Dynamical properties of a cosmological model with diffusion

M. P. Ramos, A. J. Soares

Abstract The description of the dynamics of particles undergoing diffusion in general relativity has been an object of interest in the last years. Most recently a new cosmological model with diffusion has been studied in which the evolution of the particle system is described by a Fokker-Planck equation. This equation is then coupled to a modified system of Einstein equations, in order to satisfy the energy conservation condition. Continuing with this work, we study in the present paper a spatially homogeneous and isotropic spacetime model with diffusion velocity. We write the system of ordinary differential equations of this particular model and obtain the solutions for which the scale factor in the Robertson Walker metric is linear in time. We analyse the asymptotic behavior of the subclass of spatially flat solutions. The system representing the homogeneous and isotropic model with diffusion is rewritten using dynamical variables. For the subclass of spatially flat solutions we were able to determine all equilibrium points and analyse their local stability properties.

1 Introduction

A new model to describe the dynamics of particles undergoing diffusion in general relativity is given in [3]. In this model the evolution of the particle system is described by a Fokker-Planck equation without friction on the tangent bundle of the spacetime (M,g). In general relativity, the matter field is specified by the energy-momentum tensor $\mathbf{T}_{\mu\nu}$ and the geometry of the spacetime is given by the Einstein field equations,

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$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \mathbf{T}_{\mu\nu}, \tag{1}$$

where $R_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar respectively. In kinetic theory, the matter field is described by the one-particle distribution function f in the phase space and the energy momentum tensor is given by a suitable integral of f over the velocity (or momentum) variable. When considering the existence of diffusion, f is the solution of the Fokker-Planck equation. On the other hand, a matter source with diffusion cannot be the only contributor to the Einstein equations (1), because the kinetic energy of the particles is not preserved when undergoing diffusion. Consequently, the energy-momentum tensor will not satisfy the energy conservation condition $\nabla_\mu \mathbf{T}^{\mu\nu} = 0$ (∇_μ denoting the Levi Civita covariant derivative), which is a requirement of the Bianchi identity and Einstein equations (1). In order to overcome this difficulty, a cosmological scalar field term ϕ is added on the left hand side of (1) and the corresponding energy-momentum tensor $T_{\mu\nu}$ satisfies the modified Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \phi g_{\mu\nu} = T_{\mu\nu}, \tag{2}$$

The scalar field ϕ is not an ordinary matter field but rather a background field interacting with the fluid particles and therefore causing their diffusion. It is determined by the one particle distribution function f, via the equation $\nabla_\nu \phi = \nabla^\mu T_{\mu\nu}$. In doing so, the coupling of Fokker-Planck equation to the Einstein's equations respects the Bianchi identity $\nabla^\mu (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 0$ and the energy momentum tensor satisfies the energy conservation condition $\nabla_\mu T^{\mu\nu} = 0$.

It is shown in [4] that the evolution of the particle system described by a Fokker-Planck equation without friction leads to an energy-momentum tensor satisfying

$$T^{\mu\nu} = \rho u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu), \quad J^\mu = n u^\mu, \tag{3}$$

$$\nabla_\mu T^{\mu\nu} = \sigma n u^\nu, \quad \nabla_\mu (n u^\mu) = 0. \tag{4}$$

The scalar functions ρ, p, n, σ represent respectively the rest-frame energy density, the pressure, the number density of the fluid and the diffusion constant, while the vector u^μ satisfying the condition $u^\mu u_\mu = -1$ is the four-velocity of the fluid. Equations in (3) are projected along the direction of u^μ and onto a hypersurface orthogonal to the direction of u^μ, giving

$$\nabla_\mu (\rho u^\mu) + p \nabla_\mu u^\mu = \sigma n, \tag{5}$$

$$(\rho + p) u^\mu \nabla_\mu u^\nu + u^\nu u^\mu \nabla_\mu p + \nabla^\nu p = 0, \tag{6}$$

$$\nabla_\mu (n u^\mu) = 0. \tag{7}$$

The Euler equation (6) is the same as in the diffusion-free case. The continuity equation (5) is affected by the presence of diffusion, due to the fact that the diffusion force $\sigma n u^\mu$ acts on the direction of the matter flow.

In the model studied in [4], the pressure and the energy density satisfy the following linear relation,

$$p = (\gamma - 1)\rho, \tag{8}$$

where $\frac{2}{3} \leq \gamma < 2$. In order to transform the system (5)-(6) into a complete system we introduce the following equation of state,

$$\rho = n^{\gamma} S, \tag{9}$$

where the entropy S satisfies the condition

$$u^{\mu} \nabla_{\mu} S = \sigma n^{1-\gamma}. \tag{10}$$

Equations (5), (6), (8), (9) and (10) constitute a complete system for describing the evolution of the matter field variables (n, S, u^{μ}). These equations coupled to the modified Einstein equations (1) are the equations of the cosmological model with fluid matter undergoing velocity diffusion given in [4].

In this work we study the solutions to the cosmological model described above that represent a spatially homogeneous and isotropic spacetime. Continuing with the work done in [4], in section 2 we write the ODE system that describes these solutions and we solve this system explicitly for the case where the scale factor $a(t)$ in the metric is linear in time, for all values of the spatial curvature and all values of γ. In section 3 we study the asymptotic behavior of the subclass of spatially flat solutions of the model, we find conditions on the initial data for which singularities may or may not occur. A dynamical system formulation for the model is given in section 4. In particular we write the dynamical system representing the spatially flat solutions of this model in this section. The dynamical system in question has two ordinary differential equations. We determine all the fixed points of this system and show that the interior point is associated to the self similar solution given in section 2. We obtain the phase portrait of the two dimensional system and study the local stability of the fixed points on the boundary. Finally, in section 5 we state the conclusions and make some closing remarks. For sake of completeness, we include an appendix with some tools from dynamical systems theory that are used in the analysis developed in section 4 of the present paper.

2 Spatially homogeneous and isotropic solutions

In this section we present the model equations and we solve them explicitly when the scale factor is linear in time. We consider the Robertson-Walker metric [6], [8],

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right], \tag{11}$$

where $k = 0$ corresponds to spatially flat solutions, $k = -1$ corresponds to a space with negative spatial curvature, and $k = 1$ corresponds to a space with positive spatial curvature. In the case $k = 0$, we may introduce a cartesian system of coordinates such that

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2), \quad \text{and} \quad a_0 := a(0) = 1.$$

For any $k = 0, \pm 1$, the equations for the scale factor $a(t)$, the entropy $S(t)$ of the fluid and the cosmological scalar field $\phi(t)$ are

$$\dot{a} = Ha, \tag{12a}$$

$$\dot{S} = \sigma n_0^{1-\gamma}\left(\frac{a_0}{a(t)}\right)^{3-3\gamma}, \tag{12b}$$

$$\dot{\phi} = -\sigma n_0 \left(\frac{a_0}{a(t)}\right)^3, \tag{12c}$$

$$\dot{H} = \frac{1}{3}\left[\phi - \left(\frac{3}{2}\gamma - 1\right)\rho\right] - H^2, \tag{12d}$$

where

$$H^2 = \frac{1}{3}(\rho + \phi) - \frac{k}{a(t)^2} \tag{12e}$$

is the Hubble function, and

$$\rho(t) = \left(\frac{n_0 a_0^3}{a(t)^3}\right)^\gamma S(t) \tag{12f}$$

is the rest energy density of the fluid. Moreover, γ is the parameter of the equation of state, $\sigma > 0$ is the diffusion constant, and $n_0 > 0$ is the initial (at time $t = 0$) particle density. The initial data set consists of (a_0, H_0, S_0, ϕ_0), where $a_0 = 1$ for $k = 0$, and H_0, S_0, ϕ_0 are *positive* numbers such that (12e) is satisfied at time $t = 0$, i.e.,

$$H_0^2 = \frac{1}{3}(n_0^\gamma S_0 + \phi_0) - \frac{k}{a_0^2}. \tag{13}$$

Using (12e) we may rewrite (12d) in the following two forms

$$\dot{H} = -\frac{\gamma}{2}\rho + \frac{k}{a^2}, \tag{14}$$

$$\dot{H} = \frac{\gamma}{2}\phi - \frac{3\gamma}{2}H^2 - \frac{3\gamma - 2}{2a^2}k. \tag{15}$$

Equations (12) constitute the model equations studied in the present paper. In general, such equations can not be solved explicitly. However, in the particular case that the scale factor $a(t)$ is linear in time, we find the following explicit solution

$$a(t) = a_0 + \alpha_k t, \tag{16a}$$

$$\phi(t) = \frac{\sigma n_0 a_0^3}{2\alpha_k} a(t)^{-2}, \tag{16b}$$

$$S(t) = \left(\frac{\sigma n_0^{1-\gamma} a_0^{3-3\gamma}}{\alpha_k (3\gamma - 2)} \right) a(t)^{3\gamma-2}, \tag{16c}$$

where α_k is the real solution of the polynomial equation

$$\alpha^3 + k\alpha - \frac{\gamma\sigma n_0 a_0^3}{2(3\gamma-2)} = 0.$$

Note that $\alpha_k > 0$, for all $k = 0, \pm 1$. In particular, for $k = 0$ the solution (16) becomes

$$a(t) = 1 + \left(\frac{\gamma\sigma n_0}{2(3\gamma-2)} \right)^{1/3} t, \tag{17a}$$

$$\phi(t) = \left(\sqrt{\frac{3\gamma-2}{\gamma}} \frac{\sigma n_0}{2} \right)^{2/3} a(t)^{-2}, \tag{17b}$$

$$S(t) = \left(\frac{2\sigma^2 n_0^{2-3\gamma}}{\gamma(3\gamma-2)^2} \right)^{1/3} a(t)^{3\gamma-2}. \tag{17c}$$

3 Asymptotic behavior of spatially flat solutions

In this section we consider the model equations (12) written in the previous section, and we analyse the asymptotic behavior of the subclass of spatially flat solutions, corresponding to the case $k = 0$. Since $a(0) = 1$, the scale factor is positive in some maximal interval $[0,T)$ and, by (14), H is strictly decreasing on $[0,T)$ (which implies that $a(t)$ cannot blow up in finite time). If $T = +\infty$, the solution is singularity free in the future; if $T < \infty$, i.e., $a(T) = 0$, the solution is singular at the time $t = T$. In this section we look for sufficient conditions on the initial data for which one of the two possibilities (existence or absence of a future singularity) may occur.

First, we observe that ϕ is strictly decreasing on $[0,T)$. If $\phi(t)$ is positive on $[0,T)$, then by (12e) H cannot vanish in this interval. Whence H is positive, i.e., $a(t)$ is increasing, on $[0,T)$ and therefore $a(T) > 0$. Therefore a singularity cannot form in the interval of time in which ϕ remains positive. In contrast, if ϕ vanishes at some time, then a singularity will form. In fact, by (15) we infer that if there exists $\bar{t}$ such that $\phi(\bar{t}) < 0$, then $\dot{H} \le \gamma/2\phi(\bar{t}) - 3\gamma H^2/2$, whence there exists $\bar{t} < t_* < +\infty$ such that $H \to -\infty$ as $t \to t_*$ (which implies $\lim_{t\to t_*} a(t) = 0$).

Let $\bar{t}$ be the maximal time such that $\phi(t) > 0$ for $t \in [0,\bar{t})$. If $\bar{t} < +\infty$, then $\phi(\bar{t}) = 0$ must hold. By (15), $\dot{H} \ge -3\gamma H^2/2$, for all $t \in [0,\bar{t})$. Since $H > 0$ in this interval, we obtain

$$H(t) \geq \frac{H_0}{1+\frac{3}{2}\gamma H_0 t}, \quad t \in [0,\bar{t}),$$

whence, for $t \in [0,\bar{t})$,

$$a(t) \geq (1+\frac{3}{2}\gamma H_0 t)^{\frac{2}{3\gamma}}. \tag{18}$$

On the other hand, since $H(t) \leq H_0$, we have

$$a(t) \leq \exp(H_0 t). \tag{19}$$

By the previous discussion, in order to see whether or not a singularity is formed in the future, we may equivalently check whether ϕ vanishes or not in finite time.

Proposition 1. *For*

$$\phi_0 \geq \frac{2\sigma n_0}{3H_0(2-\gamma)}, \tag{20}$$

there is no future singularity. On the other hand, for

$$\phi_0 < \frac{\sigma n_0}{3H_0} \tag{21}$$

a singularity forms in finite time in the future.

Proof. Considering (18) then by (12c) we obtain

$$\begin{aligned}\phi(t) &= \phi_0 - \sigma n_0 \int_0^t \frac{ds}{a(s)^3} \geq \phi_0 - \sigma n_0 \int_0^t \frac{ds}{(1+\frac{3}{2}\gamma H_0 s)^{\frac{2}{\gamma}}} \\ &= \phi_0 + \frac{2\sigma n_0}{3H_0(2-\gamma)}\left[(1+\frac{3}{2}\gamma H_0 t)^{1-\frac{2}{\gamma}} - 1\right] \\ &\geq \phi_0 + \frac{2\sigma n_0}{3H_0(2-\gamma)}\left[(1+\frac{3}{4}\gamma H_0 \bar{t})^{1-\frac{2}{\gamma}} - 1\right], \quad t \in \left(\frac{\bar{t}}{2},\bar{t}\right).\end{aligned}$$

It follows that when ϕ_0 satisfies (20), $\phi(t)$ is uniformly strictly positive on $[0,\bar{t})$ and therefore it cannot vanish at $t=\bar{t}$. Thus $\bar{t}$ cannot be finite, therefore no future singularity is formed. Suppose now that $\phi(t) \geq 0$ for all $t \geq 0$. Then, by (19), we find that

$$\phi(t) \leq \phi_0 - \sigma n_0 \int_0^t e^{-3H_0 s} ds = \phi_0 + \frac{\sigma n_0}{3H_0}(\exp(-3H_0 t) - 1).$$

If (21) holds, then $\phi(t) \to \phi_\infty < 0$ as $t \to +\infty$, which is a contradicion. Whence a singularity must form in finite time. □

Note that by using (13), we may rewrite (20) as

$$S_0 \geq \frac{4\sigma^2 n_0^{2-\gamma}}{3(2-\gamma)^2}\phi_0^{-2} - \frac{\phi_0}{n_0^\gamma} \tag{22}$$

which is always satisfied for

$$\phi_0 \geq \left(\frac{4\sigma^2 n_0^2}{3(2-\gamma)^2} \right)^{1/3}. \tag{23}$$

Thus we have the following result.

Corollary 1. *If the initial datum for the cosmological scalar field verifies* (23)*, there is no singularity in the future.*

Note that the arguments used so far do not consider the equation (12b) for the entropy S. In order to improve our singularity analysis, it is crucial to use (12b).

The qualitative dynamics of the cosmological model in the past depends on which of the following two mutually exclusive behaviours the solution verifies.

Case 1. The entropy $S(t)$ vanishes at some negative time t_0, while $a(t_0)$ is still positive.

Case 2. The factor $a(t)$ vanishes at some negative time t_0, while $S(t_0)$ is still positive.

Observe that in case 1, the solution is unphysical for $t < t_0$, even if the metric of spacetime remains smooth, because ρ becomes negative. This unphysical region of spacetime can be avoided by matching the metric at the time t_0 with the de Sitter solution $a_{DS}(t) = C\exp(\sqrt{\phi(t_0)/3}\,t)$, where the constant C is such that $a_{DS}(t_0) = a(t_0)$. The resulting cosmological model has no big-bang singularity and is vacuum up to the time t_0 (since $\rho = 0$ and $a(t) = a_{DS}(t)$ for $t \leq t_0$), at which time the vacuum energy $\phi(t_0)$ starts to be converted into matter energy ρ. On the other hand, in case 2, a big-bang singularity forms in the past.

We give sufficient conditions for the occurrence of each of the two possible scenarios described in cases 1 and 2. Let us introduce the times t_a and t_S defined by

$$t_a = \inf\{t < 0 : a(\tau) > 0, \text{ for all } t < \tau < 0\},$$
$$t_S = \inf\{t < 0 : S(\tau) > 0, \text{ for all } t < \tau < 0\},$$

with $t_a, t_S < 0$. Note that, from (18) and (19) we have

$$\left(1 + \frac{3}{2}\gamma H_0 t\right)^{\frac{2}{3\gamma}} \leq a(t) \leq e^{H_0 t}, \quad \text{for all } t \in (\max\{t_a, t_S\}, 0) \tag{24}$$

In particular, the lower bound in (24) implies that

$$t_a > -\frac{2}{3\gamma H_0}. \tag{25}$$

In the following proposition, we state the qualitative dynamics of the cosmological model when the solution behaves as in case 1.

Proposition 2. *If $t_a \leq t_S$ (case 1 above) then one of the following conditions holds*

(1a) *When $\gamma > 1$, we have* $S_0 < \dfrac{\sigma n_0^{1-\gamma}}{3H_0(\gamma-1)} \left[1 - \exp\Big(2(1-\gamma)/\gamma\Big)\right]$*;*

(1b) *When* $\gamma \leq 1$*, we have* $S_0 < \dfrac{2\sigma n_0^{1-\gamma}}{3(3\gamma-2)H_0}$.

Proof. If we consider $t_a \leq t_S$, then $S(t_a) \leq 0$. For $\gamma > 1$, using the upper bound of (24) in (12b) we have

$$S(t) \geq S_0 - \frac{\sigma n_0^{1-\gamma}}{3H_0(\gamma-1)} \Big(1 - \exp(3H_0(\gamma-1)t)\Big).$$

On the other hand, if we consider (25), this last inequality leads to

$$S(t_a) > S_0 - \frac{\sigma n_0^{1-\gamma}}{3H_0(\gamma-1)} \Big(1 - \exp(2(1-\gamma)/\gamma)\Big),$$

which then gives (1a). The proof when $\gamma \leq 1$ is similar. In this case we use the lower bound of (24) in (12b) to get

$$S(t) \geq S_0 - \frac{\sigma n_0^{1-\gamma}}{3(3\gamma-2)H_0} \Big(1 - (1 + 3\gamma/2H_0 t)^{3-2/\gamma}\Big),$$

so that, by (25), this last condition implies that

$$S(t_a) > S_0 - \frac{\sigma n_0^{1-\gamma}}{3(3\gamma-2)H_0}$$

which in turn gives (1b). □

4 Qualitative dynamics of solutions

In this section we introduce dynamical variables, that will be used to transform the system (12) into an autonomous dynamical system. Then, for $k = 0$, we develop a qualitative analysis of the resulting system in order to study the dynamics of spatially flat solutions of (12). The fixed points of the dynamical system represent solutions of (12) and the stability properties of the fixed points give qualitative information about the evolution of the solutions of the cosmological system (12).

4.1 The dynamical system

For each $k = -1, 0, 1$ we define the variable by $D = \sqrt{\rho}$, that is

$$D = \sqrt{3H^2 - \phi + \frac{3k}{a^2(t)}}, \tag{26}$$

and introduce the dimensionless expansion-normalized variables

$$\chi_\rho=\arctan\left(\frac{\rho}{D^2}\right),\ \chi_H=\arctan\left(\frac{H}{D}\right),\ \chi_\psi=\arctan\left(\frac{\dot\phi}{D^3}\right),\ \chi_a=\arctan\left(\frac{k}{D^2a^2}\right). \quad (27)$$

The constraint equation (12e) implies that $\tan\chi_\rho = 1$, that is $\chi_\rho = \pi/4$. Let us also define a new time variable τ, defined in terms of the cosmological time t by

$$\tau = \frac{Dt}{\cos\chi_H \cos\chi_\psi \cos\chi_a}\ .$$

The new time derivative is therefore given by

$$\frac{d}{d\tau}(\cdot) = \frac{1}{D}\cos\chi_H \cos\chi_\psi \cos\chi_a \frac{d}{dt}(\cdot)$$

and we use the simple notation $(\cdot)' = \frac{d}{d\tau}(\cdot)$. The system (12), in terms of the new variables χ_H, χ_ψ and χ_a, transforms to

$$\chi_H' = \frac{1}{2\sqrt{2}}\cos\chi_H\left[\sin\chi_H \sin\chi_\psi \cos\chi_H \cos\chi_a + 2\sin\chi_a \cos^2\chi_H \cos\chi_\psi\ ,\right. \quad (28a)$$
$$\left. +\gamma\cos\chi_\psi\cos\chi_a(3\sin^2\chi_H - \cos^2\chi_H)\right],$$
$$\chi_\psi' = \frac{3}{2\sqrt{2}}\sin\chi_\psi\cos\chi_\psi\Big[\sin\chi_\psi\cos\chi_H\cos\chi_a + (3\gamma-2)\sin\chi_H\cos\chi_\psi\cos\chi_a\Big], \quad (28b)$$

$$\chi_a' = \frac{\sqrt{2}}{2}\sin\chi_a\cos^2\chi_a\Big((3\gamma-2)\sin\chi_H\cos\chi_\psi + \sin\chi_\psi\cos\chi_H\Big). \quad (28c)$$

Finally we introduce the state space $\mathscr{X}$, for the dynamical system (28), defined by

$$\mathscr{X} = \left\{(\chi_H, \chi_\psi\ , \chi_a) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, 0\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}. \quad (29)$$

The dynamical system (28) admits a smooth extension to the closure $\overline{\mathscr{X}}$ of the state space. From (28c) we see that the 2-dimensional plane $\chi_a = 0$ is an invariant plane. The flow induced on this plane describes the dynamics of spatially flat solutions ($k = 0$). In the next subsection, we study this flow in some detail.

4.2 Dynamics of spatially flat solutions

The flow induced on the "roof" of $\mathscr{X}$ is described by the reduced dynamical system obtained by setting $\chi_a = 0$ in (28), that is

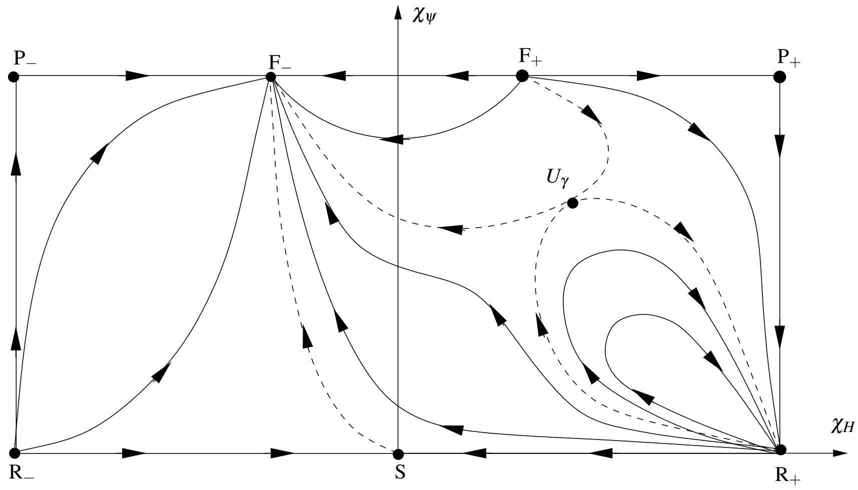

Fig. 1 Phase portrait of the dynamical system (30). Full lines represent typical orbits and dashed lines represent isolated orbits.

$$\chi_H' = \frac{1}{2\sqrt{2}}\cos\chi_H\Big(\sin\chi_H\cos\chi_H\sin\chi_\psi + \gamma\cos\chi_\psi(3\sin^2\chi_H - \cos^2\chi_H)\Big), \quad (30a)$$

$$\chi_\psi' = \frac{3}{2\sqrt{2}}\sin\chi_\psi\cos\chi_\psi\Big[\sin\chi_\psi\cos\chi_H + (3\gamma-2)\sin\chi_H\cos\chi_\psi\Big]. \quad (30b)$$

The state space for this dynamical system is $\overline{\mathscr{X}_{\rm up}}$, where

$$\mathscr{X}_{\rm up} = \Big\{(\chi_H,\chi_\psi)\in\Big(-\frac{\pi}{2},\frac{\pi}{2}\Big)\times\Big(-\frac{\pi}{2},0\Big)\Big\}.$$

The dynamical system (30) describes the flow of the spatially flat solutions of (12) which we will study in what follows. We shall say that an orbit Γ of the dynamical system (30) is typical if there exists a one parameter family of orbits having the same α- and ω-limit set of Γ. If no orbit other than Γ admits the same limit sets of Γ, we shall say that Γ is isolated. The qualitative behavior of the orbits of the dynamical system (30) is depicted in Figure 1. Our next goal is to prove the principal features of this behavior as well as to analyse the physical interpretation of the flow depicted in Figure 1 (in terms of solutions of the Einstein equations).

Fixed points. The dynamical system (30) possesses eight fixed points, seven of which are located on the boundary and one in the interior. They are represented in Figure 1 and listed in Table 1. The interior fixed point U_γ is associated to the self-similar solution (17) which has been characterized in section 2, with $a(t)$ being a linear function on time. Since the other fixed points are located at the boundary of the state space, they no longer correspond to exact solutions of (12), but to limiting states when one or more variables take an extreme value.

Fixed point	χ_H	χ_ψ
P_-	$-\frac{\pi}{2}$	0
F_-	$-\frac{\pi}{6}$	0
F_+	$\frac{\pi}{6}$	0
P_+	$\frac{\pi}{2}$	0
R_+	$\frac{\pi}{2}$	$-\frac{\pi}{2}$
S	0	$-\frac{\pi}{2}$
R_-	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$
U_γ	$\arctan\sqrt{\frac{\gamma}{2}}$	$-\arctan\sqrt{\frac{\gamma}{2}}(3\gamma-2)$

Table 1 Fixed points of the dynamical system (30) in the state space $\overline{\mathscr{X}_{\mathrm{up}}}$. U_γ is the unique interior fixed point.

We state the following conclusions about the fixed points U_γ, $F_\pm$ and $P_\pm$ of the dynamical system (30) and the corresponding solutions of the cosmological system (12).

- U_γ: At this fixed point we have $\chi_\psi = -\arctan\sqrt{\frac{\gamma}{2}}(3\gamma-2)$ and $\chi_H = \arctan\sqrt{\frac{\gamma}{2}}$. Hence, $H/D = \sqrt{\frac{\gamma}{2}}$ and $\dot{\phi}/D^3 = -\sqrt{\frac{\gamma}{2}}(3\gamma-2)$. Substituting these relations, together with (26), into equations (12a), (12b) and (12c), we obtain a differential system whose solution is given by (17).

- $F_\pm$: Since $H/D = \pm 1/\sqrt{3}$ at these fixed points, the cosmological scalar field ϕ is identically zero. Thus the orbits that converge to (resp. emanate from) the fixed point F_- (resp. F_+) identify the diffusion-free perfect fluid solutions with zero cosmological constant, i.e., the Friedmann-Lemaître solutions

$$a_{H_0}(t) = \left(1 + \frac{3}{2}\gamma H_0 t\right)^{\frac{2}{3\gamma}}. \tag{31}$$

  In particular, the fixed point F_+ is associated to the one-parameter family of expanding solutions $\{a_{H_0}(t)\}_{H_0>0}$, while F_- is associated to the one-parameter family of contracting solutions $\{a_{H_0}(t)\}_{H_0<0}$.

- $P_\pm$: At these fixed points we have $\chi_\psi = 0$ and $\chi_H = \pm\frac{\pi}{2}$. Hence, $\phi = const = \Lambda$ and $H/D \to \pm\infty$ as the point $P_\pm$ is approached, which implies that $D \to 0$.
  Thus $H^2 \to \Lambda/3$ and we obtain that P_+ is associated to the expanding de Sitter vacuum solution

$$a_\Lambda^+(t) = \exp\left(\sqrt{\frac{\Lambda}{3}}t\right), \tag{32}$$

  while P_- is associated to the contracting de Sitter vacuum solution

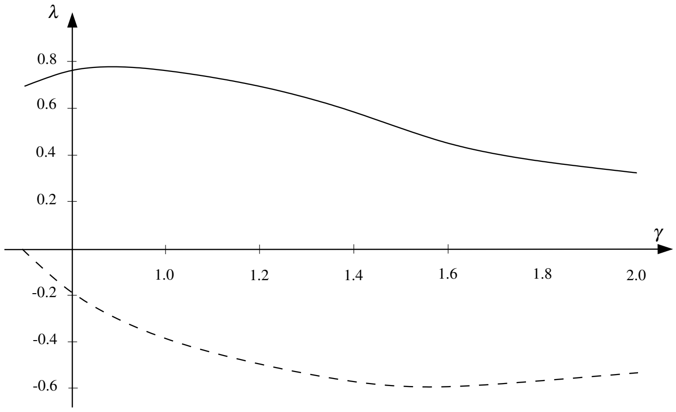

Fig. 2 Eigenvalues λ associated to the fixed point U_γ, versus different values of γ. Full line for the positive eigenvalue and dashed line for the negative eigenvalue.

$$a^-_\Lambda(t) = \exp\left(-\sqrt{\frac{\Lambda}{3}}t\right). \tag{33}$$

The identification of the solutions corresponding to the fixed points $\mathrm{R}_\pm$ is not as straightforward. Unfortunately we were unable to identify the orbits that emanate and converge to the fixed points $\mathrm{R}_\pm$ and S with solutions of system (12). Posteriorly, this has led us to reformulate the dynamical variables in order to overcome this difficulty. However, at the time of the presentation of this work at the conference *Particle Systems and PDEs II*, the dynamical variables used to study the model under investigation here were in fact the ones we describe in this section. A new version of the dynamical treatment of the system (12) can be found in [2].

The flow on the boundary of $\mathscr{X}_{\mathrm{up}}$. The analysis of flow induced on the one-dimensional boundary of $\mathscr{X}_{\mathrm{up}}$ is straightforward. The left (right) side $\chi_H = -\pi/2$ ($\chi_H = \pi/2$) consists of an orbit starting at R_- (P_+) and ending at P_- (R_+); the bottom side $\chi_\psi = -\pi/2$ consists of two orbits, connecting the points $\mathrm{R}_\pm$ to S; finally the top side $\chi_\psi = 0$ consists of three orbits, one connecting P_- to F_-, one connecting F_+ to F_- and one connecting F_+ to P_+. Next we discuss the local stability properties of the fixed points. This analysis is straightforward, except for the fixed points $\mathrm{R}_\pm$, which are not hyperbolic.

The interior fixed point U_γ. Point U_γ is a saddle: the matrix of the linearized dynamical system around this fixed point has real eigenvalues of opposite sign, see Figure 2. Hence there exist exactly two interior orbits that have U_γ as ω-limit point and exactly two orbits that have U_γ as α-limit point. This implies, in particular, that the solution (17) is unstable in the class of spatially flat Robertson-Walker solutions.

4.3 Stability properties of spatially flat solutions

Next we analyze the local stability properties of some of the fixed points on the boundary. To this purpose it is convenient to consider the dynamical system (30) in the extended state space $\mathscr{X}_\varepsilon$ defined by

$$\mathscr{X}_\varepsilon = \Big\{(\chi_H,\chi_\psi) : -\frac{\pi}{2}-\varepsilon < \chi_H < \frac{\pi}{2}+\varepsilon,\ -\frac{\pi}{2}-\varepsilon < \chi_\psi < \varepsilon\Big\}, \quad \text{with } \overline{\mathscr{X}} \subset \mathscr{X}_\varepsilon,$$

where $\varepsilon > 0$ is small. This extension allows us to perform the linearization procedure to study the local stability of the fixed points at the boundary of $\mathscr{X}$. A straightforward calculation shows that the fixed points $\mathrm{P}_\pm$ and S are hyperbolic saddles in the extended state space, while F_- and F_+ are respectively a hyperbolic sink and a hyperbolic source in the extended state space. Combining this information with the structure of the flow on the boundary we obtain the following result.

Lemma 1. *The fixed point* F_- *(resp.* F_+*) is the* ω*-limit (resp.* α*-limit) of a one parameter family of interior orbits, while* S *is the* α*-limit of exactly one interior orbit. There exists no interior orbit whose* α *or* ω*-limit set contain the points* $\mathrm{P}_\pm$.

The fixed points R_- and R_+ are not hyperbolic. In fact, the matrix of the linearized system around these two fixed points vanishes. In general this feature is the signal of a possible complicated behavior of the dynamical system near the fixed point. This is not so however for the fixed point R_-, which we will prove to be the source of a one parameter set of interior orbits. The latter statement is part of claim (iii) in the following theorem.

Theorem 1. *The following holds:*

(i) $\chi_H' < 0$ *for* $\chi_H = 0$*, and* $-\pi/2 < \chi_\psi \leq 0$*. In particular interior orbits can cross the line* $\chi_H = 0$ *only from the right to the left. Moreover, the* ω*-limit of each interior orbit which intersects the region* $\chi_H < 0$ *is the fixed point* F_-*;*

(ii) There exists exactly one orbit $\mathrm{S} \to \mathrm{F}_-$ *whose* α*-limit is* S *and whose* ω*-limit is* F_-*;*

(iii) If an orbit Γ *intersects the region* $\mathcal{L}$ *on the left of* $\mathrm{S} \to \mathrm{F}_-$*, then the whole orbit* Γ *is contained in this region. Moreover, the* ω*-limit of* Γ *is* F_-*, while the* α*-limit is* R_-*. There exist a one parameter family of orbits having the same properties as* Γ*;*

(iv) If an orbit Γ *intersects the region* $\mathcal{R}$ *on the right of* $\mathrm{S} \to \mathrm{F}_-$*, then the whole orbit* Γ *is contained in this region. The* α*-limit of* Γ *is contained in the region* $\chi_H > 0$*. There exists a one parameter family of orbits having the same behavior as* Γ*.*

Proof. We have $\chi'_{H|_{\chi_H=0}} = -\frac{\gamma}{2}\cos\chi_\psi$, by which the first part of the claim (i) follows. To prove the second part, we observe that $\chi_\psi' > 0$, for $(\chi_H,\chi_\psi) \in \big(-\frac{\pi}{2},0\big) \times \big(-\frac{\pi}{2},0\big)$. It follows that for any $0 < \varepsilon < \pi/2$, the set

$$S_\varepsilon = \Big\{(\chi_H,\chi_\psi) \in \Big[-\frac{\pi}{2},0\Big] \times \Big[-\frac{\pi}{2}+\varepsilon,0\Big]\Big\}$$

is future invariant. Since any interior orbit that crosses the line $\chi_H = 0$ must intersect S_ε, for some ε, it follows by LaSalle invariance principle [7, Th. 4.11] that the ω-limit set of any such orbit must be contained in the set $\{x \in S_\varepsilon : \chi'_\psi = 0\}$, by which it follows immediately that it must coincide with the point F_-. Next we prove (ii). We have shown in Lemma 1 that there is exactly one interior orbit that has S as α-limit. To show that its ω-limit is the point F_-, we use that the eigenvector corresponding to the positive eigenvalue of the linearized system around S in the extended state space is given by

$$v = \left(-\frac{\gamma}{4}, 1\right).$$

Since $v_1 = -\frac{\gamma}{4} < 0$, the unstable manifold of S intersects the region $\chi_H < 0$ and hence, since this region is future invariant, the whole orbit starting from S must be contained in it. Thus by (i) its ω-limit set must coincide with F_-. As to (iii), we notice that the region to the left (as well as the region on the right) of the orbit $\mathrm{S} \to \mathrm{F}_-$ is an invariant set. Since $-\chi_\psi$ is strictly monotone decreasing on this region, the Monotonicity Principle [7, Th. 4.12] gives that the ω-limit set of orbits in $\mathcal{L}$ is contained in the set

$$\left\{x \in \overline{\mathcal{L}} \setminus \mathcal{L} : \lim_{y \to x} \chi_\psi \neq -\pi/2\right\},$$

while the α-limit set is contained in the set

$$\left\{x \in \overline{\mathcal{L}} \setminus \mathcal{L} : \lim_{y \to x} \chi_\psi \neq 0\right\}.$$

The claim (iii) follows immediately from the structure of the flow on the boundary of $\mathcal{R}$. Finally, (iv) follows by the fact that $\mathcal{R}$ is an invariant set and that χ_H is monotone decreasing on $\chi_H < 0$. □

In preparation for the global analysis of orbits that intersect the region $\chi_H > 0$, we describe the behavior of the flow near the point R_+. We define a *corner neighborhood* of R_+ to be the intersection of a neighborhood of R_+ in the extended state space with the interior of $\mathscr{X}_{\mathrm{up}}$.

Proposition 3. *There exists a corner neighborhood $\mathscr{U}$ of the fixed point* R_+ *where the qualitative behavior of the flow is as depicted in Figure 3. In particular, for each $x \in \mathscr{U}$, the orbit Γ_x passing through x verifies one (and only one) of the following statements:*

1. *the α- and ω-limit set of Γ_x consist of the point* R_+*;*
2. *the α-limit set of Γ_x consists of the point* R_+*, and $\mathscr{U}$ contains no ω-limit points of Γ_x;*
3. *the ω-limit set of Γ_x consists of the point* R_+*, and $\mathscr{U}$ contains no α-limit points of Γ_x.*

Proof. Let us first shift the fixed point R_+ to the origin by introducing the new variables $\overline{\chi}_H = \chi_H - \pi/2$ and $\overline{\chi}_\psi = \chi_\psi + \pi/2$. The state space becomes

$$\overline{\mathscr{X}}_{\mathrm{up}} = \left\{(\overline{\chi}_H, \overline{\chi}_\psi) \in (-\pi, 0) \times \left(0, \frac{\pi}{2}\right)\right\}$$

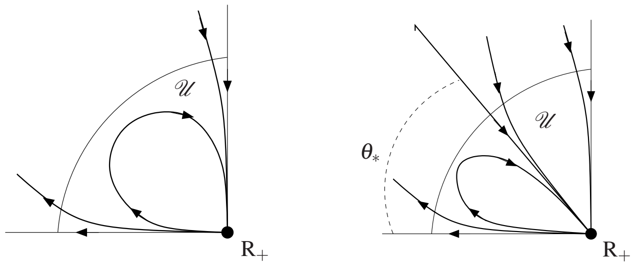

Fig. 3 Phase portrait of the dynamical system (30) in a corner neighborhood $\mathscr{U}$ of R_+ for $2/3 < \gamma \le 1$ (left) and $1 < \gamma < 2$ (right). Each orbit is typical.

while the dynamical system becomes

$$\begin{aligned}\overline{\chi}_H' &= -\frac{1}{2\sqrt{2}}\sin\overline{\chi}_H\Big(\sin\overline{\chi}_H\cos\overline{\chi}_H\cos\overline{\chi}_\psi + \gamma\sin\overline{\chi}_\psi(3\cos^2\overline{\chi}_H - \sin^2\overline{\chi}_H)\Big)\\ &:= P(\overline{\chi}_H,\overline{\chi}_\psi),\\ \overline{\chi}_\psi' &= -\frac{3}{2\sqrt{2}}\sin\overline{\chi}_\psi\cos\overline{\chi}_\psi\Big[\sin\overline{\chi}_H\cos\overline{\chi}_\psi + (3\gamma-2)\cos\overline{\chi}_H\sin\overline{\chi}_\psi\Big]\\ &:= Q(\overline{\chi}_H,\overline{\chi}_\psi).\end{aligned}$$

We study the previous dynamical system in a neighborhood of $(\overline{\chi}_H,\overline{\chi}_\psi) = (0,0)$. Let

$$\begin{aligned}P^{(2)}(\overline{\chi}_H,\overline{\chi}_\psi) &:= \frac{1}{2}\left[\frac{\partial^2 P}{\partial\overline{\chi}_H^2}(0,0)\overline{\chi}_H^2 + \frac{\partial^2 P}{\partial\overline{\chi}_\psi^2}(0,0)\overline{\chi}_\psi^2 + 2\frac{\partial^2 P}{\partial\overline{\chi}_H\partial\overline{\chi}_\psi}(0,0)\overline{\chi}_H\overline{\chi}_\psi\right]\\ &= -\frac{1}{2\sqrt{2}}\overline{\chi}_H\Big(\overline{\chi}_H + 3\gamma\overline{\chi}_\psi\Big),\\ Q^{(2)}(\overline{\chi}_H,\overline{\chi}_\psi) &:= \frac{1}{2}\left[\frac{\partial^2 Q}{\partial\overline{\chi}_H^2}(0,0)\overline{\chi}_H^2 + \frac{\partial^2 Q}{\partial\overline{\chi}_\psi^2}(0,0)\overline{\chi}_\psi^2 + 2\frac{\partial^2 Q}{\partial\overline{\chi}_H\partial\overline{\chi}_\psi}(0,0)\overline{\chi}_H\overline{\chi}_\psi\right]\\ &= -\frac{1}{\sqrt{2}}\overline{\chi}_H\overline{\chi}_\psi\Big(\overline{\chi}_H + 3(\gamma-1)\overline{\chi}_\psi\Big),\\ \mathcal{S}(\theta) &:= \Big[xQ^{(2)}(x,y) - yP^{(2)}(x,y)\Big]_{\substack{x=\cos\theta\\ y=\sin\theta}}\\ &= -\frac{1}{\sqrt{2}}\sin\theta\cos\theta\Big[\cos\theta + 3\sin\theta(\gamma-1)\Big]\end{aligned}$$

Since $\mathcal{S}(\theta)$ is not identically null, it follows by [5, Th. 2, pag. 140] that the solutions of $\mathcal{S}(\theta) = 0$ identify the directions along which an orbit may converge to or emanate

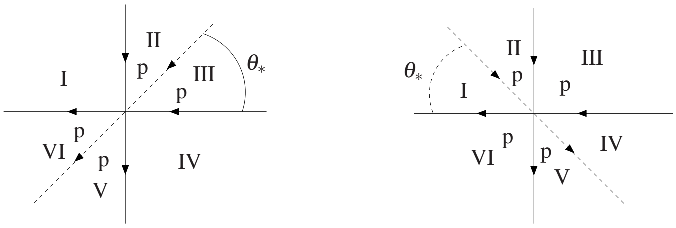

Fig. 4 The six sectors in a neighborhood of R_+ in the extended state space for $2/3 < \gamma < 1$ (left) and $1 < \gamma < 2$ (right). For $\gamma = 1$ there exist only four sectors, separated by the axes. The sectors labelled by "p" are parabolic, while the sectors I and IV may be both hyperbolic or both elliptic. Bendixson formula (34) entails that they are both elliptic.

from the fixed point $R_+ = (0,0)$. For $\gamma = 1$ the only such directions are the axes $\overline{\chi}_\psi = 0$, $\overline{\chi}_H = 0$. For $\gamma \neq 1$ we have the additional direction

$$\theta_* = -\arctan\left(\frac{1}{3(\gamma-1)}\right).$$

Notice that for $\gamma < 1$, this direction does not intersect the state space $\overline{\mathscr{X}}_{\mathrm{up}}$, which is the reason for the different behavior depicted in Figure 3. Let us assume $\gamma \neq 1$ (the case $\gamma = 1$ can be analyzed similarly). The direction θ_* and the axes $\overline{\chi}_\psi = 0$, $\overline{\chi}_H = 0$ divide any neighborhood of $(0,0)$ into six sectors, as shown in Figure 4. By the direction of the flow on the separatrixes, the sectors *II*, *III*, *V* and *VI* are parabolic. To identify the character of the sectors *I* and *IV*, we use that, by Theorem [5, Th. 7, pag. 305], the number e of elliptic sectors and the number h of hyperbolic sectors in the neighborhood of a fixed point satisfy the relation (proved by Bendixson in [1])

$$e - h = 2(i-1), \tag{34}$$

where i is the (Morse) index of the fixed point. For the fixed point R_+ we have $i = 2$ (computed with Mathematica), by which it follows that $e = 2$ and $h = 0$. Hence the sectors *I* and *IV* are elliptic and the qualitative behavior of the orbits is depicted in Figure 4. □

5 Summary

In this work we studied a model developed within the Einstein theory of relativity with a cosmological scalar field. This scalar field can be viewed as a background medium in which diffusion takes place. In particular we considered the Robertson-Walker spacetime, so that the model studied here is homogeneous and isotropic. The matter field variables are solutions of a non linear system of ordinary differential equations on the variable time. All solutions for which the scalar function of the metric is linear in time were obtained explicitly. In order to further understand the dynamical nature of other solutions the system was rewritten in terms of normalized dynamical variables. We obtained all fixed points of the dynamical system, one interior fixed point and seven fixed points on the boundary of the phase space. In particular the only interior equilibrium point is associated to the solution in which the scalar function of the metric is linear in time. The seven equilibrium points on the boundary of the state space correspond to limiting states when one or more variables take an extreme value, these limiting states being the Friedmann-Lemaître metrics and the de Sitter vacuum metrics.

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Appendix – Tools from dynamical systems

In this appendix we include some concepts and tools from the theory of dynamical systems that are used in the paper. The content of this appendix is mainly based on the books [5] and [7].

We consider an autonomous dynamical system of the form

$$\dot{x} = f(x), \tag{35}$$

where $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a $\mathscr{C}^1$ vector field and $x\colon \mathbb{R}_0^+ \longrightarrow \mathbb{R}^n$ is a function, $x = x(t)$. In the particular case of a planar dynamical system ($n = 2$) we introduce the following notation

$$x = (x,y)^{\mathrm{T}}, \quad f_1(x) = P(x,y), \quad f_2(x) = Q(x,y), \tag{36}$$

and rewrite Eq. (35) in the form

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y). \tag{37}$$

Theorem 2. *Consider the planar dynamical system (37) with $P(x,y)$ and $Q(x,y)$ being analytic functions of (x,y) in some open subset $E \subset \mathbb{R}^2$ containing the origin. Assume that the Taylor expansions of P and Q about the origin begin with mth-degree terms $P^{(m)}(x,y)$ and $Q^{(m)}(x,y)$ with $m \geq 1$. Then any orbit of the planar dynamical system (37) that approaches the origin as $t \to +\infty$ either spirals toward the origin as $t \to +\infty$ or it tends toward the origin in a definite direction $\theta = \theta_0$ as $t \to +\infty$. If the function $g(x,y) = xQ^{(m)}(x,y) - yP^{(m)}(x,y)$ is not identically null, then all directions of the approach $\theta = \theta_0$ satisfy the relation*

$$\cos\theta_0 Q^{(m)}(\cos\theta_0, \sin\theta_0) - \sin\theta_0 P^{(m)}(\cos\theta_0, \sin\theta_0) = 0. \tag{38}$$

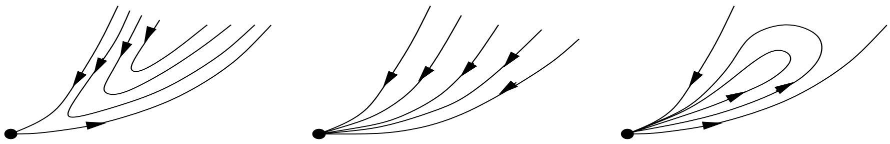

Fig. 5 Sector in the neighborhood of the origin of hyperbolic type (a), parabolic type (b) and elliptic type (c).

Moreover, if one orbit of the system (37) spirals toward the origin as $t \to +\infty$ then all trajectories of (37) in a deleted neighborhood of the origin spiral toward the origin as $t \to +\infty$.

For details on Theorem 2, see Ref. [5], page 140, Th. 2.

Now, we recall the definition of a sector, with reference to the planar dynamical system (37), as well as the possible character of a sector. If the Taylor expansions of P and Q about the origin begin with mth-degree terms $P^{(m)}$ and $Q^{(m)}$ and if the function

$$g(\theta) = \cos\theta Q^{(m)}(\cos\theta, \sin\theta) - \sin\theta P^{(m)}(\cos\theta, \sin\theta)$$

is not identically null, from Theorem 2 it follows that there are at most $2(m+1)$ directions, obtained as the solutions of the equation $g(\theta) = 0$, along which a orbit of the system (37) may approach the origin. Thus, the solution curves of the system (37) that approach the origin along these directions divide a neighborhood of the origin into a finite number of open regions called *sectors*. Three types of sectors can occur, namely either a *hyperbolic*, *parabolic* or an *elliptic* sector when it is topologically equivalent to the sector represented in Figure 4, picture (a), (b) and (c), respectively, where the directions of the flow need not to be preserved. Moreover, the trajectories that lie on the boundary of a hyperbolic sector are called *separatrixes*.

Another important concept that is used in this paper is the Morse index of a fixed point of the dynamical system (37). We begin with the definition of the index of a Jordan curve. Let $f = (P,Q)^{\mathrm{T}}$ be a $\mathscr{C}^1$ vector field on an open subset $E \subset \mathbb{R}^2$ and let $\mathscr{C}$ be a Jordan curve contained in E, such that the system (37) has no fixed point on $\mathscr{C}$. The *index of $\mathscr{C}$ relative to f* is the integer $i(\mathscr{C})$ computed as

$$i(\mathscr{C}) = \frac{1}{2\pi} \oint_{\mathscr{C}} \frac{PdQ - QdP}{P^2 + Q^2}.$$

Let x_0 be an isolated fixed point of system (37) and assume that the Jordan curve $\mathscr{C}$ contains x_0 and no other fixed points of (37) on its interior. The *Morse index of x_0 with respect to f* is defined by

$$i(x_0) = i(\mathscr{C}).$$

The following result is very convenient for the evaluation of the Morse index of a fixed point. It is stated for a fixed point at the origin but it is also valid for an arbitrary fixed point x_0.

Theorem 3. *Consider the planar dynamical system (37) with $P(x,y)$ and $Q(x,y)$ being analytic functions of (x,y) in some open subset $E \subset \mathbb{R}^2$ containing the origin. If the origin is an isolated fixed point of the system (37) then the Morse index of the origin, say i, satisfies the relation*

$$i = 1 + \frac{1}{2}(e - h), \tag{39}$$

where e and h indicate the number of elliptic and hyperbolic sectors, respectively, in a neighborhood of the origin.

For details on Theorem 3, see Ref. [5], page 305, Th. 7.

As a consequence of Theorem 3, it follows that the number h of hyperbolic sectors and the number e of elliptic sectors have the same parity.

Further concepts and properties that are used in the analysis developed in section 4 are those related to the α- and ω- limit sets of and orbit. We come back to the dynamical system (35) and assume that, for each $x_0 \in \mathbb{R}^n$, the system has a unique global solution $x \in \mathscr{C}^1(\mathbb{R})$ such that $x(0) = x_0$. We say that an equilibrium point x^* is an ω-limit point of the solution $x(t)$ if there exists a sequence $t_n \to +\infty$ such that $\lim_{n\to+\infty} x(t_n) = x^*$. The set of all ω-limit points of the solution $x(t)$ is called its ω-limit set. Analogously, by considering a sequence $t_n \to -\infty$, such that $\lim_{n\to-\infty} x(t_n) = x^*$, we define the concepts of an α-limit point and the α-limit set of a solution $x(t)$. Since solutions with the same orbit have equal ω- and α-limit sets, we will refer to ω- and α-limit sets of an orbit γ, and we will denote them by $\omega(\gamma)$ and $\alpha(\gamma)$, respectively. The following two theorems state important results on the limit sets $\omega(\gamma)$ and $\alpha(\gamma)$, that are used in section 4.

Theorem 4. *(LaSalle Invariance Principle) Let $S \subset \mathbb{R}^n$ be a compact and positively invariant subset of the dynamical system (35), and $Z : S \longrightarrow \mathbb{R}$ a $\mathscr{C}^1$ monotone function along the flow of the dynamical system. Let γ be an orbit in S. Then*

$$\omega(\gamma) \subseteq \Big\{ x \in S : \; Z'(x) = 0 \Big\},$$

where $Z' = \Delta Z \cdot f$.

For details on Theorem 4, see Ref. [7], page 103, Th. 4.11.

Theorem 5. *(Monotonicity Principle) Let $S \subset \mathbb{R}^n$ be an invariant subset of the dynamical system (35) and $Z : S \longrightarrow \mathbb{R}$ a $\mathscr{C}^1$ strictly monotonically decreasing function along the flow of the dynamical system. Let a and b be defined by $a = \inf\{Z(x) : x \in S\}$ and $b = \sup\{Z(x) : \; x \in S\}$. Let γ be an orbit in S. Then*

$$\alpha(\gamma) \subseteq \Big\{ s \in \partial S : \; \lim_{x\to s} Z(x) \neq a \Big\}, \quad \omega(\gamma) \subseteq \Big\{ s \in \partial S : \; \lim_{x\to s} Z(x) \neq b \Big\},$$

For details on Theorem 5, see Ref. [7], page 103, Th. 4.12. Finally, the following theorem states a crucial result about the limit sets $\omega(\gamma)$ and $\alpha(\gamma)$ in the particular case of a dynamical system in $\mathbb{R}^2$.

Theorem 6. *(Generalized Poincaré-Bendixson) Consider the dynamical system (35) on $\mathbb{R}^2$, and suppose that the dynamical system has only a finite number of equilibrium points. Then, for any orbit γ of the dynamical system, each one of the limit sets $\omega(\gamma)$ and $\alpha(\gamma)$ can only be one of the following: an equilibrium point; a periodic orbit the union of equilibrium points and heteroclinic cycles.*

For details on Theorem 6, see Ref. [5], page 101, Th. 4.10, or Ref. [7], page 245, Th. 2.

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