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The Extreme Value Birnbaum-Saunders Model, its Moments and an Application in Biometry

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Abstract

The Birnbaum-Saunders (BS) model is a life distribution that has been largely studied and applied. Recently, a new version of the BS distribution based on extreme value theory has been introduced, which is named extreme value Birnbaum-Saunders (EVBS) distribution. In this article, we provide some further details on the EVBS models that can be useful as a supplement to the already existing results. We use these models to analyze real survival time data of patients treated with alkylating agents for multiple myeloma. This analysis allow us to show the adequacy of these new statistical distributions and identify them as models useful for medical practitioners in order to obtain a prediction of the survival times of these patients and evaluate changes in the dose of their treatment.

1 Introduction and preliminaries

The Birnbaum-Saunders (BS) model is a life distribution that was introduced and studied by Birnbaum and Saunders (1969). This distribution has been largely applied in recent decades. BS and standard normal random variables (RVs), now denoted respectively by T and Z, are related by the formula

$$T = \delta(\alpha\, Z/2 + \sqrt{\{\alpha\, Z/2\}^2 + 1}\ )^2 \quad \text{i.e.,} \quad Z = (\sqrt{T/\delta} - \sqrt{\delta/T})/\alpha,$$

with $\alpha > 0$ and $\delta > 0$ being shape and scale parameters. Thus, when a RV T follows a BS distribution with parameters α and δ, the notation $T \sim \text{BS}(\alpha, \delta)$ is used. In addition, let us consider the usual notations ϕ and Φ for the standard normal probability density function (PDF) and cumulative distribution function (CDF), respectively, and let

$$a_t = (\sqrt{t/\delta} - \sqrt{\delta/t})/\alpha, \quad \text{so that} \quad a_t' = da_t/dt = (\sqrt{t/\delta} + \sqrt{\delta/t})/(2\alpha t). \quad (1)$$

Then, the PDF and the CDF of T are respectively

$$f_T(t) = \phi(a_t)\, a_t' \quad \text{and} \quad F_T(t) = \Phi(a_t),\ \ t > 0, \tag{2}$$

with a_t and a_t' defined in (1).

The assumption of a normal RV Z can be obviously relaxed supposing that it follows any other distribution with PDF f_Z. We then obtain a general BS type (BST) RV, denoted by

$$T \sim \mathrm{BST}(\alpha, \delta; f_Z), \quad \text{with a PDF} \quad f_T(t) = a_t'\, f_Z(a_t),\ t > 0,$$

again with a_t and a_t' given in (1). Among those models, we mention the extreme value Birnbaum-Saunders (EVBS) distributions, recently introduced by Ferreira *et al.* (2012), essentially based on results from extreme value theory (EVT).

In Section 2 of this article, we present a few results on EVT. In Section 3, we introduce the EVBS models providing information on their moments. In Section 4, we discuss about estimation and model checking for this type of models. In Section 5, we make some comments on the importance of a hazard analysis. In Section 6, we provide an application to biometrical data. Finally, in Section 7, we sketch some concluding remarks.

2 Limiting results in EVT

The main limiting result in EVT dates back to the papers by Fréchet (1927), Fisher and Tippett (1928), von Mises (1936) and Gnedenko (1943). These authors fully characterized the possible non-degenerate limit laws of the sequence of maximum values, $X_{n:n}$, suitably normalized, as $n \to \infty$, proving the so-called Gnedenko's extremal types theorem. More specifically, all possible non-degenerate weak limit distributions of the normalized partial maxima, $X_{n:n}$, of independent and identically distributed (IID) RVs, $X_1, \ldots, X_n$, are generalized extreme value (GEV) distributions. That is, if there are normalizing constants $a_n > 0$, $b_n \in \mathbb{R}$ and some non-degenerate CDF, G, such that, for all $x \in \mathcal{C}(G)$, the set of continuity points of G,

$$\lim_{n\to\infty} \mathrm{P}\left\{\frac{X_{n:n} - b_n}{a_n} \le x\right\} = G(x), \tag{3}$$

we can redefine the constants in such a way that

$$G(x) \equiv G_\gamma(x) := \begin{cases} \exp\left(-(1+\gamma x)^{-1/\gamma}\right),\ 1+\gamma x > 0, & \text{if} \quad \gamma \neq 0 \\ \\ \exp(-\exp(-x)),\ x \in \mathbb{R}, & \text{if} \quad \gamma = 0, \end{cases} \tag{4}$$

given in the von Mises-Jenkinson form (see von Mises, 1936; Jenkinson, 1955) and denoted by $\mathrm{EV_M} \equiv \mathrm{EV_M}(\gamma)$ laws. We then say that the CDF F underlying the RVs $X_1, X_2, \ldots,$ is in the max-domain of attraction (MDA) of G_γ, in (4), and use the notation $F \in \mathcal{D}_{\text{M}}(G_\gamma)$. The limiting CDFs, G, in (3), are then max-stable (MS), i.e., they are indeed the unique laws S such that the functional equation $S^n(\alpha_n x + \delta_n) = S(x)$, for $n \geq 1$, holds for some $\alpha_n > 0$ and $\delta_n \in \mathbb{R}$. The real parameter γ in (4), corresponding to the primary parameter of interest in EVT, is the so-called extreme value index. This index rules the behaviour of the right-tail of F. The GEV distribution, a unified version of all possible non-degenerate weak limits of maxima of sufficiently long sequences of IID or more generally weakly dependent and stationary RVs, reduces indeed to the Fréchet ($\gamma > 0$), max-Weibull ($\gamma < 0$) and Gumbel ($\gamma = 0$) CDFs, respectively. In fact, the GEV distribution in (4), is often separated in the three following types:

$$\begin{aligned}
&\text{Type I (Gumbel)}: && \Lambda(x) = \exp(-\exp(-x)),\ x \in \mathbb{R},\\
&\text{Type II (Fréchet)}: && \Phi_\alpha(x) = \exp(-x^{-\alpha}),\ x \geq 0,\\
&\text{Type III (max-Weibull)}: && \Psi_\alpha(x) = \exp(-(-x)^{\alpha}),\ x \leq 0,
\end{aligned}$$

with $\gamma = 0$, $\gamma = 1/\alpha > 0$ and $\gamma = -1/\alpha < 0$, respectively. We have $\Lambda(x) = G_0(x)$, $\Phi_\alpha(x) = G_{1/\alpha}(\alpha(1-x))$ and $\Psi_\alpha(x) = G_{-1/\alpha}(\alpha(x+1))$, with G_γ being the GEV distribution given in (4). For a recent overview of similar topics in the field of EVT, see Gomes *et al.* (2008).

Remark 1. *All results developed for maxima can easily be reformulated for minima since* $X_{1:n} := \min\{X_1, \ldots, X_n\} = -\max\{-X_1, \ldots, -X_n\}$. *If we are interested in the left-tails, i.e., on the limiting behaviour of the sequence of minimum values, we have for a linearly normalized minimum, a limiting* CDF, $G^*_\gamma(x) = 1 - G_\gamma(-x)$, *with* $G_\gamma(\cdot)$ *provided in* (4), *often referred to as a* $\text{EV}_\text{m} \equiv \text{EV}_\text{m}(\gamma)$ *law, i.e.,*

$$G^*_\gamma(x) = \begin{cases} 1 - \exp(-(1-\gamma x)^{-1/\gamma}),\ 1 - \gamma x > 0, & \textit{if} \quad \gamma \neq 0,\\ 1 - \exp(-\exp(x)),\ x \in \mathbb{R}, & \textit{if} \quad \gamma = 0. \end{cases} \tag{5}$$

We then say that F *belongs to the min-domain of attraction of* G^*_γ, *in short* $F \in \mathcal{D}_\text{m}(G^*_\gamma)$. *The parameter* γ, *in* (5), *determines the left-tail behavior of* F, *such as the parameter* γ, *in* (4), *determines the right-tail behavior of* F, *being so both crucial parameters in* EVT.

In Figure 1, we represent the right-tails of truncated positive EV_M and normal PDFs, as well the BS(1,1) PDF. If $\gamma < 0$, we have the so-called Weibull MDA, i.e., light right-tails, with a finite right endpoint. In addition, $\gamma = 0$ corresponds to the Gumbel MDA (exponential right-tails). And if $\gamma > 0$, we have the Fréchet MDA corresponding to heavy right-tails (polynomial tail decay, with an infinite right endpoint). Moreover, as proved in Ferreira *et al.* (2012), the BS CDF, $F_T(\cdot)$, given in (2), belongs to $\mathcal{D}_\text{M}(G_0)$, and this can be heuristically inferred from Figure 1.

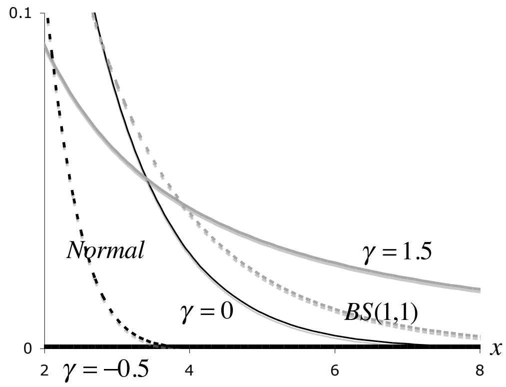

Figure 1: right-tails of positive truncated $\mathrm{EV_M}$ PDFs for $\gamma = -0.5$, $\gamma = 0$ and $\gamma = 1.5$, jointly with the right-tails of positive truncated normal and BS(1,1) PDFs.

3 Moments in EVBS models

The $\mathrm{EVBS_M}$ (and $\mathrm{EVBS_m}$) distributions based on limiting EV models for maxima, $\mathrm{EV_M}$, (and for minima, $\mathrm{EV_m}$), have been introduced in Ferreira *et al.* (2012). Specifically, consider that the RV Z follows the EV distribution for maxima given in (4), i.e., $Z \sim \mathrm{EV_M}(\gamma)$. Then, we use the notation $\mathrm{EVBS_M}(\alpha, \delta, \gamma)$ for the RV

$$T = \delta(\alpha Z/2 + \sqrt{\alpha^2 Z^2/4 + 1}\,)^2. \tag{6}$$

Analogously, if we consider that Z follows the EV distribution for minima given in (5), i.e., $Z \sim \mathrm{EV_m}(\gamma)$, and the same expression for T, i.e., that given in (6), we use the corresponding notation $T \sim \mathrm{EVBS_m}(\alpha, \delta, \gamma)$.

The shapes for the $\mathrm{EVBS_M}$ and the $\mathrm{EVBS_m}$ PDFs are quite diverse. As expected, the parameter α can modify drastically the shapes of these distributions. In the case of the parameter γ, we detect changes in the kurtosis and tail heaviness, as also expected. The EVBS models are thus very flexible and with extremely diversified left and right-tails; see Ferreira *et al.* (2012).

The following result comes directly from Theorem 2.6 of Vilca and Leiva (2006), which allows us to state the moments of the RV T given in (6).

Theorem 1. *Let the RV T be as given in* (6)*. Then, the rth moment of T exists if* $\mathrm{E}[Z^{k+l}\big(\{\alpha Z\}^2 + 4\big)^{(k-l)/2}] < \infty$*, with $k = 0, \ldots, r$ and $l = 0, \ldots, k$, and we have*

$$\mathrm{E}[T^r] = \delta^r \sum_{k=0}^{r} \binom{r}{k} \sum_{l=0}^{k} \binom{k}{l} 2^k \mathrm{E}\left[\left(\frac{\alpha Z}{2}\right)^{k+l} \left(\left\{\frac{\alpha Z}{2}\right\}^2 + 1\right)^{(k-l)/2}\right]. \tag{7}$$

Particular moments of T that are of interest correspond to the mean, $\mu[T] = \mathrm{E}[T]$, the variance, $\sigma^2[T] = \mathrm{V}[T]$, the standard deviation (SD), $\sigma[T] = \sqrt{\mathrm{V}[T]}$, and the coefficients of variation (CV), $\delta[T] = \sigma[T]/\mu[T]$, of skewness (CS), $\beta_1[T] = \mathrm{E}[(\{T - \mu[T]\}/\sigma[T])^3]$ and of kurtosis (CK), $\beta_2[T] = \mathrm{E}[(\{T - \mu[T]\}/\sigma[T])^4]$, as well as the excess kurtosis (EK), $\alpha_4[T] = \delta_2[T] - 3$.

In order to obtain these moments for the $\mathrm{EVBS_M}$ and $\mathrm{EVBS_m}$ RVs, we need to have information on moments of the $\mathrm{EV_M}$ and $\mathrm{EV_m}$ models given in (4) and (5), respectively, to be sketched next.

For an RV, X_M, with an $\mathrm{EV_M}(\gamma)$ distribution, we have

$$\mu[X_\mathrm{M}] = \begin{cases} \{\Gamma(1-\gamma) - 1\}/\gamma, & \text{if} \quad \gamma < 1 \ (\neq 0), \\ -\psi(1), & \text{if} \quad \gamma = 0, \end{cases}$$

$$\sigma^2[X_\mathrm{M}] = \begin{cases} (l_2 - l_1^2)/\gamma^2, & \text{if} \quad \gamma < 1/2 \ (\neq 0), \\ \pi^2/6, & \text{if} \quad \gamma = 0, \end{cases}$$

where $l_k = \Gamma(1 - k\gamma)$, for $k = 1, \ldots, 4$, and $-\psi(1)$ is the Euler constant, with $\psi = \Gamma'/\Gamma$ being the digamma function, i.e., the logarithmic derivative of the gamma function, denoted as usual by Γ, with Γ' being its derivative. In addition,

$$\beta_1[X_\mathrm{M}] = \begin{cases} (l_3 - 3l_1l_2 + 2l_1^3)/(l_2 - l_1^2)^{3/2}, & \text{if} \quad \gamma < 1/3 \ (\neq 0), \\ 12\sqrt{6}\,\zeta(3)/\pi^3, & \text{if} \quad \gamma = 0, \end{cases}$$

$$\alpha_4[X_\mathrm{M}] = \begin{cases} \{l_4 - 4l_1l_3 + 6l_1^2l_2 - 3l_1^4\}/(l_2 - l_1^2)^2, & \text{if} \quad \gamma < 1/4 \ (\neq 0), \\ 27/5, & \text{if} \quad \gamma = 0, \end{cases}$$

where

$$\zeta(k) = \sum_{j=0}^{\infty} j^{-k}$$

is the Riemann zeta function, for $k > 1$.

Similarly, given the relation $G_\gamma^*(x) = 1 - G_\gamma(-x)$ mentioned in Remark 1, if we consider X_m, with an $\mathrm{EV_m}(\gamma)$ distribution, we have

$$\mathrm{E}[X_\mathrm{m}^k] = (-1)^k \mathrm{E}[X_\mathrm{M}^k] \quad \text{for all} \quad k \geq 1.$$

Values of the mean, SD, CS and EK for $\mathrm{EVBS_M}$ and $\mathrm{EVBS_m}$ distributions can be seen in Table 1.

Example. The Pareto distribution is very popular in the domain of heavy-tailed models, i.e., models in the Fréchet MDA, necessarily with $\gamma > 0$. Let

Table 1: values of mean, SD, CS and EK for the indicated distributions when $\delta = 1$.

		EVBS_m distribution				EVBS_M distribution			
γ	α	$\mu[T]$	$\sigma[T]$	$\beta_1[T]$	$\alpha_4[T]$	$\mu[T]$	$\sigma[T]$	$\beta_1[T]$	$\alpha_4[T]$
1.50	0.05	0.850	0.272	-1.964	2.789	–	–	–	–
	0.10	0.800	0.320	-1.385	0.610	–	–	–	–
	0.50	0.693	0.472	-0.217	-1.471	–	–	–	–
	1.00	0.709	0.618	0.288	-1.418	–	–	–	–
	1.50	0.776	0.787	0.599	-1.101	–	–	–	–
1.00	0.05	0.898	0.205	-2.535	6.419	–	–	–	–
	0.10	0.850	0.263	-1.696	2.115	–	–	–	–
	0.50	0.740	0.463	-0.183	-1.294	–	–	–	–
	1.00	0.767	0.658	0.449	-1.094	–	–	–	–
	1.50	0.860	0.893	0.841	-0.501	–	–	–	–
0.50	0.05	0.943	0.122	-2.927	11.913	–	–	–	-
	0.10	0.905	0.186	-1.842	3.942	–	–	–	–
	0.50	0.804	0.452	0.003	-0.983	–	–	–	–
	1.00	0.852	0.735	0.803	-0.213	–	–	–	–
	1.50	0.989	1.096	1.311	1.099	–	–	–	-
0.25	0.05	0.961	0.084	-2.177	8.277	1.052	–	–	–
	0.10	0.930	0.145	-1.459	3.074	1.121	–	–	–
	0.50	0.843	0.449	0.233	-0.651	2.443	–	–	–
	1.00	0.910	0.803	1.126	0.872	6.236	–	–	–
	1.50	1.083	1.272	1.706	2.953	12.496	–	–	–
0.00	0.05	0.974	0.060	-0.863	1.242	1.031	0.069	1.461	4.219
	0.10	0.952	0.114	-0.624	0.519	1.068	0.148	1.828	6.937
	0.50	0.889	0.453	0.630	0.125	1.606	1.370	4.652	45.629
	1.00	0.985	0.911	1.624	3.160	2.994	4.766	5.920	69.648
	1.50	1.208	1.546	2.285	6.588	5.243	10.380	6.299	77.338
-0.25	0.05	0.983	0.050	0.046	-0.249	1.020	0.052	0.219	-0.204
	0.10	0.968	0.098	0.181	-0.192	1.044	0.107	0.349	-0.106
	0.50	0.941	0.476	1.257	2.230	1.353	0.693	1.260	1.864
	1.00	1.081	1.092	2.402	8.428	2.093	1.981	1.935	4.586
	1.50	1.379	1.993	3.146	14.199	3.264	4.017	2.248	6.150
-0.50	0.05	0.990	0.047	0.763	0.591	1.013	0.046	-0.506	-0.028
	0.10	0.982	0.094	0.903	1.020	1.027	0.093	-0.389	-0.238
	0.50	0.998	0.538	2.205	8.156	1.229	0.501	0.338	-0.607
	1.00	1.211	1.412	3.629	21.511	1.699	1.223	0.877	0.032
	1.50	1.623	2.760	4.439	31.184	2.425	2.284	1.173	0.663
-1.00	0.05	1.001	0.053	2.331	8.969	1.001	0.048	-1.728	4.029
	0.10	1.005	0.112	2.728	13.369	1.005	0.092	-1.505	2.695
	0.50	1.151	0.940	6.519	89.537	1.100	0.383	-0.546	-0.664
	1.00	1.654	3.144	8.746	152.367	1.346	0.755	-0.028	-1.235
	1.50	2.535	6.760	9.507	175.900	1.715	1.223	0.258	-1.252
-1.50	0.05	1.014	0.080	5.598	73.340	0.991	0.060	-2.888	11.851
	0.10	1.033	0.190	8.346	200.962	0.986	0.109	-2.366	7.009
	0.50	1.456	2.706	20.475	1130.34	1.027	0.358	-0.988	-0.069
	1.00	2.749	10.369	22.496	1317.63	1.181	0.610	-0.458	-1.157
	1.50	4.929	23.121	22.958	1361.20	1.413	0.893	-0.180	-1.435

$Z \sim \text{Pareto}(\gamma)$, i.e., $F_Z(x) = 1 - x^{-1/\gamma}$, with $x \geq 1$. Then, for s non-null and if $\gamma < 1/(s+1)$,

$$\mathrm{E}[Z^s \sqrt{(\alpha Z)^2 + 4}\,] = \frac{\alpha \;\, {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}\left(s + 1 - \frac{1}{\gamma}\right), \frac{1-\gamma(s+1)}{2\gamma}, -\frac{4}{\alpha^2}\right)}{1 - \gamma(s+1)},$$

where ${}_2F_1$ is the hypergeometric function (see Abramowitz and Stegun, 1972) given by

$${}_2F_1(a, b, c, x) = \sum_{k=0}^{\infty} \frac{a(a+1)\cdots(a+k-1)b(b+1)\cdots(b+k-1)}{c(c+1)\cdots(c+k-1)} \frac{x^k}{k!}.$$

The computations of the moments given in (7) of an associated BST RV are then much simpler.

4 Estimation and validation in EVBS models

Estimation aspects and model checking for EVBS distributions have been dealt with in Ferreira *et al.* (2012). The system of likelihood equations does not produce an explicit solution so that a numerical procedure is necessary. An `R` package named `evbs` to analyze data from EVBS models is being developed, and its "in progress" version is already available through the authors. This package contains diverse indicators, as well as methodologies useful for EVBS distributions, as for example, maximum likelihood (ML) estimation of the unknown parameters of this distribution.

Once the EVBS distribution parameters have been estimated, a natural question that arises is checking how good is the fit of the model to the data. In order to compare the EVBS distributions to other distributions, we can use model selection criteria based on loss of information, such as Akaike (AIC) and Schwarz's Bayesian (BIC) information criteria. These criteria are given by

$$\text{AIC} = -2\ell(\widehat{\boldsymbol{\theta}}) + 2d \quad \text{and} \quad \text{BIC} = -2\ell(\widehat{\boldsymbol{\theta}}) + d\log(n),$$

where $\ell(\boldsymbol{\theta})$ is the log-likelihood function for the parameter $\boldsymbol{\theta}$ associated with the model, $\widehat{\boldsymbol{\theta}}$ is its ML estimate, n is the sample size and d is the dimension of the parameter space.

Remark 2. AIC *and* BIC *are based on a penalization of the likelihood function that allows us to compare models with different numbers of parameters, because, as is known, models with more parameters provide always a better fit. Thus, a model whose* AIC *or* BIC *has the smallest value is better; see* Sanhueza *et al.* (2008) *and* Ferreira *et al.* (2012). *This is an important point, because the* EVBS *distribution has more parameters than the more close competitors, such as* BS *and* EV *distributions.*

Because, in general, differences between two values of the BIC are not very noticeable, the Bayes factor (BF) can be used to highlight such differences, if they exist. An interpretation of a transformation of the BF (B_{12}), denoted by $2\log(B_{12})$, which allows us to detect the degree of superiority of one model (Model 1) with respect to another Model 2, is displayed in Table 2. For details, see Ferreira *et al.* (2012) and references therein.

Table 2: interpretation of $2\log(B_{12})$ associated with the BF.

$2\log(B_{12})$	Evidence in favor of M_1
< 0	Negative (M_2 is accepted)
$[0, 2)$	Weak
$[2, 6)$	Positive
$[6, 10)$	Strong
≥ 10	Very strong

5 Hazard analysis in EVBS models

A hazard may be considered as a dangerous event that can lead to an emergency or disaster. A hazard analysis can be statistically conducted by the hazard rate (HR), also known as chance function, failure rate, intensity function, or risk rate, among other names. A nice property of the HR is that it allows us to better characterize the behavior of statistical distributions, and to differentiate models with very similar CDFs. For example, the HR may have several different shapes such as increasing (IHR), constant (exponential distribution), decreasing (DHR), bathtube (BT), inverse bathtube (IBT) approaching to a non-null constant or IBT approaching to zero. The HR of T is given in general by

$$h_T(t) := \frac{f_T(t)}{1 - F_T(t)}, \quad t > 0, \quad 0 < F_T(t) < 1,$$

where f_T and F_T are the PDF and the CDF of T.

A simple manner for exploring the shape of the HR of a RV T is by its corresponding scaled total time on test (TTT) function given by

$$W_T(u) = \int_0^{F_T^{-1}(u)} (1 - F_T(y))\, dy / H_T^{-1}(1), \quad 0 \leq u \leq 1,$$

which can be empirically approximated, allowing to construct the empirical scaled TTT curve by plotting the consecutive points $\big[k/n, W_n(k/n)\big]$, where $W_n(k/n) = \{\sum_{i=1}^{k} T_{i:n} + (n-k)\, T_{k:n}\}/\sum_{i=1}^{n} T_{i:n}$, for $k = 1, \ldots, n$, with $T_{i:n}$ being the corresponding ith ascending order statistic, for $1 \leq i \leq n$. From Figure 2, we detect several theoretical shapes for the scaled TTT curve. Thus, a TTT curve that is concave (or convex) is related to the IHR (or DHR) class. A concave (or convex) and then convex (or concave) TTT curve is related to a

BT (or IBT) HR. Finally, a TTT curve expressed by a straight line corresponds to the exponential distribution.

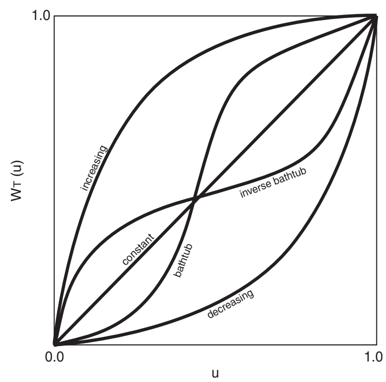

Figure 2: theoretical scaled TTT curves for a general model with the indicated HR shape.

The EVBS_{M} and EVBS_{m} distributions provide very rich models, in the sense that they can attain all types of TTT curves given above. For more details, about a hazard analysis for EVBS models, see Ferreira *et al.* (2012).

6 An application to Biometry

For illustration purposes, we consider the uncensored part of a data set analyzed by Leiva *et al.* (2007) corresponding to the survival times (T, in months) of 48 patients who were treated with alkylating agents for multiple myeloma. These data (that we call from now on `myeloma`) are: 1, 1, 2, 2, 2, 2, 3, 5, 5, 6, 6, 6, 6, 7, 7, 7, 9, 11, 11, 11, 11, 11, 13, 14, 15, 16, 16, 17, 17, 18, 19, 19, 24, 25, 26, 32, 35, 37, 41, 42, 51, 52, 54, 58, 66, 67, 88, 89, 92.

For analyzing `myeloma` data, we use the implementation in `R` code of the EVBS models considered in Ferreira *et al.* (2012). An exploratory data analysis (EDA) of such data is first produced and then estimation and EVBS model checking are carried out. The EDA of `myeloma`, based on descriptive summary presented in Table 3 and in Figure 3, allows us to detect a positively skewness distribution with a moderate to high kurtosis level and a shape for the HR, all of which can be modeled well by a EVBS distribution. Thus, we propose this distribution for describing `myeloma` data.

Table 3: descriptive statistics for myeloma data (in months).

Median	Mean	SD	CV	CS	CK	Range	Min.	Max.	n
15.500	24.440	24.672	100%	1.364	6.220	18	1	92	48

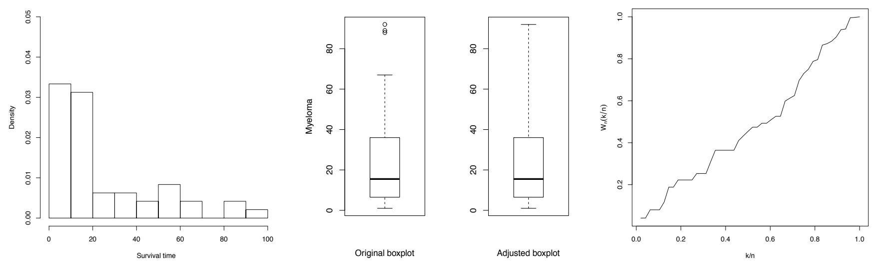

Figure 3: histogram (left) and indicated boxplots (center) and TTT plot (right) for myeloma.

As the EVBS_{m} distribution based on the $\text{Gumbel}_{\text{min}}$ model, $\Lambda^*(x) = G_0^*(x)$, given in (5), belongs to the Gumbel min-domain of attraction, we apply a semi-parametric EV test to analyze whether `myeloma` data belongs to this domain or not. We test H_0: $F \in \mathcal{D}_{\text{m}}(G_\gamma^*)$, with $\gamma \geq 0$, against H_1: $F \notin \mathcal{D}_{\text{m}}(G_\gamma^*)$, with $\gamma \geq 0$. From Figure 4, we see the sample path of the test statistic as a function of the k largest order statistics and the critical value (horizontal line) above which we reject H_0. For `myeloma`, we do not reject the null hypothesis for $1 \leq k \leq 48$, which is a credible result in EVT to keep such a hypothesis. Note also that we cannot have $\gamma > 0$ due to the fact that a infinite left endpoint does not make sense for these data. We have just restricted to the $\text{EVBS}_{\text{m}}(\alpha, \delta) \equiv \text{EVBS}_{\text{m}}(\alpha, \delta, 0)$ model.

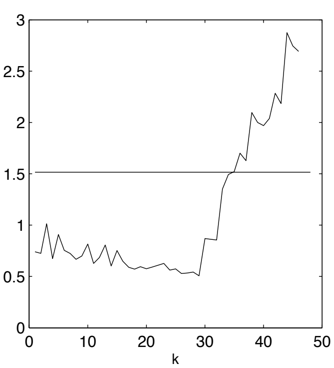

Figure 4: sample path of the EV condition test applied to myeloma (horizontal line: critical value above which we reject $F \in \mathcal{D}_{\text{m}}(G_\gamma^*)$, with $\gamma \geq 0$).

Once we have detected the type of EVBS model to be used, we estimate the model parameters using the `evbs` package and `myeloma` data. These results along with the negative value of the log-likelihood function and values for AIC, BIC and BF are displayed in Table 4.

We do a comparison among the EVBS_{m} and BS, $\text{EV}_{\text{M}}(\mu, \sigma, \gamma) := \mu + \sigma \text{EV}_{\text{M}}(\gamma)$ and generalized Pareto (GP) models using these criteria. The

GP(σ, γ) CDF is related with the GEV CDF, in (4), through $P_\gamma(x;\sigma) = 1 + \ln G_\gamma(x/\sigma)$, $1 + \gamma x/\sigma > 0, x > 0$. This comparison indicates us that the EVBS$_{\rm m}$ model is strongly superior to the GEV model for these data and with a positive evidence in its favor in relation to BS and GP models. The excellent agreement between the EVBS$_{\rm m}$ distribution and the `myeloma` data can be observed in Figure 5 by the histogram of the data with estimated EVBS$_{\rm m}$ PDF (left), by the empirical CDF plot with estimated EVBS$_{\rm m}$ CDF (center) and by the QQ plot (right).

Based in this analysis, we can use the EVBS$_{\rm m}$ model for obtaining different indicators useful in survival analysis, such as the hazard rate and survival function, in order to predict times of survival and evaluate changes in the dose of the treatment.

Table 4: ML estimates, AIC, BIC and BF for the indicated models using myeloma.

Distribution	$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\theta}_3$	$-\ell$	AIC	BIC	$2\log(B_{12})$
EVBS$_{\rm m}(\alpha,\delta)$	1.115	23.684	–	199.981	403.962	409.469	–
EV$_{\rm M}(\mu,\sigma,\gamma)$	10.182	10.202	0.612	203.364	412.728	420.989	11.520
GP(σ,γ)	–	24.467	-0.001	201.414	406.828	412.335	2.866
BS(α,δ)	1.323	12.719	–	201.494	406.988	412.495	3.026

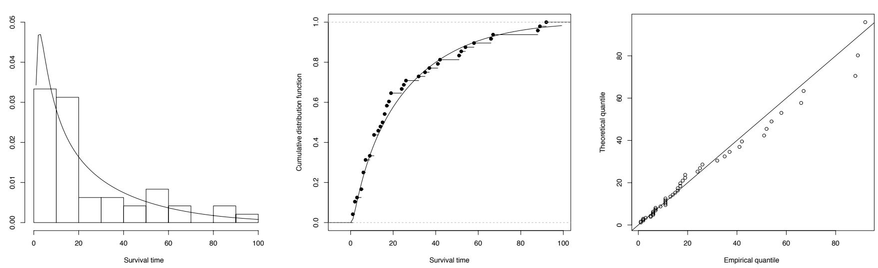

Figure 5: histogram with estimated EVBS$_{\rm m}$ (Gumbel$_{\rm min}$) PDF (left), empirical CDF plot with estimated (theoretical) EVBS$_{\rm m}$ CDF (center) and QQ plot (right) for myeloma.

7 Concluding remarks

We have dealt with extreme value versions of the Birnbaum-Saunders distribution, which were introduced by Ferreira *et al.* (2012). A description of the moments and hazard analysis of extreme value Birnbaum-Saunders distributions has been carried out. We have used the `R` package initiated in Ferreira *et al.* (2012) and have used it for analyzing a real data set corresponding to survival times of patients who were treated with alkylating agents for multiple myeloma. Such an analysis has allowed us to show the adequacy of these new statistical distributions, in a pure parametric framework, and identify them as models that can be useful for diverse medical practitioners in order to obtain a prediction of times of survival of these patients and evaluate changes in the dose of their treatment.

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