2D ELEMENTARY CELLULAR AUTOMATA WITH FOUR NEIGHBORS

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This paper is concerned with the study of square boolean synchronous four-neighbor peripheral cellular automata. It is first shown that, due to conjugation and plane reflection symmetry transformations, the number of dynamically nonequivalent such automata is equal to 4 856. The cellular automata for which the homogeneous final states play a significant role are then identified. Finally, it is shown that, contrary to what happens in the case of one-dimensional boolean three-neighbor cellular automata, for some peripheral automata there is coexistence between a homogeneous final state and other dynamics.

1. Introduction

Despite their simple basic components, cellular automata can exhibit a variety of complex dynamical behavior. This became apparent with the pioneering work of Stephen Wolfram, who, around 1980, made extensive simulations with one-dimensional boolean three-neighbor cellular automata, usually known as *elementary cellular automata* (ECA). Since then, many studies of more complicated cellular automata are still concerned with fitting their time evolution into one of the typical behaviors known for the simplest 1D situation. However, it is our belief that the complexity of one-dimensional and higher dimensional automata can differ significantly, and that it is worth investigating the dynamics of 2D cellular automata searching for some kind of behavior not yet seen with line lattices.

The present work is concerned with the study of a special class of 2D automata: square boolean four-neighbor cellular automata. It is shown that, due to the plane reflection symmetry transformations, the number of dynamically nonequivalent rules for this type of automata is "sufficiently small" to enable a detailed study of all of them. In particular, we are able to identify all the cellular automata of this type for which homogeneous configurations play a significant role. Moreover, our computational experiments show that some of these cellular automata have a singular characteristic: they exhibit coexistence between a homogeneous final state and other dynamics.

2. Two-dimensional boolean four-neighbor cellular automata

We consider finite $n \times m$ boolean synchronous cellular automata with a *peripheral* neighborhood, i.e. automata in which the state of a cell at time $t+1$ depends on the states of its four closest neighbors at the previous time t. If we denote by

$$\mathsf{A}(t) = a_{i,j}(t), \qquad i = 1, \cdots, n,\ j = 1, \cdots, m, \tag{1}$$

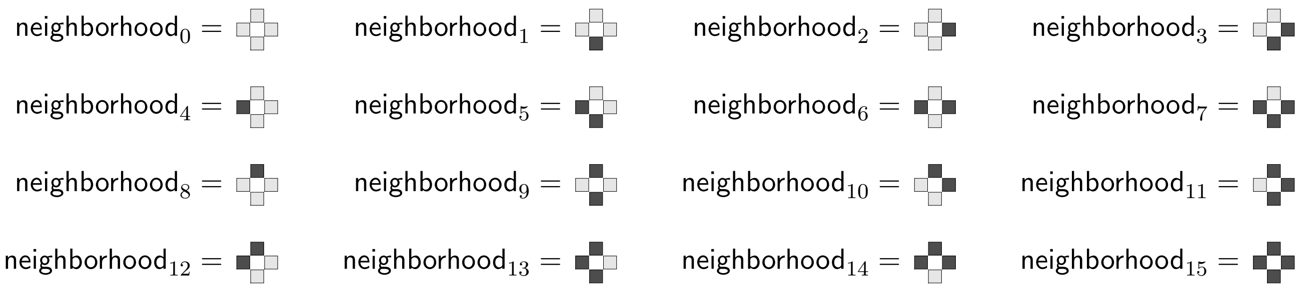
the system state configuration at time t, then the state of the site (i, j) at time $t+1$, $a_{i,j}(t+1)$, is a boolean function ϕ (the so-called *local update rule*) of four variables:

$$a_{i,j}(t+1) = \phi(a_{i-1,j}(t), a_{i,j-1}(t), a_{i,j+1}(t), a_{i+1,j}(t)). \tag{2}$$

Also, we prescribe periodic boundary conditions when updating the cells at the boundaries of the rectangle.

Each configuration is, in this case, a $n \times m$ binary matrix. If we denote by Σ the set of all such configurations, formula (2) defines the so-called global transition function $\Phi : \Sigma \to \Sigma$.

Following [Wolfram, 1984b], we can associate a code number with each cellular automaton. First, we fix the following order for the 16 different possible neighborhoods, with light gray meaning 0 and black meaning 1:



With this ordering of the different possible neighborhoods, we then associate, to each boolean function ϕ, the integer number $N(\phi)$ given by:

$$N(\phi) = \sum_{k=0}^{15} \phi(\mathsf{neighborhood}_k)\, 2^k. \tag{3}$$

In what follows, we will indistinctly refer to a cellular automaton by the associated boolean function ϕ, the global function Φ, or the integer code $N(\phi)$.

3. Dynamically equivalent cellular automata

The characterization of the time evolution of a cellular automaton must be independent of the chosen color scheme and point of view; hence, one can introduce some basic transformations between configurations and declare as dynamically equivalent those cellular automata that preserve these transformations. In the case of one-dimensional ECA, these transformations can be a conjugacy, a left-right reflection or the composition of both. The use of these transformations allows us to consider only 88 dynamically nonequivalent rules, instead of the total number of 256 different rules; see [Walker & Aadryan, 1971], [Li & Packard, 1990], [Wuensche & Lesser, 1992], [Chua *et al.*, 2004, 2005], [Chua *et al.*, 2007], and [Guan *et al.*, 2007]. In the plane case we are studying here, there are other transformations to be taken into account: besides the conjugacy and the left-right reflection, we also have an up-down reflection and, for square lattices, a diagonal reflection may also be added. Naturally, we also have to consider all the possible compositions of any of these transformations.

In what follows, we restrict our study to square $n \times n$ cellular automata.

Definition 3.1. We say that two configurations A and A' are *conjugate*, and write $\mathsf{A} \sim_{\mathsf{c}} \mathsf{A}'$, if $a'_{i,j} = \bar{a}_{i,j}$, for $i, j = 1, \ldots, n$, with $\bar{0} = 1$ and $\bar{1} = 0$ the usual conjugacy boolean operation.

Next, we introduce the basic plane symmetry transformations.

Definition 3.2. Given two configurations A and A':

- we say that they are *left-right symmetric*, and write $\mathsf{A} \sim_{\mathsf{lr}} \mathsf{A}'$, if $a'_{i,j} = a_{n+1-i,j}$, for $i, j = 1, \dots, n$;
- we say that they are *up-down symmetric*, and write $\mathsf{A} \sim_{\mathsf{ud}} \mathsf{A}'$, if $a'_{i,j} = a_{i,n+1-j}$, for $i, j = 1, \dots, n$;
- we say that they are *diagonal symmetric*, and write $\mathsf{A} \sim_{\mathsf{d}} \mathsf{A}'$, if $a'_{i,j} = a_{j,i}$, for $i, j = 1, \dots, n$.

It should be noted that there is no need to consider the anti-diagonal symmetry transformation since it can be obtained as a composition of the other three.

Definition 3.3. Given two cellular automata, ϕ and ϕ', we say that they are *conjugate equivalent*, and write $\phi \sim_{\mathsf{c}} \phi'$, if, for any two conjugate configurations $\mathsf{A} \sim_{\mathsf{c}} \mathsf{A}'$, we have $\Phi(\mathsf{A}) \sim_{\mathsf{c}} \Phi'(\mathsf{A}')$.

Definition 3.4. Given two cellular automata, ϕ and ϕ':

- we say that they are *left-right equivalent*, and write $\phi \sim_{\mathsf{lr}} \phi'$ if, given any two left-right symmetric configurations $\mathsf{A} \sim_{\mathsf{lr}} \mathsf{A}'$, we have $\Phi(\mathsf{A}) \sim_{\mathsf{lr}} \Phi'(\mathsf{A}')$;
- we say that they are *up-down equivalent*, and write $\phi \sim_{\mathsf{ud}} \phi'$ if, given any two up-down symmetric configurations $\mathsf{A} \sim_{\mathsf{ud}} \mathsf{A}'$, we have $\Phi(\mathsf{A}) \sim_{\mathsf{ud}} \Phi'(\mathsf{A}')$;
- we say that they are *diagonal equivalent*, and write $\phi \sim_{\mathsf{d}} \phi'$ if, given any two diagonal symmetric configurations $\mathsf{A} \sim_{\mathsf{d}} \mathsf{A}'$, we have $\Phi(\mathsf{A}) \sim_{\mathsf{d}} \Phi'(\mathsf{A}')$.

In what follows, given two cellular automata, ϕ and ϕ', we consider the binary representation of their integer codes, $N(\phi) = (b_{15} \dots b_0)_2$ and $N(\phi') = (b'_{15} \dots b'_0)_2$. The following four propositions characterize the above basic equivalences of cellular automata in terms of the binary representation of their integer codes. Since the proofs of the propositions are all very similar, we will only present in detail the proof of the last one.

Proposition 1. *Two cellular automata, ϕ and ϕ', are conjugate equivalent, $\phi \sim_{\mathsf{c}} \phi'$, if and only if the digits b_i and b'_i in the binary representation of their integer codes satisfy*

$$\begin{array}{llll}
b'_0 = \bar{b}_{15} & b'_1 = \bar{b}_{14} & b'_2 = \bar{b}_{13} & b'_3 = \bar{b}_{12} \\
b'_4 = \bar{b}_{11} & b'_5 = \bar{b}_{10} & b'_6 = \bar{b}_9 & b'_7 = \bar{b}_8 \\
b'_8 = \bar{b}_7 & b'_9 = \bar{b}_6 & b'_{10} = \bar{b}_5 & b'_{11} = \bar{b}_4 \\
b'_{12} = \bar{b}_3 & b'_{13} = \bar{b}_2 & b'_{14} = \bar{b}_1 & b'_{15} = \bar{b}_0
\end{array}$$

Proposition 2. *Two cellular automata, ϕ and ϕ', are left-right equivalent, $\phi \sim_{\mathsf{lr}} \phi'$, if and only if the digits b_i and b'_i in the binary representation of their integer codes satisfy*

$$\begin{array}{llll}
b'_0 = b_0 & b'_1 = b_1 & b'_2 = b_4 & b'_3 = b_5 \\
b'_4 = b_2 & b'_5 = b_3 & b'_6 = b_6 & b'_7 = b_7 \\
b'_8 = b_8 & b'_9 = b_9 & b'_{10} = b_{12} & b'_{11} = b_{13} \\
b'_{12} = b_{10} & b'_{13} = b_{11} & b'_{14} = b_{14} & b'_{15} = b_{15}
\end{array}$$

Proposition 3. *Two cellular automata, ϕ and ϕ', are up-down equivalent, $\phi \sim_{\mathsf{ud}} \phi'$, if and only if the digits b_i and b'_i in the binary representation of their integer codes satisfy*

$$\begin{array}{llll}
b'_0 = b_0 & b'_1 = b_8 & b'_2 = b_2 & b'_3 = b_{10} \\
b'_4 = b_4 & b'_5 = b_{12} & b'_6 = b_6 & b'_7 = b_{14} \\
b'_8 = b_1 & b'_9 = b_9 & b'_{10} = b_3 & b'_{11} = b_{11} \\
b'_{12} = b_5 & b'_{13} = b_{13} & b'_{14} = b_7 & b'_{15} = b_{15}
\end{array}$$

Proposition 4. *Two cellular automata, ϕ and ϕ', are diagonal equivalent, $\phi \sim_{\mathsf{d}} \phi'$, if and only if the digits b_i and b'_i in the binary representation of their integer codes satisfy*

$$\begin{array}{llll} b'_0 = b_0 & b'_1 = b_2 & b'_2 = b_1 & b'_3 = b_3 \\ b'_4 = b_8 & b'_5 = b_{10} & b'_6 = b_9 & b'_7 = b_{11} \\ b'_8 = b_4 & b'_9 = b_6 & b'_{10} = b_5 & b'_{11} = b_7 \\ b'_{12} = b_{12} & b'_{13} = b_{14} & b'_{14} = b_{13} & b'_{15} = b_{15} \end{array}$$

Proof. Let $\mathsf{A} = (a_{i,j})$ and $\mathsf{A}' = (a'_{i,j})$ be two diagonally equivalent configurations, i.e., $a'_{i,j} = a_{j,i}$, and consider their images by the automata Φ and Φ', respectively, $\tilde{\mathsf{A}} = \Phi(\mathsf{A}) = (\tilde{a}_{i,j})$ and $\tilde{\mathsf{A}}' = \Phi'(\mathsf{A}') = (\tilde{a}'_{i,j})$.

If the automata are diagonally equivalent, then $\tilde{\mathsf{A}} \sim_d \tilde{\mathsf{A}}'$, i.e. we must have $\tilde{a}'_{i,j} = \tilde{a}_{j,i}$. But,

$$\begin{aligned} \tilde{a}'_{i,j} = \tilde{a}_{j,i} \;&\Leftrightarrow\; \phi'(a'_{i-1,j},\, a'_{i,j-1},\, a'_{i,j+1},\, a'_{i+1,j}) = \phi(a_{j-1,i},\, a_{j,i-1},\, a_{j,i+1},\, a_{j+1,i}) \\ &\Leftrightarrow\; \phi'(a'_{i-1,j},\, a'_{i,j-1},\, a'_{i,j+1},\, a'_{i+1,j}) = \phi(a'_{i,j-1},\, a'_{i-1,j},\, a'_{i+1,j},\, a'_{i,j+1}). \end{aligned}$$

Hence, if the automata ϕ and ϕ' are diagonally equivalent, we must have

$$\phi'(x, y, z, w) = \phi(y, x, w, z), \qquad x, y, z, w \in \{0, 1\}. \tag{4}$$

From (4), it follows that:

$$\begin{array}{ll} b'_0 = \phi'(0,0,0,0) = \phi(0,0,0,0) = b_0 & b'_1 = \phi'(0,0,0,1) = \phi(0,0,1,0) = b_2 \\ b'_2 = \phi'(0,0,1,0) = \phi(0,0,0,1) = b_1 & b'_3 = \phi'(0,0,1,1) = \phi(0,0,1,1) = b_3 \\ b'_4 = \phi'(0,1,0,0) = \phi(1,0,0,0) = b_8 & b'_5 = \phi'(0,1,0,1) = \phi(1,0,1,0) = b_{10} \\ b'_6 = \phi'(0,1,1,0) = \phi(1,0,0,1) = b_9 & b'_7 = \phi'(0,1,1,1) = \phi(1,0,1,1) = b_{11} \\ b'_8 = \phi'(1,0,0,0) = \phi(0,1,0,0) = b_4 & b'_9 = \phi'(1,0,0,1) = \phi(0,1,1,0) = b_6 \\ b'_{10} = \phi'(1,0,1,0) = \phi(0,1,0,1) = b_5 & b'_{11} = \phi'(1,0,1,1) = \phi(0,1,1,1) = b_7 \\ b'_{12} = \phi'(1,1,0,0) = \phi(1,1,0,0) = b_{12} & b'_{13} = \phi'(1,1,0,1) = \phi(1,1,1,0) = b_{14} \\ b'_{14} = \phi'(1,1,1,0) = \phi(1,1,0,1) = b_{13} & b'_{15} = \phi'(1,1,1,1) = \phi(1,1,1,1) = b_{15}. \end{array} \tag{5}$$

Conversely, if the digits b_i and b'_i satisfy the relations (5), then relation (4) holds and this, in turn, is all we need to conclude that $\tilde{a}'_{i,j} = \tilde{a}_{j,i}$, i.e. that the automata are equivalent. ■

Definition 3.5. Given two cellular automata, ϕ and ϕ', we say that they are *dynamically equivalent* if, given any two configurations A and A' such that A' is obtained from A by a successive application of any number of the four basic transformations then, $\Phi(\mathsf{A})$ and $\Phi'(\mathsf{A}')$ are related by exactly the same transformations.

The following result is important because it identifies, which, among all different compositions of the four referred basic transformations, are different. For simplicity, we introduce the notation $\mathsf{x} \cdot \mathsf{y}$ to denote the successive application of any basic transformations $\sim_{\mathsf{x}}$, $\sim_{\mathsf{y}}$.

Proposition 5. *There are 15 different dynamical equivalences between cellular automata, which can be written as follows:*

$$\begin{array}{llllll} \mathsf{c} & \mathsf{lr} & \mathsf{ud} & \mathsf{d} & & \\ \mathsf{c}\cdot\mathsf{lr} & \mathsf{c}\cdot\mathsf{ud} & \mathsf{c}\cdot\mathsf{d} & \mathsf{lr}\cdot\mathsf{ud} & \mathsf{lr}\cdot\mathsf{d} & \mathsf{ud}\cdot\mathsf{d} \\ \mathsf{c}\cdot\mathsf{lr}\cdot\mathsf{ud} & \mathsf{c}\cdot\mathsf{lr}\cdot\mathsf{d} & \mathsf{c}\cdot\mathsf{ud}\cdot\mathsf{d} & \mathsf{lr}\cdot\mathsf{ud}\cdot\mathsf{d} & & \\ \mathsf{c}\cdot\mathsf{lr}\cdot\mathsf{ud}\cdot\mathsf{d} & & & & & \end{array} \tag{6}$$

Proof. By using Propositions 1–4, one can easily verify that each of the basic transformations is its own inverse:

$$\mathsf{c}\cdot\mathsf{c}=\mathsf{id}\qquad \mathsf{lr}\cdot\mathsf{lr}=\mathsf{id}\qquad \mathsf{ud}\cdot\mathsf{ud}=\mathsf{id}\qquad \mathsf{d}\cdot\mathsf{d}=\mathsf{id}, \tag{7}$$

where id denotes the identity transformation, and also that the following identities hold:

$$\begin{array}{llll} \mathsf{lr}\cdot\mathsf{c}=\mathsf{c}\cdot\mathsf{lr} & \mathsf{ud}\cdot\mathsf{c}=\mathsf{c}\cdot\mathsf{ud} & \mathsf{d}\cdot\mathsf{c}=\mathsf{c}\cdot\mathsf{d} & \mathsf{ud}\cdot\mathsf{lr}=\mathsf{lr}\cdot\mathsf{ud} \\ \mathsf{d}\cdot\mathsf{lr}=\mathsf{ud}\cdot\mathsf{d} & \mathsf{d}\cdot\mathsf{ud}=\mathsf{lr}\cdot\mathsf{d} & & \end{array} \tag{8}$$

It follows from (8) that any composition $\mathsf{x}_1\cdot\mathsf{x}_2\cdot\ \ldots\ \cdot\mathsf{x}_p$ with $\mathsf{x}_i\in\{\mathsf{c},\mathsf{lr},\mathsf{ud},\mathsf{d}\}$ can be rearranged in the form

$$\underbrace{\mathsf{c}\cdot\ldots\cdot\mathsf{c}}_{p_1}\cdot\underbrace{\mathsf{lr}\cdot\ldots\cdot\mathsf{lr}}_{p_2}\cdot\underbrace{\mathsf{ud}\cdot\ldots\cdot\mathsf{ud}}_{p_3}\cdot\underbrace{\mathsf{d}\cdot\ldots\cdot\mathsf{d}}_{p_4},$$

with $p_i\geq 0$ and $p_1+p_2+p_3+p_4=p$. With the use of (7), it becomes clear that $\mathsf{x}_1\cdot\mathsf{x}_2\cdot\ \ldots\ \cdot\mathsf{x}_p$ is equal to one of the transformations listed in (6). Finally, it is a trivial exercise to show that these transformations are, indeed, different; see, e.g. the example below. ■

Example 3.1. Consider the cellular automaton ϕ with integer code $N(\phi)=123$. From the previous results, we know that there are at most 15 cellular automata dynamically equivalent to ϕ. Their codes are:

$$\begin{array}{lll} 123\sim_{\mathsf{c}}8703 & 123\sim_{\mathsf{lr}}111 & 123\sim_{\mathsf{ud}}5457 \\ 123\sim_{\mathsf{d}}1805 & 123\sim_{\mathsf{c\cdot lr}}2559 & 123\sim_{\mathsf{c\cdot ud}}30039 \\ 123\sim_{\mathsf{c\cdot d}}20255 & 123\sim_{\mathsf{lr\cdot ud}}5445 & 123\sim_{\mathsf{lr\cdot d}}1551 \\ 123\sim_{\mathsf{ud\cdot d}}4913 & 123\sim_{\mathsf{c\cdot lr\cdot ud}}23895 & 123\sim_{\mathsf{c\cdot lr\cdot d}}3999 \\ 123\sim_{\mathsf{c\cdot ud\cdot d}}29495 & 123\sim_{\mathsf{lr\cdot ud\cdot d}}4659 & 123\sim_{\mathsf{c\cdot lr\cdot ud\cdot d}}13239 \end{array}$$

In Appendix A, we list all the dynamically nonequivalent cellular automata rules, obtained by applying the 15 equivalence transformations referred to in Proposition 5 to the 65 536 different automata. As a result of these computations, we can state the following result:

Theorem 1. *There are 4 856 dynamically nonequivalent square boolean synchronous peripheral cellular automata.*

We claim that the number 4 856 is sufficiently small to allow a detailed study of the dynamics of these automata, in a manner similar to what was done for the case of ECA. Moreover, this is almost surely the only family of two-dimensional cellular automata that we may still be able to investigate explicitly. Note that, according to Proposition 5, the number of nonequivalent 2D five-neighbor boolean cellular automata is, already, at least 286 331 153.

Next, we will identify the cellular automata for which a homogeneous configuration is dynamically relevant.

4. Class-1 homogeneous cellular automata

In 1984, Wolfram [Wolfram, 1984a] proposed a classification of one-dimensional boolean three-neighbor cellular automata into four different classes, based on the analysis of the behavior of patterns generated by their time evolution. Although this classification was given for a particular type of system, it became widely accepted also for more general cellular automata. The first class identified by Wolfram corresponds to the following behavior: *starting from typical initial configurations, the cellular automaton evolves to homogeneous final states* [Packard & Wolfram, 1985], where these final states can be either a fixed point, a pair of fixed points, or a 2-cycle. Since, for any automaton, one of the three situations cited above is always an attractor, the key point here is the *starting from typical initial conditions*: class-1 cellular automata are those rules for which the relative size of the set of configurations leading to the homogeneous final state, i.e. the relative size of the basin of attraction of the homogeneous final state, tends to 1 as the number of sites of the system increases. Our computational experiments indicate that of the 2D square boolean synchronous peripheral cellular automata with periodic boundary conditions studied in this paper, 353 correspond to class-1 dynamics.

First, we list the cellular automata codes corresponding to a fixed point homogeneous final state:

Table 1. Class-1 cellular automata codes with fixed point homogeneous final state.

0	8	40	64	72	104	128	136	168	192	200	232	552	576
584	616	640	648	680	704	712	744	1056	1064	1120	1128	1152	1160
1184	1192	1216	1224	1248	1256	1632	1640	1664	1672	1696	1704	1728	1736
1760	1768	2176	2184	2208	2216	2240	2248	2272	2280	2720	2728	2752	2760
2784	2792	3232	3240	3296	3304	3808	3816	5248	5256	5312	5320	5760	5768
5824	5832	6272	6280	6304	6312	6336	6344	6368	6376	6816	6824	6848	6856
6880	6888	7328	7392	7904	10240	10248	10280	10304	10312	10344	10368	10376	10408
10432	10440	10472	10752	10760	10792	10816	10824	10856	10880	10888	10920	10944	10952
10984	11272	11296	11304	11336	11360	11368	11392	11400	11424	11432	11456	11464	11488
11496	11784	11808	11816	11848	11872	11880	11904	11912	11936	11944	11968	11976	12000
15488	15496	15552	15560	16000	16008	16064	26752	26760	26792	26816	26824	26856	27304
27328	27336	27368	27808	27816	27872	27880	28384						

Next, we list the cellular automata codes for which there is coexistence of two fixed points as homogeneous final states:

Table 2. Class-1 cellular automata codes with coexistence of two fixed points as homogeneous final states.

32768	32776	32808	32832	32840	32872	32896	32904	32936	32960	32968	33000	33320	33344
33352	33384	33408	33416	33448	33472	33480	33512	33824	33832	33888	33896	33920	33928
33952	33960	33984	33992	34016	34024	34400	34408	34432	34440	34464	34472	34496	34504
34528	34536	34944	34952	34976	34984	35008	35016	35040	35048	35488	35496	35520	35528
35552	35560	36000	36008	36064	36072	36576	38016	38024	38080	38088	38528	38536	38592
38600	39040	39048	39072	39080	39104	39112	39136	39144	39584	39592	39616	39624	39648
40096	40160	40672	43008	43016	43048	43072	43080	43112	43136	43144	43176	43200	43208
43520	43528	43560	43584	43592	43624	43648	43656	43688	43712	43720	44040	44064	44072
44104	44128	44136	44160	44168	44192	44200	44224	44232	44256	44552	44576	44584	44616
44640	44672	44680	44704	44712	44736	44744	44768	48256	48264	48320	48328	48768	48776
48832	59520	59528	59560	59584	59592	60096	60104	60576	60640				

Finally, we list the cellular automata codes corresponding to a 2-cycle homogeneous final state:

Table 3. Class-1 cellular automata codes with 2-cycle homogeneous final state.

3	7	23	25	27	31	63	67	69	71	87	89	91	95	127	287
297	303	317	319	327	329	333	335	351	367	415	447	479	579	583	599
603	607	815	829	863	879	991	927	1063	1127	1339	1403	1639			

It should be mentioned that, although belonging to the same class, some of these automata show a linear growing process of their basin of attraction of the homogeneous final state, in contrast with the exponential growth behavior displayed by all the elementary cellular automata.

We now describe the computational procedure used to obtain all the results that follow. Given an automaton, we denote by B_h the basin of attraction of its homogeneous final state and by $\%B_h$ the relative size of B_h. To obtain a first approximation, M_h, to the length of B_h, we used 2 000 random initial configurations and computed the maximum of the transient times of all of those configurations that led to the homogeneous final state; in this computation, we allowed the system to evolve for a time much larger than M_h; then, using 12 000 random initial configurations, we estimated $\%B_h$ from the number of initial configurations that, for a time $\triangle t = 1.2\, M_h$, reached the homogeneous final state.

Example 4.1. We considered the cellular automaton $N(\phi) = 44648$ and computed $\%B_h$ as indicated above. The results are displayed in the next figure, which clearly shows an extremely slow linear growth of $\%B_h$ with the size of the automaton, specially for even values of d.

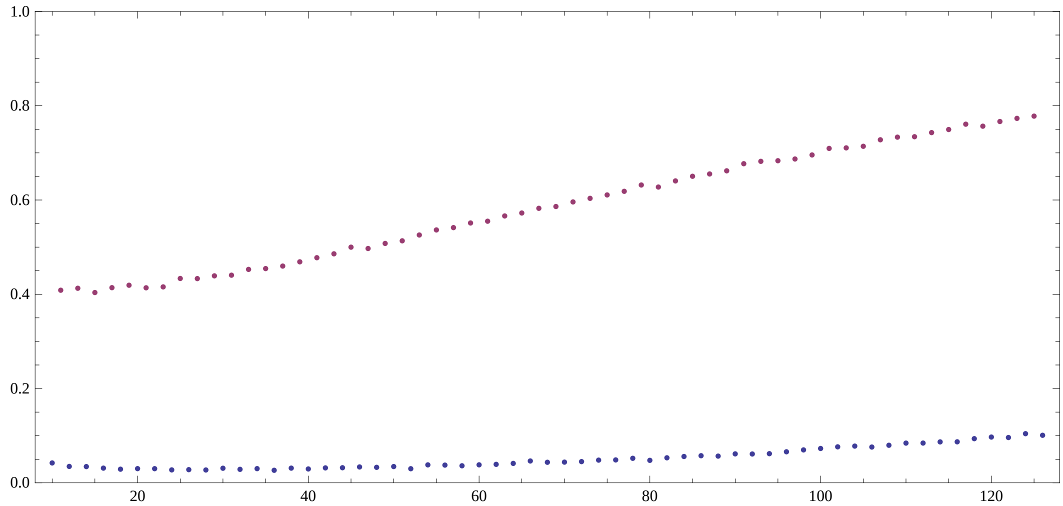


Fig. 1. Change with d of the relative size of the basin of attraction of the homogeneous null final state for the $d \times d$ cellular automaton $N(\phi) = 44648$.

5. Coexistence of dynamics

Although relevant for any computational simulation, the linear growth of the importance of the homogeneous final state with the size of the system still satisfies the original idea that defined automata of class-1. Yet, we found cellular automata for which coexistence between a homogeneous final state and other dynamics is intrinsic to the system, in the sense that, no matter how large we choose the number of their elements to be, there is always coexistence of dynamics.

Example 5.1. We considered the cellular automaton $N(\phi) = 383$ and obtained the results shown in the following figure, where we also plotted the linear fits, L_{odd} and L_{even}, of the points corresponding to odd and even values of d, respectively.

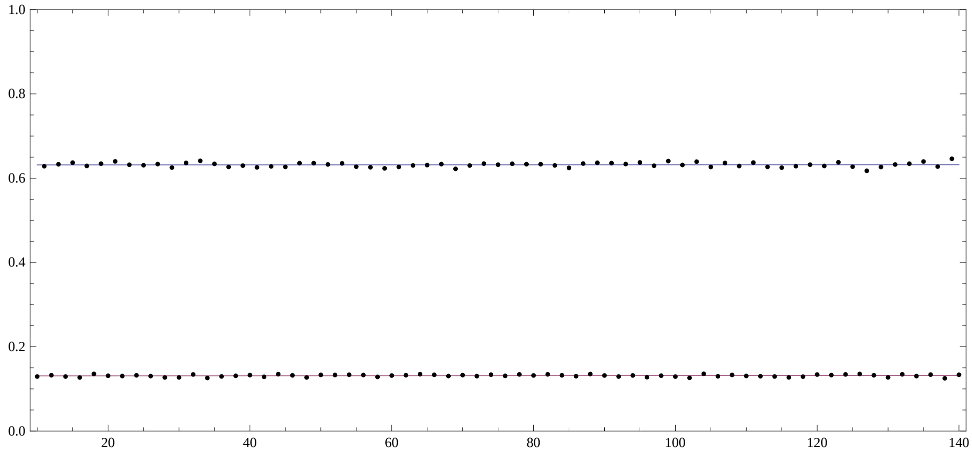


Fig. 2. Change with d of the relative size of the basin of attraction of the homogeneous final state for the $d \times d$ cellular automaton $N(\phi) = 383$.

The expressions found for the linear fits are given by $L_{\mathsf{odd}}(d) = 0.631536 + 2.45484 \times 10^{-6} d$ and $L_{\mathsf{even}}(d) = 0.130931 + 5.87447 \times 10^{-6} d$. The very small values of both slopes allow us to say that, for each case, d even and d odd, the relative size of the basin of attraction of the homogeneous final state $\%B_h$ does not depend on d.

Other computational experiments led us to conclude that there exist six square boolean peripheral cellular automata with periodic boundary conditions for which the relative size of the basin of attraction of the homogeneous final state remains constant. Their codes are given in the following table.

Table 4. The square peripheral cellular automata that exhibit a homogeneous final state coexisting with other dynamics.

383	575	831	43240	59624	60072

It is worth noticing that, of the listed six rules showing coexistence of dynamics, the first three have a 2-cycle as homogeneous final state, while the other three have a pair of fixed points as homogeneous final states.

6. Conclusions

The possibility to do a detailed analysis of a family of cellular automata, as Wolfram did for the ECA, gives us a global perception of the diversity of its dynamics. However, for more complicated systems than the ones considered by Wolfram, the attempt to systematically scrutinize all the dynamics has obvious computational difficulties, due to the exponential growth of the number of elements of the family. We have shown that, due to plane symmetry transformations, the family of 2D square boolean peripheral automata has only 4 856 dynamically nonequivalent rules. Since there are a total of 65 536 different rules, this implies a saving of nearly 93% of computer time. This reasonable number of rules enabled us to analyze all of them in order to identify which ones are of class-1. We also showed that there are systems for which the relative size of the basin of attraction of the homogeneous final state does not depend on the number of sites of the system. This is a very surprising result, not seen for the ECA case nor, as far as we are aware, referred to for other systems and gives us the conviction that this family of automata deserves further investigation.

Appendix A

In the following table we list the 4 856 dynamically nonequivalent square boolean synchronous four-neighbor peripheral cellular automata codes. As usual, we choose the cellular automaton with the smallest code as the equivalence class representative.

Table A.1. Dinamically nonequivalent square boolean peripheral cellular automata codes; cellular automata with the smallest code were chosen as the equivalence class representatives.

0	1	2	3	6	7	8	9	10	11	14	15	20	21
22	23	24	25	26	27	28	29	30	31	40	41	42	43
44	45	46	47	60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79	84	85	86	87
88	89	90	91	92	93	94	95	104	105	106	107	108	109
110	111	124	125	126	127	128	129	130	131	132	133	134	135
136	137	138	139	140	141	142	143	148	149	150	151	152	153
154	155	156	157	158	159	168	169	170	171	172	173	174	175
188	189	190	191	192	193	194	195	196	197	198	199	200	201
202	203	204	205	206	207	212	213	214	215	216	217	218	219
220	221	222	223	232	233	234	235	236	237	238	239	252	253
254	255	278	279	280	281	282	283	286	287	296	297	298	299
300	301	302	303	316	317	318	319	322	323	326	327	328	329
330	331	332	333	334	335	342	343	344	345	346	347	348	349
350	351	360	361	362	363	364	365	366	367	380	381	382	383
384	385	386	387	388	389	390	391	392	393	394	395	396	397
398	399	404	405	406	407	408	409	410	411	412	413	414	415
424	425	426	427	428	429	430	431	444	445	446	447	448	449
450	451	452	453	454	455	456	457	458	459	460	461	462	463
468	469	470	471	472	473	474	475	476	477	478	479	488	489
490	491	492	493	494	495	508	509	510	552	553	554	555	556
557	558	559	572	573	574	575	576	577	578	579	582	583	584
585	586	587	590	591	596	597	598	599	600	601	602	603	604
605	606	607	616	617	618	619	620	621	622	623	636	637	638
640	641	642	643	644	645	646	647	648	649	650	651	652	653
654	655	660	661	662	663	664	665	666	667	668	669	670	671
680	681	682	683	684	685	686	687	700	701	702	703	704	705
706	707	708	709	710	711	712	713	714	715	716	717	718	719
724	725	726	727	728	729	730	731	732	733	734	735	744	745
746	747	748	749	750	751	764	765	766	808	809	810	811	812
813	814	815	828	829	830	831	854	855	1006	1007	1020	1021	1022
1056	1057	1058	1059	1062	1063	1064	1065	1066	1067	1070	1071	1074	1075

Table A.2. Table 1A. (continued).

1076	1077	1078	1079	1080	1081	1082	1083	1084	1085	1086	1120	1121	1122
1123	1124	1125	1126	1127	1128	1129	1130	1131	1132	1133	1134	1135	1138
1139	1140	1141	1142	1143	1144	1145	1146	1147	1148	1149	1150	1152	1153
1154	1155	1156	1157	1158	1159	1160	1161	1162	1163	1164	1165	1166	1167
1168	1169	1170	1171	1172	1173	1174	1175	1176	1177	1178	1179	1180	1181
1182	1183	1184	1185	1186	1187	1188	1189	1190	1191	1192	1193	1194	1195
1196	1197	1198	1199	1200	1201	1202	1203	1204	1205	1206	1207	1208	1209
1210	1211	1212	1213	1214	1216	1217	1218	1219	1220	1221	1222	1223	1224
1225	1226	1227	1228	1229	1230	1231	1232	1233	1234	1235	1236	1237	1238
1239	1240	1241	1242	1243	1244	1245	1246	1247	1248	1249	1250	1251	1252
1253	1254	1255	1256	1257	1258	1259	1260	1261	1262	1263	1264	1265	1266
1267	1268	1269	1270	1271	1272	1273	1274	1275	1276	1277	1278	1334	1335
1336	1337	1338	1339	1342	1378	1379	1382	1383	1384	1385	1386	1387	1388
1389	1390	1398	1399	1400	1401	1402	1403	1404	1405	1406	1408	1409	1410
1411	1412	1413	1414	1415	1416	1417	1418	1419	1420	1421	1422	1424	1425
1426	1427	1428	1429	1430	1431	1432	1433	1434	1435	1436	1437	1438	1440
1441	1442	1443	1444	1445	1446	1447	1448	1449	1450	1451	1452	1453	1454
1456	1457	1458	1459	1460	1461	1462	1463	1464	1465	1466	1467	1468	1469
1470	1472	1473	1474	1475	1476	1477	1478	1479	1480	1481	1482	1483	1484
1485	1486	1488	1489	1490	1491	1492	1493	1494	1496	1497	1498	1499	1500
1501	1502	1504	1505	1506	1507	1508	1509	1510	1511	1512	1513	1514	1515
1516	1517	1518	1520	1521	1522	1523	1524	1525	1526	1528	1529	1530	1531
1532	1534	1632	1633	1634	1635	1638	1639	1640	1641	1642	1643	1646	1647
1650	1651	1652	1653	1654	1656	1657	1658	1659	1660	1661	1662	1664	1665
1666	1667	1668	1669	1670	1671	1672	1673	1674	1675	1676	1677	1678	1679
1680	1681	1682	1683	1684	1685	1686	1687	1688	1689	1690	1691	1692	1693
1694	1695	1696	1697	1698	1699	1700	1701	1702	1703	1704	1705	1706	1707
1708	1709	1710	1711	1712	1713	1714	1715	1716	1717	1718	1719	1720	1721
1722	1723	1724	1725	1726	1728	1729	1730	1731	1732	1733	1734	1735	1736
1737	1738	1739	1740	1741	1742	1743	1744	1745	1746	1747	1748	1749	1750
1751	1752	1753	1754	1755	1756	1757	1758	1760	1761	1762	1763	1764	1765
1766	1767	1768	1769	1770	1771	1772	1773	1774	1775	1776	1777	1778	1779
1780	1781	1782	1784	1785	1786	1787	1788	1789	1790	1910	1912	1913	1914
1918	1920	1921	1922	1923	1924	1925	1926	1928	1929	1930	1931	1932	1933
1934	1936	1937	1938	1939	1940	1941	1942	1944	1945	1946	1947	1948	1949
1950	1952	1953	1954	1955	1956	1957	1958	1960	1961	1962	1963	1964	1965

Table A.3. Table 1A. (continued).

1966	1968	1969	1970	1972	1973	1974	1976	1977	1978	1980	1982	1984	1985
1986	1987	1988	1989	1990	1992	1993	1994	1995	1996	1997	1998	2000	2001
2002	2003	2004	2006	2008	2009	2010	2011	2012	2014	2016	2017	2018	2019
2020	2021	2022	2024	2025	2026	2027	2028	2029	2030	2032	2033	2034	2036
2038	2040	2041	2042	2044	2046	2176	2177	2178	2179	2182	2183	2184	2185
2186	2187	2190	2191	2192	2193	2194	2195	2196	2197	2198	2199	2200	2201
2202	2203	2204	2205	2206	2208	2209	2210	2211	2212	2213	2214	2215	2216
2217	2218	2219	2220	2221	2222	2223	2224	2225	2226	2227	2228	2229	2230
2232	2233	2234	2235	2236	2237	2238	2240	2241	2242	2243	2244	2245	2246
2247	2248	2249	2250	2251	2252	2253	2254	2255	2256	2257	2258	2259	2260
2261	2262	2263	2264	2265	2266	2267	2268	2269	2270	2272	2273	2274	2275
2276	2277	2278	2279	2280	2281	2282	2283	2284	2285	2286	2287	2288	2289
2290	2291	2292	2293	2294	2296	2297	2298	2299	2300	2301	2302	2448	2449
2450	2451	2454	2455	2456	2457	2458	2459	2462	2464	2465	2466	2467	2468
2469	2470	2471	2472	2473	2474	2475	2476	2477	2478	2480	2481	2482	2484
2485	2486	2488	2489	2490	2492	2493	2494	2496	2497	2498	2499	2500	2501
2502	2503	2504	2505	2506	2507	2508	2509	2510	2512	2513	2514	2515	2516
2517	2518	2519	2520	2521	2522	2523	2524	2525	2526	2528	2529	2530	2531
2532	2533	2534	2535	2536	2537	2538	2539	2540	2541	2542	2544	2545	2546
2548	2549	2550	2552	2553	2554	2556	2557	2558	2720	2721	2722	2723	2724
2725	2726	2727	2728	2729	2730	2731	2732	2733	2734	2735	2736	2737	2738
2739	2740	2741	2742	2744	2745	2746	2747	2748	2749	2750	2752	2753	2754
2755	2758	2759	2760	2761	2762	2763	2766	2767	2768	2769	2770	2771	2772
2773	2774	2775	2776	2777	2778	2779	2780	2781	2782	2784	2785	2786	2787
2788	2789	2790	2791	2792	2793	2794	2795	2796	2797	2798	2800	2801	2802
2803	2804	2805	2806	2808	2809	2810	2812	2813	2814	2976	2977	2978	2979
2980	2981	2982	2983	2984	2985	2986	2987	2988	2989	2990	2992	2993	2994
2996	2997	2998	3000	3001	3002	3004	3005	3006	3024	3025	3026	3027	3030
3031	3032	3033	3034	3035	3038	3040	3041	3042	3043	3044	3045	3046	3047
3048	3049	3050	3051	3052	3053	3054	3056	3057	3058	3060	3061	3062	3064
3065	3066	3068	3069	3070	3232	3233	3234	3235	3238	3239	3240	3241	3242
3243	3246	3248	3249	3250	3251	3252	3253	3254	3256	3257	3258	3260	3261
3262	3296	3297	3298	3299	3300	3301	3302	3303	3304	3305	3306	3307	3308
3309	3310	3312	3313	3314	3315	3316	3317	3318	3320	3321	3322	3324	3325
3326	3504	3505	3506	3510	3512	3513	3514	3518	3552	3553	3554	3555	3556
3557	3558	3560	3561	3562	3563	3564	3566	3568	3569	3570	3572	3574	3576

Table A.4. Table 1A. (continued).

3577	3578	3580	3582	3808	3809	3810	3811	3814	3815	3816	3817	3818	3819
3822	3824	3825	3826	3828	3829	3830	3832	3833	3834	3836	3837	3838	4080
4081	4082	4086	4088	4089	4090	4094	5160	5161	5162	5163	5166	5180	5181
5182	5224	5225	5226	5227	5228	5229	5230	5244	5245	5246	5248	5249	5250
5251	5252	5253	5254	5255	5256	5257	5258	5259	5260	5261	5262	5268	5269
5270	5271	5272	5273	5274	5275	5276	5277	5278	5288	5289	5290	5291	5292
5293	5294	5308	5309	5310	5312	5313	5314	5315	5316	5317	5318	5319	5320
5321	5322	5323	5324	5325	5326	5332	5333	5334	5335	5336	5337	5338	5339
5340	5341	5342	5352	5353	5354	5355	5356	5357	5358	5372	5373	5374	5438
5482	5483	5486	5502	5504	5505	5506	5507	5508	5510	5512	5513	5514	5515
5516	5518	5524	5526	5528	5530	5532	5534	5544	5545	5546	5547	5548	5550
5564	5566	5568	5569	5570	5571	5572	5574	5576	5577	5578	5579	5580	5582
5588	5590	5592	5594	5596	5598	5608	5609	5610	5611	5612	5614	5628	5630
5736	5737	5738	5739	5742	5756	5757	5758	5760	5761	5762	5763	5764	5765
5766	5767	5768	5769	5770	5771	5772	5773	5774	5780	5781	5782	5783	5784
5785	5786	5787	5788	5789	5790	5800	5801	5802	5803	5804	5805	5806	5820
5821	5822	5824	5825	5826	5827	5828	5829	5830	5831	5832	5833	5834	5835
5836	5837	5838	5844	5845	5846	5848	5849	5850	5851	5852	5853	5854	5864
5865	5866	5867	5868	5869	5870	5884	5885	5886	6014	6016	6017	6018	6019
6020	6022	6024	6025	6026	6027	6028	6030	6036	6038	6040	6042	6044	6046
6056	6057	6058	6059	6060	6062	6076	6078	6080	6081	6082	6083	6084	6086
6088	6089	6090	6091	6092	6094	6100	6102	6104	6106	6108	6110	6120	6121
6122	6123	6124	6126	6140	6142	6272	6273	6274	6275	6278	6279	6280	6281
6282	6283	6286	6288	6289	6290	6291	6292	6293	6294	6296	6297	6298	6299
6300	6301	6302	6304	6305	6306	6307	6308	6309	6310	6311	6312	6313	6314
6315	6316	6317	6318	6320	6321	6322	6324	6325	6326	6328	6329	6330	6332
6333	6334	6336	6337	6338	6339	6340	6341	6342	6343	6344	6345	6346	6347
6348	6349	6350	6352	6353	6354	6355	6356	6357	6358	6360	6361	6362	6363
6364	6365	6366	6368	6369	6370	6371	6372	6373	6374	6375	6376	6377	6378
6379	6380	6381	6382	6384	6385	6386	6388	6389	6390	6392	6393	6394	6396
6397	6398	6544	6546	6550	6552	6554	6558	6560	6561	6562	6563	6564	6565
6566	6568	6569	6570	6571	6572	6573	6574	6576	6578	6580	6582	6584	6586
6588	6590	6592	6593	6594	6595	6596	6597	6598	6600	6601	6602	6603	6604
6605	6606	6608	6610	6612	6614	6616	6618	6620	6622	6624	6625	6626	6627
6628	6629	6630	6632	6633	6634	6635	6636	6638	6640	6642	6644	6646	6648
6650	6652	6654	6816	6817	6818	6819	6820	6821	6822	6823	6824	6825	6826

Table A.5. Table 1A. (continued).

6827	6828	6829	6830	6832	6833	6834	6836	6837	6838	6840	6841	6842	6844
6845	6846	6848	6849	6850	6851	6854	6855	6856	6857	6858	6859	6862	6864
6865	6866	6868	6869	6870	6872	6873	6874	6876	6877	6878	6880	6881	6882
6883	6884	6885	6886	6888	6889	6890	6891	6892	6893	6894	6896	6897	6898
6900	6901	6902	6904	6905	6906	6908	6909	6910	7072	7073	7074	7075	7076
7077	7078	7080	7081	7082	7083	7084	7086	7088	7090	7092	7094	7096	7098
7100	7102	7120	7122	7126	7128	7130	7134	7136	7137	7138	7139	7140	7142
7144	7145	7146	7147	7148	7150	7152	7154	7156	7158	7160	7162	7164	7166
7328	7329	7330	7331	7334	7336	7337	7338	7339	7342	7344	7345	7346	7348
7349	7350	7352	7353	7354	7356	7357	7358	7392	7393	7394	7395	7396	7397
7398	7400	7401	7402	7403	7404	7405	7406	7408	7409	7410	7412	7413	7414
7416	7417	7418	7420	7421	7422	7600	7602	7606	7608	7610	7614	7648	7649
7650	7651	7652	7654	7656	7657	7658	7659	7660	7662	7664	7666	7668	7670
7672	7674	7676	7678	7904	7905	7906	7907	7910	7912	7913	7914	7915	7918
7920	7921	7922	7924	7925	7926	7928	7929	7930	7932	7933	7934	8176	8178
8182	8184	8186	8190	10240	10241	10242	10243	10244	10245	10246	10248	10249	10250
10251	10252	10253	10254	10260	10261	10262	10264	10265	10266	10268	10269	10270	10280
10281	10282	10283	10284	10285	10286	10300	10301	10302	10304	10305	10306	10307	10308
10309	10310	10312	10313	10314	10315	10316	10317	10318	10324	10325	10326	10328	10329
10330	10332	10333	10334	10344	10345	10346	10347	10348	10349	10350	10364	10365	10366
10368	10369	10370	10371	10372	10373	10374	10376	10377	10378	10379	10380	10381	10382
10388	10389	10390	10392	10393	10394	10396	10397	10398	10408	10409	10410	10411	10412
10413	10414	10428	10429	10430	10432	10433	10434	10435	10436	10437	10438	10440	10441
10442	10443	10444	10445	10446	10452	10453	10454	10456	10457	10458	10460	10461	10462
10472	10473	10474	10475	10476	10477	10478	10492	10493	10494	10498	10499	10502	10504
10505	10506	10507	10508	10510	10518	10520	10522	10524	10526	10536	10537	10538	10539
10540	10542	10556	10558	10562	10563	10566	10568	10569	10570	10571	10572	10574	10582
10584	10586	10588	10590	10600	10601	10602	10603	10604	10606	10620	10622	10624	10625
10626	10627	10628	10630	10632	10633	10634	10635	10636	10638	10644	10646	10648	10650
10652	10654	10664	10665	10666	10667	10668	10670	10684	10686	10688	10689	10690	10691
10692	10694	10696	10697	10698	10699	10700	10702	10708	10710	10712	10714	10716	10718
10728	10729	10730	10732	10734	10748	10750	10752	10753	10754	10755	10756	10757	10758
10760	10761	10762	10763	10764	10765	10766	10772	10773	10774	10776	10777	10778	10780
10781	10782	10792	10793	10794	10795	10796	10797	10798	10812	10813	10814	10816	10817
10818	10819	10820	10821	10822	10824	10825	10826	10827	10828	10829	10830	10836	10837
10838	10840	10841	10842	10844	10845	10846	10856	10857	10858	10860	10861	10862	10876

Table A.6. Table 1A. (continued).

10877	10878	10880	10881	10882	10883	10884	10885	10886	10888	10889	10890	10891
10892	10893	10894	10900	10901	10902	10904	10905	10906	10908	10909	10910	10920
10921	10922	10923	10924	10925	10926	10940	10941	10942	10944	10945	10946	10947
10948	10949	10950	10952	10953	10954	10955	10956	10957	10958	10964	10965	10966
10968	10969	10970	10972	10973	10974	10984	10985	10986	10988	10989	10990	11004
11005	11006	11010	11011	11014	11016	11017	11018	11019	11020	11022	11030	11032
11034	11036	11038	11048	11049	11050	11051	11052	11054	11068	11070	11074	11075
11078	11080	11081	11082	11083	11084	11086	11094	11096	11098	11100	11102	11112
11113	11114	11116	11118	11132	11134	11136	11137	11138	11139	11140	11142	11144
11145	11146	11147	11148	11150	11156	11158	11160	11162	11164	11166	11176	11177
11178	11180	11182	11196	11198	11200	11201	11202	11203	11204	11206	11208	11209
11210	11211	11212	11214	11220	11222	11224	11226	11228	11230	11240	11241	11242
11244	11246	11260	11262	11272	11273	11274	11275	11276	11278	11288	11289	11290
11292	11294	11296	11297	11298	11299	11300	11301	11302	11304	11305	11306	11308
11310	11314	11316	11317	11318	11320	11321	11322	11324	11326	11336	11337	11338
11339	11340	11342	11352	11353	11354	11356	11358	11360	11361	11362	11363	11364
11365	11366	11368	11369	11370	11372	11374	11378	11380	11381	11382	11384	11385
11386	11388	11390	11392	11393	11394	11395	11396	11397	11398	11400	11401	11402
11403	11404	11406	11408	11409	11410	11412	11413	11414	11416	11417	11418	11420
11422	11424	11425	11426	11427	11428	11429	11430	11432	11433	11434	11436	11438
11440	11441	11442	11444	11445	11446	11448	11449	11450	11452	11454	11456	11457
11458	11459	11460	11461	11462	11464	11465	11466	11467	11468	11470	11472	11473
11474	11476	11477	11478	11480	11481	11482	11484	11486	11488	11489	11490	11491
11492	11493	11494	11496	11497	11498	11500	11502	11504	11505	11506	11508	11509
11510	11512	11513	11514	11516	11518	11530	11531	11534	11546	11550	11554	11555
11558	11560	11561	11562	11564	11566	11574	11576	11578	11580	11582	11594	11595
11598	11610	11614	11618	11619	11622	11624	11625	11626	11628	11630	11638	11640
11642	11644	11646	11648	11649	11650	11651	11652	11654	11656	11657	11658	11659
11660	11662	11664	11666	11668	11670	11672	11674	11676	11678	11680	11681	11682
11683	11684	11686	11688	11689	11690	11692	11694	11696	11698	11700	11702	11704
11706	11708	11710	11712	11713	11714	11716	11718	11720	11721	11722	11724	11726
11728	11730	11732	11734	11736	11738	11740	11742	11744	11745	11746	11748	11750
11752	11754	11756	11758	11760	11762	11764	11766	11768	11770	11772	11774	11784
11785	11786	11787	11788	11790	11800	11801	11802	11804	11806	11808	11809	11810
11811	11812	11814	11816	11817	11818	11820	11822	11826	11828	11830	11832	11834
11836	11838	11848	11849	11850	11852	11854	11864	11865	11866	11868	11870	11872

Table A.7. Table 1A. (continued).

11873	11874	11876	11878	11880	11881	11882	11884	11886	11890	11892	11894	11896
11898	11900	11902	11904	11905	11906	11907	11908	11910	11912	11913	11914	11915
11916	11918	11920	11921	11922	11924	11926	11928	11929	11930	11932	11934	11936
11937	11938	11939	11940	11942	11944	11945	11946	11948	11950	11952	11954	11956
11958	11960	11962	11964	11966	11968	11969	11970	11971	11972	11974	11976	11977
11978	11980	11982	11984	11985	11986	11988	11990	11992	11993	11994	11996	11998
12000	12001	12002	12004	12006	12008	12009	12010	12012	12014	12016	12018	12020
12022	12024	12026	12028	12030	12042	12043	12046	12058	12062	12066	12067	12070
12072	12073	12074	12076	12078	12086	12088	12090	12092	12094	12106	12110	12122
12126	12130	12134	12136	12138	12140	12142	12150	12152	12154	12156	12158	12160
12161	12162	12164	12166	12168	12169	12170	12172	12174	12176	12178	12180	12182
12184	12186	12188	12190	12192	12193	12194	12196	12198	12200	12202	12204	12206
12208	12210	12212	12214	12216	12218	12220	12222	12224	12226	12228	12230	12232
12234	12236	12238	12240	12242	12244	12246	12248	12250	12252	12254	12256	12258
12260	12262	12264	12266	12268	12270	12272	12274	12276	12278	12280	12282	12284
12286	15400	15401	15402	15404	15406	15420	15422	15464	15465	15466	15468	15470
15484	15486	15488	15489	15490	15491	15492	15494	15496	15497	15498	15500	15502
15508	15510	15512	15514	15516	15518	15528	15529	15530	15532	15534	15548	15550
15552	15553	15554	15555	15556	15558	15560	15561	15562	15564	15566	15572	15574
15576	15578	15580	15582	15592	15593	15594	15596	15598	15612	15614	15658	15662
15678	15722	15726	15742	15744	15746	15748	15750	15752	15754	15756	15758	15764
15766	15768	15770	15772	15774	15784	15786	15788	15790	15804	15806	15808	15810
15812	15814	15816	15818	15820	15822	15828	15830	15832	15834	15836	15838	15848
15850	15852	15854	15868	15870	15912	15913	15914	15916	15918	15932	15934	15976
15977	15978	15980	15982	15996	15998	16000	16001	16002	16003	16004	16006	16008
16009	16010	16012	16014	16020	16022	16024	16026	16028	16030	16040	16041	16042
16044	16046	16060	16062	16064	16065	16066	16068	16070	16072	16073	16074	16076
16078	16084	16086	16088	16090	16092	16094	16104	16105	16106	16108	16110	16124
16126	16170	16174	16190	16234	16238	16254	16256	16258	16260	16262	16264	16266
16268	16270	16276	16278	16280	16282	16284	16286	16296	16298	16300	16302	16316
16318	16320	16322	16324	16326	16328	16330	16332	16334	16340	16342	16344	16346
16348	16350	16360	16362	16364	16366	16380	16382	26752	26753	26754	26758	26760
26761	26762	26766	26772	26774	26776	26778	26780	26782	26792	26793	26794	26796
26798	26812	26814	26816	26817	26818	26820	26822	26824	26825	26826	26828	26830
26836	26838	26840	26842	26844	26846	26856	26857	26858	26860	26862	26876	26878
27030	27032	27034	27038	27048	27050	27052	27054	27068	27070	27074	27078	27080

Table A.8. Table 1A. (continued).

27082	27084	27086	27094	27096	27098	27100	27102	27112	27114	27116	27118	27132
27134	27304	27305	27306	27308	27310	27324	27326	27328	27329	27330	27334	27336
27337	27338	27342	27348	27350	27352	27354	27356	27358	27368	27370	27372	27374
27388	27390	27560	27562	27564	27566	27580	27582	27606	27608	27610	27614	27624
27626	27628	27630	27644	27646	27808	27809	27810	27814	27816	27818	27822	27826
27828	27830	27832	27834	27836	27838	27872	27873	27874	27876	27878	27880	27882
27884	27886	27890	27892	27894	27896	27898	27900	27902	28086	28088	28090	28094
28130	28134	28136	28138	28140	28142	28150	28152	28154	28156	28158	28384	28386
28390	28392	28394	28398	28402	28404	28406	28408	28410	28412	28414	28662	28664
28666	28670	31912	31914	31918	31932	31934	31976	31978	31980	31982	31996	31998
32190	32234	32238	32254	32488	32490	32494	32508	32510	32766	32768	32770	32774
32776	32778	32782	32788	32790	32792	32794	32796	32798	32808	32810	32812	32814
32828	32830	32832	32834	32836	32838	32840	32842	32844	32846	32852	32854	32856
32858	32860	32862	32872	32874	32876	32878	32892	32894	32896	32898	32900	32902
32904	32906	32908	32910	32916	32918	32920	32922	32924	32926	32936	32938	32940
32942	32956	32958	32960	32962	32964	32966	32968	32970	32972	32974	32980	32982
32984	32986	32988	32990	33000	33002	33004	33006	33020	33022	33046	33048	33050
33054	33064	33066	33068	33070	33084	33086	33090	33094	33096	33098	33100	33102
33110	33112	33114	33116	33118	33128	33130	33132	33134	33148	33150	33152	33154
33156	33158	33160	33162	33164	33166	33172	33174	33176	33178	33180	33182	33192
33194	33196	33198	33212	33214	33216	33218	33220	33222	33224	33226	33228	33230
33236	33238	33240	33242	33244	33246	33256	33258	33260	33262	33276	33320	33322
33324	33326	33340	33342	33344	33346	33350	33352	33354	33358	33364	33366	33368
33370	33372	33374	33384	33386	33388	33390	33404	33408	33410	33412	33414	33416
33418	33420	33422	33428	33430	33432	33434	33436	33438	33448	33450	33452	33454
33468	33470	33472	33474	33476	33478	33480	33482	33484	33486	33492	33494	33496
33498	33500	33502	33512	33514	33516	33518	33532	33576	33578	33580	33582	33596
33598	33622	33624	33626	33630	33640	33642	33644	33646	33660	33664	33666	33668
33670	33672	33674	33676	33678	33684	33686	33688	33690	33692	33694	33704	33706
33708	33710	33724	33728	33730	33732	33734	33736	33738	33740	33742	33748	33750
33752	33754	33756	33758	33768	33770	33772	33774	33788	33824	33826	33830	33832
33834	33838	33842	33844	33846	33848	33850	33852	33888	33890	33892	33894	33896
33898	33900	33902	33906	33908	33910	33912	33914	33916	33920	33922	33924	33926
33928	33930	33932	33934	33936	33938	33940	33942	33944	33946	33948	33950	33952
33954	33956	33958	33960	33962	33964	33966	33968	33970	33972	33974	33976	33978
33980	33984	33986	33988	33990	33992	33994	33996	33998	34000	34002	34004	34006

Table A.9. Table 1A. (continued).

34008	34010	34012	34014	34016	34018	34020	34022	34024	34026	34028	34030	34032
34034	34036	34038	34040	34042	34044	34102	34104	34106	34146	34150	34152	34154
34156	34166	34168	34170	34172	34176	34178	34180	34182	34184	34186	34188	34192
34194	34196	34198	34200	34202	34204	34208	34210	34212	34214	34216	34218	34220
34224	34226	34228	34230	34232	34234	34236	34240	34242	34244	34246	34248	34250
34252	34256	34258	34260	34264	34266	34268	34272	34274	34276	34278	34280	34282
34284	34288	34290	34292	34296	34298	34400	34402	34406	34408	34410	34414	34418
34420	34424	34426	34428	34432	34434	34436	34438	34440	34442	34444	34446	34448
34450	34452	34454	34456	34458	34460	34462	34464	34466	34468	34470	34472	34474
34476	34478	34480	34482	34484	34486	34488	34490	34492	34496	34498	34500	34502
34504	34506	34508	34510	34512	34514	34516	34518	34520	34522	34524	34528	34530
34532	34534	34536	34538	34540	34542	34544	34546	34548	34552	34554	34556	34680
34688	34690	34692	34696	34698	34700	34704	34706	34708	34712	34714	34716	34720
34722	34724	34728	34730	34732	34736	34740	34744	34752	34754	34756	34760	34762
34764	34768	34770	34776	34778	34784	34786	34788	34792	34794	34796	34800	34808
34944	34946	34950	34952	34954	34958	34960	34962	34964	34966	34968	34970	34972
34976	34978	34980	34982	34984	34986	34988	34990	34992	34994	34996	35000	35002
35004	35008	35010	35012	35014	35016	35018	35020	35022	35024	35026	35028	35030
35032	35034	35036	35040	35042	35044	35046	35048	35050	35052	35054	35056	35058
35060	35064	35066	35068	35216	35218	35222	35224	35226	35232	35234	35236	35238
35240	35242	35244	35248	35252	35256	35260	35264	35266	35268	35270	35272	35274
35276	35280	35282	35284	35286	35288	35290	35292	35296	35298	35300	35302	35304
35306	35308	35312	35316	35320	35324	35488	35490	35492	35494	35496	35498	35500
35502	35504	35506	35508	35512	35514	35516	35520	35522	35526	35528	35530	35534
35536	35538	35540	35542	35544	35546	35548	35552	35554	35556	35558	35560	35562
35564	35568	35570	35572	35576	35580	35744	35746	35748	35750	35752	35754	35756
35760	35764	35768	35772	35792	35794	35798	35800	35802	35808	35810	35812	35814
35816	35818	35820	35824	35828	35832	35836	36000	36002	36006	36008	36010	36016
36018	36020	36024	36028	36064	36066	36068	36070	36072	36074	36076	36080	36082
36084	36088	36092	36272	36280	36320	36322	36324	36328	36330	36336	36344	36576
36578	36582	36584	36586	36592	36596	36600	36604	36848	36856	37928	37930	37948
37992	37994	37996	38012	38016	38018	38020	38022	38024	38026	38028	38036	38038
38040	38042	38044	38056	38058	38060	38076	38080	38082	38084	38086	38088	38090
38092	38100	38102	38104	38106	38108	38120	38122	38124	38140	38250	38272	38274
38280	38282	38312	38314	38336	38338	38344	38346	38376	38378	38504	38506	38524
38528	38530	38532	38534	38536	38538	38540	38548	38550	38552	38554	38556	38568

Table A.10. Table 1A. (continued).

38570	38572	38588	38592	38594	38596	38598	38600	38602	38604	38612	38616	38618
38620	38632	38634	38636	38652	38784	38786	38792	38794	38824	38826	38848	38850
38856	38858	38888	38890	39040	39042	39046	39048	39050	39056	39058	39060	39064
39066	39068	39072	39074	39076	39078	39080	39082	39084	39088	39092	39096	39100
39104	39106	39108	39110	39112	39114	39116	39120	39122	39124	39128	39130	39132
39136	39138	39140	39142	39144	39146	39148	39152	39156	39160	39164	39328	39330
39332	39336	39338	39340	39360	39362	39364	39368	39370	39372	39392	39394	39396
39400	39402	39584	39586	39588	39590	39592	39594	39596	39600	39604	39608	39612
39616	39618	39622	39624	39626	39632	39636	39640	39644	39648	39650	39652	39656
39658	39660	39664	39668	39672	39676	39840	39842	39844	39848	39850	39904	39906
39912	39914	40096	40098	40104	40106	40112	40116	40120	40124	40160	40162	40164
40168	40170	40172	40176	40180	40184	40188	40416	40418	40424	40426	40672	40674
40680	40682	40688	40692	40696	40700	43008	43010	43012	43016	43018	43020	43028
43032	43036	43048	43050	43052	43068	43072	43074	43076	43080	43082	43084	43092
43096	43100	43112	43114	43116	43132	43136	43138	43140	43144	43146	43148	43156
43160	43164	43176	43178	43180	43196	43200	43202	43204	43208	43210	43212	43220
43224	43228	43240	43242	43244	43260	43266	43272	43274	43304	43306	43330	43336
43338	43368	43370	43392	43394	43400	43402	43432	43434	43456	43458	43464	43466
43496	43520	43522	43524	43528	43530	43532	43540	43544	43548	43560	43562	43564
43580	43584	43586	43588	43592	43594	43596	43604	43608	43612	43624	43628	43644
43648	43650	43652	43656	43658	43660	43668	43672	43676	43688	43690	43692	43708
43712	43714	43716	43720	43722	43724	43732	43736	43740	43752	43756	43772	43778
43784	43786	43816	43818	43842	43848	43850	43880	43904	43906	43912	43914	43944
43968	43970	43976	43978	44008	44040	44042	44056	44064	44066	44068	44072	44084
44088	44104	44106	44120	44128	44130	44132	44136	44148	44152	44160	44162	44164
44168	44170	44176	44180	44184	44192	44194	44196	44200	44208	44212	44216	44224
44226	44228	44232	44234	44240	44244	44248	44256	44258	44260	44264	44272	44276
44280	44298	44322	44328	44362	44386	44392	44416	44418	44424	44426	44448	44450
44456	44480	44488	44512	44552	44554	44568	44576	44578	44584	44616	44632	44640
44648	44672	44674	44680	44682	44688	44696	44704	44706	44712	44736	44738	44744
44752	44760	44768	44776	44810	44834	44840	44928	44936	44960	48168	48232	48256
48258	48264	48296	48320	48322	48328	48360	48680	48744	48768	48770	48776	48808
48832	48840	48872	59520	59528	59560	59584	59592	59624	60072	60096	60104	60576
60640												

References

Chua, L. O., Guan, J., Sbitnev, V. I. & Shin, J. [2007] "A nonlinear dynamics perspective of Wolfram's New Kind of Science. Part VII: Isles of Eden," *Intern. Journal of Bifurcation and Chaos* **17**, 2839–3012.

Chua, L. O., Sbitnev, V. I. & Yoon, S. [2004] "A nonlinear dynamics perspective of Wolfram's New Kind of Science. Part III: Predicting the unpredictable." *Intern. Journal of Bifurcation and Chaos* **14**, 3689–3820.

Chua, L. O., Sbitnev, V. I. & Yoon, S. [2005] "A nonlinear dynamics perspective of Wolfram's New Kind of Science. Part IV: From Bernoulli shift to $1/f$ spectrum," *Intern. Journal of Bifurcation and Chaos* **15**, 1045–1183.

Guan, J., Shen, S., Tang, C. & Chen, F. [2007] "Extending Chua's global equivalence theorem on Wolfram's New Kind of Science," *Intern. Journal of Bifurcation and Chaos* **17**, 4245–4259.

Li, W. & Packard, N. [1990] "The structure of the elementary cellular automata rule," *Complex Systems* **4**, 281–297.

Packard, N. & Wolfram, S. [1985] "Two-dimensional cellular automata," *Journal of Statistical Physics* **38**, 901–946.

Walker, C. C. & Aadryan, A. A. [1971] "Amount of computation preceding externally detectable steady sate behavior in a class of complex systems," *J. Bio-Med. Comput.* **2**, 85–94.

Wolfram, S. [1984a] "Computation theory of cellular automata," *Commun. Math.Phys.* **96**, 15–57.

Wolfram, S. [1984b] "Universality and complexity in cellular automata," *Physica D* **10**, 1–35.

Wuensche, A. & Lesser, M. [1992] *The Global Dynamics of Cellular Automata*, Santa Fe Institute Studies in the Sciences of Complexity (Addison-Wesley).