinger representation $\rho_h : \mathsf{H}^g \to U(L^2(\mathbb{R}^g, \mathrm{d}y))$ is the space of Schwartz functions $\mathscr{S}(\mathbb{R}^g) \subset L^2(\mathbb{R}^g, \mathrm{d}y)$ [44]. By differentiating the Schrödinger representation ρ_h we obtain a representation of the Lie algebra $\mathfrak{h}^g$ on $\mathscr{S}(\mathbb{R}^g)$ by essentially skew-adjoint operators on $L^2(\mathbb{R}^g, \mathrm{d}y)$; this representation is called the *infinitesimal Schrödinger representation with parameter h*. With an obvious abuse of notation, we denote it by the same symbol ρ_h; the action of $X \in \mathfrak{h}^g$ on a function f will be denoted $\rho_h(X)f$ or $X.f$ when no ambiguity can arise. Differentiating the formulas (2.4) we see that, for all $k = 1, 2, \dots, g$, we have

$$\rho_h(X_k^0) = |h|^{1/2} \frac{\partial}{\partial y_k}, \qquad \rho_h(\Xi_k^0) = -i\epsilon|h|^{1/2}\, y_k, \qquad \rho_h(T) = ih,$$

where (y_i) are the coordinates in $\mathbb{R}^g$ relative to the standard basis (X_i^0) and $\epsilon = \operatorname{sign}(h)$. More generally, by the Stone-von Neumann theorem quoted above, given any Heisenberg basis (X_i, Ξ_j, T) of $\mathfrak{h}^g$, the formulas

$$\rho_h(X_k) = |h|^{1/2} \frac{\partial}{\partial y_k}, \qquad \rho_h(\Xi_k) = -i\epsilon|h|^{1/2}\, y_k, \qquad \rho_h(T) = ih, \tag{2.5}$$

define, via the exponential map, a Schrödinger representation ρ_h with parameter h on $L^2(\mathbb{R}^g, \mathrm{d}y)$ such that

$$\begin{aligned} \rho_h(e^{x_1X_1+\cdots+x_gX_g})f(y) &= f(y + |h|^{1/2}x), \\ \rho_h(e^{\xi_1\Xi_1+\cdots+\xi_g\Xi_g})f(y) &= e^{-i\epsilon|h|^{1/2}\xi\cdot y}f(y), \\ \rho_h(e^{tT})f(y) &= e^{ith}f(y). \end{aligned} \tag{2.6}$$

2.2. **Siegel symplectic geometry.**

Symplectic group and moduli space. Let $\mathrm{Sp}_{2g}(\mathbb{R})$ be the group of symplectic automorphisms of the standard symplectic space $(\mathbb{R}^{2g}, \omega)$. The group of those automorphisms of H^g that are trivial on the center is the semi-direct product $\mathrm{Aut}_0(\mathsf{H}^g) = \mathrm{Sp}_{2g}(\mathbb{R}) \ltimes \mathbb{R}^{2g}$ of the symplectic group with the group of inner automorphisms $\mathsf{H}^g/\mathsf{Z}(\mathsf{H}^g) \approx \mathbb{R}^{2g}$.

The group of automorphisms of H^g acts simply transitively on the set of Heisenberg bases, hence we may identify the set of Heisenberg bases of $\mathfrak{h}^g$ with the group of automorphisms of H^g. However since we are interested in the action of subgroups defined in terms of a choice of a Heisenberg basis and since the dynamical properties of such action are invariant under inner automorphisms, we may restrict our attention to bases which are obtained by applying automorphisms $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$ to the standard Heisenberg basis.

Explicitly, the symplectic matrix written in block form $\alpha = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{R})$, with the $g \times g$ real matrices A, B, C and D satisfying $C^\top A = A^\top C$, $A^\top D - C^\top B = 1$, and $D^\top B = B^\top D$, acts as the automorphism

$$(x, \xi, t) \mapsto \alpha(x, \xi, t) := (Ax + B\xi, Cx + D\xi, t)\,.$$

Siegel symplectic geometry. The stabilizer of the standard lattice $\Gamma < \mathsf{H}^g$ inside $\mathrm{Sp}_{2g}(\mathbb{R})$ is exactly the group $\mathrm{Sp}_{2g}(\mathbb{Z})$. We call $\mathfrak{M}_g = \mathrm{Sp}_{2g}(\mathbb{R})/\mathrm{Sp}_{2g}(\mathbb{Z})$ the *moduli space* of the standard Heisenberg manifold. We may regard $\mathrm{Sp}_{2g}(\mathbb{R})$ as the *deformation (or Teichmüller) space* of the standard Heisenberg manifold $\mathsf{M} = \mathsf{H}^g/\Gamma$ and $\mathfrak{M}_g$ as the moduli space of the standard nilmanifold, in analogy with the 2-torus case.

The *Siegel modular variety*, the moduli space of principally polarized abelian varieties of dimension g, is the double coset space

$$\Sigma_g := \mathsf{K}_g \backslash \mathrm{Sp}_{2g}(\mathbb{R})/\mathrm{Sp}_{2g}(\mathbb{Z}),$$

where K_g is the maximal compact subgroup $\mathrm{Sp}_{2g}(\mathbb{R}) \cap \mathrm{SO}_{2g}(\mathbb{R})$ of $\mathrm{Sp}_{2g}(\mathbb{R})$, isomorphic to the unitary group $U_g(\mathbb{C})$. Thus, $\mathfrak{M}_g$ fibers over Σ_g with compact fibers K_g.

The quotient space $\mathsf{K}_g\backslash\mathrm{Sp}_{2g}(\mathbb{R})/\pm\mathbf{1}_{2g}$ may be identified with Siegel upper half-space in the following way. Recall that the *Siegel upper half-space* of degree/genus g [46] is the complex manifold

$$\mathfrak{H}_g := \left\{ Z \in \mathrm{Sym}_g(\mathbb{C}) \,\middle|\, \Im(Z) > 0 \right\}$$

of symmetric complex $g \times g$ matrices $Z = X + iY$ with positive definite symmetric imaginary part $\Im(Z) = Y$ and arbitrary (symmetric) real part X.

The symplectic group $\mathrm{Sp}_{2g}(\mathbb{R})$ acts on the Siegel upper half-space $\mathfrak{H}_g$ as generalized Möbius transformations. The left action of the block matrix $\alpha = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in $\mathrm{Sp}_{2g}(\mathbb{R})$ is defined as

$$Z \mapsto \alpha(Z) := (AZ+B)(CZ+D)^{-1}. \tag{2.7}$$

This action leaves invariant the Riemannian metric $ds^2 = \mathrm{tr}(dZ\,Y^{-1}d\overline{Z}\,Y^{-1})$.

As the the kernel of this action is given by $\pm\mathbf{1}_{2g}$ and the stabilizer of the point $i := i\mathbf{1}_g \in \mathfrak{H}_g$ coincides with K_g, the map

$$\alpha \in \mathrm{Sp}_{2g}(\mathbb{R}) \mapsto \alpha^{-1}(i) \in \mathfrak{H}_g$$

induces an identification $\mathsf{K}_g\backslash\mathrm{Sp}_{2g}(\mathbb{R})/\pm\mathbf{1}_{2g} \approx \mathfrak{H}_g$ and consequently an identification of the Siegel modular variety $\Sigma_g \approx \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathfrak{H}_g$.

NOTATION 2.1. For $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$ we denote by $[\alpha] := \alpha\,\mathrm{Sp}_{2g}(\mathbb{Z})$ its projection on the moduli space $\mathfrak{M}_g$. We denote by $[[\alpha]] := \mathsf{K}_g\,\alpha\,\mathrm{Sp}_{2g}(\mathbb{Z})$ the projection of α to the Siegel modular variety Σ_g. We remark that under the previous identification $[[\alpha]]$ coincides with the point $\mathrm{Sp}_{2g}(\mathbb{Z})\,\alpha^{-1}(i) \in \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathfrak{H}_g$.

3. COHOMOLOGY WITH VALUES IN H^g-MODULES

Here we discuss the cohomology of the action of a subgroup $\mathsf{P} \subset \mathsf{H}^g$ on a Fréchet H^g-module F, that is to say the Lie algebra cohomology of $\mathfrak{p} = \mathrm{Lie}(\mathsf{P})$ with values in the H^g-module F. We assume that P is a connected Abelian Lie subgroup of H^g contained in a Legendrian subgroup L.

The modules interesting for us are, in particular, those arising from the regular representation of H^g on the space $C^\infty(\mathsf{M})$ of smooth functions on a (standard) nilmanifold $\mathsf{M} := \mathsf{H}^g/\Gamma$. As mentioned in the introduction, the fact that H^g acts on M by left translations, implies that the space $F = C^\infty(\mathsf{M})$ is a $\mathfrak{p}$-module: in fact for all $V \in \mathfrak{p}$ and $f \in F$ one defines (cf. formula (1.2))

$$(V.f)(m) = \frac{\mathrm{d}}{\mathrm{d}t} f(\exp(-tV).m)\Big|_{t=0}, \quad (m \in \mathsf{M}).$$

As P is an Abelian group, the differential on the cochain complex $A^*(\mathfrak{p}, F) = \Lambda^*\mathfrak{p} \otimes F$ of F-valued alternating forms on $\mathfrak{p}$ is given, in degree k, by the usual formula

$$\mathrm{d}\omega(V_0, \dots, V_k) = \sum_{j=0}^{k} (-1)^j\, V_j.\omega(V_0, \dots, \widehat{V_j}, \dots, V_k).$$

NOTATION 3.1. When F is the space of C^∞-vectors of a representation π of H^g we may denote the complex $A^*(\mathfrak{p}, F)$ also by the symbol $A^*(\mathfrak{p}, \pi^\infty)$.

In order to study the cohomology of the complex $A^*(\mathfrak{p}, C^\infty(\mathsf{M}))$, it is convenient to observe that the projection p of M onto the quotient torus $\mathbb{T}^{2g}$ (see (2.3)) yields a H^g-invariant decomposition of all the interesting function spaces on M into functions with zero average along the fibers of p — we denote such function spaces with a suffix 0 — and functions that are constant along such fibers; these latter functions can be thought of as pull-backs of functions defined on the quotient torus $\mathbb{T}$; hence we write, for example,

$$C^\infty(\mathsf{M}) = C_0^\infty(\mathsf{M}) \oplus p^*(C^\infty(\mathbb{T}^{2g})) \approx C_0^\infty(\mathsf{M}) \oplus C^\infty(\mathbb{T}^{2g}), \tag{3.1}$$

and we have similar decompositions for $L^2(\mathsf{M})$ and — when a suitable Laplacian is used to define them — for the L^2-Sobolev spaces $W^s(\mathsf{M})$.

If we denote by $\bar{\mathsf{P}}$ the projection of P into $\mathbb{T}^{2g}$ and by $\bar{\mathfrak{p}}$ its Lie algebra, we obtain that we may split the complex $A^*(\mathfrak{p}, C^\infty(\mathsf{M}))$ into the sum of $A^*(\mathfrak{p}, C_0^\infty(\mathsf{M}))$ and $A^*(\mathfrak{p}, p^*(C^\infty(\mathbb{T}^{2g}))) \approx A^*(\bar{\mathfrak{p}}, C^\infty(\mathbb{T}^{2g}))$. The action of $\bar{\mathsf{P}}$ on $\mathbb{T}^{2g}$ being linear, the computation of the cohomology of this latter complex is elementary and folklore when $\dim \bar{\mathsf{P}} = 1$. For lack of references we review it in the next Section 3.1 for any $\dim \bar{\mathsf{P}}$. In Section 3.2 we shall consider the cohomology of $C^*(\mathfrak{p}, C_0^\infty(\mathsf{M}))$.

REMARK 3.2. To define the norm of the Hilbert Sobolev spaces $W^s(\mathsf{M})$, we fix a basis (V_i) of the Lie algebra $\mathfrak{h}^g$, set $\Delta = -\sum V_i^2$, and define $\|f\|_s^2 = \langle f, (1+\Delta)^s f\rangle$ where $\langle\cdot,\cdot\rangle$ is the ordinary L^2 Hermitian product. This has the advantage that for any Hilbert sum decomposition $L^2(\mathsf{M}) = \bigoplus_i H_i$ of $L^2(\mathsf{M})$ into closed H^g-invariant subspaces we also have a Hilbert sum decomposition $W^s(\mathsf{M}) = \bigoplus_i W^s(H_i)$ of $W^s(\mathsf{M})$ into closed H^g-invariant subspaces $W^s(H_i) := W^s(\mathsf{M}) \cap H_i$.

Currents. Let F be any tame Fréchet $\mathfrak{h}^g$-module, graded by increasing norms $(\|\cdot\|_s)_{s\ge 0}$, defining Banach spaces $W^s \subset F$.

The space of continuous linear functionals on $A^k(\mathfrak{p}, F) = \Lambda^k\mathfrak{p} \otimes F$ will be called *the space of currents of dimension* k and will be denoted $A_k(\mathfrak{p}, F')$, where F' is the strong dual of F; the notation is justified by the fact that the natural pairing $(\Lambda_k\mathfrak{p}, \Lambda^k\mathfrak{p})$ between k vectors and k-forms allows us to write $A_k(\mathfrak{p}, F') \approx \Lambda^k\mathfrak{p}\otimes F'$. Endowed with the strong topology, $A_k(\mathfrak{p}, F')$ is the inductive limit of the spaces $\Lambda^k\mathfrak{p}\otimes(W^s)'$.

The *boundary operators* $\partial : A_k(\mathfrak{p}, F') \to A_{k-1}(\mathfrak{p}, F')$ are, as usual, the adjoint of the differentials d; hence they are defined by $\langle\partial T, \omega\rangle = \langle T, \mathrm{d}\omega\rangle$. A *closed* current T is one such that $\partial T = 0$. We denote by $Z_k(\mathfrak{p}, F')$ the space of closed currents of dimension k and by $Z_k(\mathfrak{p}, (W^s)')$ the space of closed currents with coefficients in $(W^s)'$.

3.1. **Cohomology of a linear $\mathbb{R}^d$ action on a torus.** Let Λ be a lattice subgroup of $\mathbb{R}^\ell$ and let $\mathbb{R}^\ell$ act on the torus $\mathsf{T}^\ell = \mathbb{R}^\ell/\Lambda$ by translations. We consider the restriction of this action to a subgroup $\mathsf{Q} < \mathbb{R}^\ell$ isomorphic to $\mathbb{R}^d$, with Lie algebra $\mathfrak{q}$. Then the Fréchet space $C^\infty(\mathsf{T}^\ell)$ is a $\mathfrak{q}$-module. In this section we consider the cohomology of the associated complex $A^*(\mathfrak{q}, C^\infty(\mathsf{T}^\ell))$.

Let $\Lambda^\perp = \{\lambda \in (\mathbb{R}^\ell)' \,|\, \lambda\cdot n = \mathbb{Z}\ \forall\ n \in \Lambda\}$ denote the dual lattice of Λ. We say that *the subspace* $\mathfrak{q}$ *satisfies a Diophantine condition of exponent* $\tau > 0$ with respect to the lattice Λ, and we write $\mathfrak{q} \in \mathrm{DC}_\tau(\Lambda)$, if

$$(3.2)\qquad \exists\, C > 0 \quad \text{such that} \quad \sup_{V\in\mathfrak{q}\smallsetminus\{0\}} \frac{|\lambda\cdot V|}{\|V\|} \geq C\|\lambda\|^{-\tau}, \quad \forall\ \lambda \in \Lambda^\perp \smallsetminus \{0\}.$$

We set

$$\mu(\mathfrak{q}, \Lambda) = \inf\{\tau \,|\, \mathfrak{q} \in \mathrm{DC}_\tau(\Lambda)\}\,.$$

REMARK 3.3. The Diophantine condition considered here is dual to the Diophantine condition on subspaces of $(\mathbb{R}^\ell)' \approx \mathbb{R}^\ell$ considered by Moser in [41]. In fact, if we set $\mathfrak{q}^\perp = \{\lambda \in (\mathbb{R}^\ell)' \,|\, \ker\lambda \supset \mathfrak{q}\}$, the condition (3.2) is equivalent to

$$\exists\, C > 0 \quad \text{such that} \quad \mathrm{dist}(\lambda, \mathfrak{q}^\perp) \geq C\|\lambda\|^{-\tau}, \qquad \forall\ \lambda \in \Lambda^\perp \smallsetminus \{0\}.$$

Thus, by Theorem 2.1 of [41], the inequalities (3.2) are possible only if $\tau \geq \ell/d - 1$, and the set of subspaces $\mathfrak{q}^\perp$ with $\mu(\mathfrak{q}, \Lambda) = \ell/d - 1$ has full Lebesgue measure in the Grassmannian $\mathrm{Gr}(\mathbb{R}^d; \mathbb{R}^\ell)$.

We say that $\mathfrak{q}$ is *resonant (with respect to* Λ*)* if, for some $\lambda \in \Lambda^\perp \smallsetminus \{0\}$, we have $\mathfrak{q} \subset \ker\lambda$; in this case the closure of the orbits of Q on $\mathbb{R}^\ell/\Lambda$ are contained in lower dimensional tori, the orbits of the rational subspace $\ker\lambda$, and we may understand this case by considering a lower dimensional ambient space $\mathbb{R}^{\ell'}$ with $\ell' < \ell$.

Thus we may limit ourselves to non-resonant $\mathfrak{q}$; in this case, if $\mathfrak{q}$ is not Diophantine, we have $\mu(\mathfrak{q}, \Lambda) = +\infty$ and we say that $\mathfrak{q}$ is *Liouvillean (with respect to* Λ*)*.

THEOREM 3.4 (Folklore)**.** *Let* $\mathfrak{q} \in \mathrm{Gr}(\mathbb{R}^d; \mathbb{R}^\ell)$ *be a non-resonant subspace with respect to the lattice* $\Lambda < \mathbb{R}^\ell$*. Then the action of* $\mathsf{Q} = \exp\mathfrak{q}$ *on the torus* $\mathsf{T}^\ell := \mathbb{R}^\ell/\Lambda$ *is cohomologically* C^∞*-stable if and only if* $\mathfrak{q} \in DC_\tau(\Lambda)$ *for some* $\tau > 0$*. In this case we have*

$$H^*(\mathfrak{q}, C^\infty(\mathsf{T}^\ell)) \approx \Lambda^*\mathfrak{q},$$

the cohomology classes being represented by forms with constant coefficients. Furthermore, the $\mathfrak{q}$*-module* $C^\infty(\mathsf{T}^\ell)$ *is tamely cohomologically* C^∞*-stable and has tame splitting in all degrees.*

Proof. Without loss of generality we may assume $\Lambda = \mathbb{Z}^\ell$. The s-Sobolev norm of a function $f \in C^\infty(\mathsf{T}^\ell)$ with Fourier series representation

$$f(x) = \sum_{n\in\mathbb{Z}^\ell} \hat f(n)\, e^{2\pi i n\cdot x}$$

is given by

$$\|f\|_s^2 = \sum_{n\in\mathbb{Z}^\ell} \left(1+\|n\|^2\right)^s |\hat f(n)|^2 .$$

We have a direct sum decomposition $C^\infty(\mathsf{T}^\ell) = \mathbb{C}\langle 1\rangle \oplus C_0^\infty(\mathsf{T}^\ell)$, where $\mathbb{C}\langle 1\rangle$ is the space of constant functions and $C_0^\infty(\mathsf{T}^\ell)$ is the space of zero mean smooth functions on T^ℓ. An analogous *orthogonal* decomposition $W^s(\mathsf{T}^\ell) = \mathbb{C}\langle 1\rangle \oplus W_0^s(\mathsf{T}^\ell)$ holds for Sobolev spaces. Hence every $\omega \in Z^k(\mathfrak{q}, C^\infty(\mathsf{T}^\ell))$ splits (tamely) into a sum $\omega = \omega_0 + \omega_c$ of a form $\omega_0 \in Z^k(\mathfrak{q}, C_0^\infty(\mathsf{T}^\ell))$ and a constant coefficient form $\omega_c \in \Lambda^k\mathfrak{q}$. Consequently, the cohomology $H^*(\mathfrak{q}, C^\infty(\mathsf{T}^\ell))$ splits into the sum of cohomology classes represented by forms with constant coefficients and $H^*(\mathfrak{q}, C_0^\infty(\mathsf{T}^\ell))$. We now show that, under the assumption (3.2) on $\mathfrak{q}$, we have $H^*(\mathfrak{q}, C_0^\infty(\mathsf{T}^\ell)) = 0$.

By Fourier analysis, $C_0^\infty(\mathsf{T}^\ell)$ splits into a L^2-orthogonal sum of one-dimensional modules $\mathbb{C}_n \approx \mathbb{C}$, $n \in \mathbb{Z}^\ell \smallsetminus \{0\}$; the space $\mathfrak{q}$ acts on $\mathbb{C}_n$ by

$$V.z = i\,(n\cdot V)\,z, \qquad \forall\ z \in \mathbb{C}_n,\ \forall\ V \in \mathfrak{q};$$

hence, for $\omega \in \Lambda^k\mathfrak{q}\otimes\mathbb{C}_n$ and $V_0,\dots,V_k \in \mathfrak{q}$,

$$\mathrm{d}\omega(V_0,\dots,V_k) = \sum_{j=0}^{k} i\,(n\cdot V_j)\,\omega(V_0,\dots,\widehat{V}_j,\dots,V_k).$$

Let $X_1, X_2, \dots, X_d$ be a basis of $\mathfrak{q}$, and define the co-differential d^* by

$$\mathrm{d}^*\eta(V_1,\dots,V_k) := -\sum_{m=1}^{d} i\,(n\cdot X_m)\,\eta(X_m, V_1,\dots,V_k).$$

We have $H = \mathrm{d}^*\circ\mathrm{d} + \mathrm{d}\circ\mathrm{d}^* = \left(\sum_{m=1}^d |n\cdot X_m|^2\right)\mathrm{Id}_{\Lambda^*\mathfrak{q}}$. It follows that if $\omega \in \Lambda^k\mathfrak{q}\otimes\mathbb{C}_n$ is closed then $\omega = \mathrm{d}\Omega$ with

$$\Omega = H^{-1}\mathrm{d}^*\omega.$$

We conclude that the map $\mathrm{d}_{-1} := H^{-1}\mathrm{d}^*$ is a right inverse of d on the space $Z^k(\mathfrak{q}, \mathbb{C}_n)$ of closed forms. From the definitions of the maps d^* and H we obtain the estimate

$$\|\mathrm{d}_{-1}\omega\|_0 \le \Big(\sum_{m=1}^{d} |n\cdot X_m|^2\Big)^{-\frac{1}{2}} \|\omega\|_0, \quad \forall\ \omega \in Z^k(\mathfrak{q}, \mathbb{C}_n).$$

It is easily seen that the Diophantine condition (3.2) is equivalent to the existence of a constant $C > 0$ such that $\sum_{m=1}^d |n\cdot X_m|^2 > C\,\|n\|^{-2\tau}$ for all $n \in \mathbb{Z}^\ell$.

Hence, for some constant $C > 0$ we have $\|\mathrm{d}_{-1}\omega\|_0 \le C^{-1}\|n\|^{\tau}\|\omega\|_0$, and therefore

$$\|\mathrm{d}_{-1}\omega\|_s \le C^{-1}\|\omega\|_{s+\tau}$$

for all $s \in \mathbb{R}$ and all $\omega \in Z^k(\mathfrak{q}, \mathbb{C}_n)$.

Since the Sobolev space $(W_0^s(\mathsf{T}^\ell), \|\cdot\|_s)$ is equal to the Hilbert direct sum $\bigoplus_{n\neq 0}(\mathbb{C}_n, \|\cdot\|_s)$, the map d_{-1} extends to a tame map

$$\mathrm{d}_{-1}\colon Z^k(\mathfrak{q}, C_0^\infty(\mathsf{T}^\ell)) \to A^{k-1}(\mathfrak{q}, C_0^\infty(\mathsf{T}^\ell))$$

satisfying a tame estimate of degree τ with base 0 and associating a primitive to each closed form.

Combining these results with the previous remark on constant coefficient forms, we conclude that under the Diophantine assumption (3.2) the $\mathfrak{q}$-module $C^\infty(\mathsf{T}^\ell)$ is tamely cohomologically C^∞-stable and has a tame splitting in all degrees.

The “only if” part of the statement may be proved as in the case $\dim \mathsf{Q} = 1$ (see Katok [28, page 71]). □

3.2. **Cohomology with values in $C_0^\infty(\mathsf{M})$.** The previous section settles the study of the cohomology of the action of a abelian subgroup $\mathsf{P} \subset \mathsf{H}^g$ with values in the H^g-sub-module $p^*(C^\infty(\mathbb{T}^{2g}))$. We are left to consider the action of P with values in the H^g-sub-module $C_0^\infty(\mathsf{M})$.

Since the center $Z(\mathsf{H}^g)$ has spectrum $2\pi\mathbb{Z} \smallsetminus \{0\}$ on $L_0^2(\mathsf{M})$, the space $L_0^2(\mathsf{M})$ splits as a Hilbert sum of Schrödinger H^g-modules H_i equivalent to ρ^h, with $h \in 2\pi\mathbb{Z} \smallsetminus \{0\}$. The same remark applies to the Sobolev space $W_0^s(\mathsf{M})$, which splits as a Hilbert sum of the (non-unitary) H^g-modules $W_0^s(H_i) = H_i \cap W_0^s(\mathsf{M})$.

The space $C^\infty(\mathsf{M}) \cap H_i$ can be characterized as the space $C^\infty(H_i)$ of C^∞ vectors in the H^g-module H_i; it is a tame graded Fréchet space topologized and graded by the increasing family of Sobolev norms. This leads us to consider the action of P with values in the space of smooth vectors of a Schrödinger H^g-module.

Thus let P be an isotropic subgroup of H^g of dimension d. Fix a Legendrian subgroup L such that $\mathsf{P} \le \mathsf{L} < \mathsf{H}^g$. Let $|h| > h_0 > 0$.

Since the group of automorphisms of H^g acts transitively on Heisenberg bases, we may assume that we have fixed a Heisenberg basis (X_i, Ξ_j, T) of $\mathfrak{h}^g$ such that $(X_1, \dots, X_d)$ forms a basis of $\mathfrak{p}$ and $(X_1, \dots, X_g)$ is a basis of Lie(L). This yields isomorphisms $\mathsf{L} \approx \mathbb{R}^g$ and $\mathsf{P} \approx \mathbb{R}^d$, with the latter group embedded in $\mathbb{R}^g$ via the first d coordinates. With these assumptions, the formulas yielding the representation ρ_h on $L^2(\mathbb{R}^g)$ are given by the equations (2.6). The space ρ_h^∞ of C^∞ vectors for the representation ρ_h is identified with $\mathscr{S}(\mathbb{R}^g)$, on which $\mathfrak{h}^g$ acts by the formulas (2.5).

Homogeneous Sobolev norms. The infinitesimal representation extends to a representation of the enveloping algebra $\mathfrak{U}(\mathfrak{h}^g)$ of $\mathfrak{h}^g$; this allows us to define the “sub-Laplacian” as the image via ρ_h of the element

$$H_g = -(X_1^2 + \cdots + X_g^2 + \Xi_1^2 + \cdots + \Xi_g^2) \in \mathfrak{U}(\mathfrak{h}^g).$$

Formulas (2.5) yield

$$\rho_h(H_g) = |h| \left(|x|^2 - \sum_{k=1}^{g} \frac{\partial^2}{\partial x_k^2} \right) = |h|\, \rho_1(H_g). \tag{3.3}$$

Since H_g is a positive operator with (discrete) spectrum bounded below by $g|h|$, we define the space $W^s(\rho_h, \mathbb{R}^g)$ of functions of Sobolev order s as the Hilbert space of vectors φ of finite *homogeneous* Sobolev norm

$$|||\varphi|||_{s,h}^2 := \left\langle (\rho_h(H_g))^s \varphi, \varphi \right\rangle. \tag{3.4}$$

This makes explicit the fact that the space ρ_h^∞ of C^∞ vectors for the representation ρ_h coincides with $\mathscr{S}(\mathbb{R}^g)$.

The homogeneous Sobolev norms (3.4) are not the standard ones (later on we shall make a comparison with standard Sobolev norms). They have, however, the advantage that the norm on $W^s(\rho_h, \mathbb{R}^g)$ is obtained by rescaling by the factor $|h|^{s/2}$ the norm on $W^s(\rho_1, \mathbb{R}^g)$. For this reason we can limit ourselves to studying the case $h = 1$; later we shall consider the appropriate rescaling. Thus we denote $\rho = \rho_1$ and, to simplify, we write H_g for $\rho(H_g)$ and $W^s(\mathbb{R}^g)$ for $W^s(\rho_1, \mathbb{R}^g)$; we also set

$$|||\varphi|||_s := |||\varphi|||_{s,1} = \|H_g^{s/2} \varphi\|_0.$$

The cochain complex $A^*(\mathfrak{p}, \rho^\infty)$. It will be convenient to use the identification $\mathbb{R}^g \approx \mathbb{R}^d \times \mathbb{R}^{g-d}$ and, accordingly, to write $\varphi(x, y)$, with $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^{g-d}$, for a function φ defined on $\mathbb{R}^g$. We also write $\mathrm{d}x = \mathrm{d}x_1 \cdots \mathrm{d}x_d$. Then, by the formula (2.4), the group element $q \in \mathsf{P} \approx \mathbb{R}^d$ acts on $\varphi \in \mathscr{S}(\mathbb{R}^g)$ according to

$$\varphi(x, y) \mapsto \varphi(x + q, y).$$

Thus the complex $A^*(\mathfrak{p}, \rho^\infty)$ is identified with the complex of differential forms on $\mathfrak{p} \approx \mathbb{R}^d$ with coefficients in $\mathscr{S}(\mathbb{R}^g)$. It will be also convenient to define the operators $H'_d = \left(|x|^2 - \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2} \right)$ and $H''_{g-d} = \left(|y|^2 - \sum_{k=1}^{g-d} \frac{\partial^2}{\partial y_k^2} \right)$ on $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}(\mathbb{R}^{g-d})$, respectively; they may be also considered as operators on $\mathscr{S}(\mathbb{R}^g)$, and then $H_g = H'_d + H''_{g-d}$.

LEMMA 3.5. *Consider $\mathscr{S}(\mathbb{R}^g)$ as a H^g-module with parameter $h = 1$. Define the distribution $\mathscr{I}_g \in \mathscr{S}'(\mathbb{R}^g)$ by*

$$\mathscr{I}_g(f) := \int_{\mathbb{R}^g} f(x)\, \mathrm{d}x$$

for $f \in \mathscr{S}(\mathbb{R}^g)$. Then, for any $s > g/2$, $\mathscr{I}_g$ extends to a bounded linear functional on $W^s(\mathbb{R}^g)$, that is $\mathscr{I}_g \in W^{-s}(\mathbb{R}^g)$.

Proof. Using Cauchy-Schwartz inequality we have

$$|\mathscr{I}_g(f)|^2 \le \int_{\mathbb{R}^g} |(g + |x|^2)|^{-s}\, \mathrm{d}x \cdot \int_{\mathbb{R}^g} (g + |x|^2)^s |f(x)|^2\, \mathrm{d}x$$

As $g + |x|^2 \le 2H_g$, the second integral is bounded by a constant times $|||f|||_s^2$, and the result follows. □

For the next lemma we adopt the convention $\mathbb{R}^0 = \{0\}$ and $\mathscr{S}(\mathbb{R}^0) = W^s(\mathbb{R}^0) = \mathbb{C}$ with the usual norm.

LEMMA 3.6. *For $1 \le d \le g$, consider the map $\mathscr{I}_{d,g} : \mathscr{S}(\mathbb{R}^g) \mapsto \mathscr{S}(\mathbb{R}^{g-d})$ defined by*

$$(\mathscr{I}_{d,g} f)(y) := \int_{\mathbb{R}^d} f(x, y)\, \mathrm{d}x.$$

We consider $\mathscr{S}(\mathbb{R}^g)$ and $\mathscr{S}(\mathbb{R}^{g-d})$ as H^g and H^{g-d}-modules, respectively, with parameter $h = 1$. Then, for any $\varepsilon > 0$ and $s \ge 0$, the map $\mathscr{I}_{d,g}$ extends to a bounded linear map from $W^{s+d/2+\varepsilon}(\mathbb{R}^g)$ to $W^s(\mathbb{R}^{g-d})$, i.e.,

$$|||\mathscr{I}_{d,g} f|||_s \le C\, |||\, f|||_{s+d/2+\varepsilon}$$

for some constant $C = C(s, \varepsilon, d, g)$. In particular this proves the inclusion

$$\mathscr{I}_{d,g}(\mathscr{S}(\mathbb{R}^g)) \subset \mathscr{S}(\mathbb{R}^{g-d}).$$

Proof. For $d = g$ we have $\mathscr{I}_{g,g} = \mathscr{I}_g$ and the result is a restating of the previous lemma.

Now suppose $d < g$. The operators H'_d and H''_{g-d}, considered as operators on $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}^{g-d})$, have discrete spectrum (they are independent d-dimensional and $(g-d)$-dimensional harmonic oscillators); thus identifying $L^2(\mathbb{R}^g) \approx L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^{g-d})$ their joint spectral measure on $L^2(\mathbb{R}^g)$ is the product of the spectral measures on $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}^{g-d})$ respectively. Clearly $H_g \ge H'_d$ and $H_g \ge H''_{g-d}$.

Let (v_m) and (w_n) be orthonormal bases of $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}^{g-d})$ of eigenvectors of H'_d and H''_{g-d} with eigenvalues (λ_m) and (μ_n), respectively. We may choose these bases so that $\{v_m\} \subset \mathscr{S}(\mathbb{R}^d)$ and $\{w_n\} \subset \mathscr{S}(\mathbb{R}^{g-d})$.

Writing for $f \in \mathscr{S}(\mathbb{R}^g)$, $f = \sum f_{mn} v_m \otimes w_n$ and letting $d_m = \mathscr{I}_d(v_m)$ we have $\mathscr{I}_{d,g} f = \sum_n (\sum_m (d_m) f_{mn}) w_n$. It follows that

$$|||\mathscr{I}_{d,g}\, f|||_s^2 = \sum_n \mu_n^s \Big| \sum_m d_m f_{mn} \Big|^2 \le \Big(\sum_m |d_m|^2 \lambda_m^{-d/2-\varepsilon} \Big) \Big(\sum_{m,n} \mu_n^s \lambda_m^{d/2+\varepsilon} |f_{mn}|^2 \Big).$$

The first term in this product equals $\|\mathscr{I}_d\|^2_{-(d/2+\varepsilon)}$, which is bounded by Lemma 3.5; the second term is majorated by $|||f|||^2_{s+d/2+\varepsilon}$, since $H_g \ge H'_d$ and $H_g \ge H''_{g-d}$. □

The proof of the following corollary is immediate.

COROLLARY 3.7. *We use the notation of the previous lemma. Suppose $d < g$. For all $t \ge 0$ and all $s > t + d/2$ the map*

$$D \in W^{-t}(\mathbb{R}^{g-d}) \mapsto D \circ \mathscr{I}_{d,g} \in W^{-s}(\mathbb{R}^g)$$

is continuous. In particular, if $f \in W^s(\mathbb{R}^g)$ with $s > d/2$ then $\mathscr{I}_{d,g}(f) = 0$ if and only if $T \circ \mathscr{I}_{d,g}(f) = 0$ for all $T \in L^2(\mathbb{R}^{g-d})'$.

Let $\varphi_d \in \mathscr{S}(\mathbb{R}^d)$ be the ground state of H_d normalized by the condition $\mathscr{I}_d(\varphi_d) = 1$, namely

$$\varphi_d(x) := (2\pi)^{-d/2} e^{-|x|^2/2}, \quad (x \in \mathbb{R}^d);$$

we have $|||\varphi_d|||_s = \pi^{-d/4} d^{s/2}$.

LEMMA 3.8. *For $1 \le d < g$, let $\mathscr{E}_{d,g} : \mathscr{S}(\mathbb{R}^{g-d}) \mapsto \mathscr{S}(\mathbb{R}^g)$ be defined by*

$$(\mathscr{E}_{d,g} f)(x, y) := \varphi_d(x) f(y).$$

We consider $\mathscr{S}(\mathbb{R}^g)$ and $\mathscr{S}(\mathbb{R}^{g-d})$ as H^g and H^{g-d}-modules, respectively, with parameter $h = 1$. Then, for any $s \ge 0$, the map $\mathscr{E}_{d,g}$ extends to a bounded linear map from $W^s(\mathbb{R}^{g-d})$ to $W^s(\mathbb{R}^g)$, i.e.,

$$|||\mathscr{E}_{d,g} f|||_s \le C\, |||f|||_s$$

for some constant $C = C(s, d)$.

Proof. Consider $H'_d =$ and H''_{g-d} as operators on $\mathscr{S}(\mathbb{R}^g)$. For all integers n, from the binomial identity for $(H'_d + H''_{g-d})^n$, we obtain $|||\mathscr{E}_{d,g} f|||_n^2 = \sum_j \binom{n}{j} |||\,\varphi_d\,|||_j^2$ $|||f|||_{n-j}^2 \le 2^n\, |||\,\varphi_d^2\,|||_n\, |||f|||_n^2$, where for the last inequality we used $H'_d \ge 1$ and $H''_{g-d} \ge 1$. This proves the lemma for integer s; the general claim follows by interpolation. □

LEMMA 3.9. *Let $d = 1$. Let f be an element of the H^g-module $\mathscr{S}(\mathbb{R}^g)$ with parameter $h = 1$. Suppose that $\mathscr{I}_{1,g} f = 0$. Set*

$$(\mathscr{P} f)(x, y) := \int_{-\infty}^{x} f(t, y)\, \mathrm{d}t.$$

For all $t \ge 0$ and all $\varepsilon > 0$ there exists a constant $C = C(t, \varepsilon)$ such that

$$|||\mathscr{P} f|||_t \le C\, |||\,f\,|||_{t+1+\varepsilon}\,. \tag{3.5}$$

In particular this proves that $\mathscr{P}(\mathscr{S}(\mathbb{R}^g)) \subset \mathscr{S}(\mathbb{R}^{g-d})$.

Proof. When $g = 1$, the lemma is a variation on the statement of Lemma 6.1 in [14], which can be easily proved by use of the Cauchy-Schwartz inequality as in Lemma 3.5.

Suppose now that $g > 1$ and consider the decomposition $H_g = H'_1 + H''_{g-1}$. The condition $\mathscr{I}_{1,g} f = 0$ implies that $\mathscr{I}_{1,g}(H''_{g-1})^w f = 0$ for any $w \ge 0$; furthermore $\mathscr{P}(H''_{g-1})^w f = (H''_{g-1})^w \mathscr{P} f$. Using the result for the case $g = 1$ and the definition of the norm $\|\cdot\|_0$ we have for all $t \ge 0$ and all $\varepsilon > 0$

$$\|(H_1)^{t/2} (H''_{g-1})^{w/2} \mathscr{P} f\|_0 \le C(t, \varepsilon) \|(H'_1)^{(t+1+\varepsilon)/2} (H''_{g-1})^{w/2} f\|_0.$$

For integer values of the Sobolev order, using the above inequality and the binomial formula, we may write, for any $\varepsilon > 0$ and $n \in \mathbb{N}$,

$$\begin{aligned}|||\mathscr{P} f|||_n^2 = \langle \mathscr{P} f, H_g^n \mathscr{P} f\rangle_0 &= \sum_{k=0}^{n} \binom{n}{k} \|(H_1')^{k/2} (H_{g-1}'')^{(n-k)/2} \mathscr{P} f\|_0^2 \\ &\le C(\varepsilon, n) \sum_{k=0}^{n} \binom{n}{k} \|(H_1')^{(k+1+\varepsilon)/2} (H_{g-1}'')^{(n-k)/2} f\|_0^2 \\ &\le C(\varepsilon, n) \|(H_g)^{n/2} (H_1')^{(1+\varepsilon)/2} f\|_0^2 \\ &\le C(\varepsilon, n) \|H_g^{(n+1+\varepsilon)/2} f\|_0^2 = C(\varepsilon, n)\, |||\, f\, |||_{n+1+\varepsilon}^2 .\end{aligned}$$

The general inequality follows by interpolation of the family of norms $|||\cdot|||_n$. □

Sobolev cocycles and coboundaries. Having fixed a Euclidean product on $\mathfrak{h}^g$, we obtain, by restriction, a Euclidean product on $\mathfrak{p} \subset \mathfrak{h}^g$ and, by duality and extension to the exterior algebra, a Euclidean product on $\Lambda^k \mathfrak{p}'$. The spaces $A^k(\mathfrak{p}, \rho^\infty) \approx \Lambda^k \mathfrak{p}' \otimes \mathscr{S}(\mathbb{R}^g)$ of cochains of degree k are endowed with the Hermitian products obtained as tensor product of the Euclidean product on $\Lambda^k \mathfrak{p}'$ and the Hermitian products $\|\cdot\|_s$ or $|||\cdot|||_s$ on $\mathscr{S}(\mathbb{R}^g)$. Completing with respect to these norms, we define the Sobolev spaces $\Lambda^k \mathfrak{p}' \otimes W^s(\mathbb{R}^g)$ of cochains of degree k and use the same notations for the norms.

It is clear that, for $k < d$, the cohomology groups are $H^k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) = 0$. Here we estimate the Sobolev norm of a primitive $\Omega \in A^{k-1}(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ of a coboundary $\omega = \mathrm{d}\Omega \in B^k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) = Z^k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ in terms of the Sobolev norm of ω.

PROPOSITION 3.10. *Let $s \ge 0$ and $1 \le k < d \le g$. Consider $\mathscr{S}(\mathbb{R}^g)$ as a H^g-module with parameter $h = 1$. For every $\varepsilon > 0$ there exist a constant $C = C(s, \varepsilon, g, d) > 0$ and a linear map*

$$\mathrm{d}_{-1} \colon Z^k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) \to A^{k-1}(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$$

associating to every $\omega \in Z^k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ a primitive $\Omega = \mathrm{d}_{-1}\omega \in A^{k-1}(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ satisfying the estimate

$$|||\Omega|||_s \le C\, |||\, \omega\, |||_{s+(k+1)/2+\varepsilon} . \tag{3.6}$$

Proof. We denote points of $\mathbb{R}^g \approx \mathfrak{p} \times \mathbb{R}^{g-d} \approx \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}^{g-d}$ as triples (t, x, y) with $t \in \mathbb{R}$, $x \in \mathbb{R}^{d-1}$, and $y \in \mathbb{R}^{g-d}$. For $0 \le k \le d \le g$, one defines linear maps

$$A^k(\mathbb{R}^d, \mathscr{S}(\mathbb{R}^g)) \underset{\mathscr{E}}{\overset{\mathscr{I}}{\rightleftarrows}} A^{k-1}(\mathbb{R}^{d-1}, \mathscr{S}(\mathbb{R}^{g-1}))$$

as follows. For a monomial $\omega = f(t, x, y)\, dt \wedge \mathrm{d}x^a$ in $A^k(\mathbb{R}^d, \mathscr{S}(\mathbb{R}^g))$, where a is a multi-index in the set $\{1, 2, \dots, d-1\}$, we define

$$\mathscr{I}\omega := \left(\int_{-\infty}^{\infty} f(t, x, y)\, \mathrm{d}t\right) \mathrm{d}x^a = (\mathscr{I}_{1,g} f)\, \mathrm{d}x^a ; \tag{3.7}$$

if $\mathrm{d}t$ does not divide ω we define instead $\mathscr{I}\omega = 0$. For a monomial $\omega = f(x,y)\,\mathrm{d}x^a$ in $A^{k-1}(\mathbb{R}^{d-1}, \mathscr{S}(\mathbb{R}^{g-1}))$, we define

$$\mathscr{E}\omega := \varphi(t) f(x,y)\,\mathrm{d}t \wedge \mathrm{d}x^a = (\mathscr{E}_{1,g} f)\,\mathrm{d}t \wedge \mathrm{d}x^a. \tag{3.8}$$

By Lemma 3.6 we obtain that for any $t \geq 0$ and $\varepsilon > 0$ we have

$$|||\mathscr{I}\omega|||_t \leq C\,|||\omega|||_{t+1/2+\varepsilon}, \qquad C = C(t,\varepsilon,g). \tag{3.9}$$

It follows from this inequality that the image of $\mathscr{I}$ lies in $A^{k-1}(\mathbb{R}^{d-1}, \mathscr{S}(\mathbb{R}^{g-1}))$. For the map $\mathscr{E}$ the inclusion $\mathscr{E}(A^{k-1}(\mathbb{R}^{d-1}, \mathscr{S}(\mathbb{R}^{g-1}))) \subset A^k(\mathbb{R}^d, \mathscr{S}(\mathbb{R}^g))$ is obvious, and by Lemma 3.8 we have, for any $s \geq 0$,

$$|||\mathscr{E}\eta|||_s \leq C\,|||\eta|||_s, \qquad C = C(s,d). \tag{3.10}$$

From (3.9) and (3.10) it follows that, for any $s \geq 0$,

$$|||\mathscr{E}\mathscr{I}\omega|||_s \leq C\,|||\omega|||_{s+1/2+\varepsilon}. \tag{3.11}$$

The maps $\mathscr{I}$ and $\mathscr{E}$ commute with the differential d. It is well known that $\mathscr{I}$ and $\mathscr{E}$ are homotopy inverses of each other. In fact, it is clear that $\mathscr{I}\mathscr{E}$ is the identity.

We claim that the usual homotopy operator

$$\mathscr{K} : A^k(\mathbb{R}^d, \mathscr{S}(\mathbb{R}^g)) \to A^{k-1}(\mathbb{R}^d, \mathscr{S}(\mathbb{R}^g))$$

satisfying $1 - \mathscr{E}\mathscr{I} = \mathrm{d}\mathscr{K} - \mathscr{K}\mathrm{d}$ also satisfies tame estimates. For a monomial ω not divisible by $\mathrm{d}t$, $\mathscr{K}$ is defined as $\mathscr{K}\omega = 0$; for a monomial $\omega = f(t,x,y)\,\mathrm{d}t \wedge \mathrm{d}x^a$ it is defined as $\mathscr{K}\omega = g(t,x,y)\,\mathrm{d}x^a$ where

$$\begin{aligned} g(t,x,y) &= \int_{-\infty}^{t} \left[f(r,x,y) - \varphi(r) \left(\int_{\mathbb{R}} f(u,x,y)\,\mathrm{d}u \right) \right] \mathrm{d}r \\ &= \mathscr{P}(f - \mathscr{E}_{1,g}\mathscr{I}_{1,g} f). \end{aligned} \tag{3.12}$$

Then by Lemma 3.9 and (3.11) we have that for all $s \geq 0$

$$|||\mathscr{K}\omega|||_s \leq C(s,\varepsilon,g,d)\,|||\omega|||_{s+3/2+\varepsilon}, \tag{3.13}$$

unless $\mathscr{I}\omega = 0$, in which case we have

$$|||\mathscr{K}\omega|||_s \leq C(s,\varepsilon,g,d)\,|||\omega|||_{s+1+\varepsilon}. \tag{3.14}$$

This proves the claim.

Let $\omega \in A^1(\mathbb{R}^d, \mathscr{S}(\mathbb{R}^g))$ be closed and $1 < d \leq g$. Then $\mathscr{I}\omega = 0$ (by homotopying the integral in (3.7) with an integral with $x \to \infty$) and therefore $\Omega = \mathscr{K}\omega$ in $A^0(\mathbb{R}^d, \mathscr{S}(\mathbb{R}^g)) \approx \mathscr{S}(\mathbb{R}^{g+1})$ is a primitive of ω, i.e., $\mathrm{d}\Omega = \omega$, and by (3.14) it satisfies the estimate $|||\Omega|||_s \leq C(s) \cdot |||\omega|||_{s+1+\varepsilon}$ for all $s > 1/2$. Thus the proposition is proved in this case.

Assume, by induction, that the proposition is true for all $g \geq 1$, all $d \leq g$ and all $k \leq \min\{n,d\} - 1$. Let $\omega \in A^n(\mathbb{R}^d, \mathscr{S}(\mathbb{R}^g))$, with $n < d$, be closed. Then the $(n-1)$-form $\mathscr{I}\omega \in A^{n-1}(\mathbb{R}^{d-1}, \mathscr{S}(\mathbb{R}^{g-1}))$ is also closed. By the induction assumption, $\mathscr{I}\omega = \mathrm{d}\eta$ for a primitive $\eta \in A^{n-2}(\mathbb{R}^{d-1}, \mathscr{S}(\mathbb{R}^{g-1}))$ satisfying the estimate

$$|||\eta|||_s \leq C\,|||\mathscr{I}\omega|||_{s+n/2+\varepsilon}. \tag{3.15}$$

Since $\mathscr{E}\mathscr{I}\omega = \mathscr{E}\mathrm{d}\eta$ and $\mathscr{E}$ commutes with d, we obtain that a primitive of ω is given by $\mathrm{d}_{-1}\omega := \Omega := \mathscr{K}\omega + \mathscr{E}\eta$. Therefore, from Lemma 3.6 and the estimates (3.9), (3.10), (3.13), and (3.15), we have, for some constants C's which only depend on $s \geq 0$ and $\varepsilon > 0$,

$$\begin{aligned} |||\Omega|||_s &\leq |||\mathscr{K}\omega|||_s + |||\mathscr{E}\eta|||_s \\ &\leq C' \, |||\omega|||_{s+3/2+\varepsilon} + C'' \, |||\eta|||_s \\ &\leq C' \, |||\omega|||_{s+3/2+\varepsilon} + C''' \, |||\mathscr{I}\omega|||_{s+n/2+\varepsilon/2} \\ &\leq C' \, |||\omega|||_{s+3/2+\varepsilon} + C'''' \, |||\omega|||_{s+n/2+1/2+\varepsilon} \\ &\leq C \, |||\omega|||_{s+(n+1)/2+\varepsilon} \, . \end{aligned} \tag{3.16}$$

Thus the estimate (3.6) holds also for $k = n$. This concludes the proof. □

We are left to consider the space $H^k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ when $k = d := \dim \mathfrak{p}$.

The map $\mathscr{I}_{d,g}$ extends to a map

$$\mathscr{I}_{d,g} \colon A^d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) \to \mathscr{S}(\mathbb{R}^{g-d}) \tag{3.17}$$

by setting for a form $\omega = f(x,y)\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d$

$$(\mathscr{I}_{d,g}\omega)(y) := \int_{\mathbb{R}^d} f(x,y)\,\mathrm{d}x.$$

PROPOSITION 3.11. *Let $s \geq 0$ and $1 \leq d \leq g$. Consider $\mathscr{S}(\mathbb{R}^g)$ as a H^g-module with parameter $h = 1$ and let $\omega \in A^d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$. The form ω is exact if and only if $\mathscr{I}_{d,g}\omega = 0$. Furthermore, for every $\varepsilon > 0$ there exist a constant $C = C(s,\varepsilon,g,d) > 0$ and a linear map*

$$\mathrm{d}_{-1} \colon \ker \mathscr{I}_{d,g} \subset A^d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) \to A^{d-1}(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$$

associating to every $\omega \in \ker \mathscr{I}_{d,g}$ a primitive Ω of ω satisfying the estimate

$$|||\Omega|||_s \leq C \, |||\omega|||_{s+(d+1)/2+\varepsilon} \, . \tag{3.18}$$

Proof. The "only if" part of the statement is obvious. For $d = 1$ and any $g \geq 1$, this is Lemma 3.9. Indeed, a primitive of the 1-form $\omega = f(x,y)\,\mathrm{d}x$ is the 0-form $\Omega := (\mathscr{P}f)(x,y)$, and the estimate for the norms comes from (3.5).

Assume, by recurrence, that the Proposition is true for all $g' < g$ and all $d \leq g'$. Let $\omega \in A^d(\mathbb{R}^d, \mathscr{S}(\mathbb{R}^g))$ be a d-form such that $\mathscr{I}_{d,g}\omega = 0$. Consider $\mathscr{I}\omega$ in $A^{d-1}(\mathbb{R}^{d-1}, \mathscr{S}(\mathbb{R}^{g-1}))$, where $\mathscr{I}$ is the operator defined in the previous proof (see (3.7)). It is clear from the definitions that $\mathscr{I}_{d,g}(\omega) = 0$ implies $\mathscr{I}_{d-1,g-1}\mathscr{I}\omega = 0$. By recurrence, $\mathscr{I}\omega = \mathrm{d}\eta$ for a primitive $\eta \in A^{k-1}(\mathbb{R}^k, \mathscr{S}(\mathbb{R}^g))$ satisfying the estimate

$$|||\eta|||_s \leq C \, |||\mathscr{I}\omega|||_{s+d/2+\varepsilon} \, . \tag{3.19}$$

As in the previous proof, one verifies that the form $\mathrm{d}_{-1}\omega := \Omega := \mathscr{K}\omega + \mathscr{E}\eta$ in $A^{d-1}(\mathbb{R}^d, \mathscr{S}(\mathbb{R}^g))$ is a primitive of ω (where the operators $\mathscr{E}$ and $\mathscr{K}$ are defined in the previous proof, see (3.8) and (3.12)). Therefore, from Lemma 3.6 and the estimates (3.9), (3.10), (3.13), and (3.19), we have, for some constants C's which only depend on $s \geq 0$ and $\varepsilon > 0$,

$$
\begin{aligned}
|||\Omega|||_s &\leq |||\mathscr{K}\omega|||_s + |||\mathscr{E}\eta|||_s \\
&\leq C' \, |||\omega|||_{s+3/2+\varepsilon} + C'' \, |||\eta|||_s \\
&\leq C' \, |||\omega|||_{s+3/2+\varepsilon} + C''' \, |||\mathscr{I}\omega|||_{s+d/2+\varepsilon/2} \\
&\leq C' \, |||\omega|||_{s+3/2+\varepsilon} + C'''' \, |||\omega|||_{s+d/2+1/2+\varepsilon} \\
&\leq C \, |||\omega|||_{s+(d+1)/2+\varepsilon}.
\end{aligned}
\tag{3.20}
$$

The proof is complete. □

PROPOSITION 3.12. *Let $s \geq 0$ and $1 \leq d \leq g$. Consider $\mathscr{S}(\mathbb{R}^g)$ as a H^g-module with parameter $h = 1$. For any $k = 0, \dots, d$, the space of coboundaries $B^d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ is a tame direct summand of $A^k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$. In fact, there exist linear maps*

$$M^k : A^k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) \to B^k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$$

satisfying the following properties:

- *the restriction of M^k to $B^k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ is the identity map;*
- *the map M^k satisfies, for any $\varepsilon > 0$, tame estimates of degree $(k+3)/2 + \varepsilon$ if $k < d$ and of degree $d/2 + \varepsilon$ if $k = d$.*

Proof. For $\omega = f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d \in A^d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ let

$$M^d(\omega) = \omega - (\mathscr{E}_{d,g} \circ \mathscr{I}_{d,g} f) \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d.$$

Lemmas 3.6 and 3.8 show that M^d is a linear tame map of degree $d/2 + \varepsilon$ for every $\varepsilon > 0$. Clearly for $\omega \in B^d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ we have $M^d(\omega) = \omega$. Since the map M^d maps $A^d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ into $B^d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$, we have proved that $B^d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ is a direct summand of $A^d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$.

Now consider the case where $k < d$. We have $B^k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) = Z^k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$. For $\omega \in A^k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ let

$$M^k(\omega) = \omega - \mathrm{d}_{-1} \circ \mathrm{d}(\omega).$$

The map M^k is a linear tame map of degree $(k+3)/2 + \varepsilon$ for every $\varepsilon > 0$.

Clearly for $\omega \in Z^k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ we have $M(\omega) = \omega$. Furthermore $\mathrm{d} \circ M = 0$. Thus the map M^k sends $A^k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ into $Z^k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$. We have proved that $Z^d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ is a direct summand of $A^d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$. □

P-invariant currents of dimension $\dim \mathsf{P}$. Recall that the space of currents of dimension k is the space $A_k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ of continuous linear functionals on $A^k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ and that $A_k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ is identified with $\Lambda^k \mathfrak{p} \otimes \mathscr{S}'(\mathbb{R}^g)$. For any $s \geq 0$, the space $\Lambda^k \mathfrak{p} \otimes W^{-s}(\mathbb{R}^g)$ is identified with the space of currents of dimension k and Sobolev order s.

It is clear, from Lemma 3.5, that $\mathscr{I}_g = \mathscr{I}_{g,g} \in W^{-s}(\mathbb{R}^g)$ for any $s > g/2$, i.e., it is a closed current of dimension g and Sobolev order $g/2 + \varepsilon$, for any $\varepsilon > 0$.

For $d < g$ and $t > 0$, consider the currents $D \circ \mathscr{I}_{d,g}$ with $D \in W^{-t}(\mathbb{R}^{g-d})$. It follows from Lemma 3.6 that such currents belong to $\Lambda^d \mathfrak{p} \otimes W^{-s}(\mathbb{R}^g)$ for any $s > t + d/2$. It is also easily seen that they are closed.

We have the following proposition, whose proof follows immediately from Lemma 3.6 and Proposition 3.11.

PROPOSITION 3.13. *For any $s > \dim \mathsf{P}/2$, the space of P-invariant currents of dimension $d := \dim \mathsf{P}$ and order s is a closed subspace of $\Lambda^d \mathfrak{p} \otimes W^{-s}(\mathbb{R}^g)$ and it coincides with the space of closed currents of dimension d. It is*

- *a one dimensional space spanned by $\mathscr{I}_g$, if $\dim \mathsf{P} = g$;*
- *an infinite-dimensional space generated by*

$$I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) = \left\{ D \circ \mathscr{I}_{d,g} \,\middle|\, D \in L^2(\mathbb{R}^{g-d})' \right\}$$

if $\dim \mathsf{P} < g$. We have $I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) \subset W^{-d/2-\varepsilon}(\mathbb{R}^g)$, for all $\varepsilon > 0$.

Let $\omega \in \Lambda^d \mathfrak{p}' \otimes W^s(\mathbb{R}^g)$ with $s > (d+1)/2$. Then ω admits a primitive Ω if and only if $T(\omega) = 0$ for all $T \in I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$; under this hypothesis we may have $\Omega \in \Lambda^{d-1} \mathfrak{p}' \otimes W^t(\mathbb{R}^g)$ for any $t < s - (d+1)/2$.

Bounds uniform in the parameter *h*. Here we observe that the estimates in Propositions 3.10 and 3.11 are uniform in the Planck constant h, provided that this constant is bounded away from zero.

PROPOSITION 3.14. *Let $s \geq 0$ and $1 \leq k \leq d \leq g$, and consider the H^g-module $\mathscr{S}(\mathbb{R}^g)$ with parameter h such that $|h| \geq h_0 > 0$. Let $B^k = Z^k(\mathbb{R}^d, \mathscr{S}(\mathbb{R}^g))$ if $k < d$ and $B^d = \ker \mathscr{I}_{d,g}$ if $k = d$. For every $\varepsilon > 0$ there exist a positive constant $C = C(s, \varepsilon, g, d, h_0)$ and a linear map*

$$\mathrm{d}_{-1} \colon B^k \to A^{k-1}(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$$

associating to every $\omega \in B$ a primitive $\Omega = \mathrm{d}_{-1}\omega \in A^{k-1}(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ satisfying the estimate

$$|||\Omega|||_s \leq C\, |||\, \omega\, |||_{s+(k+1)/2+\varepsilon}\,. \tag{3.21}$$

Furthermore, for any $\varepsilon > 0$ there exists a constant $C' = C'(s, \varepsilon, g, d, h_0) > 0$ such that the splitting linear maps of Proposition 3.12

$$M^k : A^k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) \to B^k(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$$

satisfy tame estimates

$$|||\, M^k(\omega)\, |||_s \leq C'\, |||\, \omega |||_{s+w}$$

where $w = (k+3)/2 + \varepsilon$ if $k < d$ and $w = d/2 + \varepsilon$ if $k = d$.

Proof. From (2.5) we see that the boundary operators in the Schrödinger representation with Planck constant h are ${}^{\hbar}\mathrm{d} := \rho^h(\mathrm{d}) = |h|^{1/2}\,\mathrm{d}$. Therefore, if $\omega = \mathrm{d}\Omega$, then $\omega = {}^{\hbar}\mathrm{d}\Omega'$ with $\Omega' = |h|^{-1/2}\Omega$. Consequently, by (3.3), the estimates (3.6) and (3.18) imply

$$\begin{aligned} |||\Omega'|||_{s,h} &= |h|^{-1/2}\, |||\, \Omega|||_{s,h} = |h|^{s/2-1/2}\, |||\, \Omega|||_s \\ &\leq C\, |h|^{s/2-1/2}\, |||\, \omega|||_{s+(k+1)/2+\varepsilon} \\ &= C\, |h|^{-(k+1+\varepsilon)/2}\, |||\, \omega|||_{s+(k+1)/2+\varepsilon,h} \\ &\leq C'\, |||\, \omega\, |||_{s+t+\varepsilon,h}\,. \end{aligned} \tag{3.22}$$

for some C' depending also on h_0. The second statement is proved in an analogous manner. □

Comparison with the usual Sobolev norms. The standard Sobolev norms associated with a Heisenberg basis (X_i, Ξ_j, T) of $\mathfrak{h}^g$ were defined in Remark 3.2. For a H^g-module $\mathscr{S}(\mathbb{R}^g)$ with parameter h, the image of the Laplacian

$$-(X_1^2 + \cdots + X_g^2 + \Xi_1^2 + \cdots + \Xi_g^2 + T^2) \in \mathfrak{U}(\mathfrak{h}^g)$$

under ρ_h is $\Delta_g = H_g + h^2$. Thus

$$\|f\|_s^2 = \langle f, (1+\Delta_g)^s f\rangle = \langle f, (1+h^2+H_g)^s f\rangle.$$

Here we claim that the uniform bound as in Proposition 3.14 continues to hold with respect to the usual Sobolev norms. This is a consequence of the following easy lemma which applies to $\mathscr{S}(\mathbb{R}^g)$ but also to any tensor product of $\mathscr{S}(\mathbb{R}^g)$ with some finite dimensional Euclidean space.

LEMMA 3.15. *Let $L\colon \mathscr{S}(\mathbb{R}^g) \to \mathscr{S}(\mathbb{R}^g)$ be a linear map satisfying, for some $t \geq 0$ and every $s \geq 0$, the estimate*

$$|||L(f)|||_s \leq C(s)\, |||\, f\, |||_{s+t}\,.$$

Then for every $s \geq 0$ we have

$$\|L(f)\|_s \leq C_1(s)\|f\|_{s+t},$$

where $C_1(s) = \max_{u\in[0,s+1]} C(u)$.

Proof. For integer $s = n$, using the binomial formula, we get

$$\begin{aligned}\|L(f)\|_n^2 &:= \left\langle L(f), (H_g+1+h^2)^n L(f)\right\rangle_0 \\ &= \sum_{j=0}^n \binom{n}{j} \|(1+h^2)^{(n-j)/2} H_g^{j/2} L(f)\|_0^2 \\ &\leq C'(n) \sum_{j=0}^n \binom{n}{k} \|(1+h^2)^{(n-j)/2} H_g^{(j+t)/2} f\|_0^2 \\ &= C'(n)\, \|(1+\Delta_g)^n H_g^{t/2} f\|_0^2 \\ &\leq C'(n)\, \|f\|_{n+t}^2,\end{aligned}$$

with $C'(n) := \max_{j\in[0,n]} C(j)^2$. For non integer s the lemma follows by interpolation. □

3.3. **Proofs of Theorems 1.5 and 1.6.** We are now in a position to integrate over Schrödinger representations and obtain our main result on the cohomology of $\mathsf{P} < \mathsf{H}^g$ with values in Fréchet H^g-modules.

THEOREM 3.16. *Let P be a d-dimensional isotropic subgroup of H^g, and let F^∞ be the Fréchet space of C^∞-vectors of a unitary H^g-module F. Let $F = \int F_\alpha\, \mathrm{d}\alpha$ be the direct integral decomposition of F into irreducible sub-modules. Suppose that*

1. *F does not contain any one-dimensional sub-modules;*

2. *A generator of the center* $\mathsf{Z}(\mathsf{H}^g)$ *acting on* F *has a spectral gap.*

Then the reduced and the ordinary cohomology of the complex $A^*(\mathfrak{p}, F^\infty)$ *coincide. In fact, for all* $k = 1, \dots, d$*, there are linear maps*

$$\mathrm{d}_{-1} : B^k(\mathfrak{p}, F^\infty) \to A^{k-1}(\mathfrak{p}, F^\infty)$$

associating to each $\omega \in B^k(\mathfrak{p}, F^\infty)$ *a primitive of* ω *and satisfying tame estimates of degree* $(k+1)/2 + \varepsilon$ *for any* $\varepsilon > 0$.

We have $H^k(\mathfrak{p}, F^\infty) = 0$ *for* $k < d$*; in degree* d*, we have that* $H^d(\mathfrak{p}, F^\infty)$ *is finite dimensional only if* $d = g$ *and the measure* $d\alpha$ *has finite support.*

For any $k = 0, \dots, d$ *and any* $\varepsilon > 0$*, there exist a constant* C *and a linear map*

$$M^k : A^k(\mathfrak{p}, F^\infty) \to B^k(\mathfrak{p}, F^\infty)$$

such that the restriction of M^k *to* $B^k(\mathfrak{p}, F^\infty)$ *is the identity map and the following estimate holds:*

$$\|M^k \omega\|_s \le C \|\omega\|_{s+w}, \quad \forall\ \omega \in A^k(\mathfrak{p}, F^\infty),$$

where $w = (k+3)/2 + \varepsilon$ *if* $k < d$ *and* $w = d/2 + \varepsilon$ *if* $k = d$*. Hence the space of coboundaries* $B^k(\mathfrak{p}, F^\infty)$ *is a tame direct summand of* $A^k(\mathfrak{p}, F^\infty)$.

(The hypotheses 1 and 2 of the above theorem could be stated more briefly by saying that F satisfies the following property: any non-trivial unitary H^g-module weakly contained in F is infinite dimensional.)

Proof. Let F^∞ be the Fréchet space of C^∞-vectors of a unitary H^g-module (ρ, F). Let $F = \int F_\alpha \mathrm{d}\alpha$ be the direct integral decomposition of F into irreducible submodules (ρ_α, F_α). The hypotheses of Theorem 3.16 imply that there exists $h_0 > 0$ such that for almost every α the H^g-module F_α is unitarily equivalent to a Schrödinger module with parameter h satisfying $|h| \ge h_0$.

For any $s \in \mathbb{R}$, we also have a decomposition of the Sobolev spaces $W^s(F, \rho)$ as direct integrals $\int W^s(F_\alpha, \rho_\alpha)\mathrm{d}\alpha$; this is because the operator $1 + \Delta_g$ defining the Sobolev norms is an element of the enveloping algebra $\mathfrak{U}(\mathfrak{h}^g)$ and because the spaces F_α are $\mathfrak{U}(\mathfrak{h}^g)$-invariant. It follows that any form $\omega \in A^k(\mathfrak{p}, F^\infty)$ has a decomposition $\omega = \int \omega_\alpha \,\mathrm{d}\alpha$ with $\omega_a \in A^k(\mathfrak{p}, F^\infty_\alpha)$ and

$$\|\omega\|^2_{W^s(F,\rho)} = \int \|\omega_\alpha\|^2_{W^s(F_\alpha, \rho_\alpha)} \,\mathrm{d}\alpha. \tag{3.23}$$

For the same reason mentioned above, we have

$$\mathrm{d}\omega = \int (\mathrm{d}\omega_\alpha) \,\mathrm{d}\alpha. \tag{3.24}$$

Hence ω is closed if and only if ω_α is closed for almost all α, that is, $Z^k(\mathfrak{p}, W^s(F, \rho)) = \int Z^k(\mathfrak{p}, W^s(F_\alpha, \rho_\alpha)) \,\mathrm{d}\alpha$.

For $k < d$ we set $B^k_\alpha = Z^k(\mathfrak{p}, F^\infty_\alpha)$. For $k = d$ we set $B^d_\alpha = \ker I_{d,g,\alpha}$, where $I_{d,g,\alpha} : A^d(\mathfrak{p}, F^\infty_\alpha) \to \mathscr{S}(\mathbb{R}^{g-d})$ are the tame maps defined, for each α, as in (3.17).

By Proposition 3.14 and Lemma 3.15, we have a constant $C = C(s, \varepsilon, g, d, h_0)$ and, for each α, a linear map

$$\mathrm{d}_{-1,\alpha} : B^k_\alpha \to A^{k-1}(\mathfrak{p}, F^\infty_\alpha)$$

associating to each $\omega \in B^k_\alpha(\mathfrak{p}, F^\infty_\alpha)$ a primitive $\Omega = \mathrm{d}_{-1}\omega$ of ω satisfying the estimates

$$\|\mathrm{d}_{-1,\alpha}\omega\|_{W^s(F_\alpha,\rho_\alpha)} \le C\,\|\omega\|_{W^{s+(k+1)/2+\varepsilon}(F_\alpha,\rho_\alpha)}. \tag{3.25}$$

Let B^k be the graded Fréchet subspace of $A^k(\mathfrak{p}, F^\infty)$ defined as $\int B^k_\alpha \,\mathrm{d}\alpha$. For $k < d$ we have $B^k = Z^k(\mathfrak{p}, F^\infty)$ and, in degree d, we have $B^d \supset B^d(\mathfrak{p}, F^\infty)$.

The above estimate shows that it is possible for one to define a linear map $\mathrm{d}_{-1} : B^k \to A^{k-1}(\mathfrak{p}, F^\infty)$ by setting, for $\omega = \int \omega_\alpha \,\mathrm{d}\alpha \in B^k$,

$$\mathrm{d}_{-1}\omega := \int \mathrm{d}_{-1,\alpha}\omega_\alpha \,\mathrm{d}\alpha.$$

By (3.23) and (3.24), the estimates (3.25) are still true if we replace $\mathrm{d}_{-1,\alpha}$ by d_{-1}.

This shows that d_{-1} is a tame map of degree $(k+1)/2+\varepsilon$ for all $\varepsilon > 0$ associating to each $\omega \in B^k$ a primitive of ω.

Thus $H^k(\mathfrak{p}, F^\infty) = 0$ if $k < d$. For $k = d$, we have $H^d(\mathfrak{p}, F^\infty) = \int H^d(\mathfrak{p}, F^\infty_\alpha)\,\mathrm{d}\alpha$. By Proposition 3.11, we have $H^d(\mathfrak{p}, F^\infty_\alpha) \approx \mathscr{S}(\mathbb{R}^{g-d})$, hence the top degree cohomology is infinite dimensional if $d < g$ and one-dimensional if $d = g$. This shows that $H^d(\mathfrak{p}, F^\infty)$ is finite dimensional if and only if $d = g$ and the measure $\mathrm{d}\alpha$ has finite support.

Finally for each α, we have tame maps M^k_α given by Proposition 3.12. Setting $M^k = \int M^k_\alpha \,\mathrm{d}\alpha$ we obtain maps M^k satisfying the Theorem's conclusion. □

Proof of Theorem 1.5. The proof is immediate as the space $F = L^2_0(\mathsf{M})$ formed by the L^2 functions on M of average zero along the fibers of the central fibration of M satisfy the hypothesis of the theorem above. In fact $L^2_0(\mathsf{M})$ is a direct sum of irreducible representations of H^g on which the generator Z of the center $\mathsf{Z}(\mathsf{H}^g)$ acts as scalar multiplication by $2\pi n$, with $n \in \mathbb{Z} \smallsetminus \{0\}$. □

Proof of Theorem 1.6. The theorem follows from the theorem above and the "folklore" Theorem 3.4, as explained at the beginning of Section 3. □

4. SOBOLEV STRUCTURES AND BEST SOBOLEV CONSTANT

4.1. **Sobolev bundles.**

Sobolev spaces. The group $\mathrm{Sp}_{2g}(\mathbb{R}) < \mathrm{Aut}(\mathsf{H}^g) \approx \mathrm{Aut}(\mathfrak{h}^g)$ acts (on the right) on the enveloping algebra $\mathfrak{U}(\mathfrak{h}^g)$ in the following way: we identify $\mathfrak{U}(\mathfrak{h}^g)$ with the algebra of right invariant differential operators on H^g; if $V \in \mathfrak{U}(\mathfrak{h}^g)$ and $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$, the action of α on V yields the differential operator V_α defined by

$$V_\alpha(f) := \alpha^* V\big((\alpha^{-1})^* f\big), \qquad f \in C^\infty(\mathsf{H}^g). \tag{4.1}$$

Let $\Delta = -(X_1^2 + \cdots + X_g^2 + \Xi_1^2 + \cdots + \Xi_g^2 + T^2) \in \mathfrak{U}(\mathfrak{h}^g)$ denote the Laplacian on H^g defined via the "standard" basis (X_i, Ξ_j, T) (cf. sect. 2.1). Then $\Delta_\alpha = -((\alpha^{-1}X_1)^2 + \cdots + (\alpha^{-1}\Xi_g)^2 + T^2)$, that is, Δ_α is the Laplacian on H^g defined by the basis $(\alpha^{-1}(X_i), \alpha^{-1}(\Xi_j), T)$.

Let Γ' be any lattice of H^g and $\mathsf{M}' := \mathsf{H}^g/\Gamma'$ the corresponding nilmanifold. For each $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$, the operator Δ_α is an elliptic, positive and essentially self-adjoint operator on $L^2(\mathsf{M}')$. Recall that $L^2_0(\mathsf{M}')$ denotes the space of ell-two functions on M' with zero average along the fibers of the toral projection. Its norm is defined via the ell-two Hermitian product $\langle\cdot,\cdot\rangle$ with integration done with respect to the normalized Haar measure. Setting $L_\alpha = 1+\Delta_\alpha$ we define the Sobolev spaces

$$W^s_\alpha(\mathsf{M}') := L_\alpha^{-s/2} L^2_0(\mathsf{M}'), \tag{4.2}$$

which are Hilbert spaces equipped with the inner product

$$\langle f_1, f_2\rangle_{s,\alpha} := \langle L_\alpha^{s/2} f_1, L_\alpha^{s/2} f_2\rangle = \langle f_1, L^s_\alpha f_2\rangle.$$

For simplicity, we denote by $W^s(\mathsf{M}')$ the Sobolev spaces defined via the operator $1+\Delta$. The space $W^{-s}_\alpha(\mathsf{M}')$ is canonically isomorphic to the dual Hilbert space of $W^s_\alpha(\mathsf{M}')$.

REMARK 4.1. It is useful to notice that, since the Laplacian Δ is invariant under the above action of the maximal compact subgroup K_g of $\mathrm{Sp}_{2g}(\mathbb{R})$, the Sobolev space $W^{-s}_\alpha(\mathsf{M}')$ depends only on the class $\mathsf{K}_g\alpha \in \mathfrak{H}_g$ in the Siegel upper half-space.

Let Γ be the standard lattice of H^g and $\mathsf{M} := \mathsf{H}^g/\Gamma$. For $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$, let $\Gamma_\alpha := \alpha(\Gamma)$ and $\mathsf{M}_\alpha := \mathsf{H}^g/\Gamma_\alpha$ the corresponding nilmanifold. The automorphism α induces a diffeomorphism (denoted with the same symbol) according to the formula

$$\alpha : \mathsf{M} \to \mathsf{M}_\alpha, \qquad h\Gamma \mapsto \alpha(h)\Gamma_\alpha, \quad \forall\ h \in \mathsf{H}^g.$$

It is immediate that the pull-back map $\alpha^* : C^\infty(\mathsf{M}_\alpha) \to C^\infty(\mathsf{M})$ satisfies

$$\alpha^*(\Delta f) = \Delta_\alpha(\alpha^* f), \quad f \in C^\infty(\mathsf{M}_\alpha);$$

since α^* preserves the volume, we obtain an isometry

$$\alpha^* : W^s(\mathsf{M}_\alpha) \to W^s_\alpha(\mathsf{M}).$$

Observe that, as topological vector spaces, the spaces $W^s_\alpha(\mathsf{M})$, with $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$, are all isomorphic to $W^s(\mathsf{M})$. Only their Hilbert structure varies as α ranges in $\mathrm{Sp}_{2g}(\mathbb{R})$. In fact we have the following lemma, whose proof is omitted.

LEMMA 4.2. *For every $R > 0$ there exists a constant $C(s) > 0$ such that for all $\alpha, \beta \in \mathrm{Sp}_{2g}(\mathbb{R})$ with $\mathrm{dist}(\alpha,\beta) < R$ we have*

$$\|\varphi\|_{s,\alpha} \le C(s)\,(1+\mathrm{dist}(\alpha,\beta)^2)^{|s|/2} \cdot \|\varphi\|_{s,\beta}.$$

Here, $\mathrm{dist}(\cdot,\cdot)$ *is some left-invariant distance on* $\mathrm{Sp}_{2g}(\mathbb{R})$.

LEMMA 4.3. *Let $s \ge 0$. For $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$, the pull-back map γ^* is an isometry of $W^s_\alpha(\mathsf{M})$ onto $W^s_{\alpha\gamma}(\mathsf{M})$. Hence $\gamma_* : W^{-s}_{\alpha\gamma}(\mathsf{M}) \to W^{-s}_\alpha(\mathsf{M})$ is an isometry.*

Proof. By the above, we have isometries $(\alpha\gamma)^* : W^s(\mathsf{M}_{\alpha\gamma}) \to W^s_{\alpha\gamma}(\mathsf{M})$ and $\alpha^* : W^s(\mathsf{M}_\alpha) \to W^s_\alpha(\mathsf{M})$. However, $\mathsf{M}_{\alpha\gamma} = \mathsf{M}_\alpha$, since $\Gamma_{\alpha\gamma} = \Gamma_\alpha$. It follows that $\gamma^* = (\alpha\gamma)^*(\alpha^*)^{-1}$ is an isometry of $W^s_\alpha(\mathsf{M})$ onto $W^s_{\alpha\gamma}(\mathsf{M})$. □

The Sobolev bundle over the moduli space and its dual. For $s \geq 0$, let us consider $W^s(\mathsf{M})$ as a topological vector space. The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on the right on the trivial bundles $\mathrm{Sp}_{2g}(\mathbb{R}) \times W^s(\mathsf{M}) \to \mathrm{Sp}_{2g}(\mathbb{R})$ according to

$$(\alpha, \varphi) \mapsto (\alpha, \varphi)\gamma := (\alpha\gamma, \gamma^* \varphi),$$

for all $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$, and all $(\alpha, \varphi) \in \mathrm{Sp}_{2g}(\mathbb{R}) \times W^s(\mathsf{M})$. By Lemma 4.3, the norms

$$\|(\alpha, \varphi)\|_s := \|\varphi\|_{s,\alpha}$$

are $\mathrm{Sp}_{2g}(\mathbb{Z})$-invariant. In fact, by that lemma we have $\|\gamma^* \varphi\|_{s,\alpha\gamma} = \|\varphi\|_{s,\alpha}$. Consequently, we obtain a quotient flat bundle of Sobolev spaces over the moduli space:

$$(\mathrm{Sp}_{2g}(\mathbb{R}) \times W^s(\mathsf{M}))/\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathfrak{M}_g = \mathrm{Sp}_{2g}(\mathbb{R})/\mathrm{Sp}_{2g}(\mathbb{Z});$$

the fiber over $[\alpha] \in \mathfrak{M}_g$ may be locally identified with the space $W^s_\alpha(\mathsf{M})$ normed by $\|\cdot\|_{s,\alpha}$. We denote this bundle by $\mathfrak{W}^s$ and the class of (α, φ) by $[\alpha, \varphi]$.

By the duality paring, we also have a flat bundle of distributions $\mathfrak{W}^{-s}$ whose fiber over $[\alpha] \in \mathfrak{M}_g$ may be locally identified with the space $W^{-s}_\alpha(\mathsf{M})$ normed by $\|\cdot\|_{-s,\alpha}$. Observe that for this bundle $(\alpha, \mathscr{D}) \equiv (\alpha\gamma^{-1}, \gamma_* \mathscr{D})$ for all $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and $(\alpha, \mathscr{D}) \in \mathrm{Sp}_{2g}(\mathbb{R}) \times W^{-s}(\mathsf{M})$. We denote the class of $(\alpha, \mathscr{D})$ by $[\alpha, \mathscr{D}]$.

4.2. **Best Sobolev constant.**

The best Sobolev constant. The Sobolev embedding theorem implies that for any $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$ and any $s > g + 1/2$ there exists a constant $B_s(\alpha) > 0$ such that any $f \in W^s_\alpha(\mathsf{M})$ has a continuous representative such that

$$\|f\|_\infty \leq B_s(\alpha) \cdot \|f\|_{s,\alpha}. \tag{4.3}$$

For any Sobolev order $s > g + 1/2$, the *best Sobolev constant* is defined as the function on the group of automorphisms $\mathrm{Sp}_{2g}(\mathbb{R})$ given by

$$B_s(\alpha) := \sup_{f \in W^s_\alpha(\mathsf{M}) \setminus \{0\}} \frac{\|f\|_\infty}{\|f\|_{s,\alpha}}. \tag{4.4}$$

LEMMA 4.4. *The best Sobolev constant B_s is a $\mathrm{Sp}_{2g}(\mathbb{Z})$-modular function on $\mathfrak{H}_g$, i.e., $B_s(\alpha) = B_s(\kappa\alpha\gamma)$ for all $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$, all $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and all $\kappa \in \mathsf{K}_g$.*

Proof. The K_g invariance is an immediate consequence of Remark 4.1. By Lemma 4.3, the the pull-back map γ^* is an isometry of $W^s_\alpha(\mathsf{M})$ onto $W^s_{\alpha\gamma}(\mathsf{M})$. As the map γ^* is also an isometry for the sup-norm, the lemma follows. □

Thus, we may regard B_s as a function on the Siegel modular variety $\Sigma_g = \mathsf{K}_g \backslash \mathrm{Sp}_{2g}(\mathbb{R})/\mathrm{Sp}_{2g}(\mathbb{Z})$ or as a $\mathrm{Sp}_{2g}(\mathbb{Z})$-invariant function on the Siegel upper half-space $\mathfrak{H}_g$. Recalling that $[[\alpha]]$ denotes the class of $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$ in Σ_g, we shall write $B_s([[\alpha]])$ or $B_s([\alpha])$ for $B_s(\alpha)$.

Let $\mathsf{A} \subset \mathrm{Sp}_{2g}(\mathbb{R})$ denote the Cartan subgroup of diagonal symplectic matrices, $\mathsf{A}^+ \subset \mathsf{A}$ the subgroup of positive matrices, and $\mathfrak{a} \subset \mathfrak{sp}_{2g}$ the Lie algebra of A.

For $\alpha = \left(\begin{smallmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{smallmatrix}\right) \in \mathsf{A}^+$, where $\delta = \operatorname{diag}(\delta_1, \dots, \delta_g)$, we define

$$\daleth(\alpha) := \prod_{i=1}^{g} (\delta_i + \delta_i^{-1}).$$

PROPOSITION 4.5. *For any order $s > g + 1/2$ and any $\alpha \in \mathsf{A}^+$ there exists a constant $C = C(s) > 0$ such that*

$$B_s([[\alpha]]) \le C \, \daleth(\alpha)^{1/2}.$$

Proof. Let $\alpha = \left(\begin{smallmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{smallmatrix}\right) \in \mathsf{A}^+$, where $\delta = \operatorname{diag}(\delta_1, \dots, \delta_g)$ as above. Since the map $\alpha^* : W^s(\mathsf{M}_\alpha) \to W^s_\alpha(\mathsf{M})$ is an isometry, the best s-Sobolev constant $B_s([\alpha])$ for the operator $1 + \Delta_\alpha$ on the Heisenberg manifold M is equal to the best s-Sobolev constant for the operator $1 + \Delta$ on the Heisenberg manifold M_α, namely

$$B_s([\alpha]) = \sup_{f \in W^s(\mathsf{M}_\alpha) \setminus \{0\}} \frac{\|f\|_\infty}{\|(1+\Delta)^{s/2} f\|_{L^2(\mathsf{M}_\alpha)}}. \tag{4.5}$$

We fix the fundamental domain $F = [0,1]^g \times [0,1]^g \times [0,1/2]$ for the action of the lattice Γ on H^g. By the standard Sobolev embedding theorem, for any $s > g + 1/2$ there exists a constant $C(s)$ such that for any $f \in W^s_{\mathrm{loc}}(\mathsf{H}^g)$ we have

$$|f(I)|^2 \le C(s) \int_F |(1+\Delta)^{s/2} f(x)|^2 \, \mathrm{d}x,$$

where $I = (0,0,0)$ is the identity of H^g and $\mathrm{d}x$ is the Haar measure assigning volume 1 to F. Since left and right translation commute and since $(1+\Delta)$ operates on the left, for every $f \in W^s_{\mathrm{loc}}(\mathsf{H}^g)$ and every $h \in \mathsf{H}^g$ we have

$$|f(h)|^2 \le C(s) \int_{Fh} |(1+\Delta)^{s/2} f(x)|^2 \, \mathrm{d}x. \tag{4.6}$$

It easy to see that, for any $h \in \mathsf{H}^g$, the set Fh is also a fundamental domain for Γ. Furthermore, if we let $p_\alpha : h \in \mathsf{H}^g \mapsto h\Gamma_\alpha \in \mathsf{M}_\alpha$ denote the natural projection, the projection $p_\alpha((Fh)^o)$ of the interior of Fh covers each point of $\mathsf{M}_{\alpha^{-1}}$ at most

$$2^g \prod_{i=1}^{g} \max\{\delta_i, {\delta_i}^{-1}\} \le 2^g \, \daleth(\alpha) \tag{4.7}$$

times.

Given any $f \in W^s(\mathsf{M}_\alpha)$, let $\tilde f = f \circ p_\alpha$. Then, for any $h \in \mathsf{H}^g$ and any integer $n \ge 0$

$$\begin{aligned} \int_{Fh} \left|(1+\Delta)^{n/2} \tilde f(x)\right|^2 \mathrm{d}x &\le 2^g \, \daleth(\alpha) \int_{\mathsf{M}_\alpha} \left|(1+\Delta)^{n/2} f(x)\right|^2 \mathrm{d}x \qquad \text{(by (4.7))} \\ &= 2^g \, \daleth(\alpha) \, \|(1+\Delta)^{n/2} f\|^2_{L^2(\mathsf{M}_\alpha)}. \end{aligned}$$

We deduce, by interpolation and by (4.6), that for any $s \ge g + 1/2$ there exists a constant C such that

$$\sup_{h \in \mathsf{M}_\alpha} |f(h)| \le C \, (\daleth(\alpha))^{1/2} \, \|f\|_{W^s(\mathsf{M}_\alpha)}. \tag{4.8}$$

This concludes the proof. □

4.3. **Best Sobolev constant and height function.** The height of a point $Z \in \mathfrak{H}_g$ is the positive number

$$\mathrm{hgt}(Z) := \det \Im(Z). \tag{4.9}$$

Let $F_g \subset \mathfrak{H}_g$ denote the Siegel fundamental domain for the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathfrak{H}_g$ (see [31]). We define the *height function* $\mathrm{Hgt}\colon \Sigma_g \to \mathbb{R}^+$ to be the maximal height of a $\mathrm{Sp}_{2g}(\mathbb{Z})$-orbit (which is attained by Proposition 1 of [5]), or, equivalently, the height of the unique representative of an orbit inside F_g. Thus, if $[Z] \in \Sigma_g$ denotes the class of $Z \in \mathfrak{H}_g$ in the Siegel modular variety,

$$\mathrm{Hgt}([Z]) := \max_{\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})} \mathrm{hgt}(\gamma(Z)) = \max_{\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})} \det \Im(\gamma(Z)). \tag{4.10}$$

Any point in $\mathfrak{H}_g$ may be uniquely written as $Z = X + iW^\top DW$, where $X = (x_{ij})$ is a symmetric real matrix, $W = (w_{ij})$ is an upper triangular real matrix with ones on the diagonal, and $D = \mathrm{diag}(\delta_1, \dots, \delta_g)$ is a diagonal positive matrix. The coordinates $(x_{ij})_{1\le i\le j\le g}$, $(w_{ij})_{1\le i<j\le g}$, and $(\delta_i)_{1\le i\le g}$ thus defined are called Iwasawa coordinates on the Siegel upper half-space. For $t > 0$, define $S_g(t) \subset \mathfrak{H}_g$ as the set of those $Z = X + iW^\top DW \in \mathfrak{H}_g$ such that

$$|x_{ij}| < t \qquad (1 \le i, j \le g) \tag{4.11}$$

$$|w_{ij}| < t \qquad (i < j) \tag{4.12}$$

$$1 < t\delta_1 \qquad \text{and} \qquad 0 < \delta_k < t\delta_{k+1} \qquad (1 \le k \le g-1). \tag{4.13}$$

For all t sufficiently large, $S_g(t)$ is a "fundamental open set" for the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathfrak{H}_g$, containing the Siegel fundamental domain F_g (see [5] or [31]). We will need the following Lemma, which is an easy consequence of the expression

$$ds^2 = \mathrm{tr}\left(dXY^{-1}dXY^{-1} + dDD^{-1}dDD^{-1} + 2(W^\top)^{-1}dW^\top DdWW^{-1}D^{-1}\right) \tag{4.14}$$

for the Siegel metric in Iwasawa coordinates, where $Y = W^\top DW$.

LEMMA 4.6. *Any point $Z = X + iW^\top DW$ inside a Siegel fundamental open set $S_g(t)$ is at a bounded distance from the point iD.*

Proof. Let $Z = X + iW^\top DW$, with W and D as explained above, be a point in $S_g(t)$. In the sequel of the proof we denote by C_1, C_2 etc., positive constants depending only on t and the dimension g.

We first observe that (4.12) says that the entries of the matrices W and $W^\top$ are bounded by t. Since these matrices are unipotent, their inverses are also bounded by a constant C_1. Consider the path $Z(\tau) = X + iW(\tau)^\top D\,W(\tau)$, with $W(\tau) := \tau W$ and $\tau \in [0,1]$. The entries of $(W^\top)^{-1}\mathrm{d}W^\top D\,\mathrm{d}W\,W^{-1}D^{-1}$ along this path are all proportional to $C_2\,(\delta_i/\delta_j)(\mathrm{d}\tau)^2$, where $j > i$. Since $\delta_i/\delta_j < t^{j-i}$ by (4.13), it follows from (4.14) that the length of the path is bounded by a constant C_3. Thus, the arbitrary point $Z = X + iW^\top DW \in S_g(t)$ is within a bounded distance from $X + iD$.

But $X+iD$ is within a bounded distance from iD. Indeed, fixed any pair of indices $1 \le i \le j \le g$, we may consider the path $Z_{(ij)}(\tau) = X_{(ij)}(\tau) + iD$, $(\tau \in [0,1])$, where $X_{(ij)}(\tau)$ is the symmetric matrix with entries $x_{ij}(\tau) = x_{ji}(\tau) = \tau x_{ij}$ and all other entries constant and equal to those of X. It follows from (4.14) that the length of any such path is

$$\int_0^1 \frac{|x_{ij}|}{\sqrt{\delta_i \delta_j}} \mathrm{d}\tau,$$

which is bounded by some constant C_4 because of (4.11) and (4.13). The claim follows by choosing successively all pair of indices, thus constructing a sequence of paths joining $X+iD$ to iD. □

The Siegel volume form $\mathrm{d}X\mathrm{d}Y/(\det Y)^{g+1}$ in Iwasawa coordinates is

$$\mathrm{dVol}_g = \prod_{i\le j} \mathrm{d}x_{ij} \cdot \prod_{i<j} \mathrm{d}w_{ij} \cdot \prod_k \delta_k^{-(k+1)} \mathrm{d}\delta_k. \tag{4.15}$$

A computation, using again the fundamental open set $S_g(t)$, gives the following.

LEMMA 4.7. *The logarithm of the height function on the Siegel modular variety is distance-like with exponent $k_g = \frac{g+1}{2}$. More precisely, for any $\tau \gg 0$*

$$\mathrm{Vol}_g \left\{ [Z] \in \Sigma_g \,\middle|\, \mathrm{Hgt}([Z]) \ge \tau \right\} \asymp e^{-\frac{g+1}{2}\tau}.$$

Proof. A change of variable as in page 67 of [31] shows that this volume is within a bounded ratio of

$$\int_{e^\tau}^{\infty} t^{-(g+3)/2} \mathrm{d}t. \qquad \square$$

PROPOSITION 4.8. *For any $s > g+1/2$ there exists a constant $C(s) > 0$ such that the best Sobolev constant satisfies the estimate*

$$B_s([[\alpha]]) \le C(s) \cdot \left(\mathrm{Hgt}([[\alpha]])\right)^{1/4}.$$

Proof. Let $Z = X + iW^\top DW \in F_g$ be the representative of $[[\alpha]] \in \Sigma_g$ inside the Siegel fundamental domain, so that $B_s(Z) = B_s([[\alpha]])$. Since the Siegel fundamental domain F_g is contained in a fundamental open set $S_g(t)$, by Lemma 4.6, the point Z is within a uniformly bounded distance from the point iD. Thus, by Lemma 4.2, there exists a constant $C = C(s) > 0$ such that

$$B_s(Z) \le C\, B_s(iD).$$

Since $iD = \beta^{-1}(i)$, with $\beta = \begin{pmatrix} D^{-1/2} & 0 \\ 0 & D^{1/2} \end{pmatrix}$, we have $B_s(iD) = B_s(\beta)$ and, by Proposition 4.5, $B_s(\beta) \le C \urcorner(\beta)^{1/2} \le C'(t) \det(D)^{1/4} = C'(t)\,\mathrm{hgt}([[\alpha]])^{1/4}$. The middle inequality above follows from the definition of $\urcorner(\beta)$ and the observation that, for Z in a fundamental open set set $S_g(t)$, the entries δ_i of the matrix D are bounded below by t^{-i}. □

4.4. **Diophantine conditions and logarithm law.** We will need, in the final renormalization argument, some control on the best Sobolev constant $B_s([[\rho\alpha]])$, hence, by Proposition 4.8, on $\mathrm{Hgt}([[\rho\alpha]])$, when ρ are certain automorphisms in the Cartan subgroup $\mathsf{A} \subset \mathrm{Sp}_{2g}(\mathbb{R})$ of diagonal symplectic matrices. This control is the higher-dimensional analogue of the escape rate of geodesics into the cusp of the modular surface.

Diophantine conditions. Let $\mathfrak{a}^+ \subset \mathfrak{sp}_{2g}$ be the cone of those $\widehat{\delta} = \left(\begin{smallmatrix} \delta & 0 \\ 0 & -\delta \end{smallmatrix}\right) \in \mathfrak{sp}_{2g}$ where $\delta = \mathrm{diag}(\delta_1, \dots, \delta_g)$ is a non-negative diagonal matrix. We consider the corresponding one-parameter subgroup of diagonal symplectic matrices $e^{t\widehat{\delta}}$ in $\mathsf{A} \subset \mathrm{Sp}_{2g}(\mathbb{R})$, and also denote by $e^{-t\widehat{\delta}}$ the corresponding automorphisms $(x, \xi, z) \mapsto (e^{-t\delta} x, e^{t\delta}\xi, t)$ of the Heisenberg group.

We recall that under the left action of the symplectic matrix $\beta = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{R})$, the height on $\mathfrak{H}_g$ transforms according to

$$\mathrm{hgt}(\beta(Z)) = |\det(CZ + D)|^{-2}\,\mathrm{hgt}(Z). \tag{4.16}$$

LEMMA 4.9. *Let $\delta = \mathrm{diag}(\delta_1, \delta_2, \dots, \delta_g)$ be a non-negative diagonal matrix and let $\widehat{\delta} = \left(\begin{smallmatrix} \delta & 0 \\ 0 & -\delta \end{smallmatrix}\right) \in \mathfrak{a}$ be the generator of the one-parameter group $(e^{t\widehat{\delta}})_{t\in\mathbb{R}} < \mathrm{Sp}_{2g}(\mathbb{R})$. For any $[\alpha] \in \mathfrak{M}_g$ and any $t \geq 0$ we have the trivial bound*

$$\mathrm{Hgt}([[e^{-t\widehat{\delta}}\alpha]]) \leq (\det e^{t\delta})^2\, \mathrm{Hgt}([[\alpha]]).$$

Proof. We recall that Hgt is the maximal hgt of a $\mathrm{Sp}_{2g}(\mathbb{Z})$ orbit. Therefore, we may take the representative $\beta = \alpha\gamma$, with $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$, such that $(e^{-t\widehat{\delta}}\beta)^{-1}(i) \in \mathfrak{H}_g$ realizes the maximal height, that is,

$$\mathrm{Hgt}([[e^{-t\widehat{\delta}}\alpha]]) = \mathrm{hgt}((e^{-t\widehat{\delta}}\beta)^{-1}(i)),$$

and prove the inequality for the function hgt, namely

$$\mathrm{hgt}((e^{-t\widehat{\delta}}\beta)^{-1}(i)) \leq (\det e^{t\delta})^2\, \mathrm{hgt}(\beta^{-1}(i)),$$

since then $\mathrm{hgt}(\beta^{-1}(i)) \leq \mathrm{Hgt}([[\alpha]])$. By the Iwasawa decomposition, any symplectic matrix $\beta \in \mathrm{Sp}_{2g}(\mathbb{R})$ sending the base point $i := i\mathbf{1}_g$ into the point $\beta^{-1}(i) = X + iW^\top DW$ may be written as $\beta^{-1} = \nu\eta\kappa$ with $\nu = \left(\begin{smallmatrix} W^\top & XW^{-1} \\ 0 & -W^{-1} \end{smallmatrix}\right)$, $\eta = \left(\begin{smallmatrix} \sqrt{D} & 0 \\ 0 & \sqrt{D}^{-1} \end{smallmatrix}\right)$, and $\kappa \in \mathsf{K}_g$. By the formula (4.16),

$$\mathrm{hgt}(\nu\eta\kappa(Z)) = \mathrm{hgt}(\eta\kappa(Z)) = (\det D)\,\mathrm{hgt}(\kappa(Z))$$

(because $\det W = 1$) for all $Z \in \mathfrak{H}_g$. Therefore, since $\mathrm{hgt}(\kappa(i)) = 1$, we only need to prove

$$\mathrm{hgt}(\kappa e^{t\widehat{\delta}}(i)) \leq \det e^{2t\delta}.$$

Let $\kappa = \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right) \in \mathsf{K}_g$, i.e., with $A^\top A + B^\top B = \mathbf{1}_g$ and $A^\top B$ symmetric. Since $e^{t\widehat{\delta}}(i) = ie^{2t\delta}$, using formula (4.16), the above inequality is equivalent to

$$|\det(-iBe^{2t\delta} + A)|^{-2} \cdot \det e^{2t\delta} \leq \det e^{2t\delta},$$

that is, to

$$|\det(A - iBe^{2t\delta})|^2 \geq 1,$$

and therefore to

$$|\det(AA^\top + Be^{4t\delta}B^\top)| \geq 1.$$

But, by our hypothesis on δ and t, the norm of $e^{2t\delta}$ is $\|e^{2t\delta}\| \geq 1$, and therefore

$$\left\langle x, (A^\top A + B^\top e^{4t\delta}B)x \right\rangle \geq \langle x, (A^\top A + B^\top B)x \rangle = \|x\|^2$$

for any vector $x \in \mathbb{R}^g$. Hence, all the eigenvalues of the symmetric matrix $A^\top A + B^\top e^{4t\delta}B$ are ≥ 1, and the same occurs for the determinant. □

DEFINITION 4.10. Let $\delta = \mathrm{diag}(\delta_1, \ldots, \delta_g)$ be a non-negative diagonal matrix, and $\widehat{\delta} = \left(\begin{smallmatrix} \delta & 0 \\ 0 & -\delta \end{smallmatrix}\right) \in \mathfrak{a}^+ \subset \mathfrak{sp}_{2g}$. We say that an automorphism $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$, or, equivalently, a point $[\alpha] \in \mathfrak{M}_g$ in the moduli space,

- is $\widehat{\delta}$-*Diophantine* of *type* σ if there exists a $\sigma > 0$ and a constant $C > 0$ such that

$$\mathrm{Hgt}([[e^{-t\widehat{\delta}}\alpha]]) \leq C\, \mathrm{Hgt}([[e^{-t\widehat{\delta}}]])^{(1-\sigma)}\, \mathrm{Hgt}([[\alpha]]) \qquad \forall\ t \gg 0; \tag{4.17}$$

- satisfies a $\widehat{\delta}$-*Roth condition* if for any $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$\mathrm{Hgt}([[e^{-t\widehat{\delta}}\alpha]]) \leq C\, \mathrm{Hgt}([[e^{-t\widehat{\delta}}]])^{\varepsilon}\, \mathrm{Hgt}([[\alpha]]) \qquad \forall\ t \gg 0, \tag{4.18}$$

  that is, if it is Diophantine of every type $0 < \sigma < 1$;
- is of *bounded type* if there exists a constant $C > 0$ such that

$$\mathrm{Hgt}([[e^{-t\widehat{\delta}}\alpha]]) \leq C \tag{4.19}$$

  for all $\widehat{\delta} \in \mathfrak{a}^+$ and all $t \geq 0$.

REMARK 4.11. In the final section, dealing with theta sums, we will be interested in Diophantine properties in the direction of the particular $\widehat{\delta} = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right) \in \mathfrak{a}$. For such $\widehat{\delta}$, the Diophantine properties of an automorphism $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$ only depend on the right T class of α^{-1}, where $\mathrm{T} \subset \mathrm{Sp}_{2g}(\mathbb{R})$ is the subgroup of block-triangular symplectic matrices of the form $\left(\begin{smallmatrix} A & B \\ 0 & (A^\top)^{-1} \end{smallmatrix}\right)$. In particular, those α in the full measure set of those automorphisms such that $\alpha^{-1} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with $A \in \mathrm{GL}_g(\mathbb{R})$ are in the same Diophantine class of $\beta = \left(\begin{smallmatrix} I & 0 \\ -X & I \end{smallmatrix}\right)$, where X is the symmetric matrix $X = CA^{-1}$. For such lower-triangular block matrices β, the Height in the Diophantine conditions above is (see (4.16))

$$\mathrm{Hgt}([[e^{-t\widehat{\delta}}\beta]]) = \max \left| \det(QQ^\top e^{-2t} + (QX+P)(QX+P)^\top e^{2t}) \right|^{-1}, \tag{4.20}$$

the maximum being over all $\left(\begin{smallmatrix} N & M \\ P & Q \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{Z})$. When $g = 1$, we recover the classical relation between Diophantine properties of a real number X and geodesic excursion into the cusp of the modular orbifold Σ_1, or the behaviour of a certain flow in the space $\mathfrak{M}_1 = \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ of unimodular lattices in the plane. Indeed, our (4.20) coincides with the function $\delta(\Lambda_t) = \max_{v \in \Lambda_t \setminus \{0\}} \|v\|_2^{-2}$, where Λ_t is the unimodular lattice made of $\left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} P \\ Q \end{smallmatrix}\right)$, with $P, Q \in \mathbb{Z}$. The maximizers, for increasing time t, define a sequence of relatively prime integers P_n and Q_n which give best approximants P_n/Q_n to X in the sense of continued fractions. In particular, our definitions of Diophantine, Roth, and bounded type coincide with the classical notions.

This same function $\delta(\Lambda_t)$, extended to the space $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ of unimodular lattices in $\mathbb{R}^n$, has been used by Lagarias [36], or, more recently, by Chevallier [6] to understand simultaneous Diophantine approximations. A similar function, $\Delta(\Lambda_t) = \max_{v\in\Lambda_t\setminus\{0\}} \log(1/\|v\|_\infty)$, has been considered by Dani [10] in his correspondence between Diophantine properties of systems of linear forms and certain flows on the space $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$, or more recently by Kleinbock and Margulis [32] to prove a "higher-dimensional multiplicative Khinchin theorem".

Khinchin-Sullivan-Kleinbock-Margulis logarithm law. A stronger control on the best Sobolev constant comes from the following generalization of the Kinchin-Sullivan logarithm law for geodesic excursion [47], due to Kleinbock and Margulis [32].

Let $\mathsf{X} = \mathsf{G}/\Lambda$ be a homogeneous space, equipped with the probability Haar measure μ. A function $\phi : \mathsf{X} \to \mathbb{R}$ is said to be k-DL (for "distance-like") for some exponent $k > 0$ if it is uniformly continuous and if there exist constants $c_\pm > 0$ such that

$$c_- e^{-kt} \le \mu\big(\big\{x \in \mathsf{X} \,\big|\, \phi(x) \ge t\big\}\big) \le c_+ e^{-kt}.$$

Theorem 1.7 of [32] says the following.

PROPOSITION 4.12 (Kleinbock-Margulis)**.** *Let G be a connected semisimple Lie group without compact factors, μ its normalized Haar measure, $\Lambda \subset \mathsf{G}$ an irreducible lattice, $\mathfrak{a}$ a Cartan subalgebra of the Lie algebra of G, $\mathbf{z}$ a non-zero element of $\mathfrak{a}$. If $\phi : \mathsf{G}/\Lambda \to \mathbb{R}$ is a k-DL function for some $k > 0$, then for μ-almost all $x \in \mathsf{G}/\Lambda$ one has*

$$\limsup_{t\to\infty} \frac{\phi(e^{t\mathbf{z}}x)}{\log t} = 1/k.$$

We have seen in Proposition 4.7 that the logarithm of the height function Hgt is a DL-function with exponent $\frac{g+1}{2}$ on the Siegel variety Σ_g, hence it induces a DL-function on the homogeneous space $\mathfrak{M}_g = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathrm{Sp}_{2g}(\mathbb{R})$. Thus, the following proposition is a consequence of the easy part of Proposition 4.12 and of Proposition 4.8.

PROPOSITION 4.13. *Let $s > g + 1/2$. For any non-zero vector $\widehat{\delta} \in \mathfrak{a}$ in the Cartan subalgebra of diagonal symplectic matrices there exists a full measure set $\Omega_g(\widehat{\delta}) \subset \mathfrak{M}_g$ such that for all $[\alpha] \in \Omega_g(\widehat{\delta})$ we have*

$$\limsup_{t\to\infty} \frac{\log \mathrm{Hgt}([[e^{-t\widehat{\delta}}\alpha]])}{\log t} \le \frac{2}{g+1}.$$

In particular, any such $[\alpha]$ satisfies a $\widehat{\delta}$-Roth condition.

5. EQUIDISTRIBUTION

In this section we consider only functional spaces "built up" from the space of functions with zero average along the fibers of the central fibration of the standard nilmanifold M. Thus, all smooth forms have coefficients in $C_0^\infty(\mathsf{M})$, all Sobolev forms and currents have coefficients in some $W^s_\alpha(\mathsf{M})$, $s \in \mathbb{R}$ (see Definition 4.2).

5.1. **Birkhoff sums and renormalization.** Let $(X_1^0, \dots, X_g^0, \Xi_1^0, \dots, \Xi_g^0, T)$ be the "standard" Heisenberg basis defined in Section 2.1.

For $1 \le d \le g$, we define the sub-algebra $\mathfrak{p}^{d,0} \subset \mathfrak{h}^g$ generated by the first d base elements $X_1^0, \dots, X_d^0$, and then the Abelian subgroup $\mathsf{P}^{d,0} := \exp \mathfrak{p}^{d,0}$.

According to (4.1), the group $\mathrm{Sp}_{2g}(\mathbb{R})$ acts on the right on the enveloping algebra $\mathscr{U}(\mathfrak{h}^g)$ and in particular for $V \in \mathfrak{h}^g$, $V_\alpha = \alpha^{-1}(V)$. For simplicity we set, for any $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$, $(X_i^\alpha, \Xi_j^\alpha, T) := (\alpha^{-1}(X_i^0), \alpha^{-1}(\Xi_j^0), T)$. Then $\mathfrak{p}^{d,\alpha} := \alpha^{-1}(\mathfrak{p}^{d,0})$ and $\mathsf{P}^{d,\alpha} = \alpha^{-1}(\mathsf{P}^{d,0})$ are respectively the algebra and the subgroup generated by $(X_i^\alpha, \Xi_j^\alpha, T)$. Every isotropic subgroup of H^g is obtained in this way, i.e., given by some $\mathsf{P}^{d,\alpha}$ defined as above.

It is immediate that for every $\alpha, \beta \in \mathrm{Sp}_{2g}(\mathbb{R})$ we have

$$\alpha^{-1}(\mathsf{P}^{d,\beta}) = \mathsf{P}^{d,\beta\alpha};$$

in particular, if β belongs to the diagonal Cartan subgroup A, then $\mathsf{P}^{d,\beta\alpha} = \mathsf{P}^{d,\alpha}$.

We define a parametrization of $\mathsf{P}^{d,\alpha}$, hence a $\mathbb{R}^d$-action on M subordinate to α, by setting

$$\mathsf{P}_x^{d,\alpha} := \exp(x_1 X_1^\alpha + \cdots + x_d X_d^\alpha) \quad \text{with } x = (x_1, \dots, x_d) \in \mathbb{R}^d. \tag{5.1}$$

Birkhoff sums. We define the bundle $A^j(\mathfrak{p}^d, \mathfrak{W}^s) \to \mathfrak{M}_g$ of $\mathfrak{p}$-forms of degree j and Sobolev order s as the set of pairs

$$(\alpha, \omega), \quad \alpha \in \mathrm{Sp}_{2g}(\mathbb{R}), \quad \omega \in A^j(\mathfrak{p}^{d,\alpha}, W^s_\alpha(\mathsf{M})),$$

mod the equivalence relation $(\alpha, \omega) \equiv (\alpha\gamma, \gamma^*\omega)$ for all $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$. The class of (α, ω) is denoted $[\alpha, \omega]$. We also define the dual bundle $A_j(\mathfrak{p}^d, \mathfrak{W}^{-s}) \to \mathfrak{M}_g$ of $\mathfrak{p}$-currents of dimension j and Sobolev order s as the set of pairs

$$(\alpha, \mathscr{D}), \quad \alpha \in \mathrm{Sp}_{2g}(\mathbb{R}), \quad \mathscr{D} \in A_j(\mathfrak{p}^{d,\alpha}, W^{-s}_\alpha(\mathsf{M})),$$

mod the equivalence relation $(\alpha, \mathscr{D}) \equiv (\alpha\gamma, (\gamma_*)^{-1}\mathscr{D})$ for all elements $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$. The class of $(\alpha, \mathscr{D})$ is denoted $[\alpha, \mathscr{D}]$.

The bundles $A^j(\mathfrak{p}, \mathfrak{W}^s)$ and $A_j(\mathfrak{p}, \mathfrak{W}^{-s})$ are Hilbert bundles for the dual norms

$$\| [\alpha, \omega] \|_s := \|\omega\|_{s,\alpha}, \qquad \| [\alpha, \mathscr{D}] \|_{-s} := \|\mathscr{D}\|_{-s,\alpha}.$$

In the following, it will be convenient to set $\omega^{d,\alpha} = dX_1^\alpha \wedge \cdots \wedge dX_d^\alpha$ and to identify top-dimensional currents $\mathscr{D}$ with distributions by setting $\langle \mathscr{D}, f \rangle := \langle \mathscr{D}, f\omega^{d,\alpha} \rangle$.

Given a Jordan region $U \subset \mathbb{R}^d$ and a point $m \in \mathsf{M}$, we define a top-dimensional $\mathfrak{p}$-current $\mathscr{P}_U^{d,\alpha} m$ as the Birkhoff sums given by integration along the chain $\mathsf{P}_U^{d,\alpha} m = \left\{ \mathsf{P}_x^{d,\alpha} m \,\middle|\, x \in U \right\}$. Explicitly, if $\omega = f\,\mathrm{d}X_1^\alpha \wedge \cdots \wedge \mathrm{d}X_d^\alpha$ is a top-dimensional

$\mathfrak{p}$-form, then

$$\left\langle \mathscr{P}_U^{d,\alpha} m, \omega \right\rangle := \int_{\mathsf{P}_U^{d,\alpha} m} \omega = \int_U f(\mathsf{P}_x^{d,\alpha} m)\, \mathrm{d}x_1 \dots \mathrm{d}x_d. \tag{5.2}$$

Our goal is to understand the asymptotic of these distributions as $U \nearrow \mathbb{R}^d$ in a Følner sense. A particular case is obtained when $U = Q(T) = [0,T]^d$.

We remark that the Birkhoff sums satisfy the following covariance property:

$$\gamma_*^{-1}\left(\mathscr{P}_U^{d,\alpha} m\right) = \mathscr{P}_U^{d,\alpha\gamma}(\gamma^{-1} m), \qquad \forall\ m \in \mathsf{M}, \forall\ \gamma \in \mathrm{Sp}_{2g}(\mathbb{Z}).$$

Renormalization flows. For each $1 \le i \le g$, we denote by $\widehat{\delta}_i := \left(\begin{smallmatrix} \delta_i & 0 \\ 0 & -\delta_i \end{smallmatrix}\right) \in \mathfrak{a}$ the element of the Cartan subalgebra of diagonal symplectic matrices defined by the diagonal matrix $\delta_i = \mathrm{diag}(d_1, \dots, d_g)$ with $d_i = 1$ and $d_k = 0$ if $k \ne i$. Any such $\widehat{\delta}_i$ generates a one parameter group of automorphisms $r_i^t := e^{t\widehat{\delta}_i} \in \mathsf{A}$, with $t \in \mathbb{R}$.

Left multiplication by the one parameter group (r_i^t) yields a flow on $\mathrm{Sp}_{2g}(\mathbb{R})$ that projects to the moduli space $\mathfrak{M}_g$ according to $[\alpha] \mapsto r_i^t[\alpha] = [r_i^t \alpha]$.

Above this flow, we consider its horizontal lift to the bundles $A^j(\mathfrak{p}^d, \mathfrak{W}^s)$ and $A_j(\mathfrak{p}^d, \mathfrak{W}^{-s})$ $(s \in \mathbb{R})$, defined by

$$r_i^t[\alpha, \omega] := [r_i^t \alpha, \omega] \qquad r_i^t[\alpha, \mathscr{D}] := [r_i^t \alpha, \mathscr{D}]$$

for $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$ and $\omega \in A^j(\mathfrak{p}^{d,\alpha}, \mathfrak{W}^s)$ or $\mathscr{D} \in A_j(\mathfrak{p}^{d,\alpha}, \mathfrak{W}^{-s})$. This is well defined since, as we remarked before, $\mathfrak{p}^{d,\alpha} = \mathfrak{p}^{d, r_i^t \alpha}$.

DEFINITION 5.1. For $s > 0$, let $Z_d(\mathfrak{p}^d, \mathfrak{W}^{-s})$ be the sub-bundle of the bundle $A_d(\mathfrak{p}^d, \mathfrak{W}^{-s})$ consisting of elements $[\alpha, \mathscr{D}]$ with $\mathscr{D} \in Z_d(\mathfrak{p}^{d,\alpha}, W_\alpha^{-s}(\mathsf{M}))$, i.e., with $\mathscr{D}$ a closed $\mathfrak{p}^{d,\alpha}$-current of dimension d and Sobolev order s.

We remark that the definition is well posed. In fact, if $\mathscr{D}$ is a closed $\mathfrak{p}^{d,\alpha}$-current of dimension d then, from the identities $\langle \mathscr{D}, X_i^\alpha(f) \rangle = 0$ for all test functions f and $i \in [1,d]$, we obtain $0 = \langle \gamma_* \mathscr{D}, \gamma_* X_i^\alpha(f) \rangle = \left\langle \gamma_* \mathscr{D}, X_i^{\alpha\gamma^{-1}}(f) \right\rangle$, which shows that $\gamma_* \mathscr{D}$ is a closed $\mathfrak{p}^{d,\alpha\gamma^{-1}}$-current of dimension d.

Observe that, although the subgroup $\mathsf{P}^{d,(r_i^t \alpha)}$ and $\mathsf{P}^{d,\alpha}$ coincide, the actions of $\mathbb{R}^d$ defined by their parameterizations (5.1) differ by a constant rescaling; in fact

$$\mathsf{P}_{(x_1,\dots,x_d)}^{d,(r_1^{t_1} \dots r_g^{t_g} \alpha)} = \mathsf{P}_{(e^{-t_1} x_1, \dots, e^{-t_d} x_d)}^{d,\alpha}. \tag{5.3}$$

Consequently, denoting by $(e^{-t_1}, \dots, e^{-t_d})U$ the obvious diagonal automorphism of $\mathbb{R}^d$ applied to the region U, the Birkhoff sums satisfy the identities

$$\mathscr{P}_U^{d,(r_1^{t_1} \dots r_g^{t_g} \alpha)} m = e^{t_1 + \dots + t_d} \mathscr{P}_{(e^{-t_1}, \dots, e^{-t_d})U}^{d,\alpha} m. \tag{5.4}$$

PROPOSITION 5.2. *Let $s > d/2$. The sub-bundle $Z_d(\mathfrak{p}^d, \mathfrak{W}^{-s})$ is invariant under the renormalization flows r_i^t with $1 \le i \le d$. Furthermore, for every $(t_1, \dots, t_d) \in \mathbb{R}^d$ and any $[\alpha, \mathscr{D}] \in Z_d(\mathfrak{p}^d, \mathfrak{W}^{-s})$ and any $s > d/2$, we have*

$$\| r_1^{t_1} \dots r_d^{t_d} [\alpha, \mathscr{D}] \|_{-s} = e^{-(t_1 + \dots + t_d)/2} \| [\alpha, \mathscr{D}] \|_{-s}.$$

Proof. The invariance of the sub-bundle $Z_d(\mathfrak{p}^d, \mathfrak{W}^{-s})$ is clear from (5.3).

Set, for simplicity, $r := r_1^{t_1} \dots r_d^{t_d}$. By definition $\| r[\alpha, \mathscr{D}] \|_{-s} = \| [r\alpha, \mathscr{D}] \|_{-s} = \|\mathscr{D}\|_{-s,r\alpha}$ for any $[\alpha, \mathscr{D}] \in A_d(\mathfrak{p}^d, \mathfrak{W}^{-s})$.

Without loss of generality we may assume that $\mathscr{D}$ belongs to the space $A_d(\mathfrak{p}^{d,\alpha}, W^{-s}(\rho_h))$, where ρ_h is an irreducible Schrödinger representation in which the basis $(X_i^\alpha, \Xi^\alpha, T)$ acts according to (2.5). Let $\tilde{L}_\alpha = (\rho_h)_* L_\alpha$ and $\widetilde{L}_{r_d^t \alpha} = (\rho_h)_* \widetilde{L}_{r_d^t \alpha}$ be the push-forward to $L^2(\mathbb{R}^g)$ of the operators defining the norms $\|\cdot\|_{s,\alpha}$ and $\|\cdot\|_{s,r_d^t\alpha}$.

By Proposition 3.13, the space of closed currents of dimension d is spanned by $\mathscr{I}_g$ if $d = g$ and by the dense set of currents $\mathscr{D} = D_y \circ \mathscr{I}_{d,g}$ with D_y in $L^2(\mathbb{R}^{g-d}, dy)$ if $d < g$. Any such current is given, for any test function $f \in \mathscr{S}(\mathbb{R}^g)$, by $\langle \mathscr{D}, f \rangle = \langle D_y, \int_{\mathbb{R}^d} f(x,y)\, \mathrm{d}x \rangle$. The unitary operator $U_t : L^2(\mathbb{R}^g) \to L^2(\mathbb{R}^g)$ defined, for $t = (t_1, \dots, t_d)$, by[2]

$$U_t f(x,y) := e^{(t_1+\cdots+t_d)/2} f\big((e^{t_1}, \dots, e^{t_d})x, y\big) \tag{5.5}$$

($x \in \mathbb{R}^d$, $y \in \mathbb{R}^{g-d}$), intertwines the differential operator $\widetilde{L}_\alpha$ with the operator $\widetilde{L}_{r\alpha}$, i.e., $U_t(\widetilde{L}_\alpha f) = \widetilde{L}_{r\alpha} U_t f$ for any smooth f. Thus

$$\begin{aligned}
\|\mathscr{D}\|_{-s,r\alpha} &= \sup_{\|f\|_{s,r\alpha}=1} \big|\langle \mathscr{D}, f\rangle\big| = \sup_{\|\widetilde{L}_{r\alpha}^{s/2} f\|=1} \big|\langle \mathscr{D}, f\rangle\big| \\
&= \sup_{\|\widetilde{L}_{\alpha}^{s/2} U_t^{-1} f\|=1} \big|\langle \mathscr{D}, f\rangle\big| = \sup_{\|\widetilde{L}_{\alpha}^{s/2} f\|=1} \big|\langle \mathscr{D}, U_t f\rangle\big| \\
&= \sup_{\|(L_\alpha)^{s/2} f\|=1} \left|\left\langle D_y, \int_{\mathbb{R}^g} e^{(t_1+\cdots+t_d)/2} f\big((e^{t_1}, \dots, e^{t_d})x, y\big)\, \mathrm{d}x \right\rangle\right| \\
&= \sup_{\|(L_\alpha)^{s/2} f\|=1} e^{-(t_1+\cdots+t_d)/2} \left|\left\langle D_y, \int_{\mathbb{R}^g} f(x,y)\, \mathrm{d}x \right\rangle\right| \\
&= e^{-(t_1+\cdots+t_d)/2} \|\mathscr{D}\|_{-s,\alpha}
\end{aligned}$$

□

5.2. **The renormalization argument.**

Orthogonal splittings. For any exponent $s > d/2$, the sub-bundle $Z_d(\mathfrak{p}^d, \mathfrak{W}^{-s})$ is a closed subspace of the Hilbert bundle $A_d(\mathfrak{p}^d, \mathfrak{W}^{-s})$ and therefore induces an orthogonal decomposition

$$A_d(\mathfrak{p}^d, \mathfrak{W}^{-s}) = Z_d(\mathfrak{p}^d, \mathfrak{W}^{-s}) \oplus R_d(\mathfrak{p}^d, \mathfrak{W}^{-s}), \tag{5.6}$$

where $R_d(\mathfrak{p}^d, \mathfrak{W}^{-s}) := Z_d(\mathfrak{p}^d, \mathfrak{W}^{-s})^\perp$. We denote by $\mathscr{Z}^{-s}$ and $\mathscr{R}^{-s}$ the corresponding orthogonal projections, and, given $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$, by $\mathscr{Z}_\alpha^{-s}$ and $\mathscr{R}_\alpha^{-s}$ the restrictions of these projections to the fiber over $[\alpha] \in \mathfrak{M}_g$. In particular, we obtain a decomposition of the Birkhoff sums $\mathscr{D} = \mathscr{P}_U^{d,\alpha} m$ as

$$[\alpha, \mathscr{D}] = \mathscr{Z}^{-s}[\alpha, \mathscr{D}] + \mathscr{R}^{-s}[\alpha, \mathscr{D}] = [\alpha, \mathscr{Z}_\alpha^{-s}(\mathscr{D})] + [\alpha, \mathscr{R}_\alpha^{-s}(\mathscr{D})] \tag{5.7}$$

with "boundary term" $\mathscr{Z}_\alpha^{-s}(\mathscr{D}) \in Z_d(\mathfrak{p}^{d,\alpha}, W_\alpha^{-s}(\mathsf{M}))$ and "remainder term" $\mathscr{R}_\alpha^{-s}(\mathscr{D}) \in R_d(\mathfrak{p}^{d,\alpha}, W_\alpha^{-s}(\mathsf{M}))$.

[2] This is a particular case of the *metaplectic representation* (see [49, 19]).

We will also need an estimate for the distortion of the Sobolev norms along the renormalization flow. Below, $|t|$ denotes the sup norm of a vector $t \in \mathbb{R}^d$.

LEMMA 5.3. *Let $s > d/2+2$. For $t = (t_1, \dots, t_d) \in \mathbb{R}^d$ and $\tau \in \mathbb{R}$, let $r^\tau = r_1^{-\tau t_1} \dots r_d^{-\tau t_d}$. There exists a constant $C = C(s)$ such that if $|\tau t|$ is sufficiently small then the orthogonal projection*

$$\mathcal{Z}^{-s}_{r^\tau \alpha} : R_d(\mathfrak{p}^{d,\alpha}, W^{-(s-2)}_\alpha(\mathsf{M})) \to Z_d(\mathfrak{p}^{d,\alpha}, W^{-s}_{r^\tau \alpha}(\mathsf{M}))$$

has norm bounded by $C|\tau t|$.

Proof. As in the proof of Proposition 5.2, we may restrict to a fixed Schrödinger representation ρ_h in which the basis $(X_i^\alpha, \Xi_i^\alpha, T)$ acts according to (2.5). It is also clear from Lemma 3.15 that we may use the homogeneous Sobolev norm defined in (3.4). If $H = (\rho_h)_* L_\alpha$ denotes the sub-Laplacian inducing the Sobolev structure of $W^{-s}_\alpha(\mathbb{R}^g)$, then the Sobolev structure of $W^{-s}_{r^\tau \alpha}(\mathbb{R}^g)$ is induced by

$$H_\tau = U'_{-\tau} H U'_\tau$$

where $U'_\tau = U_{\tau t}$ is the one-parameter group of unitary operators of $L^2(\mathbb{R}^g)$ defined according to (5.5). We denote by $\langle \phi, \psi \rangle_{-s,\tau} = \langle \phi, H_\tau^{-s} \psi \rangle$ the inner product in $W^{-s}_{r^\tau \alpha}(\mathbb{R}^g)$. A computation shows that the infinitesimal generator of U'_τ is i times the self-adjoint operator $A = (\rho_h)_* \left(\sum_{k=1}^d t_k (1/2 - X_k \Xi_k)\right)$. Moreover, using the Hermite basis, one can show that there exists a constant C such that $\|A\psi\| \le C|t| \, \|H\psi\|$ for ψ in the domain of A.

Now, let $\mathscr{R} \in W^{-s+2}_\alpha(\mathbb{R}^g)$ be a distribution (we identify top-dimensional currents with distributions as explained in 5.1) which is orthogonal to the subspace Z of closed distributions when $\tau = 0$, i.e., such that

$$\langle \mathscr{R}, \mathscr{D} \rangle_{-s,0} = \langle \mathscr{R}, H^{-s} \mathscr{D} \rangle = 0$$

for all $\mathscr{D} \in Z$. In order to bound the norm of its projection to Z w.r.t. the Sobolev structure at τ we must bound the absolute values of the scalar products $\langle \mathscr{R}, \mathscr{D} \rangle_{-s,\tau}$ for all $\mathscr{D}$ in Z. Now,

$$\langle \mathscr{R}, \mathscr{D} \rangle_{-s,\tau} = \langle \mathscr{R}, U'_{-\tau} H^{-s} U'_\tau \mathscr{D} \rangle = \langle U'_\tau \mathscr{R}, H^{-s} U'_\tau \mathscr{D} \rangle.$$

If $\mathscr{R}$ is in the domain of A, we may write

$$U'_\tau \mathscr{R} = \mathscr{R} + i \int_0^\tau U'_u A \mathscr{R} \, \mathrm{d}u.$$

According to Proposition 5.2, the group U'_τ preserves Z. Therefore, since $\mathscr{R}$ is orthogonal to $U'_\tau \mathscr{D}$ for all τ, we may write

$$\begin{aligned} \langle \mathscr{R}, \mathscr{D} \rangle_{-s,\tau} &= i \int_0^\tau \langle U'_u A \mathscr{R}, H^{-s} U'_\tau \mathscr{D} \rangle \, \mathrm{d}u \\ &= i \int_0^\tau \langle A \mathscr{R}, U'_{-u} H^{-s} U'_\tau \mathscr{D} \rangle \, \mathrm{d}u \\ &= i \int_0^\tau \langle A \mathscr{R}, U'_{\tau-u} \mathscr{D} \rangle_{-s,u} \, \mathrm{d}u. \end{aligned}$$

By Cauchy-Schwartz and Lemma 4.2, if $|\tau t|$ is sufficiently small we have

$$\begin{aligned}\left|\langle \mathscr{R}, \mathscr{D}\rangle_{-s,\tau}\right| &\le \left|\int_0^\tau \|A\mathscr{R}\|_{-s,u}\,\|U'_{\tau-u}\mathscr{D}\|_{-s,u}\,\mathrm{d}u\right| \\ &\le C'\,\|A\mathscr{R}\|_{-s,0}\left|\int_0^\tau \|U'_{\tau-u}\mathscr{D}\|_{-s,u}\,\mathrm{d}u\right| \\ &\le C''\,|t|\|\mathscr{R}\|_{-s+2,0}\left|\int_0^\tau \|U'_{\tau-u}\mathscr{D}\|_{-s,u}\,\mathrm{d}u\right|.\end{aligned}$$

But $\|U'_{\tau-u}\mathscr{D}\|_{-s,u} = \|\mathscr{D}\|_{-s,\tau}$. It follows that

$$\left|\langle \mathscr{R}, \mathscr{D}\rangle_{-s,\tau}\right| \le |\tau t|\,C''\,\|\mathscr{R}\|_{-s+2,0}\,\|\mathscr{D}\|_{-s,\tau}.$$

This says that the orthogonal projection $Z_\tau(\mathscr{R})$ of $\mathscr{R}$ onto Z w.r.t. the Sobolev structure at τ has norm

$$\|Z_\tau(\mathscr{R})\|_{-s,\tau} \le |\tau t|\,C''\,\|\mathscr{R}\|_{-s+2,0}. \qquad \square$$

NOTATION 5.4. In order to shorten our formulas, in the proofs of the following statements we drop the "initial point" $m \in M$ or the automorphism α in the symbol $\mathscr{P}_U^{d,\alpha} m$ whenever the estimates are uniform in m, in α, or both.

From the Sobolev embedding theorem and the definition (4.4) of the Best Sobolev Constant B_s we have the following trivial bound.

LEMMA 5.5. *For any Jordan region $U \subset \mathbb{R}^d$ with Lebesgue measure $|U|$, for any $s > g + 1/2$ and all $m \in \mathsf{M}$ we have*

$$\left\|[\alpha, \mathscr{P}_U^{d,\alpha} m]\right\|_{-s} \le B_s([[\alpha]])\,|U|.$$

For the remainder term we have the following estimate. Below, we denote by $\partial\mathscr{D}$ the boundary of the current $\mathscr{D}$, defined by $\langle \partial\mathscr{D}, \eta\rangle = \langle \mathscr{D}, \mathrm{d}\eta\rangle$.

LEMMA 5.6. *Let $s > g + d/2 + 1$. For any non-negative $s' < s - (d+1)/2$, there exists a constant $C = C(g, d, s', s) > 0$ such that, for all $m \in \mathsf{M}$ and $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$, we have*

$$\|\mathscr{R}^{-s}[\alpha, \mathscr{P}_U^{d,\alpha} m]\|_{-s} \le C\,\|[\alpha, \partial(\mathscr{P}_U^{d,\alpha} m)]\|_{-s'}.$$

Proof. Let $\omega : [\alpha] \to \omega([\alpha])$ be a section of $A^d(\mathfrak{p}^d, \mathfrak{W}^s)$. Writing $\omega = \omega_Z^s + \omega_R^s$ for its decomposition with ω_R^s in the annihilator of $Z_d(\mathfrak{p}^d, \mathfrak{W}^{-s})$ and ω_Z^s in the annihilator of $R_d(\mathfrak{p}^d, \mathfrak{W}^{-s})$, we have

$$\left\langle \mathscr{R}_\alpha^{-s}(\mathscr{P}_U^{d,\alpha}), \omega\right\rangle = \left\langle \mathscr{R}_\alpha^{-s}(\mathscr{P}_U^{d,\alpha}), \omega_R^s\right\rangle = \left\langle \mathscr{P}_U^{d,\alpha}, \omega_R^s\right\rangle.$$

Since $s > (d+1)/2$ and since, by definition, $\langle T, \omega_R^s\rangle = 0$ for any $T \in Z_d(\mathfrak{p}^d, \mathfrak{W}^{-s})$, by Theorem 3.16 there exists a constant $C := C(g, d, s', s)$ and a section of $(d-1)$-forms η with $\mathrm{d}\eta = \omega_R^s$ and satisfying, for all $s' < s - (d+1)/2$, the estimate $\|\eta([\alpha])\|_{s',\alpha} \le C\|\omega_R^s([\alpha])\|_{s,\alpha}$ for all α. It follows that

$$\left\langle \mathscr{P}_U^d, \omega_R^s\right\rangle = \left\langle \partial\mathscr{P}_U^d, \eta\right\rangle.$$

Hence, for $s' < s - (d+1)/2$, for all $m \in \mathsf{M}$ and $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$, we have

$$\left|\left\langle \mathscr{P}_U^d, \omega_R^s \right\rangle\right| \le C \|\partial \mathscr{P}_U^d\|_{-s'} \times \|\omega_R^s\|_s \le C \|\partial \mathscr{P}_U^d\|_{-s'} \times \|\omega\|_s. \qquad \square$$

To estimate the boundary term, we need the following recursive estimate.

LEMMA 5.7. *Let $s > d/2+2$. There exists a positive constant $C_1 = C_1(s)$ such that for all $t_1 \ge 0, \dots, t_d \ge 0$ and all $[\alpha, \mathscr{D}] \in A_d(\mathfrak{p}^d, \mathfrak{W}^{-(s-2)})$ we have*

$$\begin{aligned}\|\mathcal{Z}^{-s}[\alpha,\mathscr{D}]\|_{-s} \le\ & e^{-(t_1+\cdots+t_d)/2}\|\mathcal{Z}^{-s}[r_1^{-t_1}\dots r_d^{-t_d}\alpha,\mathscr{D}]\|_{-s}\\ &+ C_1|t_1+\cdots+t_d|\int_0^1 e^{-u(t_1+\cdots+t_d)/2}\|\mathscr{R}^{-s}[r_1^{-ut_1}\dots r_d^{-ut_d}\alpha,\mathscr{D}]\|_{-(s-2)}\,\mathrm{d}u.\end{aligned}$$

Proof. Set for simplicity $r^u = r_1^{-ut_1}\dots r_d^{-ut_d}$ and $t = t_1+\cdots+t_d$. Consider the orthogonal decomposition

$$\mathscr{D} = \mathcal{Z}^{-s}_{r^{-u}\alpha}(\mathscr{D}) + \mathscr{R}^{-s}_{r^{-u}\alpha}(\mathscr{D}), \quad u \in [0,1].$$

If we apply the projection $\mathcal{Z}^{-s}_{r^{\tau-u}\alpha}$, since by Proposition 5.2 we have the identity $\mathcal{Z}^{-s}_{r^{\tau-u}\alpha}\mathcal{Z}^{-s}_{r^{-u}\alpha}(\mathscr{D}) = \mathcal{Z}^{-s}_{r^{-u}\alpha}(\mathscr{D})$, we obtain

$$\mathcal{Z}^{-s}_{r^{\tau-u}\alpha}(\mathscr{D}) = \mathcal{Z}^{-s}_{r^{-u}\alpha}(\mathscr{D}) + \mathcal{Z}^{-s}_{r^{\tau-u}\alpha}(\mathscr{R}^{-s}_{r^{-u}\alpha}(\mathscr{D}))$$

and therefore we may write

$$\begin{aligned}[r^{\tau-u}\alpha, \mathcal{Z}^{-s}_{r^{\tau-u}\alpha}(\mathscr{D})] &= [r^{\tau-u}\alpha, \mathcal{Z}^{-s}_{r^{-u}\alpha}(\mathscr{D})] + [r^{\tau-u}\alpha, \mathcal{Z}^{-s}_{r^{\tau-u}\alpha}(\mathscr{R}^{-s}_{r^{-t}\alpha}(\mathscr{D}))]\\ &= r^\tau\mathcal{Z}^{-s}[r^{-u}\alpha,\mathscr{D}] + \mathcal{Z}^{-s}[r^{\tau-u}\alpha, \mathscr{R}^{-s}_{r^{-u}\alpha}(\mathscr{D})].\end{aligned}$$

Now, we compute the norm with exponent $-s$. By Proposition 5.2, the first term on the right has norm

$$\|r^\tau\mathcal{Z}^{-s}[r^{-u}\alpha,\mathscr{D}]\|_{-s} = e^{-\frac{t}{2}\tau}\|\mathcal{Z}^{-s}[r^{-u}\alpha,\mathscr{D}]\|_{-s}.$$

To estimate the norm of the second term on the right, we observe that $\mathcal{Z}^{-s}_{r^{\tau-u}}$ is an orthogonal projection, and that by Lemma 5.3 the projection

$$R_d(\mathfrak{p}^{d,\alpha}, W^{-(s-2)}_{r^{\tau-u}\alpha}(\mathsf{M})) \to Z_d(\mathfrak{p}^{d,\alpha}, W^{-s}_{r^{-u}\alpha}(\mathsf{M}))$$

has norm bounded by $C(s)\,t\tau$. Therefore

$$\begin{aligned}\|\mathcal{Z}^{-s}[r^{\tau-u}\alpha,\mathscr{D}]\|_{-s} \le\ & e^{-\frac{t}{2}\tau}\|\mathcal{Z}^{-s}[r^{-u}\alpha,\mathscr{D}]\|_{-s}\\ &+ C(s)\,t\tau\,\|\mathscr{R}^{-s}[r^{-u}\alpha,\mathscr{D})\|_{-(s-2)}.\end{aligned}$$

Let $n \in \mathbb{N}^+$, and set $\tau = 1/n$, $u = k\tau$, with $k \in \mathbb{N}\cap[0,n]$. By finite induction on k we obtain

$$\begin{aligned}\|\mathcal{Z}^{-s}[\alpha,\mathscr{D}]\|_{-s} \le\ & e^{-\frac{t}{2}}\|\mathcal{Z}^{-s}[r^{-1}\alpha,\mathscr{D}]\|_{-s}\\ &+ C(s)\frac{t}{n}\sum_{k=1}^n e^{-\frac{tk}{2n}}\|\mathscr{R}^{-s}[r^{-k/n}\alpha,\mathscr{D}]\|_{-(s-2)}.\end{aligned}$$

The lemma follows by taking the limit as $n \to \infty$. $\square$

Next, we consider the case $d = 1$.

THEOREM 5.8. *Let $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$ and $s > g + 7/2$. Let $\mathsf{P}^{1,\alpha}$ be the 1-dimensional Abelian subgroup of H^g generated by the base vector field $X_1^\alpha \in \mathfrak{h}^g$. Let $U_T = [0, T]$ and $\mathscr{P}^{1,\alpha}_{U_T} m$ be the Birkhoff sum associated to some $m \in \mathsf{M}$ for the action of $\mathsf{P}^{1,\alpha}_x$ ($x \in \mathbb{R}$). There exists a constant $C_2 = C_2(s) > 0$ such that for all $T \geq 1$ and all $m \in \mathsf{M}$ we have*

$$\left\| [\alpha, \mathscr{P}^{1,\alpha}_{U_T} m] \right\|_{-s} \leq \quad C_2\, T^{1/2}\, \mathrm{Hgt}\Big([[r_1^{-\log T}\alpha]]\Big)^{1/4} + C_2 \int_0^{\log T} e^{u/2}\, \mathrm{Hgt}\big([[r_1^{-u}\alpha]]\big)^{1/4}\, \mathrm{d}u.$$

Proof. For simplicity we set $r^t = r_1^t$. To start, we observe that, according to (5.4) and Lemma 5.6, we have

$$\begin{aligned}\| \mathscr{R}^{-s}[r^{-t}\alpha, \mathscr{P}^{1,\alpha}_{U_{e^tT}}] \|_{-(s-2)} &= e^t \, \| \mathscr{R}^{-s}[r^{-t}\alpha, \mathscr{P}^{1,r^{-t}\alpha}_{U_T}] \|_{-(s-2)} \\ &\leq e^t \left\| [r^{-t}\alpha, \partial(\mathscr{P}^{1,r^{-t}\alpha}_{U_T})] \right\|_{-s'}\end{aligned}$$

provided $g + 1/2 < s' < s - 3$. The boundary $\partial(\mathscr{P}^{1,r^{-t}\alpha}_{U_T})$ is a 0-dimensional current given by

$$\left\langle \partial(\mathscr{P}^{1,r^{-t}\alpha}_{U_T}, f \right\rangle = f(\mathsf{P}^{r^{-t}\alpha}_T(m)) - f(m),$$

hence, by the Sobolev embedding theorem and the definition (4.4) of the Best Sobolev Constant, we have

$$\left\| [r^{-t}\alpha, \partial(\mathscr{P}^{1,r^{-t}\alpha}_{U_T})] \right\|_{-s'} \leq 2\, B_{s'}([[r^{-t}\alpha]]).$$

It follows from Proposition 4.8 that

$$\left\| \mathscr{R}^{-s}[r^{-t}\alpha, \mathscr{P}^{1,\alpha}_{U_{e^tT}}] \right\|_{-(s-2)} \leq 2\, e^t\, B_{s'}([[r^{-t}\alpha]]) \leq C(s')\, e^t\, \mathrm{Hgt}([[r^{-t}\alpha]])^{1/4}.$$

Using Lemma 5.7 with $\mathscr{D} = \mathscr{P}^{1,\alpha}_{U_{e^tT}} m$ and $t = n\tau$, we may estimate the boundary term in the decomposition (5.7) as

$$\left\| \mathscr{Z}^{-s}[\alpha, \mathscr{P}^{1,\alpha}_{U_{e^tT}}] \right\|_{-s} \leq e^{-t/2} \left\| \mathscr{Z}^{-s}[r^{-t}\alpha, \mathscr{P}^{1,\alpha}_{U_{e^tT}}] \right\|_{-s} + C(s,s') \int_0^t e^{u/2}\, \mathrm{Hgt}([[r^{-u}\alpha]])^{1/4}\, \mathrm{d}u.$$

By the covariance formula (5.4), the Proposition 4.8 and Lemma 5.5, we have

$$\begin{aligned}\left\| \mathscr{Z}^{-s}[r^{-t}\alpha, \mathscr{P}^{1,\alpha}_{U_{e^tT}}] \right\|_{-s} &= e^t \left\| \mathscr{Z}^{-s}[r^{-t}\alpha, \mathscr{P}^{1,r^{-t}\alpha}_{U_T}] \right\|_{-s} \\ &\leq e^t\, C(s)\, T\, \mathrm{Hgt}([[r^{-t}\alpha]])^{1/4}.\end{aligned}$$

It follows that

$$\left\| \mathscr{Z}^{-s}[\alpha, \mathscr{P}^{1,\alpha}_{U_{e^tT}}] \right\|_{-s} \leq e^{t/2}\, C(s)\, T\, \mathrm{Hgt}([[r_1^{-t}\alpha]])^{1/4} + C(s,s') \int_0^t e^{u/2}\, \mathrm{Hgt}([[r^{-u}\alpha]])^{1/4}\, \mathrm{d}u.$$

If we take first $T = 1$, then rename $e^t := T \geq 1$, we finally get

$$\left\| \mathcal{Z}^{-s}[\alpha, \mathcal{P}^{1,\alpha}_{U_T} m] \right\|_{-s} \leq C(s)\, T^{1/2}\, \mathrm{Hgt}([[r^{-\log T}\alpha]])^{1/4} + C(s,s') \int_0^{\log T} e^{t/2}\, \mathrm{Hgt}([[r^{-t}\alpha]])^{1/4}\, \mathrm{d}t.$$

The reminder term in the decomposition (5.7) is estimated as at the beginning of the proof, using Lemma 5.6, Proposition 4.8 and Lemma 4.9, and is bounded by

$$\begin{aligned}\left\| \mathcal{R}^{-s}[\alpha, \mathcal{P}^{1,\alpha}_{U_T}] \right\|_{-s} &\leq C(s)\, \mathrm{Hgt}([\alpha])^{1/4} \\ &= C(s)\, \mathrm{Hgt}([[r^{\log T} r^{-\log T}\alpha]])^{1/4} \\ &\leq C(s)\, T^{1/2}\, \mathrm{Hgt}([[r^{-\log T}\alpha]])^{1/4}.\end{aligned}$$

The theorem follows. □

The next result follows immediately from the above Theorem 5.8 and the Kleinbock-Margulis logarithm law, i.e., from Proposition 4.13.

PROPOSITION 5.9. *Let the notation as in Theorem 5.8. There exists a full measure set $\Omega_g(\widehat{\delta}_1) \subset \mathfrak{M}_g$ such that if $[\alpha] \in \Omega_g(\widehat{\delta}_1)$ and $\varepsilon > 0$ there exists a constant $C = C(s,\varepsilon) > 0$ such that for all $T \gg 1$ and all $m \in \mathsf{M}$ we have*

$$\left\| [\alpha, \mathcal{P}^{1,\alpha}_{U_T} m] \right\|_{-s} \leq C\, T^{1/2} (\log T)^{1/(2g+2)+\varepsilon}.$$

Now we may use induction on the dimension of the isotropic group $\mathsf{P}^d \subset \mathsf{H}^g$. Let $(s_d)_{d\in\mathbb{N}}$ be the solution of the recursive equation $s_{d+1} = s_d + 3 + d/2$ with initial condition $s_1 = g + 7/2$, that is, $s_d = d(d+11)/4 + g + 1/2$.

THEOREM 5.10. *Let $s > s_d$. There exists a constant $C_3 = C_3(s,d) > 0$ such the following holds true. Let $\alpha \in \mathrm{Sp}_{2g}(\mathbb{R})$ and let $\mathsf{P}^{d,\alpha} \subset \mathsf{H}^g$ be the d-dimensional Abelian subgroup of H^g generated by the base vector fields $X^\alpha_1, \ldots, X^\alpha_d \in \mathfrak{h}^g$. Let $U_d(t) := [0, e^t]^d$. Let $\mathcal{P}^{d,\alpha}_{U_d(t)} = \mathcal{P}^{d,\alpha}_{U_d(t)} m$ be the Birkhoff sum associated to some $m \in M$ for the action of $\mathsf{P}^{d,\alpha}_x$, $(x \in \mathbb{R}^d)$. Then, for all $t > 0$ and all $m \in \mathsf{M}$, we have*

$$\left\| [\alpha, \mathcal{P}^{d,\alpha}_{U_d(t)} m] \right\|_{-s} \leq C_3 \sum_{k=0}^{d} \sum_{1\leq i_1<\cdots<i_k\leq d} \int_0^t \cdots \int_0^t \exp\Big(\tfrac{d}{2} t - \tfrac{1}{2}\sum_{\ell=1}^{k} u_\ell\Big) \times \mathrm{Hgt}\Big([[\prod_{1\leq j\leq d} r_j^{-t} \prod_{\ell=1}^{k} r_{i_\ell}^{u_\ell}\alpha]]\Big)^{1/4} \mathrm{d}u_1 \ldots \mathrm{d}u_k. \tag{5.8}$$

Proof. We argue by induction. The case $d = 1$ is Theorem 5.8. We assume the result holds for $d - 1 \geq 1$.

Set for simplicity $r^u = r^u_1 \ldots r^u_d$.

Decomposing $\mathcal{P}^{d,\alpha}_{U_d(t)} m$ as in (5.7) as a sum of the currents $\mathcal{Z}^{-s}[\alpha, \mathcal{P}^{d,\alpha}_{U_d(t)}]$ and $\mathcal{R}^{-s}[\alpha, \mathcal{P}^{d,\alpha}_{U_d(t)}]$, we first estimate the boundary term $\| \mathcal{Z}^{-s}[\alpha, \mathcal{P}^{d,\alpha}_{U_d(t)}] \|_{-s}$. Using

Lemma 5.7 we have

$$
\begin{aligned}
\left\| \mathcal{Z}^{-s}[\alpha, \mathcal{P}^{d,\alpha}_{U_d(t)}] \right\|_{-s} &\le e^{-dt/2} \left\| \mathcal{Z}^{-s}[r^{-1}\alpha, \mathcal{P}^{d,\alpha}_{U_d(t)}] \right\|_{-s} \\
&\quad + C_1(s) \int_0^t e^{-ud/2} \| \mathcal{R}^{-s}[r^{-u}\alpha, \mathcal{P}^{d,\alpha}_{U_d(t)}] \|_{-(s-2)} \, \mathrm{d}u \\
&= I + II.
\end{aligned} \tag{5.9}
$$

By the covariance (5.4), Lemma 5.5, and Proposition 4.8, we have

$$
\begin{aligned}
\left\| \mathcal{Z}^{-s}[r^{-1}\alpha, \mathcal{P}^{d,\alpha}_{U_d(t)}] \right\|_{-s} &= e^{dt} \left\| \mathcal{Z}^{-s}[r^{-1}\alpha, \mathcal{P}^{d,r^{-t}\alpha}_{U_d(0)}] \right\|_{-s} \\
&\le C\, e^{dt} \operatorname{Hgt}([[r^{-t}\alpha]])^{1/4}.
\end{aligned}
$$

Hence

$$
I \le C\, e^{dt/2} \operatorname{Hgt}([[r^{-t}\alpha]])^{1/4}, \tag{5.10}
$$

corresponding to the term with $k = 0$ in the statement of the theorem.

To estimate the term II, we start observing that, provided $s' < s - 2 - (d+1)/2$, using (5.4) and Lemma 5.6, we have

$$
\begin{aligned}
\| \mathcal{R}^{-s}[r^{-u}\alpha, \mathcal{P}^{d,\alpha}_{U_d(t)}] \|_{-(s-2)} &= \| e^{ud} \mathcal{R}^{-s}[r^{-u}\alpha, \mathcal{P}^{d,r^{-u}\alpha}_{U_d(t-u)}] \|_{-(s-2)} \\
&\le C(s', s)\, e^{ud} \left\| [r^{-u}\alpha, \partial(\mathcal{P}^{d,r^{-u}\alpha}_{U_d(t-u)})] \right\|_{-s'}.
\end{aligned} \tag{5.11}
$$

The boundary $\partial(\mathcal{P}^{d,r^{-u}\alpha}_{U_d(t-u)})$ is the sum of $2d$ currents of dimension $d-1$. These currents are Birkhoff sums of d "face" subgroups $\mathsf{P}^{d-1,r^{-u}\alpha}_j$, ($j = 1, \dots, d$), obtained from $\mathsf{P}^{d,r^{-u}\alpha}$ by omitting one of the base vector fields $X^\alpha_1, \dots, X^\alpha_d$. For each $j = 1, \dots, d$ there are two Birkhoff sums of $\mathsf{P}^{d-1,r^{-u}\alpha}_j$ for points $m_{\pm j}$ along the $(d-1)$-dimensional cubes $U_{d-1,j}(t-u)$ obtained from $U_d(t-u)$ by omitting the j-th factor interval $[0, e^{t-u}]$.

If $s' > s_{d-1}$ (and therefore $s > s_{d-1} + (d+1)/2 + 2 = s_d$), denoting by $\mathcal{P}^{d-1,r^{-u}\alpha}_{U_{d-1}(t-u)}$ the generic summand of $\partial(\mathcal{P}^{d,r^{-u}\alpha}_{U_d(t-u)})$, we may estimate the norm of each such boundary term using the inductive hypothesis (5.8). For the j-face we obtain

$$
\begin{aligned}
\left\| [r^{-u}\alpha, \mathcal{P}^{d-1,r^{-u}\alpha}_{U_{d-1}(t-u)}] \right\|_{-s'} \le\ & C_3(s', d-1) \sum_{k=0}^{d-1} \sum_{\substack{1 \le i_1 < \dots < i_k \le d \\ i_\ell \ne j}} \\
& \times \int_0^{t-u} \mathrm{d}u_{i_1} \cdots \int_0^{t-u} \mathrm{d}u_{i_k} \exp\Big(\tfrac{d-1}{2}(t-u) - \tfrac{1}{2} \sum_{\ell=1}^{k} u_{i_\ell}\Big) \\
& \times \operatorname{Hgt}\Big([[\prod_{\substack{1 \le \ell \le d \\ \ell \ne j}} r_\ell^{-(t-u)} \prod_{\ell=1}^{k} r_{i_\ell}^{u_{i_\ell}}\, r^{-u}\alpha]]\Big)^{1/4}.
\end{aligned}
$$

From (5.9) and (5.11) we obtain the following estimate for the term II:

$$
\begin{aligned}
II \le\ & C_4(s,d) \sum_{j=1}^{d} \sum_{k=0}^{d-1} \sum_{\substack{1\le i_1<\dots<i_k\le d\\ i_\ell\neq j}} \\
& \times \int_0^t \mathrm{d}u \int_0^{t-u} \mathrm{d}u_{i_1} \cdots \int_0^{t-u} \mathrm{d}u_{i_k} \exp\left(\tfrac{d-1}{2}t + \tfrac{1}{2}u - \tfrac{1}{2}\sum_{\ell=1}^{k} u_{i_\ell}\right) \\
& \times \operatorname{Hgt}\left([[\prod_{1\le \ell\le d} r_\ell^{-t} \prod_{\ell=1}^{k} r_{i_\ell}^{u_{i_\ell}} r_j^{-u+t}\alpha]]\right).
\end{aligned}
\tag{5.12}
$$

Applying the change of variable $u_j = t-u$, majorizing the integrals $\int_0^{t-u}$ with integrals $\int_0^t$ and observing that there are at most $k+1$ integer intervals $]i_t, i_{t+1}[$ in which the integer j in the above sum may land, we obtain

$$
\begin{aligned}
II \le\ & C_4(s,d) \sum_{j=1}^{d} \sum_{k=0}^{d-1} \sum_{\substack{1\le i_1<\dots<i_k\le d\\ i_\ell\neq j}} \\
& \times \int_0^t \mathrm{d}u_j \int_0^{t-u} \mathrm{d}u_{i_1} \cdots \int_0^{t-u} \mathrm{d}u_{i_k} \exp\left(\tfrac{d}{2}t - \tfrac{1}{2}u_j - \tfrac{1}{2}\sum_{\ell=1}^{k} u_{i_\ell}\right) \\
& \times \operatorname{Hgt}\left([[\prod_{1\le \ell\le d} r_\ell^{-t} \prod_{\ell=1}^{k} r_{i_\ell}^{u_{i_\ell}} r_j^{-u_j}\alpha]]\right). \\
\le\ & C_5(s,d) \sum_{k=1}^{d} \sum_{1\le i_1<\dots<i_k\le d} \int_0^t \mathrm{d}u_{i_1} \cdots \int_0^t \mathrm{d}u_{i_k} \\
& \times \exp\left(\tfrac{d}{2}t - \tfrac{1}{2}\sum_{\ell=1}^{k} u_{i_\ell}\right) \operatorname{Hgt}\left([[\prod_{1\le \ell\le d} r_\ell^{-t} \prod_{\ell=1}^{k} r_{i_\ell}^{u_{i_\ell}} \alpha]]\right).
\end{aligned}
\tag{5.13}
$$

The remainder term $\mathscr{R}^{-s}[\alpha, \mathscr{P}^{d,\alpha}_{U_d(t)}]$ in the decomposition (5.7) is estimated using Lemma 5.6, Proposition 4.8 and Lemma 4.9. We have:

$$
\begin{aligned}
\left\| \mathscr{R}^{-s}[\alpha, \mathscr{P}^{d,\alpha}_{U_d(t)}] \right\|_{-s} &\le C(s)\operatorname{Hgt}([\alpha])^{1/4} \\
&= C(s)\operatorname{Hgt}([[r^t r^{-t}\alpha]])^{1/4} \\
&\le C(s)\, e^{td/2} \operatorname{Hgt}([[r^{-t}\alpha]])^{1/4},
\end{aligned}
\tag{5.14}
$$

producing one more term like (5.10). The theorem follows from the estimates (5.10) and (5.13) for the terms I and II and from (5.14) for the remainder. □

Different possible asymptotics are then consequences of the Diophantine conditions (4.17), (4.18), and (4.19), or the Kleinbock-Margulis logarithm law (Proposition 4.13).

Proof of Theorem 1.7. Let the notations be as in Theorem 5.10, and consider the integrals in (5.8). It follows from Lemma 4.9 that, for any $0 \le k \le d$,

$$\mathrm{Hgt}\Big([[\prod_{1\le j\le d} r_j^{-t} \prod_{\ell=1}^{k} r_{i_\ell}^{u_\ell}\alpha]]\Big)^{1/4} \le e^{\frac{1}{2}\sum_{\ell=1}^{k} u_k}\,\mathrm{Hgt}\Big([[\prod_{1\le j\le d} r_j^{-t}\alpha]]\Big)^{1/4}.$$

It follows from (5.8) that

$$\Big\| [\alpha, \mathscr{P}^{d,\alpha}_{U_d(t)}] \Big\|_{-s} \le C\, t^d\, e^{\frac{d}{2}t}\,\mathrm{Hgt}\Big([[\prod_{1\le j\le d} r_j^{-t}\alpha]]\Big)^{1/4} \tag{5.15}$$

for some constant $C = C(s,d)$. Therefore the norms of our currents depend on the Diophantine properties of α in the direction of $\widehat{\delta}(d) := \widehat{\delta}_1 + \cdots + \widehat{\delta}_d \in \mathfrak{a}$ (recall that $r_i^t = e^{t\widehat{\delta}_i}$), defined in 4.10. For example, if α satisfies a $\widehat{\delta}(d)$-Diophantine condition (4.17) of exponent $\sigma > 0$, we get

$$\Big\| [\alpha, \mathscr{P}^{d,\alpha}_{U_d(t)}] \Big\|_{-s} \le C\, t^d\, e^{d(1-\sigma/2)t} \le C' e^{d(1-\sigma'/2)t}$$

for all $\sigma' < \sigma$. If α satisfies a $\widehat{\delta}(d)$-Roth condition (4.18), we get

$$\Big\| [\alpha, \mathscr{P}^{d,\alpha}_{U_d(t)}] \Big\|_{-s} \le C\, e^{(d/2+\varepsilon)t}$$

for all $\varepsilon > 0$. If α is of bounded type, i.e., satisfies (4.19), then all the “Height” terms inside the integrals of (5.8) are bounded, and we get

$$\Big\| [\alpha, \mathscr{P}^{d,\alpha}_{U_d(t)}] \Big\|_{-s} \le C\, e^{(d/2)t}.$$

On the other side, according to the easy part of Kleinbock and Margulis theorem 4.12, there exists a full measure set $\Omega_g(\widehat{\delta}(d)) \subset \Sigma_g$ such that if $[[\alpha]] \in \Omega_g(\widehat{\delta}(d))$ and $\varepsilon > 0$ then

$$\mathrm{Hgt}\Big([[\prod_{1\le j\le d} r_j^{-t}\alpha]]\Big)^{1/4} \le C\, t^{1/(2g+2)+\varepsilon}.$$

It follows from (5.15) that for such α’s

$$\Big\| [\alpha, \mathscr{P}^{d,\alpha}_{U_d(t)}] \Big\|_{-s} \le C\, t^{d+1/(2g+2)+\varepsilon}\, e^{(d/2)t}. \qquad \square$$

5.3. **Birkhoff sums and Theta sums.**

First return map. Here it is convenient to work with the “polarized” Heisenberg group, the set $\mathsf{H}^g_{\mathrm{pol}} \approx \mathbb{R}^g \times \mathbb{R}^g \times \mathbb{R}$ equipped with the group law $(x,\xi,t)\cdot(x',\xi',t') = (x+x', \xi+\xi', t+t'+\xi x')$. The homomorphism $\mathsf{H}^g \to \mathsf{H}^g_{\mathrm{pol}}$, as well as the exponential map $\exp : \mathfrak{h}^g \to \mathsf{H}^g_{\mathrm{pol}}$, is $(x,\xi,t) \mapsto (x,\xi,t+\frac{1}{2}\xi x)$. Define the “reduced standard Heisenberg group” $\mathsf{H}^g_{\mathrm{red}} := \mathsf{H}^g_{\mathrm{pol}}/(\{0\}\times\{0\}\times\frac{1}{2}\mathbb{Z}) \approx \mathbb{R}^g \times \mathbb{R}^g \times (\mathbb{R}/\frac{1}{2}\mathbb{Z})$, and then the “reduced standard lattice” $\Gamma_{\mathrm{red}} := \mathbb{Z}^g \times \mathbb{Z}^g \times \{0\} \subset \mathsf{H}^g_{\mathrm{red}}$. It is clear that the quotient $\mathsf{H}^g_{\mathrm{red}}/\Gamma_{\mathrm{red}} \approx \mathsf{H}^g/\Gamma$ is the standard nilmanifold. The subgroup $\mathsf{N} = \big\{(0,\xi,t) \,\big|\, \xi \in \mathbb{R}^g, t \in \mathbb{R}/\frac{1}{2}\mathbb{Z}\big\}$ is a normal subgroup of $\mathsf{H}^g_{\mathrm{red}}$. The quotient $\mathsf{H}^g_{\mathrm{red}}/\mathsf{N}$ is isomorphic to the Legendrian subgroup $\mathsf{P} = \big\{(x,0,0) \,|\, x \in \mathbb{R}^g\big\}$, and we have an exact sequence $0 \to \mathsf{N} \to \mathsf{H}^g_{\text{red}} \to \mathsf{P} \to 0$. Therefore $\mathsf{H}^g_{\text{red}} \approx \mathsf{P} \ltimes \mathsf{N}$, and in particular any $(x,\xi,t) \in \mathsf{H}^g_{\text{red}}$ may be uniquely written as the product

$$(x,\xi,t) = \exp(x_1 X_1 + \cdots + x_g X_g) \cdot (0,\xi,t) = (x,0,0) \cdot (0,\xi,t).$$

Given a symmetric $g \times g$ real matrix $\mathcal{Q}$, we consider the symplectic matrix $\alpha = \left(\begin{smallmatrix} I & 0 \\ \mathcal{Q} & I \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{R})$. Then $\exp(x_1 X_1^\alpha + \cdots + x_g X_g^\alpha) = (x, -\mathcal{Q}x, -x^\top \mathcal{Q} x)$, and any element of $\mathsf{H}^g_{\text{red}}$ can be written uniquely as a product

$$\exp(x_1 X_1^\alpha + \cdots + x_g X_g^\alpha) \cdot (0,\xi,t) = (x, \xi - \mathcal{Q}x, t - \tfrac{1}{2} x^\top \mathcal{Q} x)$$

for some $x \in \mathbb{R}^g$, $\xi \in \mathbb{R}^g$ an $t \in \mathbb{R}/\frac{1}{2}\mathbb{Z}$. Given $n \in \mathbb{Z}^g$, $m \in \mathbb{Z}^g$, hence $(n,m,0) \in \Gamma_{\text{red}}$, then

(5.16)
$$\exp(x_1 X_1^\alpha + \cdots + x_g X_g^\alpha) \cdot (0,\xi,t) \cdot (n,m,0) = \exp(x_1' X_1^\alpha + \cdots + x_g' X_g^\alpha) \cdot (0,\xi',t')$$

if and only if $x' = x + n$, $\xi' = \xi + m + \mathcal{Q}n$ and $t' = t + \xi^\top n + \frac{1}{2} n^\top \mathcal{Q} n + \frac{1}{2}\mathbb{Z}$.

Birkhoff sums of certain functions on the circle. Let $\varphi \in \mathcal{S}\left(\mathbb{R}/\frac{1}{2}\mathbb{Z}\right)$, and let $\psi \in \mathcal{E}(\mathbb{R}^g)$ be a smooth function with compact support. Define a function $\phi : \mathsf{H}^g_{\text{red}} \approx \alpha^{-1}(\mathsf{P}) \ltimes \mathsf{N} \to \mathbb{C}$ as the product

$$\phi(\exp(x_1 X_1^\alpha + \cdots + x_g X_g^\alpha) \cdot (0,\xi,t)) := \psi(x) \cdot \varphi(t)$$

and then a function $\tilde{\phi} : \mathsf{M} \to \mathbb{C}$ on the quotient standard nilmanifold by summing over the lattice Γ_{red}. Namely, if $m = \exp(x_1 X_1^\alpha + \cdots + x_g X_g^\alpha) \cdot (0,\xi,t) \cdot \Gamma_{\text{red}} \in \mathsf{M}$, we set

$$\begin{aligned} \tilde{\phi}(m) &:= \sum_{(n,m,0) \in \Gamma_{\text{red}}} \phi(\exp(x_1 X_1^\alpha + \cdots + x_g X_g^\alpha) \cdot (0,\xi,t) \cdot (n,m,0)) \\ &= \sum_{n \in \mathbb{Z}^g} \psi(x+n) \cdot \varphi\left(t + \xi^\top n + \tfrac{1}{2} n^\top \mathcal{Q} n\right), \end{aligned}$$

where we used (5.16). Since ψ has compact support, this sum is finite, so that $\tilde{\phi}$ is indeed a smooth function. The Birkhoff average of $\omega = \tilde{\phi}\, \mathrm{d}X_1^\alpha \wedge \cdots \wedge \mathrm{d}X_g^\alpha$ along the current $\mathsf{P}_U^{g,\alpha} m$ with $m \in \mathsf{M}$ as above is, according to (5.2),

$$\langle \mathcal{P}_U^{g,\alpha} m, \omega \rangle = \sum_{n \in \mathbb{Z}^g} \left(\varphi\left(t + \xi^\top n + \tfrac{1}{2} n^\top \mathcal{Q} n\right) \cdot \int_U \psi(y + x + n)\, \mathrm{d}y \right).$$

Let $0 < \delta < 1/2$, and choose a test function $\psi \in \mathcal{E}(\mathbb{R}^g)$ with support in a small ball $B_\varepsilon(0) = \left\{x \in \mathbb{R}^g \,|\, |x|_\infty \le \varepsilon\right\}$ of radius $0 < \varepsilon < \delta$, and unit mass $\int_{\mathbb{R}^g} \psi(x)\, \mathrm{d}x = 1$. For N a positive integer, $U = [-\delta, N+\delta]^g$ and $x = 0$, we have

$$\langle \mathcal{P}_U^{g,\alpha} m, \omega \rangle = \sum_{n \in \mathbb{Z}^g \cap [0,N]^g} \varphi\left(t + \xi^\top n + \tfrac{1}{2} n^\top \mathcal{Q} n\right). \tag{5.17}$$

Theorem 5.11 follows from Theorem 1.7 in the Introduction and the above discussion (i.e., formula (5.17)):

THEOREM 5.11. *Let $\mathcal{Q}[x] = x^\top \mathcal{Q} x$ be the quadratic forms defined by the symmetric $g \times g$ real matrix $\mathcal{Q}$, $\alpha = \left(\begin{smallmatrix} I & 0 \\ \mathcal{Q} & I \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{R})$, $\ell(x) = \ell^\top x$ be the linear form defined by $\ell \in \mathbb{R}^g$, and $t \in \mathbb{R}$. Then,*

- *there exists a full measure set* $\Omega_g \subset \mathfrak{M}_g$ *such that if* $[\alpha] \in \Omega_g$ *and* $\varepsilon > 0$, *then*
$$\sum_{n\in\mathbb{Z}^g\cap[0,N]^g} \varphi\left(t+\ell(n)+\tfrac{1}{2}\mathscr{Q}[n]\right) = \mathscr{O}\left((\log N)^{g+1/(2g+2)+\varepsilon} N^{g/2}\right);$$
- *if* $[\alpha] \in \mathfrak{M}_g$ *satisfies a* $\widehat{\delta}(g)$*-Roth condition, then for any* $\varepsilon > 0$
$$\sum_{n\in\mathbb{Z}^g\cap[0,N]^g} \varphi\left(t+\ell(n)+\tfrac{1}{2}\mathscr{Q}[n]\right) = \mathscr{O}\left(N^{g/2+\varepsilon}\right);$$
- *if* $[\alpha] \in \mathfrak{M}_g$ *is of bounded type, then*
$$\sum_{n\in\mathbb{Z}^g\cap[0,N]^g} \varphi\left(t+\ell(n)+\tfrac{1}{2}\mathscr{Q}[n]\right) = \mathscr{O}\left(N^{g/2}\right)$$

as $N \to \infty$, *for any test function* $\varphi \in W^s(\mathbb{R}/\frac{1}{2}\mathbb{Z})$ *with Sobolev order* $s > s_g$ *and zero average* $\int_0^{1/2} \varphi(t)\,dt = 0$.

Corollary 1.8 in the Introduction follows if we take $\varphi(t) = e^{4\pi i t}$.

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