Special monogenic polynomials - properties and applications

H. R. Malonek* and M. I. Falcão†

*Universidade de Aveiro, Portugal
hrmalon@ua.pt
†Universidade do Minho, Portugal
mif@math.uminho.pt

Abstract. In Clifford Analysis, several different methods have been developed for constructing monogenic functions as series with respect to properly chosen homogeneous monogenic polynomials. Almost all these methods rely on sets of orthogonal polynomials with their origin in classical (real) Harmonic Analysis in order to obtain the desired basis of homogeneous polynomials. We use a direct and elementary approach to this problem and construct a set of homogeneous polynomials involving only products of a hypercomplex variable and its hypercomplex conjugate. The obtained set is an Appell set of monogenic polynomials with respect to the hypercomplex derivative. Its intrinsic properties and some applications are presented.

Keywords: homogeneous monogenic polynomials, quaternions, Appell sets
PACS: 02.30.-f, 02.30.Lt, 02.10.Ox

1. INTRODUCTION

Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal base of the Euclidean vector space $\mathbb{R}^n$ with a non-commutative product according to the multiplication rules $e_k e_l + e_l e_k = -2\delta_{kl},\ k,l = 1,\cdots,n$, where δ_{kl} is the Kronecker symbol. The set $\{e_A : A \subseteq \{1,\cdots,n\}\}$ with $e_A = e_{h_1}e_{h_2}\cdots e_{h_r}$, $1 \le h_1 \le \cdots \le h_r \le n$, $e_\emptyset = e_0 = 1$, forms a basis of the 2^n-dimensional Clifford algebra $Cl_{0,n}$ over $\mathbb{R}$. Let $\mathbb{R}^{n+1}$ be embedded in $Cl_{0,n}$ by identifying $(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}$ with the element $x = x_0 + \underline{x}$ of the algebra, where $\underline{x} = e_1x_1 + \cdots + e_nx_n$.The conjugate of x is $\bar{x} = x_0 - \underline{x}$ and the norm $|x|$ of x is defined by $|x|^2 = x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + \cdots + x_n^2$.

We consider functions of the form $f(z) = \sum_A f_A(z)e_A$, where $f_A(z)$ are real valued, i.e. $Cl_{0,n}$-valued functions defined in some open subset $\Omega \subset \mathbb{R}^{n+1}$. We suppose that f is hypercomplex differentiable in Ω in the sense of [1], [2], i.e. has a uniquely defined areolar derivative f' in each point of Ω (see [3]). Then f is real differentiable and f' can be expressed by the real partial derivatives as $f' = 1/2(\partial_0 - \partial_{\underline{x}})f$, where $\partial_0 := \frac{\partial}{\partial x_0}$, $\partial_{\underline{x}} := e_1\frac{\partial}{\partial x_1} + \cdots + e_n\frac{\partial}{\partial x_n}$. With $D := \partial_0 + \partial_{\underline{x}}$ as the generalized Cauchy-Riemann operator, obviously holds $f' = 1/2\overline{D}f$. Since a hypercomplex differentiable function belongs to the kernel of D, i.e. satisfies $Df = 0$ or $0 = fD$ (f is a *left resp. right monogenic function* in the sense of Clifford Analysis), then it follows that in fact $f' = \partial_0 f$ like in the complex case. We also need to consider the monogenic polynomials as functions of the hypercomplex monogenic variables $z_k = x_k - x_0e_k = -\frac{xe_k+e_kx}{2}$, $k = 1,2,\cdots,n$. This implies the use of so called *generalized powers* of degree m that are by convention symbolically written as $z_1^{\mu_1} \times \cdots z_n^{\mu_n}$ and defined as an *m-nary* symmetric product by $z_1^{\mu_1} \times \cdots z_n^{\mu_n} = \frac{1}{m!}\sum_{\pi(i_1,\ldots,i_n)} z_{i_1}\cdots z_{i_n}$, where the sum is taken over *all* $m! = |\mu|!$ permutations of $(i_1,\ldots,i_n)$, (see [2] and [3]).

2. THE CONSTRUCTION OF THE APPELL SET

The essential ideas of our approach can be shown for the case $n = 2$, since the general structure of the considered polynomial remains one and the same for different values of $n \ge 2$. The extension of the obtained relationships from $n = 2$ to an arbitrary dimension higher than 2 relates only to some combinatorial calculations on the coefficients which in more detail will be discussed and applied in [4].

We recall that a sequence of polynomials $P_0(x),\ P_1(x), \cdots$ is said to form an Appell set or Appell sequence if

i. $P_k(x)$ is of exact degree k, for each $k = 0,1,\cdots$;
ii. $P_k'(x) = kP_{k-1}(x)$, for each $k = 1,2,\cdots$.

The basic idea is that the polynomials of an Appell sequence behave like power-law functions under the differentiation operation (see e.g. [5, 6, 7]). In our case the polynomials will be monogenic and therefore the derivative should be understand as the hypercomplex derivative mentioned in the first section. As usual, the sequence will be normalized by demanding that $P_0(x) \equiv 1$. It is evident, that only the use of a hypercomplex derivative enables us to speak about an Appell sequence in the setting of Clifford Analysis. Treating monogenic polynomials exclusively as solutions of a generalized Cauchy-Riemann system would not allow to obtain an analogue to the concept of an Appell set in the real or complex case.

Independent of the dimension n, we are looking for an Appell set of monogenic polynomials $\mathscr{P}_k(x)$ of the form

$$\mathscr{P}_k(x) = \sum_{s=0}^{k} T_s^k x^{k-s} \overline{x}^s,$$

where T_s^k are suitable defined real numbers.

Notice that in the complex case, corresponding to $n = 1$ with $e_1 := i$, for polynomials $\mathscr{P}_k(x)$ normalized by $\mathscr{P}_k(1) = 1,\ k = 0,1,\dots$ follows immediately that $T_0^k \equiv 1$ and $T_s^k \equiv 0$, for $s > 0$, since holomorphic functions in $\mathbb{C}$ have a series expansion which involves only the powers of $z = x_0 + ix_1$ and not the conjugate variable $\overline{z} = x_0 - ix_1$. In the hypercomplex case, and particularly in the case $n = 2$ which is in the center of our attention, the $\mathscr{P}_k$ a priori may depend on the values of T_s^k not only for the trivial case $s = 0$. This can already be seen by the following

Theorem 1 *Consider in the case $n = 2$ the variable $x = x_0 + x_1e_1 + x_2e_2$ and its conjugate $\overline{x} = x_0 - x_1e_1 - x_2e_2$. The homogeneous polynomial of degree k; $k = 0,1,\cdots$,*

$$\mathscr{P}_k(x) = \sum_{s=0}^{k} T_s^k x^{k-s} \overline{x}^s, \tag{1}$$

normalized by

$$\mathscr{P}_k(1) = 1, \tag{2}$$

is monogenic if and only if the alternating sum

$$c_k := \sum_{s=0}^{k} T_s^k (-1)^s \tag{3}$$

satisfies

$$c_k = \left[\sum_{|\nu|=k} (-1)^k \binom{k}{\nu} \left(e_1^{\nu_1} \times e_2^{\nu_2} \right)^2 \right]^{-1}. \tag{4}$$

The explicit expression of the uniquely defined c_k relies on the fact that the polynomials $\mathscr{P}_k(x)$ in terms of the corresponding hypercomplex monogenic variables $z_k = x_k - x_0e_k,\ k = 1,2$ are obtained as

$$\mathscr{P}_k(x) = \mathbf{P}_k(z_1,z_2) = c_k \sum_{k=0}^{n} z_1^{n-k} \times z_2^k \binom{n}{k} e_1^{n-k} \times e_2^k. \tag{5}$$

The normalization condition (2), i.e. $\mathscr{P}_k(1) = \mathbf{P}_k(-e_1,-e_2) = 1$ then implies that

$$c_k = \left[\sum_{|\nu|=k} (-1)^k \binom{k}{\nu} \left(e_1^{\nu_1} \times e_2^{\nu_2} \right)^2 \right]^{-1}.$$

Suppose now that $\mathscr{P}_k'(x) = k\mathscr{P}_{k-1}(x);\ k = 1,2,\cdots$. Then it is possible to prove that the values of $T_s^k,\ s = 0,\cdots,k$, can be determined recursively from the values of $T_s^{k-1},\ j = 0,\cdots,k-1$ and c_k. In other words, we have a recursion formula for the $\mathscr{P}_k(x)$.

Theorem 2 *The coefficients $T_s^k,\ s=0,\cdots,k$ and $T_s^{k-1},\ j=0,\cdots,k-1$ satisfy the $(k+1)\times(k+1)$ system of algebraic equations*

$$M_k \begin{pmatrix} T_0^k \\ T_1^k \\ T_2^k \\ \vdots \\ T_{k-2}^k \\ T_{k-1}^k \\ T_k^k \end{pmatrix} = k \begin{pmatrix} T_0^{k-1} \\ T_1^{k-1} \\ T_2^{k-1} \\ \vdots \\ T_{k-2}^{k-1} \\ T_{k-1}^{k-1} \\ c_k \end{pmatrix}. \tag{6}$$

where

$$M_k := \begin{pmatrix} k & 1 & 0 & 0 & & 0 & 0 \\ 0 & k-1 & 2 & 0 & & 0 & 0 \\ 0 & 0 & k-2 & 3 & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & & k-1 & 0 \\ 0 & 0 & 0 & 0 & & 1 & k \\ 1 & -1 & 1 & -1 & \cdots & (-1)^{k-1} & (-1)^k \end{pmatrix}.$$

The system is uniquely solvable since

$$\det(M_k) = (-1)^k k! 2^k \neq 0,\ k=0,1,\ldots.$$

As a corollary it is possible to relate for every fixed value of $k \geq 0$ the vector $\{T_s^k\}$ to the vector $\{c_s\},\ s=0,\cdots,k$.

Corollary 1 *For every $k \geq 0$ the values of T_s^k and c_s; $s=0,1,\cdots,k$ are related by*

$$\begin{pmatrix} T_0^k \\ \vdots \\ T_k^k \end{pmatrix} = N_k \begin{pmatrix} c_0 \\ \vdots \\ c_k \end{pmatrix}, \tag{7}$$

where

$$N_k = M_k^{-1} \left(\begin{array}{c|c} kN_{k-1} & 0 \\ \hline 0 & 1 \end{array} \right);\ k=1,2,\cdots \qquad and \qquad N_0 = 1.$$

It is also possible to prove other intrinsic properties of the set $\{T_s^k,\ s=0,\cdots,k\}$, which are of own interest in combinatorial questions, since they resemble in a lot of aspects a set of non-symmetric generalized binomial coefficients.

This all together leads to the following

Theorem 3 *Monogenic polynomials of the form*

$$\mathscr{P}_k(x) = \sum_{s=0}^{k} T_s^k x^{k-s} \bar{x}^s, \quad with \quad T_s^k = \frac{1}{k+1} \frac{(\frac{3}{2})_{(k-s)} (\frac{1}{2})_{(s)}}{(k-s)!s!}, \tag{8}$$

where $a_{(r)}$ denotes the Pochhammer symbol (raising factorial) form an Appell set of monogenic polynomials.

In terms of generalized powers these polynomials are of the form

$$\mathscr{P}_k(x) = \mathbf{P}_k(z_1,z_2) = c_k \sum_{k=0}^{n} z_1^{n-k} \times z_2^k \binom{n}{k} e_1^{n-k} \times e_2^k, \tag{9}$$

where

$$c_k := \sum_{s=0}^{k} T_s^k (-1)^s = \begin{cases} \frac{k!!}{(k+1)!!}, & \text{if } k \text{ is odd} \\ c_{k-1}, & \text{if } k \text{ is even} \end{cases} \tag{10}$$

3. AN APPELL SET FOR ARBITRARY DIMENSION AND APPLICATIONS

It can be proved that the polynomials

$$\mathscr{P}_k^n(x) = \sum_{s=0}^{k} T_s^k(n)\, x^{k-s}\, \overline{x}^s, \quad \text{with} \quad T_s^k(n) = \frac{n!}{(n)_k} \frac{\left(\frac{n+1}{2}\right)_{(k-s)} \left(\frac{n-1}{2}\right)_{(s)}}{(k-s)!s!}, \tag{11}$$

form an Appell set for arbitrary $n \geq 1$.

They are used in [4] for discussing in detail a monogenic exponential function from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$, which does not rely on Fueter's mapping and auxiliary constructions through solutions to partial differential equations of higher order (c.f. [8], which gives a survey on related questions). Evidently, the constructed polynomials allow to obtain other special monogenic functions as series of the form

$$\Phi(x) = \sum_{k=0}^{\infty} a_k \mathscr{P}_k(x) \qquad \left(\text{or } \Phi(x) = \sum_{k=0}^{\infty} \mathscr{P}_k(x) a_k, \text{ resp.}\right),$$

with suitable chosen coefficients.

Of course, the $\mathscr{P}_k,\ k = 0, 1, \cdots$, form only a restricted set of homogeneous monogenic polynomials, due to the fact that a general homogeneous left monogenic (right monogenic, resp.) polynomial has the form (see [2], [3])

$$\mathscr{P}_k(x) = \sum_{|\nu|=k} \vec{z}^{\,\nu} c_\nu \qquad \left(\text{or } \mathscr{P}_k(x) = \sum_{|\nu|=k} c_\nu \vec{z}^{\,\nu}, \text{ resp.}\right).$$

Nevertheless, it can be shown that with the help of a linear combination of the partial derivatives with respect to $x_k,\ k = 1, \ldots, n$, of the unique $\mathscr{P}_k(x)$ a complete basis system of Appell sets for homogeneous polynomials can be obtained.

We would also like to point out that the Appell sequence of homogeneous monogenic polynomials (8) is particularly easy to handle and can play an important role in 3D-mapping problems. For a case study of such approach, see for example [9].

ACKNOWLEDGMENTS

The research of the first author was partially supported by the R&D unit *Matemática e Aplicações* (UIMA) of the University of Aveiro, through the Portuguese Foundation for Science and Technology (FCT). The research of the second author was partially supported by the Portuguese Foundation for Science and Technology (FCT) through the research program POCTI.

REFERENCES

1. K. Gürlebeck, and H. Malonek, *Complex Variables Theory Appl.* **39**, 199–228 (1999).
2. H. Malonek, *Complex Variables, Theory Appl.* **15**, 181–191 (1990).
3. H. Malonek, "Selected topics in hypercomplex function theory," in *Clifford algebras and potential theory*, edited by S.-L. Eriksson, 7, University of Joensuu, 2004, pp. 111–150.
4. M. I. Falcão, and H. R. Malonek, "Generalized Exponentials through Appell sets in $\mathbb{R}^{n+1}$ and Bessel functions," this issue of ICNAAM2007-International Conference on Numerical Analysis and Applied Mathematics, AIP-Proceedings, 2007.
5. P. Appell, *Ann. Sci. École Norm. Sup.* **9**, 119–144 (1880).
6. B. C. Carlson, *J. Math. Anal. Appl.* **32**, 543–558 (1970).
7. R. P. Boas, and R. C. Buck, *Polynomial Expansions of Analytic Functions*, Springer, Berlin, 1958.
8. W. Sprössig, *Journal for Analysis and its Applications* **18**, 349–360 (1999).
9. J. Cruz, M. I. Falcão, and H. R. Malonek, "3D-Mappings and their approximations by series of powers of a small parameter," IKM2006, 2006.