Some results on B-matrices and doubly B-matrices [*†]

C. Mendes Araújo[‡] Juan R. Torregrosa
CMAT - Centro de Matemática / Dpto. de Matemática Aplicada
Universidade do Minho / Universidad Politécnica de Valencia
Campus de Gualtar / Camino de Vera S/N
4710-057 Braga / 46022 Valencia
Portugal / Spain
clmendes@math.uminho.pt / jrtorre@mat.upv.es

Abstract

A real matrix with positive row sums and all its off-diagonal elements bounded above by their corresponding row means was called in [4] a B-matrix. In [5], the class of doubly B-matrices was introduced as a generalization of the previous class. We present several characterizations and properties of these matrices and for the class of B–matrices we consider corresponding questions for subdirect sums of two matrices (a general 'sum' of matrices introduced in [1] by S.M. Fallat and C.R. Johnson, of which the direct sum and ordinary sum are special cases), for the Hadamard product of two matrices and for the Kronecker product and sum of two matrices.

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1 Introduction

A square real matrix $A = (a_{ij})_{i,j=1}^n$ with positive row sums is a B–*matrix* if all its off-diagonal elements are bounded above by the corresponding row means (see [4]), that is, for all $i \in \{1, \dots, n\}$,

$$\sum_{j=1}^{n} a_{ij} > 0$$

and

$$\frac{1}{n}\sum_{j=1}^{n} a_{ij} > a_{ik}, \forall k \neq i.$$

In [2] it was proved that these matrices have positive determinants and the author provided a first application to the localization of the real eigenvalues of a real matrix. In [4] the author proved

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[‡]Corresponding author

that the class of B–matrices is a subset of the class of P–matrices and applied this property to the localization of the real parts of all eigenvalues of a real matrix.

Given a real matrix $A = (a_{ij})$ we define, for each row i, $r_{i_A} = \max\{0, a_{ij} \mid j \neq i\}$. We simply refer to r_i if the context is unambiguous. If A is a square matrix of order n, let A^+ be the following matrix

$$A^+ = \begin{bmatrix} a_{11} - r_1 & a_{12} - r_1 & \dots & a_{1n} - r_1 \\ a_{21} - r_2 & a_{22} - r_2 & \dots & a_{2n} - r_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} - r_n & a_{n2} - r_n & \dots & a_{nn} - r_n \end{bmatrix}.$$

Throughout this paper, $\mathcal{Z}_n$ will stand for the set of real square matrices of order n whose off-diagonal entries are nonpositive, that is $\mathcal{Z}_n = \{A = (a_{ij}) \in \mathcal{M}_n(\mathbb{R}) \ : \ a_{ij} \leq 0 \text{ if } i \neq j, i, j = 1, \dots, n\}$. If A is in $\mathcal{Z}_n$, we say that A is a Z–matrix (of order n).

In [4] Peña derived a characterization of B–matrices using the values r_i: he proved that a real matrix $A = (a_{ij})_{i,j=1}^n$ is a B-matrix if and only if, for all $i \in \{1, ..., n\}$,

$$\sum_{k=1}^{n} a_{ik} > n r_i. \tag{1}$$

He also proved (see [4]) that A is a B-matrix if and only if, for all $i \in \{1, ..., n\}$,

$$(a_{ii} - r_i) > \sum_{k \neq i} (r_i - a_{ik}).$$

In [5], the author defined another class of matrices, the *doubly B–matrices*, containing the B–matrices: a square real matrix $A = (a_{ij})_{i,j=1}^n$ is a doubly B-matrix if, for all i,

$$a_{ii} > r_i$$

and, for all $i \neq j$ in $\{1, \dots, n\}$,

$$(a_{ii} - r_i)(a_{jj} - r_j) > \sum_{k \neq i} (r_i - a_{ik}) \sum_{k \neq j} (r_j - a_{jk}).$$

In the mentioned paper, Peña showed that doubly B-matrices are also P-matrices and in [4] he presented the following characterization of B–matrices:

Proposition 1.1. *Let A be a matrix in $\mathcal{Z}_n$. Then the following properties are equivalent:*

1. *A is a B-matrix.*
2. *The row sums of A are positive.*
3. *A is strictly diagonally dominant by rows with positive diagonal entries.*

In Section 2 we will focus on some properties of B–matrices presented by Peña in [4] and [5] and obtain similar results for the doubly B–matrices. In Section 3 we analyze the concept of B-matrix under the subdirect sum. Finally, Sections 4 and 5 are devoted to the Hadamard product and the Kronecker sum of B–matrices.

2 B–matrices and doubly B–matrices

As it was mentioned above, in $\mathcal{Z}_n$ the concept of being a B–matrix coincides with the strict diagonal dominance by rows. This means that a Z–matrix $A = (a_{ij})$ of order n with positive diagonal entries is a B–matrix if and only if, for each $i = 1, \ldots, n$,

$$|a_{ii}| > \sum_{k \neq i} |a_{ik}|.$$

In the next result, we present a similar characterization for doubly B–matrices in $\mathcal{Z}_n$.

Proposition 2.1. *A matrix A in $\mathcal{Z}_n$ is a doubly B-matrix if and only if A is strictly doubly diagonally dominant by rows with positive diagonal entries.*

Proof. Let $A = (a_{ij})$ be a matrix in $\mathcal{Z}_n$. Recall that A is a doubly B–matrix if and only if, for all i, $a_{ii} > r_i$ and, for all $i \neq j$ in $\{1, \ldots, n\}$,

$$(a_{ii} - r_i)(a_{jj} - r_j) > \sum_{k \neq i} (r_i - a_{ik}) \sum_{k \neq j} (r_j - a_{jk}).$$

Given that A is a matrix in $\mathcal{Z}_n$, $r_i = 0$ for all i. Therefore, we can assert that A is a doubly B–matrix if and only if, for all i, $a_{ii} > 0$ and, for all $i \neq j$ in $\{1, \ldots, n\}$,

$$a_{ii} a_{jj} > \sum_{k \neq i} (-a_{ik}) \sum_{k \neq j} (-a_{jk}),$$

or, equivalently, since all diagonal elements are positive and all off-diagonal entries are nonpositive,

$$| a_{ii} || a_{jj} |> \sum_{k \neq i} | a_{ik} | \sum_{k \neq j} | a_{jk} |.$$

□

In the next two results, we establish a relation between a matrix A and the corresponding matrix A^+ in terms of belonging to the classes of B–matrices and doubly B–matrices.

Proposition 2.2. *A is a B-matrix if and only if A^+ is a B-matrix.*

Proof. Let $A = (a_{ij})$ be a square matrix of order n and consider A^+. It is clear that A^+ is in $\mathcal{Z}_n$. Moreover, for each $i \in \{1, ..., n\}$, the i*th*-row sum in A^+ is given by

$$\sum_{j=1}^{n} (a_{ij} - r_{i_A}) = \sum_{j=1}^{n} a_{ij} - n r_{i_A}.$$

Using Peña's characterization (1), we know that A is a B–matrix if and only if, for all $i \in \{1, ..., n\}$,

$$\sum_{j=1}^{n} a_{ij} - n r_{i_A} > 0,$$

which is equivalent to say that the row sums of A^+ are positive. □

Proposition 2.3. *A is a doubly B-matrix if and only if A^+ is a doubly B-matrix.*

Proof. Let $A = (a_{ij})$ be a square matrix of order n and consider $A^+ = (b_{ij})$. We have that A is a doubly B–matrix if and only if, for all i, $a_{ii} > r_{i_A}$ and, for all $i \neq j$ in $\{1, \ldots, n\}$,

$$(a_{ii} - r_{i_A})(a_{jj} - r_{j_A}) > \sum_{k \neq i}(r_{i_A} - a_{ik}) \sum_{k \neq j}(r_{j_A} - a_{jk}).$$

Hence, A is a doubly B–matrix if and only if, for all i, $b_{ii} > 0$ and, for all $i \neq j$ in $\{1, \ldots, n\}$,

$$b_{ii} b_{jj} > \sum_{k \neq i} b_{ik} \sum_{k \neq j} b_{jk},$$

which is equivalent, taking into account that A^+ is in $\mathcal{Z}_n$, to $b_{ii} > r_{i_{A^+}}$, for all i, and, for all $i \neq j$ in $\{1, \ldots, n\}$,

$$(b_{ii} - r_{i_{A^+}})(b_{jj} - r_{j_{A^+}}) > \sum_{k \neq i}(r_{i_{A^+}} - b_{ik}) \sum_{k \neq j}(r_{j_{A^+}} - b_{jk}).$$

□

The next theorem is due to Peña and it characterizes B–matrices in terms of sums of two matrices in certain classes.

Theorem 2.1. *Let A be an $n \times n$ matrix. Then the following conditions are equivalent:*

1. *A is a B-matrix.*
2. *$A = B + C$, where B is a strictly diagonally dominant by rows M-matrix and C is a nonnegative matrix of the form*

$$C = \begin{bmatrix} c_1 + \varepsilon & c_1 & \ldots & c_1 \\ c_2 & c_2 + \varepsilon & \ldots & c_2 \\ \vdots & \vdots & & \vdots \\ c_n & c_n & \ldots & c_n + \varepsilon \end{bmatrix},$$

   with $\varepsilon > 0$.
3. *$A = B + C$, where B is a Z-matrix and a B-matrix and C is a nonnegative B-matrix.*

For doubly B–matrices, we derive a similar result.

Theorem 2.2. *Let A be an $n \times n$ matrix. Then the following conditions are equivalent:*

1. *A is a doubly B-matrix.*
2. *$A = B + C$, where B is a strictly doubly diagonally dominant by rows M-matrix and C is a nonnegative matrix of the form*

$$C = \begin{bmatrix} c_1 + \varepsilon & c_1 & \ldots & c_1 \\ c_2 & c_2 + \varepsilon & \ldots & c_2 \\ \vdots & \vdots & & \vdots \\ c_n & c_n & \ldots & c_n + \varepsilon \end{bmatrix},$$

   with $\varepsilon > 0$.

To prove this theorem we need the following two simple lemmas.

Lemma 2.1. *Let A be a doubly B–matrix of order n and C a nonnegative matrix of the form*

$$C = \begin{bmatrix} c_1 & c_1 & \dots & c_1 \\ c_2 & c_2 & \dots & c_2 \\ \vdots & \vdots & & \vdots \\ c_n & c_n & \dots & c_n \end{bmatrix}.$$

Then $A + C$ is a doubly B–matrix.

Proof. Note that, for each $i \in \{1, ..., n\}$, $r_{i_{A+C}} = r_{i_A} + c_i$. Moreover, $(A + C)^+ = A^+$. Since A^+ is a doubly B–matrix by Proposition 2.3, it follows that $(A+C)^+$ is also a doubly B–matrix and, by the same result, $A + C$ is a doubly B–matrix. □

Lemma 2.2. *If A is an $n \times n$ doubly B–matrix and D is a nonnegative diagonal matrix of the same order then $A + D$ is a doubly B–matrix.*

Proof. By Proposition 2.3, we can assume, without loss of generality, that $A = (a_{ij})$ is a Z–matrix. We have $a_{ii} > 0$ and $r_{i_A} = 0$, for $i = 1, ..., n$, and

$$a_{ii}a_{jj} > \sum_{k \neq i} a_{ik} \sum_{k \neq j} a_{jk},$$

for all $i \neq j$.

Let $D = diag(d_1, d_2, ..., d_n)$ and write $A+D = (b_{ij})$. It is trivial that $b_{ii} = a_{ii}+d_i$, for $i = 1, ..., n$, and $b_{ij} = a_{ij}$ if $i \neq j$. Besides, $r_{i_{A+D}} = 0$, for all $i \in \{1, ..., n\}$.

For each $i \in \{1, ..., n\}$, $b_{ii} > 0 = r_{i_{A+D}}$ and, for all $i, j \in \{1, ..., n\}$, $i \neq j$,

$$b_{ii}b_{jj} = (a_{ii} + d_i)(a_{jj} + d_j) \geq a_{ii}a_{jj} > \sum_{k \neq i} a_{ik} \sum_{k \neq j} a_{jk} = \sum_{k \neq i} b_{ik} \sum_{k \neq j} b_{jk}.$$

Hence, $A + D$ is a doubly B–matrix. □

We are now in conditions to prove Theorem 2.2.

Proof. (of Theorem 2.2) Let $A = (a_{ij})$. Suppose A is a doubly B–matrix. Recall that, for all $i \in \{1, ..., n\}$, $a_{ii} > r_{i_A}$. Given $\varepsilon > 0$, define

$$B = \begin{bmatrix} a_{11} - r_{1_A} - \varepsilon & a_{12} - r_{1_A} & \dots & a_{1n} - r_{1_A} \\ a_{21} - r_{2_A} & a_{22} - r_{2_A} - \varepsilon & \dots & a_{2n} - r_{2_A} \\ \vdots & \vdots & & \vdots \\ a_{n1} - r_{n_A} & a_{n2} - r_{n_A} & \dots & a_{nn} - r_{n_A} - \varepsilon \end{bmatrix}$$

and C as described in the statement of the theorem taking $c_i = r_{i_A}$ for all $i \in \{1, ..., n\}$. B is obviously a Z–matrix and it has positive diagonal entries when $\varepsilon < a_{ii} - r_{i_A}$, for $i = 1, ..., n$. Our aim is to choose ε so that B is a doubly B–matrix. Let $i, j \in \{1, ..., n\}$ such that $i \neq j$. We have

$$(a_{ii} - r_{i_A})(a_{jj} - r_{j_A}) > \sum_{k \neq i}(a_{ik} - r_{i_A}) \sum_{k \neq j}(a_{jk} - r_{j_A}).$$

It is trivial that there exists $\varepsilon_{ij} > 0$ such that

$$(a_{ii} - r_{i_A} - \varepsilon_{ij})(a_{jj} - r_{j_A} - \varepsilon_{ij}) > \sum_{k \neq i}(a_{ik} - r_{i_A})\sum_{k \neq j}(a_{jk} - r_{j_A}).$$

Take ε such that

$$0 < \varepsilon < \min\{\varepsilon_{ij}, a_{ii} - r_{i_A} \ \mid \ i,j \in \{1,...,n\}, i \neq j\}.$$

For such ε, B is a doubly B–matrix by Proposition 2.1. Therefore B is a P–matrix (see [5]) and hence an M–matrix. Let us now prove that $C = (c_{ij})$ is also a doubly B–matrix. Note that

$$c_{ii} = r_{i_A} + \varepsilon > r_{i_A} = \max\{0, c_{ij} \ \mid \ j \neq i\} = r_{i_C}.$$

It is clear that

$$C^+ = \begin{bmatrix} \varepsilon & 0 & \dots & 0 \\ 0 & \varepsilon & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \varepsilon \end{bmatrix}$$

is a doubly B–matrix. By Proposition 2.3, C is also a doubly B–matrix.

Conversely, admit that $A = B + C$, where B is a strictly doubly diagonally dominant by rows M-matrix and C is a nonnegative matrix of the form

$$C = \begin{bmatrix} c_1 + \varepsilon & c_1 & \dots & c_1 \\ c_2 & c_2 + \varepsilon & \dots & c_2 \\ \vdots & \vdots & & \vdots \\ c_n & c_n & \dots & c_n + \varepsilon \end{bmatrix},$$

with $\varepsilon > 0$.

Since B is an M–matrix, B is, in particular, a Z–matrix. By Proposition 2.1, we can assert that B is a doubly B–matrix.

Note that

$$A = B + C = B + \begin{bmatrix} c_1 & c_1 & \dots & c_1 \\ c_2 & c_2 & \dots & c_2 \\ \vdots & \vdots & & \vdots \\ c_n & c_n & \dots & c_n \end{bmatrix} + \varepsilon I_n.$$

By Lemma 2.1 and Lemma 2.2, we can conclude that A is a doubly B–matrix. $\square$

When comparing Theorem 2.1 and Theorem 2.2, the natural question that arises is whether the two conditions in the latter are equivalent to a third condition, similar to the third one of Peña in Theorem 2.1. That is, are the two conditions in Theorem 2.2 equivalent to condition

3. $A = B + C$, where B is a Z-matrix and a doubly B-matrix and C is a nonnegative doubly B-matrix?

It is not difficult to show that condition *2.* implies *3.*, but condition *1.* does not follow from *3.* as the following example illustrates.

Example 2.1. *Let B and C be the following doubly B-matrices*

$$B = \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0.1 & 0.1 \\ 3 & 4 & 2 \\ 0.1 & 0.1 & 1 \end{bmatrix}.$$

B is obviously a Z–matrix and $C \geq 0$. Since

$$(B+C)^+ = \begin{bmatrix} 2 & -2.9 & -1.9 \\ 0 & 2 & -1 \\ 0 & 0 & 1.9 \end{bmatrix}$$

and $2 \times 2 \not> -4.8 \times (-1)$, $A = B + C$ is not a doubly B-matrix.

3 Sub-direct sums of B–matrices

It is well known that a direct sum is a P–matrix if and only if each of the direct summands is a P–matrix. This statement remains true for many other classes of matrices, including positive semidefinite, doubly negative, completely positive, totally nonnegative and M–matrices. It is easy to check, however, that this is not true for the case of B–matrices. Fallat and Johnson considered, for several classes of matrices, corresponding questions for a more general 'sum' of two matrices, of which the direct sum and ordinary sum are special cases - the subdirect sum ([1]).

Let $0 \leq k \leq m, n$ and suppose that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathcal{M}_m(\mathbb{C}) \textit{ and } B = \begin{bmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{bmatrix} \in \mathcal{M}_n(\mathbb{C}),$$

in which $A_{22}, B_{22} \in \mathcal{M}_k(\mathbb{C})$. Then

$$A \oplus_k B = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} + B_{22} & B_{23} \\ 0 & B_{32} & B_{33} \end{bmatrix}$$

is called the k–subdirect sum of A and B. We simply refer to a subdirect sum when the value of k is irrelevant or unambiguous. When $k = 0$ we have the familiar direct sum and we abbreviate $\oplus_0$ to $\oplus$. In many key positive classes of matrices, we have that the direct sum lies in the class if and only if each direct summand lies in the class. As we mentioned above, this does not hold for B-matrices. On the other hand, when $k = n = m$, we have the ordinary sum of two B–matrices, which we know it is still a B–matrix. It is also true that any B–matrix can be written as a sum of two B–matrices. In their paper, Fallat and Johnson address four natural questions: (I) If A and B lie in the class must a 1-subdirect sum C lie in the class?; (II) If

$$C = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & C_{23} \\ 0 & C_{32} & C_{33} \end{bmatrix}$$

lies in the class, may C be written as $C = A \oplus_1 B$, such that A and B lie in the class when C_{22} is 1×1?; (III) and (IV) the corresponding questions with 1 replaced by $k > 1$.

As we have mentioned above, regarding B–matrices, these questions have affirmative answers when $k = n = m$.

The following examples illustrate, however, that, in general, question (I) has negative answer when B–matrices are considered.

Example 3.1. *Let* $A = \begin{bmatrix} 4 & 1 \\ 2 & 2.5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. *It is clear that both matrices* A *and* B *are* B*–matrices. However,*

$$A \oplus_1 B = \begin{bmatrix} 4 & 1 & 0 \\ 2 & 3.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is not a B*–matrix.*

Moreover, if we consider question (I) for the particular case $B = A$, we still have a negative answer, as the following example shows.

Example 3.2. *Let* $A = \begin{bmatrix} 4 & 3 \\ 0.5 & 1 \end{bmatrix}$. *It is easy to see that* A *is a* B*–matrix. Nevertheless,*

$$A \oplus_1 A = \begin{bmatrix} 4 & 3 & 0 \\ 0.5 & 5 & 3 \\ 0 & 0.5 & 1 \end{bmatrix}$$

is not a B*–matrix.*

Even if we add the condition of A being symmetric, the answers to questions (I) and (III) remain negative as we can see in the next example.

Example 3.3. *Let*

$$A = \begin{bmatrix} 4 & 2 & 0.5 \\ 2 & 5 & 0.2 \\ 0.5 & 0.2 & 1 \end{bmatrix},$$

A *is a* B*-matrix, but, for* $k \in \{0, 1, 2\}$, $A \oplus_k A$ *is not.*

Even if we add the condition of A being symmetric, the answers to the last referred problems remain negative and examples are easy to find.

It is not difficult, however, to set necessary and sufficient conditions.

Theorem 3.1. *Let*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = (a_{ij}), \quad B = \begin{bmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{bmatrix} = (b_{ij})$$

be B*-matrices, with* A *of order* n, B *of order* m *and* A_{22}, B_{22} *of order* k $(0 \leq k \leq n, m)$. $A \oplus_k B$ *is a* B*-matrix if and only if the following conditions hold*

(C1) For $i = 1, \ldots, n-k$, $\frac{1}{n+m-k} \sum_{j=1}^{n} a_{ij} > a_{ir}$, *for all* $r = 1, \ldots, n$, $r \neq i$;

(C2) For $i = k+1, \ldots, m$, $\frac{1}{n+m-k} \sum_{j=1}^{m} b_{ij} > b_{ir}$, *for all* $r = 1, \ldots, m$, $r \neq i$;

(C3) For $i = n-k+1, \dots, n$, $\frac{1}{n+m-k}\Big(\sum_{j=1}^{n} a_{ij} + \sum_{j=1}^{m} b_{i-n+k,j}\Big) > a_{ir}$*, for all* $r = 1, \dots, n-k$*;*

(C4) For $i = n-k+1, \dots, n$, $\frac{1}{n+m-k}\Big(\sum_{j=1}^{n} a_{ij} + \sum_{j=1}^{m} b_{i-n+k,j}\Big) > a_{ir} + b_{i-n+k,r-n+k}$*, for all* $r = n-k+1, \dots, n$, $r \neq i$*;*

(C5) For $i = n-k+1, \dots, n$, $\frac{1}{n+m-k}\Big(\sum_{j=1}^{n} a_{ij} + \sum_{j=1}^{m} b_{i-n+k,j}\Big) > b_{i-n+k,r-n+k}$*, for all* $r = n+1, \dots, m$.

Proof. Let $A \oplus_k B = (c_{ij})_{i,j=1}^{n+m-k}$. Since the row sums of A and the row sums of B are both positive, it is obvious that the row sums of $A \oplus_k B$ are also positive.

For $i = 1, \dots, n-k$,

$$\frac{1}{n+m-k}\sum_{j=1}^{n+m-k} c_{ij} = \frac{1}{n+m-k}\Big(\sum_{j=1}^{n} a_{ij} + \sum_{j=n+1}^{n+m-k} 0\Big) = \frac{1}{n+m-k}\sum_{j=1}^{n} a_{ij}.$$

Hence, for these values of i, it is trivial that all the off-diagonal elements c_{ir} are bounded above by the corresponding row means when $r \in \{n+1, ..., n+m-k\}$. For $r \in \{1, ..., n\}$, with $r \neq i$, the off-diagonal elements c_{ir} are bounded above by the corresponding row means if and only if condition *(C1)* holds.

For $i = n+1, \dots, n+m-k$,

$$\begin{aligned}\frac{1}{n+m-k}\sum_{j=1}^{n+m-k} c_{ij} &= \frac{1}{n+m-k}\Big(\sum_{j=1}^{n-1} 0 + \sum_{j=n-k+1}^{n+m-k} b_{i-n+k,j-n+k}\Big)\\ &= \frac{1}{n+m-k}\sum_{j=1}^{m} b_{i-n+k,j}.\end{aligned}$$

It becomes clear that, for $i = n+1, \dots, n+m-k$, all the off-diagonal elements c_{ir} are bounded above by the corresponding row means when $r \in \{1, ..., n-k\}$. For $r \in \{n-k+1, ..., n+m-k\}$, all the off-diagonal elements c_{ir} are bounded above by the corresponding row means if and only if $\frac{1}{n+m-k}\sum_{j=1}^{m} b_{i-n+k,j} > c_{ir}$. This is equivalent to condition *(C2)*.

For $i = n-k+1, \dots, n$,

$$\begin{aligned}\frac{1}{n+m-k}\sum_{j=1}^{n+m-k} c_{ij} &= \frac{1}{n+m-k}\Big(\sum_{j=1}^{n} a_{ij} + \sum_{j=n-k+1}^{n+m-k} b_{i-n+k,j-n+k}\Big)\\ &= \frac{1}{n+m-k}\Big(\sum_{j=1}^{n} a_{ij} + \sum_{j=1}^{m} b_{i-n+k,j}\Big).\end{aligned}$$

Therefore, for these values of i, all the off-diagonal elements c_{ir}, with $r \in \{1, ..., n-k\}$, are bounded above by the corresponding row means if and only if condition *(C3)* holds, all the off-diagonal elements c_{ir}, with $r \in \{n-k+1, ..., n\}$ and $r \neq i$, are bounded above by the corresponding row means if and only if condition *(C4)* holds, and all the off-diagonal elements c_{ir}, with $r \in \{n+1, ..., n+m-k\}$, are bounded above by the corresponding row means if and only if condition *(C5)* holds. □

From the previous result we can derive some necessary conditions in which the elements of matrix A do not depend on the elements of matrix B and *vice versa*.

Corollary 3.1. *Let A and B be matrices as in Theorem 3.1. $A \oplus_k B$ is a B-matrix if the following conditions hold*

(S1) For $i = 1, \ldots, n$, $\frac{1}{n+m-k}\sum_{j=1}^{n} a_{ij} > a_{ir}$, for all $r = 1, \ldots, n$, $r \neq i$;

(S2) For $i = 1, \ldots, m$, $\frac{1}{n+m-k}\sum_{j=1}^{m} b_{ij} > b_{ir}$, for all $r = 1, \ldots, m$, $r \neq i$.

Corollary 3.2. *Let A and B be matrices as in Theorem 3.1. If A and B are Z-matrices then $A \oplus_k B$ is a B-matrix.*

Let us now focus on question (II) for B–matrices: If

$$C = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & C_{23} \\ 0 & C_{32} & C_{33} \end{bmatrix}$$

is a B-matrix, may C be written as $C = A \oplus_1 B$, with A and B B-matrices, when C_{22} is 1×1?

In general, the answer is negative, as the following example shows.

Example 3.4. *Let us consider the B–matrix*

$$C = \begin{bmatrix} 1 & 0.4 & 0 \\ 1 & 2 & 1 \\ 0 & 0.4 & 1 \end{bmatrix}.$$

It is not possible to write $C = A \oplus_1 B$ with A, B B-matrices. In fact, we would have

$$C = A \oplus_1 B = \begin{bmatrix} 1 & 0.4 \\ 1 & a \end{bmatrix} \oplus_1 \begin{bmatrix} b & 1 \\ 0.4 & 1 \end{bmatrix},$$

with $a + b = 2$. If A and B were B–matrices, we would have $(1 + a)/2 > 1$ and $(1 + b)/2 > 1$, which imply $a, b > 1$, a contradiction.

However, the answer is positive if C is a Z–matrix.

Proposition 3.1. *If C is a B-matrix in $\mathcal{Z}_n$ of the form*

$$C = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & C_{23} \\ 0 & C_{32} & C_{33} \end{bmatrix},$$

with C_{22} of size 1×1, then C can be written as $A \oplus_1 B$ with A, B B-matrices.

Proof. Being $C = (c_{ij})$ a B-matrix, we know $C_{22} = -c_{m1} - \ldots - c_{mm-1} - c_{mm+1} - \ldots - c_{mn} + \delta$ where $m - 1$ is the order of C_{11}, and $\delta > 0$. Take

$$A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & a \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b & C_{23} \\ C_{32} & C_{33} \end{bmatrix},$$

where $a = -c_{m1} - \ldots - c_{mm-1} + \frac{\delta}{2}$ and $b = -c_{mm+1} - \ldots - c_{mn} + \frac{\delta}{2}$. It is easy to prove that A and B are B–matrices. $\square$

This last result can be extended and we also have a positive answer to question (IV) in case C is a Z–matrix.

Theorem 3.2. *If C is a B-matrix in $\mathcal{Z}_n$ of the form*

$$C = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & C_{23} \\ 0 & C_{32} & C_{33} \end{bmatrix},$$

with C_{22} of size $k \times k$ $(0 \le k \le n)$, then C can be written as $A \oplus_k B$ with A, B B-matrices.

Proof. Let m be the order of block C_{11} and $C = (c_{ij})$. Since C is a B–matrix, we know that, for each $i \in \{m+1, ..., m+k\}$, $c_{ii} = -c_{i1} - ... - c_{i,i-1} - c_{i,i+1} - ... - c_{in} + \delta_i$ for some $\delta_i > 0$.

Consider $A = (a_{ij})$ of order $m+k$ given by $a_{ij} = c_{ij}$ for all $i \neq j$, $a_{ii} = c_{ii}$ if $i \in \{1, ..., m\}$ and $a_{ii} = -c_{i1} - ... - c_{i,i-1} - c_{i,i+1} - ... - c_{i,m+k} + \frac{\delta_i}{2}$ if $i \in \{m+1, ..., m+k\}$ and $B = (b_{ij})$ of order $n-m$ given by $b_{ij} = 0$ for all $i, j \in \{1, ..., k\}$ such that $i \neq j$, $b_{ij} = c_{i+m,j+m}$ if $i \in \{k+1, ..., n-m\}$ or $j \in \{k+1, ..., n-m\}$, and $b_{ii} = -c_{i+m,m+k+1} - ... - c_{i+m,n} + \frac{\delta_{i+m}}{2}$ if $i \in \{1, ..., k\}$. It is not difficult to prove that A and B are B–matrices such that $A \oplus_k B = C$. $\square$

As for question (II), in general the answer to question (IV) is negative. In the next example we consider $k = 3$ but it is easy to extend this to any value of $k > 1$.

Example 3.5. *Let us consider the B–matrix*

$$C = \begin{bmatrix} 2 & 0.2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0.2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 2 \end{bmatrix}.$$

It is not possible to write $C = A \oplus_3 B$ with A, B B-matrices. In fact, we would have

$$C = A \oplus_3 B = \begin{bmatrix} 2 & 0.2 & 0 & 0 \\ 1 & a_1 & a_2 & a_3 \\ 1 & a_4 & a_5 & a_6 \\ 1 & a_7 & a_8 & a_9 \end{bmatrix} \oplus_3 \begin{bmatrix} b_1 & b_2 & b_3 & 1 & 1 \\ b_4 & b_5 & b_6 & 1 & 1 \\ b_7 & b_8 & b_9 & 1 & 1 \\ 0 & 0 & 0.2 & 2 & 0 \\ 0 & 0 & 0 & 0.2 & 2 \end{bmatrix},$$

with $a_1 + a_2 + a_3 + b_1 + b_2 + b_3 = 4$. If A and B were B–matrices, we would have $1 + a_1 + a_2 + a_3 > 4$ and $2 + b_1 + b_2 + b_3 > 5$, which imply $a_1 + a_2 + a_3 + b_1 + b_2 + b_3 > 6$, a contradiction.

4 Hadamard product

It is known that the Hadamard product of two real square matrices $A = (a_{ij})$ and $B = (b_{ij})$, denoted by $A \circ B$, is a new matrix $C = (c_{ij})$ such that $c_{ij} = a_{ij} b_{ij}$ (see [3]). In this section we analyze when the Hadamard product of two B–matrices lies in the class of B–matrices.

The following example illustrate that, in general, this question has a negative answer.

Example 4.1. *Let matrix* $A = \begin{bmatrix} 4 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. *It is clear that A is a B–matrix. However,* $A \circ A = \begin{bmatrix} 16 & 9 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *is not a B–matrix.*

Theorem 4.1. *If $A = (a_{ij})$ and $B = (b_{ij})$ are nonnegative B–matrices, then $A \circ B$ is also a B–matrix.*

Proof. It is easy to observe that $r_{i_{A\circ B}} \leq r_{i_A} r_{i_B}$, for all $i = 1, 2, \ldots, n$. From (1) we only need to prove that, for all i,

$$a_{ii}b_{ii} > (n-1)r_{i_{A\circ B}} - \sum_{r\neq i,l} a_{ir}b_{ir},$$

where $r_{i_{A\circ B}} = a_{il}b_{il}$. Let $i \in \{1, 2, \ldots, n\}$. We distinguish two cases:
(I) $r_{i_{A\circ B}} = r_{i_A} r_{i_B}$
In this case we may assume, without loss of generality, that $r_{i_A} = a_{in}$, $r_{i_B} = b_{in}$ and so, $r_{i_{A\circ B}} = a_{in}b_{in}$.

We know that

$$a_{ii} > (n-1)a_{in} - \sum_{r\neq i,n} a_{ir} \geq 0 \qquad \text{and} \qquad b_{ii} > (n-1)b_{in} - \sum_{r\neq i,n} b_{ir} \geq 0,$$

so

$$\begin{aligned}
a_{ii}b_{ii} &> \left[(n-1)a_{in} - \sum_{r\neq i,n} a_{ir}\right]\left[(n-1)b_{in} - \sum_{r\neq i,n} b_{ir}\right] \\
&= \left[a_{in} + \sum_{r\neq i,n}(a_{in} - a_{ir})\right]\left[(n-1)b_{in} - \sum_{r\neq i,n} b_{ir}\right] \\
&= (n-1)a_{in}b_{in} - \sum_{r\neq i,n} a_{in}b_{ir} + \sum_{r\neq i,n}(a_{in} - a_{ir})\left[b_{in} + \sum_{r\neq i,n}(b_{in} - b_{ir})\right] \\
&= (n-1)a_{in}b_{in} - \sum_{r\neq i,n} a_{in}b_{ir} + \sum_{r\neq i,n}(a_{in} - a_{ir})b_{in} + \sum_{\substack{r\,\neq\, i,\,n \\ k\,\neq\, i,\,n}}(a_{in} - a_{ir})(b_{in} - b_{ik}) \\
&= (n-1)a_{in}b_{in} + \sum_{\substack{r\,\neq\, i,\,n \\ k\,\neq\, i,\,n}}(a_{in} - a_{ir})(b_{in} - b_{ik}) + \sum_{r\neq i,n}(a_{in} - a_{ir})b_{in} - \\
&\quad - \sum_{r\neq i,n}(a_{in} - a_{ir})b_{ir} - \sum_{r\neq i,n} a_{ir}b_{ir} \\
&= (n-1)a_{in}b_{in} + \sum_{\substack{r\,\neq\, i,\,n \\ k\,\neq\, i,\,n}}(a_{in} - a_{ir})(b_{in} - b_{ik}) + \sum_{r\neq i,n}(a_{in} - a_{ir})(b_{in} - b_{ir}) - \\
&\quad - \sum_{r\neq i,n} a_{ir}b_{ir}.
\end{aligned}$$

Taking into account that all terms of the form $(a_{in}-a_{ir})$ or $(b_{in}-b_{ir})$ $(r\neq i, r\neq n)$ are nonnegative, we can conclude that

$$\begin{aligned} a_{ii}b_{ii} &> (n-1)a_{in}b_{in} + \sum_{\substack{r\neq i,n\\ k\neq i,n}} (a_{in}-a_{ir})(b_{in}-b_{ik}) + \sum_{r\neq i,n}(a_{in}-a_{ir})(b_{in}-b_{ir}) - \\ &\quad - \sum_{r\neq i,n} a_{ir}b_{ir} \\ &\geq (n-1)a_{in}b_{in} - \sum_{r\neq i,n} a_{ir}b_{ir}. \end{aligned}$$

Since $l=n$, we can conclude that

$$a_{ii}b_{ii} > (n-1)r_{i_{A\circ B}} - \sum_{r\neq i,l} a_{ir}b_{ir}.$$

(II) $r_{i_{A\circ B}} < r_{i_A}r_{i_B}$.
In this case, let $j,k\in\{1,2,\ldots,n\}$ be such that $r_{i_A}=a_{ij}$ and $r_{i_B}=b_{ik}$. Note that, by definition, $j\neq i$ and $k\neq i$. If $j=k$, we would have $r_{i_{A\circ B}}=r_{i_A}r_{i_B}$. Therefore, $j\neq k$. Let $l\in\{1,2,\ldots,n\}$ be such that $r_{i_{A\circ B}}=a_{il}b_{il}$ $(l\neq i)$.

We know that

$$a_{ii} > (n-1)a_{ij} - \sum_{r\neq i,j} a_{ir} \geq 0 \qquad \text{and} \qquad b_{ii} > (n-1)b_{ik} - \sum_{r\neq i,k} b_{ir} \geq 0,$$

so

$$\begin{aligned} a_{ii}b_{ii} &> \left[(n-1)a_{ij} - \sum_{r\neq i,j} a_{ir}\right]\left[(n-1)b_{ik} - \sum_{r\neq i,k} b_{ir}\right] \\ &= \left[a_{ij} + \sum_{r\neq i,j}(a_{ij}-a_{ir})\right]\left[(n-1)b_{ik} - \sum_{r\neq i,k} b_{ir}\right] \\ &= (n-1)a_{ij}b_{ik} - \sum_{r\neq i,k} a_{ij}b_{ir} + \sum_{r\neq i,j}(a_{ij}-a_{ir})\left[b_{ik} + \sum_{r\neq i,k}(b_{ik}-b_{ir})\right] \\ &= (n-1)a_{ij}b_{ik} - \sum_{r\neq i,k} a_{ij}b_{ir} + \sum_{r\neq i,j}(a_{ij}-a_{ir})b_{ik} + \sum_{\substack{r\neq i,j\\ s\neq i,k}} (a_{ij}-a_{ir})(b_{ik}-b_{is}) \\ &= \sum_{\substack{r\neq i,j\\ s\neq i,k}} (a_{ij}-a_{ir})(b_{ik}-b_{is}) - \sum_{r\neq i,k}(a_{ij}-a_{ir})b_{ir} - \sum_{r\neq i,k} a_{ir}b_{ir} + \\ &\quad + \sum_{r\neq i,j}(a_{ij}-a_{ir})b_{ik} + (n-1)a_{ij}b_{ik} \\ &= \sum_{\substack{r\neq i,j\\ s\neq i,k}} (a_{ij}-a_{ir})(b_{ik}-b_{is}) + \sum_{r\neq i,j,k} (a_{ij}-a_{ir})(b_{ik}-b_{ir}) + (a_{ij}-a_{ik})b_{ik} + \\ &\quad + (a_{ij}-a_{ij})b_{ij} - \sum_{r\neq i, r\neq k} a_{ir}b_{ir} + (n-1)a_{ij}b_{ik} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{r \neq i,j \\ s \neq i,k}} (a_{ij} - a_{ir})(b_{ik} - b_{is}) + \sum_{r \neq i,j,k} (a_{ij} - a_{ir})(b_{ik} - b_{ir}) - \sum_{r \neq i,k} a_{ir} b_{ir} + \\
&\quad + (n-1) a_{ij} b_{ik} + a_{ij} b_{ik} - a_{ik} b_{ik} \\
&= \sum_{\substack{r \neq i,j \\ s \neq i,k}} (a_{ij} - a_{ir})(b_{ik} - b_{is}) + \sum_{r \neq i,j,k} (a_{ij} - a_{ir})(b_{ik} - b_{ir}) - a_{il} b_{il} - \\
&\quad - \sum_{r \neq i,l} a_{ir} b_{ir} + n a_{ij} b_{ik} \\
&= (n-1) a_{il} b_{il} - \sum_{r \neq i,l} a_{ir} b_{ir} + \sum_{\substack{r \neq i,j \\ s \neq i,k}} (a_{ij} - a_{ir})(b_{ik} - b_{is}) + \\
&\quad + \sum_{r \neq i,j,k} (a_{ij} - a_{ir})(b_{ik} - b_{ir}) + n a_{ij} b_{ik} - n a_{il} b_{il}. \qquad (2)
\end{aligned}
$$

Observe that each term of the form $(a_{ij} - a_{ir})$ or $(b_{ik} - b_{ir})$ that occurs in the last expression is nonnegative. In addition, $a_{ij} \geq a_{il} \geq 0$ and $b_{ik} \geq b_{il} \geq 0$ and so $na_{ij}b_{ik} - na_{il}b_{il} \geq 0$. Therefore,

$$a_{ii} b_{ii} > (n-1) a_{il} b_{il} - \sum_{r \neq i,l} a_{ir} b_{ir},$$

that is

$$a_{ii} b_{ii} > (n-1) r_{i_{A \circ B}} - \sum_{r \neq i,l} a_{ir} b_{ir}.$$

We have proved that, for all $i \in \{1, 2, \ldots, n\}$,

$$a_{ii} b_{ii} > (n-1) r_{i_{A \circ B}} - \sum_{r \neq i,l} a_{ir} b_{ir},$$

where $r_{i_{A \circ B}} = a_{il} b_{il}$. Hence, $A \circ B$ is a B–matrix. $\square$

We can generalize the previous result in the following terms:

Theorem 4.2. *Let $A = (a_{ij})$, $B = (b_{ij})$ be B–matrices. Let us suppose that if $a_{rs} < 0$, then $b_{rs} \geq 0$. Then $A \circ B$ is a B–matrix.*

Proof. With similar reasoning to the previous result we obtain the expression (2). In order to obtain the same conclusion that in Theorem 4.1, that is, for all $i \in \{1, 2, \ldots, n\}$

$$a_{ii} b_{ii} > (n-1) a_{il} b_{il} - \sum_{r \neq i,l} a_{ir} b_{ir},$$

we only need to analyze the sign of the term $na_{ij}b_{ik} - na_{il}b_{il}$. We consider the following cases:

a) If a_{ij} and b_{ik} are positive, then $na_{ij}b_{ik} \geq na_{il}b_{il}$.

b) If $a_{ij} = 0$, then $a_{il} \leq 0$ since $a_{ij} = r_{i_A}$, $b_{il} \geq 0$ by hypothesis, and $b_{ik} \geq 0$ since $b_{ik} = r_{i_B}$. Therefore, $na_{ij}b_{ik} - na_{il}b_{il} = -na_{il}b_{il} \geq 0$.

c) If $b_{ik} = 0$, then $b_{il} \leq 0$, $a_{il} \geq 0$ and $a_{ij} \geq 0$. So, $na_{ij}b_{ik} - na_{il}b_{il} \geq 0$.

Therefore, $A \circ B$ is a B–matrix. $\square$

From this theorem we can establish the following result.

Corollary 4.1. *Let A be a Z-matrix and B a nonnegative matrix. If A, B are B–matrices, then $A \circ B$ is a B–matrix.*

Let us note that if there exists a position (i, s) such that $a_{is} < 0$ and $b_{is} < 0$, it is possible to construct B–matrices A and B such that $A \circ B$ is not a B–matrix. The row i of A can be defined as $a_{ii} = 4$, $a_{is} = -3$ and $a_{ij} = \pm\varepsilon$, for $j \neq i, s$ and ε sufficiently small. In analogous way, $b_{ii} = 4$, $b_{is} = -3$ and $b_{ij} = \pm\varepsilon$, for $j \neq i, s$ and ε sufficiently small. The remaining rows of matrices A and B may be the corresponding rows of the identity matrix. We can show that A and B are B–matrices, but if we analyze the B-conditions of the row i of $A \circ B$

$$\frac{1}{n}(16 + 9 + q\varepsilon^2) \approx \frac{25}{n} < 9, \qquad \text{for} \qquad n \geq 3.$$

So, $A \circ B$ is not a B–matrix for $n \geq 3$.

5 Kronecker product and sum of B–matrices

If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ is the $mp \times nq$ matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}.$$

If $m = n$ and $p = q$, we define the Kronecker sum of A and B as $S_{AB} = (I_p \otimes A) + (B \otimes I_m)$ (see [3]).

In this section we consider the Kronecker product and the Kronecker sum of two B–matrices and we analyse whether they lie in the same class.

The following example illustrates that the Kronecker product of two B–matrices is not necessarily a B–matrix.

Example 5.1. *Let* $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ *and* $D = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$. *These matrices are B–matrices. Nevertheless, $A \otimes B$, $C \otimes B$ and $C \otimes D$ are not B–matrices.*

Observe that the example above shows that even the Kronecker product of two B–matrices in $\mathcal{Z}_n$ is not, in general, a B–matrix, and that if one of the matrices is nonegative and the other a Z-matrix, we can not conclude that the Kronecker product is also a B–matrix.

When it comes to the Kronecker sum of two B–matrices, it is also not true, in general, that the resulting matrix is a B–matrix, as we can conclude from the next example.

Example 5.2. *Let* $A = \begin{bmatrix} 1.25 & 1 \\ 1 & 1.25 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1.25 & 1 & 1 \\ 1 & 1.25 & 1 \\ 1 & 1 & 1.25 \end{bmatrix}$. *It is clear that both* A *and* B *are* B*–matrices. However,*

$$S_{AB} = (I_3 \otimes A) + (B \otimes I_2) = \begin{bmatrix} 2.5 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2.5 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2.5 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2.5 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2.5 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2.5 \end{bmatrix}$$

is not a B*–matrix.*

In the next result we establish a sufficient condition for the Kronecker sum of two B–matrices to be also a B–matrix.

Proposition 5.1. *Let* A *and* B *be* B*–matrices of order* n *and* m*, respectively. If, for all* $i \in \{1, ..., n\}$ *and all* $k \in \{1, ..., m\}$, $\sum_{j=1}^{n} a_{ij} > nmr_{i_A}$ *and* $\sum_{j=1}^{m} b_{kj} > nmr_{k_B}$*, then* S_{AB} *is a* B*–matrix.*

Proof. Write $I_m \otimes A = (c_{ij})$ and $B \otimes I_n = (d_{ij})$. It is clear that for each $i \in \{1, ..., nm\}$ there exist $t \in \{1, ..., n\}$ and $s \in \{1, ..., m\}$ such that

$$\sum_{j=1}^{nm} c_{ij} = \sum_{j=1}^{n} a_{tj} \quad \textit{and} \quad \sum_{j=1}^{nm} d_{ij} = \sum_{j=1}^{m} b_{sj}.$$

Hence, for each $i \in \{1, ..., nm\}$,

$$\sum_{j=1}^{nm} c_{ij} > nmr_{t_A} = nmr_{i_{I_m \otimes A}} \quad \textit{and} \quad \sum_{j=1}^{nm} d_{ij} > nmr_{s_B} = nmr_{i_{B \otimes I_n}}.$$

Therefore, $I_m \otimes A$ and $B \otimes I_n$ are B–matrices and, since the sum of two B–matrices is also a B–matrix, we can conclude that S_{AB} is a B–matrix. □

From this result we can derive the following.

Corollary 5.1. *Let* A *and* B *be* B*–matrices in* $\mathcal{Z}_n$ *and* $\mathcal{Z}_m$*, respectively. Then* S_{AB} *is a* B*-matrix.*

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