## Archivum Mathematicum

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Archivum Mathematicum, Vol. 59 (2023), No. 1, 117-123

Persistent URL: http://dml.cz/dmlcz/151556

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# UNIQUE SOLVABILITY OF FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATION ON THE BASIS OF VALLÉE-POUSSIN THEOREM 

Satyam Narayan Srivastava, Alexander Domoshnitsky, Seshadev Padhi, and Vladimir Raichik


#### Abstract

We propose explicit tests of unique solvability of two-point and focal boundary value problems for fractional functional differential equations with Riemann-Liouville derivative.


## 1. Introduction

In this paper we consider the fractional functional differential equation

$$
\begin{equation*}
\left(D_{0+}^{\alpha} x\right)(t)+\sum_{i=0}^{m}\left(T_{i} x^{(i)}\right)(t)=f(t), \quad t \in[0,1], m \leq n-2, n \geq 2 \tag{1.1}
\end{equation*}
$$

where $D_{0+}^{\alpha}$ is the Riemann-Liouville fractional derivative of the order $n-1<\alpha \leq n$ (see [11], [14]), $n$ is integer, the operators $T_{i}: C \rightarrow L_{\infty}$ are linear continuous operators acting from the space of the continuous functions $C$ to the space of essentially bounded functions $L_{\infty}, i=0, \ldots, m$, and $f \in L_{\infty}$.

We consider also the auxiliary equation

$$
\begin{equation*}
\left(D_{0+}^{\alpha} x\right)(t)+\sum_{i=0}^{m}\left(\left|T_{i}\right| x^{(i)}\right)(t)=f(t), \quad t \in[0,1], m \leq n-2, n \geq 2 \tag{1.2}
\end{equation*}
$$

where the positive operator $\left|T_{i}\right|$ is such that the following inequalities hold:

$$
\begin{equation*}
-\left(\left|T_{i}\right| 1\right)(t) \leq\left(T_{i} 1\right)(t) \leq\left(\left|T_{i}\right| 1\right)(t), \quad t \in[0,1] . \tag{1.3}
\end{equation*}
$$

Of course, it will be clear below, that we are interested in the operators $\left|T_{i}\right|$ with the minimal norms in the space of continuous functions $C$.

The operators $T_{i}: C \rightarrow L_{\infty}$ and $\left|T_{i}\right|: C \rightarrow L_{\infty}$ can be, for example, of the following forms:

[^0]1) Operators with deviations

$$
\begin{align*}
\left(T_{i} x^{(i)}\right)(t) & =\sum_{j=0}^{m_{i}} q_{i j}(t) x^{(i)}\left(t-\tau_{i j}(t)\right),  \tag{1.4}\\
\left(\left|T_{i}\right| x^{(i)}\right)(t) & =\sum_{j=0}^{m_{i}}\left|q_{i j}(t)\right| x^{(i)}\left(t-\tau_{i j}(t)\right),
\end{align*}
$$

where $\tau_{i j}:[0,1] \rightarrow \mathbb{R}, q_{i j}:[0,1] \rightarrow \mathbb{R}$, are measurable bounded functions, $\mathbb{R}=$ $(-\infty,+\infty)$. To complete the description of these operators, we have to define what has to be substituted into (1.4) instead of $x^{(i)}\left(t-\tau_{i j}(t)\right)$ in the case of $t-\tau_{i j}(t) \notin[0,1]$. Let us assume that

$$
\begin{equation*}
x^{(i)}(\xi)=0 \text { for } \xi \notin[0,1], i=0, \ldots, m \tag{1.5}
\end{equation*}
$$

that allows us to preserve the $n$-dimensional fundamental system for the homogeneous equation

$$
\begin{equation*}
\left(D_{0+}^{\alpha} x\right)(t)+\sum_{j=0}^{m_{i}} q_{i j}(t) x^{(i)}\left(t-\tau_{i j}(t)\right)=0 \tag{1.6}
\end{equation*}
$$

2) Integral operators

$$
\begin{align*}
\left(T_{i} x^{(i)}\right)(t) & =\int_{0}^{1} K_{i}(t, s) x^{(i)}(s) d s  \tag{1.7}\\
\left(\left|T_{i}\right| x^{(i)}\right)(t) & =\int_{0}^{1}\left|K_{i}(t, s)\right| x^{(i)}(s) d s
\end{align*}
$$

under the standard assumptions on the kernels $K_{i}(t, s)$ implementing that $T_{i}: C \rightarrow$ $L_{\infty}$, for example, $K_{i}(t, s)$ is a continuous function $[0,1] \times[0,1] \rightarrow \mathbb{R}$ (see, [12]).
3) Linear combinations and superpositions of the deviations and integral operators, for example, the operators

$$
\begin{align*}
\left(T_{i} x^{(i)}\right)(t) & =\int_{0}^{1} \sum_{j=1}^{m_{i}} K_{i j}(t, s) x^{(i)}\left(s-\tau_{i j}(s)\right) d s  \tag{1.8}\\
\left(\left|T_{i}\right| x^{(i)}\right)(t) & =\int_{0}^{1} \sum_{j=1}^{m_{i}}\left|K_{i j}(t, s)\right| x^{(i)}\left(s-\tau_{i j}(s)\right) d s
\end{align*}
$$

We consider the boundary value problem consisting of equation 1.1) and the boundary conditions

$$
\begin{equation*}
x^{(i)}(0)=0 \text { for } i=0,1, \ldots, n-2, x^{(k)}(1)=0 \tag{1.9}
\end{equation*}
$$

where $k$ is an integer which is between 0 and $n-1$. In the case of $k=0$, we have the classical two-point $(n-1,1)$ - problem. In the case of $k \leq n-1$, we have the sort of focal problems. We assume below that $m \leq k$.

We consider equation 1.1 in the space $D$ of functions $x:[0,1] \rightarrow \mathbb{R}$ such that $x^{(n-1)}$ is absolutely continuous on every interval $[\varepsilon, 1]$, where $\varepsilon>0$ and summable on $[0,1]$ and $x^{(n)}$ such that $t x^{(n)}$ is summable. The norm in the space $D$ define as $\|x\|_{D}=\sum_{i=0}^{n-2} \max _{0 \leq t \leq 1}\left|x^{(i)}(t)\right|+\int_{0}^{1}\left|x^{(n-1)}(t)\right| d t+\int_{0}^{1} t\left|x^{(n)}(t)\right| d t$. Considering this space $D$ looks naturally when fractional equations with the Riemann-Liouville derivatives and the boundary conditions 1.9 are considered. We say that $x \in D$ is a solution of 1.1 if it satisfies this equation for almost every $t \in[0,1]$. If the problem consisting of the homogeneous equation $\left(D_{0+}^{\alpha} x\right)(t)+\sum_{i=0}^{m}\left(T_{i} x^{(i)}\right)(t)=0$ and condition (1.9) has only the trivial solution, then problem (1.1), (1.9) has a unique solution which can be represented in the form [2]

$$
\begin{equation*}
x(t)=\int_{0}^{1} G(t, s) f(s) d s \tag{1.10}
\end{equation*}
$$

For applications of fractional differential equations in various field of science and engineering one can refer the classical books [11, 14.

The main reason for the study of fractional functional differential equations could be, in our opinion, around the following idea for the study of systems of fractional equations. Consider a boundary value problem consisting, for example, of a system of two "ordinary fractional differential equations". For its analysis, we can use the integral representations of solutions of the first equation and obtain $x_{1}(t)$ through $x_{2}(t)$. Then we substitute this representation instead of $x_{1}(t)$ into the second equation and obtain a scalar fractional functional differential equation. In the simplest case of a system of "ordinary" fractional equations, the equation, we get, includes the integral operator of type 2). If we start with a system of delay fractional differential equations, the equation, we get after the substitution into the second equation, is a fractional functional differential equation that includes the superpositions of deviation and integral operators. Thus, operators of type 3) appear. Examples of such systems can be found in [7, 8, 9].

Positivity of solutions is one of the most important properties in applications (see, for example, the book by Henderson and Luca [7]). Concerning problem (1.4), (1.9), in the case of so called ordinary linear equations, (i.e. $\tau_{i j}(t) \equiv 0$, $t \in[0,1], j=0, \ldots, m_{i}, i=1, \ldots, m$ in (1.4)) and its nonlinear generalizations, we can note the following papers [3, 8, 9, 10, 13, 15].

One of the motivations for our research is Lyapunov's inequalities for fractional differential equations which have been presented in Chapter 5 of the recent book by Agarwal, Bohner, and Ozbekler [1]. Note the following assertion was presented for the first time in [5]. Actually, the result in [5] is more general than Theorem 1.1 ] as the solution need not be assumed to be different from zero on $(0,1)$.

| $\alpha$ | In inequality (1.13) | In inequality (1.15) |
| :---: | :---: | :---: |
| 1.6 | 2.052759111 | 4.120246548 |
| 1.5999 | 2.05244883 | 4.119533208 |
| 1.5998 | 2.052138367 | 4.11819636 |
| 1.597 | 2.043474592 | 4.098884212 |
| 1.58 | 1.991943084 | 3.97506386 |
| 1.5 | 1.7724538 | 3.45372767 |

TAB. 1

Theorem 1.1 ([1] [5]). Let $1<\alpha \leq 2$ and $x$ be a solution of the boundary value problem

$$
\left\{\begin{array}{l}
\left(D_{0+}^{\alpha} x\right)(t)+q_{0}(t) x(t)=0 \quad \text { on } \quad[0,1]  \tag{1.11}\\
x(0)=x(1)=0
\end{array}\right.
$$

If $x(t) \neq 0$ for all $t \in(0,1)$, then the inequality

$$
\begin{equation*}
\int_{0}^{1}\left|q_{0}(t)\right| d t>\Gamma(\alpha) 4^{\alpha-1} \tag{1.12}
\end{equation*}
$$

holds.
Note that in [5], it was not assumed that $x(t) \neq 0$ for $t \in(0,1)$. For (1.11) with a constant coefficient $q_{0}(t)=q_{0}$, we have 1.12) in the form

$$
\begin{equation*}
\left|q_{0}\right| \geq \Gamma(\alpha) 4^{\alpha-1} \tag{1.13}
\end{equation*}
$$

Using Corollary 2.3 (one can refer [4] for proof), we get that the inequality

$$
\begin{equation*}
\left|q_{0}\right|<\frac{\alpha^{\alpha}}{(\alpha-1)^{\alpha-1}} \Gamma(\alpha+1) \tag{1.14}
\end{equation*}
$$

guarantees that the problem (1.11) has only the trivial solution. Note that the part on unique solvability coincides with the known result of [6]. Inequality (1.14) means that in the case of zeros of solution $x(t)$ at the points 0 and 1 , we obtain that

$$
\begin{equation*}
\left|q_{0}(t)\right| \geq \frac{\alpha^{\alpha}}{(\alpha-1)^{\alpha-1}} \Gamma(\alpha+1) \tag{1.15}
\end{equation*}
$$

since in the case of the coefficient $q_{0}$ satisfying inequality (1.11) we exclude the existence of zero at the point 1 , i.e. $x(1) \neq 0$. Let us compare (1.13) and (1.15), computing the right-hand sides in them, we have values in Table 1

Table 1 demonstrates the advances of our results if we compare the results of [1) 5] and ours.

## 2. Main Results

Lemma 2.1. Using the technique of [13], one can obtain the uniqueness of solution to the problem

$$
\left\{\begin{array}{l}
D_{0+}^{\alpha} x(t)=f(t)  \tag{2.1}\\
x(0)=x^{\prime}(0)=\ldots=x^{(n-2)}(0)=0 \\
x^{(k)}(1)=0
\end{array}\right.
$$

where $k$ is an integer number which is between 0 and $n-1$, in the form

$$
\begin{equation*}
x(t)=\int_{0}^{1} G_{k}(t, s) f(s) d s \tag{2.2}
\end{equation*}
$$

where $G_{k}(t, s)$ is Green's function of problem 2.1 defined by

$$
G_{k}(t, s)=\frac{1}{\Gamma(\alpha)} \begin{cases}(t-s)^{\alpha-1}-t^{\alpha-1}(1-s)^{\alpha-1-k}, & 0 \leq s \leq t \leq 1  \tag{2.3}\\ -t^{\alpha-1}(1-s)^{\alpha-1-k}, & 0 \leq t<s \leq 1\end{cases}
$$

and its $j$-th derivative is defined by

$$
\begin{align*}
& \frac{\partial^{j}}{\partial t^{j}} G_{k}(t, s)= \frac{(\alpha-1)(\alpha-2) \cdots(\alpha-j)}{\Gamma(\alpha)}  \tag{2.4}\\
& \qquad \begin{cases}(t-s)^{\alpha-j-1}-t^{\alpha-j-1}(1-s)^{\alpha-1-k}, & 0 \leq s \leq t \leq 1 \\
-t^{\alpha-j-1}(1-s)^{\alpha-1-k}, & 0 \leq t<s \leq 1\end{cases}
\end{align*}
$$

Let us define the operator $K: L_{\infty} \rightarrow L_{\infty}$ and $|K|: L_{\infty} \rightarrow L_{\infty}$ by the equalities

$$
\begin{align*}
(K z)(t) & =-\sum_{i=0}^{m} T_{i}\left[\int_{0}^{1} \frac{\partial^{i}}{\partial t^{i}} G_{k}(t, s) z(s) d s\right](t)=f(t),  \tag{2.5}\\
(|K| z)(t) & =-\sum_{i=0}^{m}\left|T_{i}\right|\left[\int_{0}^{1} \frac{\partial^{i}}{\partial t^{i}} G_{k}(t, s) z(s) d s\right](t)=f(t) .
\end{align*}
$$

We use the notation $T_{i}[\gamma(t)],\left(\left|T_{i}\right|[\gamma(t)]\right)$ meaning that the operator $T_{i}$ and $\left|T_{i}\right|$ acts on the continuous function $\gamma(t)$, i.e. $T_{i}[\gamma(t)]=\left(T_{i} \gamma\right)(t),\left|T_{i}\right|[\gamma(t)]=\left(\left|T_{i}\right| \gamma\right)(t)$.

Theorem 2.2. Assume that there exist a function $v \in D$ such that $v(t)>0, v^{\prime}(t)>$ $0, \cdots, v^{(k)}(t)>0$ for $t \in(0,1), v(0)=v^{\prime}(0)=\cdots=v^{(n-2)}(0)=0$ and

$$
\begin{equation*}
\left(D_{0+}^{\alpha} v\right)(t)+\sum_{i=0}^{m}\left(\left|T_{i}\right| v^{(i)}\right)(t) \equiv \psi(t) \leq-\varepsilon<0 \quad \text { for } \quad t \in(0,1) \tag{2.6}
\end{equation*}
$$

then the problem (1.1), (1.9) is uniquely solvable for any essentially bounded $f$ and the spectral radius of $|K|: L_{\infty} \rightarrow L_{\infty}$ is less than one.

Proof. Consider the auxiliary problem

$$
\left\{\begin{array}{l}
\left(D_{0+}^{\alpha} x\right)(t)=z(t)  \tag{2.7}\\
x^{(i)}(0)=v^{(i)}(0), x^{(k)}(1)=v^{(k)}(1), \quad i=0,1, \ldots, n-2
\end{array}\right.
$$

where $z(t)$ is a function in $L_{\infty}$ and such that there exists a positive number $\delta$ such that $z(t) \leq-\delta$ for $t \in[0,1]$. It is clear that

$$
\left\{\begin{array}{l}
x(t)=\int_{0}^{1} G_{k}(t, s) z(s) d s+u_{k}(t)  \tag{2.8}\\
x^{\prime}(t)=\int_{0}^{1} \frac{\partial}{\partial t} G_{k}(t, s) z(s) d s+u_{k}^{\prime}(t) \\
x^{\prime \prime}(t)=\int_{0}^{1} \frac{\partial^{2}}{\partial t^{2}} G_{k}(t, s) z(s) d s+u_{k}^{\prime \prime}(t) \\
\vdots \\
x^{(m)}(t)=\int_{0}^{1} \frac{\partial^{m}}{\partial t^{m}} G_{k}(t, s) z(s) d s+u_{k}^{(m)}(t)
\end{array}\right.
$$

where $u(t)$ is a solution of the homogeneous equation $D_{0+}^{\alpha} u(t)=0$ satisfying the conditions $u^{(i)}(0)=v^{(i)}(0), i=0, \ldots, n-2, u^{(k)}(1)=v^{(k)}(1)$. Let us substitute these representations instead of $v(t)$ and its derivatives into inequality 2.6):

$$
\begin{equation*}
\left.z(t)+\sum_{i=0}^{m} T_{i}\left[\int_{0}^{1} \frac{\partial^{i}}{\partial t^{i}} G_{k}(t, s) z(s) d s\right]+\sum_{i=0}^{m}\left(T_{i} u^{i}\right)(t)\right)=\psi(t) \tag{2.9}
\end{equation*}
$$

It is clear that $\left|T_{i}\right|: C \rightarrow L_{\infty}$ are positive operators for $i=0,1, \ldots, m$, and this imply that the operator $|K|: L_{\infty} \rightarrow L_{\infty}$ defined by equality (2.5) is positive.
Thus, we have the equation

$$
\begin{equation*}
z(t)-(|K| z)(t)=\Psi(t), \quad t \in[0,1] \tag{2.10}
\end{equation*}
$$

where

$$
\begin{equation*}
\Psi(t) \equiv \psi(t)-\sum_{i=0}^{m}\left(\left|T_{i}\right| u^{(i)}\right)(t) \tag{2.11}
\end{equation*}
$$

It is clear that $u^{(i)}(t)>0$ for $t \in(0,1]$. This implies that $\Psi(t) \leq-\varepsilon<0$. The function $w(t)=-z(t)$ satisfies the inequality $w(t)-(|K| w)(t)=-\Psi(t)>0$ for $t \in[0,1]$. From equality (2.10), according to [12, Theorem 5.3 on page 76] it follows that $\rho(|K|)<1$. This completes the proof of the theorem.

Corollary 2.3. If $n-1<\alpha \leq n$ and the following inequality is fulfilled

$$
\begin{equation*}
\left|T_{0}\right|\left[t^{\alpha-1}\left(\frac{\alpha}{\alpha-k}-t\right)\right] \tag{2.12}
\end{equation*}
$$

$$
+\sum_{i=1}^{m} \alpha(\alpha-1) \cdots(\alpha-i+1)\left|T_{i}\right|\left[t^{\alpha-i-1}\left(\frac{\alpha-i}{\alpha-k}-t\right)\right]<\Gamma(\alpha+1), t \in[0,1]
$$

then problem 1.1), 1.9) is uniquely solvable for any $f \in L_{\infty}$.
Proof. The proof follows from Corollary 4 of 4].
Acknowledgement. The authors would like to thank the anonymous referee for many valuable comments and suggestions, leading to a better presentation of our results.

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[^0]:    2020 Mathematics Subject Classification: primary 26A33; secondary 34A08, 34K37.
    Key words and phrases: Riemann-Liouville derivative, unique solvability, differential inequality. Received August 26, 2022, accepted December 8, 2022. Editor Z. Došlá.
    DOI: 10.5817/AM2023-1-117

