OCCUPATION TIME OF EXCLUSION PROCESSES WITH CONDUCTANCES

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ABSTRACT. We obtain the fluctuations for the occupation time of one-dimensional symmetric exclusion processes with speed change, where the transition rates (*conductances*) are driven by a general function W. The approach does not require sharp bounds on the spectral gap of the system nor the jump rates to be bounded from above or below. We present some examples and for one of them, we observe that the fluctuations of the current are trivial, but the fluctuations of the occupation time are given by a fractional Brownian Motion. This shows that, in general, the fluctuations of the current and of the occupation time are not of same order.

1. INTRODUCTION

Occupation time is the usual nomenclature for the additive functional $\int_0^t \eta_s(x)ds$, where $\eta_s(x)$ denotes the occupation variable at the site x at the time s. Namely, $\eta_s(x)$ represents how many particles stand at the site x and at the time s for some particle system $\{\eta_t\,:\,t\geq 0\}$. In this paper, we are concerned with a standard interacting particle system, the *exclusion process*. Succinctly, the exclusion process consists in a system of random walks evolving on a lattice under the rule that a particle can not jump to an already occupied site. This is the so-called *exclusion rule*. Such model is of great importance in Probability and Statistical Mechanics for several reasons. At the same time it has a simple interaction among particles but its peculiarities allow to prove deep results which are shared by many other models.

We consider here one-dimensional speed change exclusion processes. The dynamics of these process can be informally described as follows. A Poisson clock is associated to each bond of the lattice, the parameter of which is given by a function W of the position of the bond, in the same way as considered in [3, 4, 5, 6]. When a clock rings the occupation variables at the bonds are exchanged. The system is taken to start from the equilibrium state, which consists in a Bernoulli product measure with constant parameter. Our main result is the derivation of a functional central limit for the occupation time, when suitably re-scaled.

There is a vast literature on the fluctuations of the occupation time of symmetric particle systems, see for instance [8, 13, 14] and references therein. In this paper we follow the approach proposed in [8], which consists in replacing the occupation time functional by an additive functional of the density of particles. Then, as a consequence of the Central Limit Theorem for the density of particles, we deduce the corresponding result for the occupation time functional. We consider exclusion processes with speed change for which the Central Limit Theorem for the density

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of particles has been derived [2]. Therefore, to complete our goals we just need to justify the proper replacement of the aforementioned functionals. For that purpose, we introduce what we call a *Local Replacement* which allows to substitute the occupation time functional by an additive functional of the empirical average of particles on a small macroscopic box. This Local Replacement avoids performing a multi-scale analysis in order to derive a second order Boltzmann Gibbs Principle as in [8]. More than that, we do not require sharp bounds on the spectral gap, nor the boundedness of the jump rates of the system, as required in [8]. Therefore, our results are true for a general class of exclusion processes, for which the methods of [8] do not apply directly. On the other hand, our results are not as general as the results of [8], since they only hold for the occupation time functional and no other additive functional. We believe that our method can be extended to more general dynamics than the exclusion constrain, but this is left for future work.

We present here some particular cases of interest. First, we consider porous media models which were analyzed in [9] and correspond to taking W as the identity function. These models do not satisfy the spectral gap bound required in [8] but with our method we obtain the fractional Brownian Motion ruling the fluctuations of the occupation time. Second, we consider *exclusion processes with a slow bond* which were analyzed in [5]. These models do not satisfy the boundedness of the jump rates as required in [8], but our method also fits these models. We remark that *exclusion processes with a slow bond* is an interesting example of a particle system for which the fluctuations of the current and the fluctuations of the occupation time have completely different behaviors. This shows that, in general, the fluctuations of the current and of the occupation time are not of same type.

This paper has the following outline. In Section 2 we define our models and we state our main result, namely Theorem 2.2. In Section 3, we recall the hydrodynamic limit and the fluctuations of the density from [6] and [2], respectively. In Section 4, we prove our main result. Section 5 is devoted to examples: porous media models and exclusion processes with a slow bond. In the Appendix we present some technical lemmas.

2. The main result

Denote by $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0,1)$ the one-dimensional continuous torus, and by $\mathbb{T}_n = \mathbb{Z}/n\mathbb{Z} = \{0,\dots,n-1\}$ the one-dimensional discrete torus with n points.

Fix $W : \mathbb{R} \to \mathbb{R}$ a strictly increasing right continuous function with left limits (càdlàg), periodic in the sense that, for all $u \in \mathbb{R}$,

$$W(u+1) - W(u) = W(1) - W(0). \tag{2.1}$$

Consider the state space $\Omega_n := \{0,1\}^{\mathbb{T}_n}$. The *speed change exclusion process with conductances* is the Markov process $\{\eta_t : t \geq 0\}$ whose infinitesimal generator acts on local functions $f : \Omega_n \to \mathbb{R}$ as

$$(\mathscr{L}_n f)(\eta) = \sum_{x\in\mathbb{T}_n} \xi^n_{x,x+1}\, c_{x,x+1}(\eta)\, [f(\eta^{x,x+1}) - f(\eta)]\,,$$

where $\eta^{x,x+1}$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(x+1)$:

$$(\eta^{x,x+1})(y) = \begin{cases} \eta(x+1), & \text{if } y = x\,, \\ \eta(x), & \text{if } y = x+1\,, \\ \eta(y), & \text{otherwise,} \end{cases} \tag{2.2}$$

the conductantes $\xi^n_{x,x+1}$ are given by

$$\xi^n_{x,x+1} \;=\; \frac{1}{n\big(W(\frac{x+1}{n}) - W(\frac{x}{n})\big)}$$

and

$$c_{x,x+1}(\eta) \;=\; 1 + b(\eta(x-1) + \eta(x+2)),$$

with $b > -1/2$. The motivation for the choice of the conductances given as the inverse of the discrete derivative of W is explained in [6]. Under this choice the hydrodynamic limit can be obtained, the hydrodynamics being governed by a partial differential equations, which depends on W, see [1, 2, 6].

We remark that due to the choice of the state space, in all the formulas above if $x = n-1$ then $x+1 = 0$.

Throughout this paper, we assume the following technical condition on the function W: for any $n \in \mathbb{N}$ and any small $\varepsilon > 0$, there exists a constant $\theta > 0$ such that

$$\frac{1}{\varepsilon n}\sum_{y=0}^{\varepsilon n-1}\big(W(\tfrac{y}{n}) - W(0)\big) \sim O(\varepsilon^\theta), \tag{2.3}$$

where $f \sim O(g)$ means that the function f is bounded from above by a constant times the function g. Above, it is assumed that the constant θ does not depend on $n \in \mathbb{N}$.

To exemplify the assumption (2.3), if W is a θ-Hölder function in a neighborhood of zero, then (2.3) is satisfied, since for any $n \in \mathbb{N}$

$$\frac{1}{\varepsilon n}\sum_{y=0}^{\varepsilon n-1}\big(W(\tfrac{y}{n}) - W(0)\big) \le \frac{C_W}{\varepsilon n}\sum_{y=0}^{\varepsilon n-1}\frac{y^\theta}{n^\theta} \le \frac{C_W}{\varepsilon n}\sum_{y=0}^{\varepsilon n-1}\varepsilon^\theta = C_W\varepsilon^\theta\,,$$

where C_W is the Hölder constant.

The dynamics of the process $\{\eta_t : t \ge 0\}$ can be informally described as follows. At each bond $\{x, x+1\}$ of $\mathbb{T}_n$, there is an exponential clock of parameter $\xi^n_{x,x+1}$, all of them being independent. Suppose the configuration at the present is η. After a ring of the clock at the bond $\{x, x+1\}$, the occupation variables $\eta(x)$ and $\eta(x+1)$ are exchanged at rate $c_{x,x+1}(\eta)$.

We remark that the condition $b > -1/2$ is required to ensure that the system is ergodic in the following sense. First, we notice that the dynamics introduced above conserves the total number of particles. Therefore, the state space of the process can be written as $\Omega_n := \bigcup_{k=0}^n \mathscr{H}_{n,k}$, where $\mathscr{H}_{n,k}$ denotes the hyperplane of configurations in Ω_n with k particles. The ergodicity property means that on each hyperplane, with positive probability, we can reach any configuration in the same hyperplane using the allowed jumps of the dynamics. For instance, if $b = -1/2$ and for a configuration η having the sites $x-1$, x, $x+2$ occupied, and the site $x+1$ empty, then $c_{x,x+1}(\eta) = 1 + 2b = 0$. Then, for this choice of b there are blocked configurations, that is, configurations that do not evolve under the dynamics. Therefore, the system is not ergodic, in the sense given above.

Also, it is well known that the Bernoulli product measures on Ω_n with parameter $\rho \in [0,1]$, denoted by $\{\nu_\rho : 0 \le \rho \le 1\}$, are invariant for the dynamics introduced above. Moreover, they are also reversible.

Fix $T > 0$ and $\rho \in (0,1)$. The trajectories of $\{\eta_t : t \ge 0\}$ live on the space $\mathscr{D}([0,T], \Omega_n)$, that is, the path space of càdlàg trajectories with values in Ω_n. For a measure ν_ρ on Ω_n, we denote by $\mathbb{P}_{\nu_\rho}$ the probability measure on $\mathscr{D}([0,T],\Omega_n)$ induced by ν_ρ and by $\{\eta_t : t \geq 0\}$ and we denote by $\mathbb{E}_\rho$ the expectation with respect to $\mathbb{P}_{\nu_\rho}$.

Let $\mathscr{C}([0,T],\mathbb{R})$ be the path space of continuous trajectories with values in $\mathbb{R}$.

Definition 2.1. *The occupation time of the origin is defined as the additive functional*

$$\Gamma_n(t) := \frac{1}{n^{3/2}} \int_0^{tn^2} (\eta_s(0) - \rho)\, ds. \tag{2.4}$$

The definition above has already the correct scaling in terms of n, in order to $\Gamma_n(t)$ have a non trivial limit when taking n to infinity. The occupation time at a site $x \in \mathbb{T}_n$ is defined as above by replacing $\eta_s(0)$ by $\eta_s(x)$.

Our main result is the following

Theorem 2.2. *(Fluctuations of the occupation time)*
As n goes to infinity, the sequence of processes $\{\Gamma_n(t) : t \in [0,T]\}_{n\in\mathbb{N}}$ converges in distribution, with respect to the uniform topology of $\mathscr{C}([0,T],\mathbb{R})$, to a Gaussian process $\{\Gamma(t) : t \in [0,T]\}$.

Remark 2.3. We notice that the previous result also holds for the occupation time of any site $x \in \mathbb{T}_n$, by replacing condition (2.3) for

$$\frac{1}{\varepsilon n} \sum_{y=x}^{x+\varepsilon n-1} \big(W(\tfrac{y}{n}) - W(\tfrac{x}{n})\big) \sim O(\varepsilon^\theta). \tag{2.5}$$

For ease of notation we opt to present the result for $x = 0$.

3. Scaling limits: hydrodynamics and fluctuations

In this section we review the hydrodynamic limit and the equilibrium fluctuations of the density, for the models introduced above.

3.1. **Hydrodynamic Limit.** In words, the hydrodynamic limit consists in the analysis of the time evolution of the spatial density of particles. This spatial density of particles is represented by the empirical measure process $\pi_t^n(\eta, du) := \pi^n(\eta_t, du)$ defined, for $t \in [0,T]$, by

$$\pi^n(\eta_t, du) \;=\; \tfrac{1}{n} \sum_{x\in\mathbb{T}_n} \eta_{tn^2}(x)\delta_{\frac{x}{n}}(du) \;\in\; \mathscr{M},$$

where δ_y is the Dirac measure concentrated on $y \in \mathbb{T}$. Above, $\mathscr{M}$ denotes the space of positive measures on $\mathbb{T}$ with total mass bounded by one, endowed with the weak topology. To uniquely characterize the time evolution of the empirical measure, some condition must be imposed on the starting measures. This is the content of next definition.

Definition 3.1. *A sequence of probability measures $\{\mu_n\}_{n\in\mathbb{N}}$, where μ_n is a probability measure on Ω_n, is said to be associated to a profile $\psi_0 : \mathbb{T} \to [0,1]$, if for every $\delta > 0$ and every continuous function $H : \mathbb{T} \to \mathbb{R}$*

$$\lim_{n\to\infty} \mu_n \Big\{ \eta \in \Omega_n \;:\; \Big| \tfrac{1}{n} \sum_{x\in\mathbb{T}_n} H(\tfrac{x}{n})\, \eta(x) - \int_{\mathbb{T}} H(u)\, \psi_0(u) du \Big| > \delta \Big\} \;=\; 0. \tag{3.1}$$

In [6] it was proved that:

Theorem 3.2. *Fix a continuous profile $\psi_0 : \mathbb{T} \to [0,1]$. Let $\{\mu_n\}_{n\geq 1}$ be a sequence of probability measures associated to ψ_0. Then, for any $t \in [0,T]$, for every $\delta > 0$ and every continuous function $H : \mathbb{T} \to \mathbb{R}$, it holds that*

$$\lim_{n\to\infty} \mathbb{P}_{\mu_n}\Big\{\eta_\cdot : \Big|\tfrac{1}{n}\sum_{x\in\mathbb{T}_n} H(\tfrac{x}{n})\,\eta_{tn^2}(x) - \int_{\mathbb{T}} H(u)\,\psi(t,u)du\Big| > \delta\Big\} \;=\; 0\,,$$

where $\psi : [0,T]\times\mathbb{T} \to \mathbb{R}$ is the unique weak solution of

$$\begin{cases} \partial_t \psi \;=\; \mathfrak{L}_W \psi\,, \\ \psi(0,u) \;=\; \psi_0(u)\,, \forall u \in \mathbb{T}_n\,. \end{cases} \tag{3.2}$$

The operator $\mathfrak{L}_W$ is defined in next subsection, as well as the notion of weak solution of (3.2).

In order to state properly what is a weak solution of (3.2) we need to introduce some definitions.

3.2. **The operator $\mathfrak{L}_W$.** We detail here the operator $\mathfrak{L}_W : \mathfrak{D}_W \subset L^2(\mathbb{T}) \to L^2(\mathbb{T})$. We start by defining its domain $\mathfrak{D}_W$. For that purpose, we consider $W(dy)$ as the measure on the continuous torus associated to the function $W : \mathbb{R} \to \mathbb{R}$ in the usual way, or else, as the unique measure such that

$$W((a,b]) := W(b) - W(a) \quad \forall\, a,b \in \mathbb{T} \text{ with } a < b. \tag{3.3}$$

Notice that the periodicity condition given in (2.1) guarantees that the measure above is well defined.

The domain $\mathfrak{D}_W$ consists on the set of functions G in $L^2(\mathbb{T})$ such that

$$G(u) = a + b\,W(u) + \int_{(0,u]} \Big(\int_0^y g(z)\,dz\Big) W(dy), \quad \forall\, u \in \mathbb{T},$$

for some function g in $L^2(\mathbb{T})$ that satisfies

$$\int_0^1 g(z)\,dz = 0 \qquad \text{and} \qquad \int_{(0,1]} \Big(b + \int_0^y g(z)\,dz\Big) W(dy) = 0.$$

The operator $\mathfrak{L}_W$ acts on $G \in \mathfrak{D}_W$ as $\mathfrak{L}_W G = g$. An alternative definition of the operator can be stated in the following way. Denote by ∂_u the usual space derivative and define the generalized derivative ∂_W of a function $G : \mathbb{T} \to \mathbb{R}$ by

$$\partial_W G(u) = \lim_{\varepsilon\to 0} \frac{G(u+\varepsilon) - G(u)}{W(u+\varepsilon) - W(u)}, \tag{3.4}$$

when the above limit exists and is finite. Keeping this in mind, given $G \in \mathfrak{D}_W$, we have $\mathfrak{L}_W G(u) = \partial_u \partial_W G(u)$, for all $u \in \mathbb{T}$.

This operator $\mathfrak{L}_W$ is a Krein-Feller type operator (see e.g. [7] on the subject). In [6], it was proved that $\mathfrak{L}_W$ satisfies the properties stated in the ensuing theorem. Below $\langle\langle \cdot,\cdot \rangle\rangle$ denotes the inner product in $L^2(\mathbb{T})$ and $\|\cdot\|$ the corresponding norm.

Theorem 3.3. *There exists an Hilbert space $\mathscr{H}^1_W$ compactly embedded in $L^2(\mathbb{T})$ such that $\mathfrak{D}_W \subset \mathscr{H}^1_W$ and $\mathfrak{L}_W$ can be extended to $\mathscr{H}^1_W$ such that the extension enjoys the following properties:*

(a) *The domain $\mathscr{H}^1_W$ is dense in $L^2(\mathbb{T})$;*
(b) *The operator $\mathfrak{L}_W$ is self-adjoint and non-positive $\langle\langle H, -\mathfrak{L}_W H\rangle\rangle \geq 0$, for all $H \in \mathscr{H}^1_W$;*

(c) *Let $\mathbb{I}$ be the identity operator. The operator $\mathbb{I} - \mathfrak{L}_W : \mathscr{H}_W^1 \to L^2(\mathbb{T})$ is bijective and $\mathfrak{D}_W$ is a core of it;*
(d) *The operator $\mathfrak{L}_W$ is dissipative, i.e., $\|\mu H - \mathfrak{L}_W H\| \geq \mu \|H\|$, for some $\mu > 0$ and for all $H \in \mathscr{H}_W^1$;*
(e) *The eigenvalues of $-\mathfrak{L}_W$ form a countable set $0 = \mu_0 \leq \mu_1 \leq \mu_2 \leq \cdots$ with $\lim_{n\to\infty} \mu_n = \infty$, and all of them have finite multiplicity;*
(f) *There exists a complete orthonormal basis of $L^2(\mathbb{T})$ composed of eigenfunctions φ_n of $-\mathfrak{L}_W$ associated to the eigenvalues μ_n.*

In view of (a), (c) and (d), by the Hille-Yosida Theorem, $\mathfrak{L}_W$ is the generator of a strongly continuous contraction semigroup in $L^2(\mathbb{T})$.

Finally, we state what is meant to be a weak solution to (3.2).

Definition 3.4. *A bounded function $\psi : [0,T] \times \mathbb{T} \to \mathbb{R}$ is said to be a weak solution of the parabolic differential equation* (3.2) *if, for any $t \in [0,T]$ and any $H \in \mathscr{H}_W^1$, the function $\psi(t,\cdot)$ satisfies the integral equation*

$$\int_{\mathbb{T}} \psi(t,u) H(u)\, du \;-\; \int_{\mathbb{T}} \psi(0,u) H(u)\, du \;-\; \int_0^t \int_{\mathbb{T}} \psi(s,u) \mathfrak{L}_W H(u)\, du\, ds \;=\; 0.$$

3.3. **Equilibrium fluctuations and the generalized Ornstein-Uhlenbeck process.** Following the ideas of [2], we define $S_W(\mathbb{T}) = \bigcap_{n=0}^{\infty} S_n$, where S_n is the Hilbert space obtained by completing the space $\mathfrak{D}_W$ with respect to the inner product $\langle \cdot, \cdot \rangle_n$ given by

$$\langle f, g \rangle_n = \sum_{k=1}^{\infty} (1+\mu_k)^{2n} k^{2n} \int_{\mathbb{T}} P_k f(u) P_k g(u) du, \tag{3.5}$$

where P_k is the orthogonal projection on the linear space generated by the eigenfunction φ_k given in Theorem 3.3. Let $S'_W(\mathbb{T})$ denote the dual space of $S_W(\mathbb{T})$, that is, the space of the bounded linear functionals from $S_W(\mathbb{T})$ to $\mathbb{R}$.

We define the density fluctuation field, which is an element of $S'_W(\mathbb{T})$, as the linear functional acting on functions $H \in S_W(\mathbb{T})$ as

$$\mathscr{Y}_t^n(H) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{T}_n} H\Big(\frac{x}{n}\Big) \Big(\eta_{tn^2}(x) - \rho\Big). \tag{3.6}$$

We will use the more compact notation $\bar{\eta}(x)$ to denote $\eta(x) - \rho$. The equilibrium density fluctuations for these models was proved in Theorem 2.1 of [2] and is stated as follows. Denote by $\mathscr{D}([0,T], S'_W(\mathbb{T}))$ the path space of càdlàg trajectories with values in $S'_W(\mathbb{T})$.

Theorem 3.5. *As n goes to infinity, the sequence $\{\mathscr{Y}_t^n : t \in [0,T]\}_{n\in\mathbb{N}}$ converges, in the Skorohod topology of $\mathscr{D}([0,T], S'_W(\mathbb{T}))$, to $\{\mathscr{Y}_t : t \in [0,T]\}$ the generalized Ornstein-Uhlenbeck process which is the stationary solution of the stochastic partial differential equation given by*

$$d\mathscr{Y}_t = \tilde{c}\,'(\rho) \mathfrak{L}_W \mathscr{Y}_t dt + \sqrt{2\chi(\rho)\tilde{c}\,'(\rho)}\, d\mathscr{B}_t, \tag{3.7}$$

where $\chi(\rho) = \rho(1-\rho)$, $\tilde{c}\,'(\rho) = 1 + 2b\rho$ and $\mathscr{B}_t$ is a $S'_W(\mathbb{T})$-valued Brownian motion with quadratic variation given by

$$\langle \mathscr{B}(H) \rangle_t = t \int_{\mathbb{T}} (\partial_W H(x))^2\, W(dx).$$

4. Proof of Theorem 2.2

The proof of this theorem relies on two steps. First, we claim that the occupation time is close to an additive functional of the density fluctuation field $\mathscr{Y}_t^n$, this is what we called the Local Replacement. Second, we use Theorem 3.5 to prove that the additive functional of the density fluctuation field $\mathscr{Y}_t^n$ converges to a Gaussian process. Before proving these two claims we develop some crucial estimates that we need in due course.

4.1. **The Local Replacement.** For a function $g \in L^2(\nu_\rho)$, we denote by $\mathscr{D}_n(g)$ the Dirichlet form of the function g, defined as: $\mathscr{D}_n(g) = -\int_{\Omega_n} g(\eta)\mathscr{L}_n g(\eta)\nu_\rho(d\eta)$. An elementary computation shows that

$$\mathscr{D}_n(g) = \sum_{x\in\mathbb{T}_n} \frac{\xi^n_{x,x+1}}{2}\int_{\Omega_n} c_{x,x+1}(\eta)\Big(g(\eta^{x,x+1}) - g(\eta)\Big)^2 \nu_\rho(d\eta). \tag{4.1}$$

Lemma 4.1 (*Local Replacement*).
For any $\ell \geq 1$, for any $n \geq 1$ and $t \in [0,T]$, it holds that

$$\mathbb{E}_\rho\Big[\Big(\int_0^t \{\bar{\eta}_{sn^2}(0) - \bar{\eta}^\ell_{sn^2}(0)\}ds\Big)^2\Big] \leq \frac{40\,t}{n^2\ell}C(\rho)\sum_{y=0}^{\ell-1}\sum_{z=0}^{y-1}\frac{1}{\xi^n_{z,z+1}},$$

where

$$\bar{\eta}^\ell_{sn^2}(0) = \frac{1}{\ell}\sum_{y=0}^{\ell-1}\bar{\eta}_{sn^2}(y)$$

and $C(\rho)$ is a positive constant.

In order to prove the last lemma, we use the following result.

Lemma 4.2. *For any $\ell \geq 1$, for any $n \geq 1$, for any $g \in L^2(\nu_\rho)$ and for a constant $A > 0$, it holds that*

$$\int_{\Omega_n}\{\bar{\eta}(0) - \bar{\eta}^\ell(0)\}g(\eta)\nu_\rho(d\eta) \leq \frac{A}{2\ell}\sum_{y=0}^{\ell-1}\sum_{z=0}^{y-1}\frac{1}{\xi^n_{z,z+1}}\int_{\Omega_n}\frac{1}{c_{z,z+1}(\eta)}\nu_\rho(d\eta) + \frac{1}{A}\mathscr{D}_n(g).$$

Proof. By the definition of the empirical average $\bar{\eta}^\ell(0)$, we can rewrite the integral on the left hand side in the statement of the lemma as

$$\frac{1}{\ell}\sum_{y=0}^{\ell-1}\sum_{z=0}^{y-1}\int_{\Omega_n}\{\eta(z) - \eta(z+1)\}g(\eta)\nu_\rho(d\eta).$$

Writing the previous expression as twice its half and performing the change of variables $\eta \mapsto \eta^{z,z+1}$, for which the measure ν_ρ is invariant, it equals to

$$\frac{1}{2\ell}\sum_{y=0}^{\ell-1}\sum_{z=0}^{y-1}\int_{\Omega_n}(\eta(z) - \eta(z+1))(g(\eta) - g(\eta^{z,z+1}))\nu_\rho(d\eta).$$

By the Cauchy-Schwarz inequality we bound the expression above by

$$\frac{1}{2\ell}\sum_{y=0}^{\ell-1}\sum_{z=0}^{y-1}\frac{1}{\xi^n_{z,z+1}}\int_{\Omega_n}\frac{A}{c_{z,z+1}(\eta)}(\eta(z) - \eta(z+1))^2\nu_\rho(d\eta)$$

$$+\frac{1}{2\ell}\sum_{y=0}^{\ell-1}\sum_{z=0}^{y-1}\xi^n_{z,z+1}\int_{\Omega_n}\frac{c_{z,z+1}(\eta)}{A}(g(\eta) - g(\eta^{z,z+1}))^2\nu_\rho(d\eta).$$

To finish the proof it is enough to recall (4.1). □

Proof of Lemma 4.1. By Proposition A1.6.1 of [12], we have that

$$\mathbb{E}_\rho\Big[\Big(\int_0^t \{\bar\eta_{sn^2}(0)-\bar\eta^\ell_{sn^2}(0)\}ds\Big)^2\Big]$$
$$\leq 20\,t \sup_{g\in L^2(\nu_\rho)}\Big\{2\int_{\Omega_n}\{\bar\eta(0)-\bar\eta^\ell(0)\}g(\eta)\nu_\rho(d\eta)-n^2\mathscr{D}_n(g)\Big\}$$
$$\leq 20\,t \sup_{g\in L^2(\nu_\rho)}\Big\{\frac{A}{\ell}\sum_{y=0}^{\ell-1}\sum_{z=0}^{y-1}\frac{1}{\xi^n_{z,z+1}}\int_{\Omega_n}\frac{1}{c_{z,z+1}(\eta)}\nu_\rho(d\eta)+\frac{2}{A}\mathscr{D}_n(g)-n^2\mathscr{D}_n(g)\Big\}.$$

In last inequality we used the previous lemma. Taking $1/A=n^2$ we get the bound

$$\frac{40\,t}{n^2\ell}\sum_{y=0}^{\ell-1}\sum_{z=0}^{y-1}\frac{1}{\xi^n_{z,z+1}}\int_{\Omega_n}\frac{1}{c_{z,z+1}(\eta)}\nu_\rho(d\eta).$$

To conclude it is enough to observe that

$$\int_{\Omega_n}\frac{1}{c_{z,z+1}(\eta)}\nu_\rho(d\eta)=(1-\rho)^2+\frac{2}{1+b}\rho(1-\rho)+\frac{1}{1+2b}\rho^2 \;:=\; C(\rho). \tag{4.2}$$

□

Corollary 4.3. *For any $\varepsilon>0$ and any $t\in[0,T]$, it holds that*

$$\mathbb{E}_\rho\Big[\Big(\int_0^t \{\bar\eta_{sn^2}(0)-\bar\eta^{\varepsilon n}_{sn^2}(0)\}ds\Big)^2\Big]\leq\frac{40\,t}{\varepsilon n^2}C(\rho)\sum_{y=0}^{\varepsilon n-1}\big(W(\tfrac{y}{n})-W(0)\big),$$

for a positive constant $C(\rho)$.

Proof. This result is a consequence of Lemma 4.1 with $\ell=\varepsilon n$ and the fact that $\xi^n_{x,x+1}=\frac{1}{n(W(\frac{x+1}{n})-W(\frac{x}{n}))}$ so that

$$\sum_{y=0}^{\varepsilon n-1}\sum_{z=0}^{y-1}\frac{1}{\xi^n_{z,z+1}}\leq n\sum_{y=0}^{\varepsilon n-1}\big(W(\tfrac{y}{n})-W(0)\big).$$

□

Corollary 4.4. *For any $\varepsilon\geq 0$, for any $n\geq 1$, for any W satisfying* (2.3) *for some $\theta>0$ and for any $t\in[0,T]$, it holds that*

$$\mathbb{E}_\rho\Big[\Big(\sqrt{n}\int_0^t \{\bar\eta_{sn^2}(0)-\bar\eta^{\varepsilon n}_{sn^2}(0)\}ds\Big)^2\Big]\;\leq\;40\,t\,C(\rho)\,\varepsilon^\theta,$$

for a positive constant $C(\rho)$.

Proof. By the previous corollary, the expectation above is bounded from above by

$$\frac{40\,t}{\varepsilon n}C(\rho)\sum_{y=0}^{\varepsilon n-1}\big(W(\tfrac{y}{n})-W(0)\big),$$

and by the assumption (2.3) last term is smaller than $40\,t\,C(\rho)\,\varepsilon^\theta$, where $C(\rho)$ is a constant. □

At this point we are able to use the Local Replacement in order to prove that the occupation time is close to an additive functional of the density of particles. For that purpose, for $\varepsilon \in (0,1)$ we denote by ι_ε the function $y \mapsto \varepsilon^{-1}\mathbf{1}_{[0,\varepsilon]}(y)$. The sequence $\{\iota_\varepsilon\,;\, \varepsilon \in (0,1)\}$ is therefore an *approximation of the identity*.

Proposition 4.5. *Fix $t > 0$. For any $\varepsilon \geq 0$ and for any $n \geq 1$, it holds that*

$$\mathbb{E}_\rho\Big[\Big(\Gamma_n(t) - \int_0^t \mathscr{Y}_s^n(\iota_\varepsilon)ds\Big)^2\Big] \leq Ct\varepsilon^\theta, \tag{4.3}$$

for a positive constant C.

Proof. Observe that

$$\begin{aligned}\Gamma_n(t) - \int_0^t \mathscr{Y}_s^n(\iota_\varepsilon)ds =& \frac{1}{n^{3/2}}\int_0^{tn^2} \bar{\eta}_s(0)ds - \int_0^t \frac{1}{\varepsilon\sqrt{n}}\sum_{x=0}^{\varepsilon n} \bar{\eta}_{sn^2}(x)ds\\ =& \sqrt{n}\int_0^t \Big(\bar{\eta}_{sn^2}(0) - \bar{\eta}_{sn^2}^{\varepsilon n}(0)\Big)ds.\end{aligned}$$

In the first equality we used the definitions of $\Gamma_n(t)$ and $\mathscr{Y}_s^n$ given, respectively, in (2.4) and (3.6) and the definition of ι_ε given above. In the second equality, we used a change of variables in the time integral. Now, it is enough to recall Corollary 4.4 in order to finish the proof. □

4.2. **The approximation in the $S_W(\mathbb{T})$ space.** So far, we were able to show that the occupation time is close to the additive functional of the density of particles evaluated at the function ι_ε. We would like to invoke Theorem 3.5 in order to assure the convergence of the density fluctuation field $\mathscr{Y}_t^n$ to some process $\mathscr{Y}_t$, as n tends to infinity. However, the function ι_ε does not belong to the space of test functions $S_W(\mathbb{T})$, therefore, we can not apply directly the Theorem 3.5 to $\mathscr{Y}_t^n(\iota_\varepsilon)$. To overcome this problem, we approximate first the function ι_ε by a sequence of functions $\{\iota_\varepsilon^k\}_{k\in\mathbb{N}}$ in the space of test functions $S_W(\mathbb{T})$. This is the content of the next lemma.

Denote by $\mathbf{1}_A(u)$ the function that takes the value 1 if $u \in A$ and 0 if $u \notin A$.

Lemma 4.6. *For fixed $\varepsilon \in (0,1)$, there exists a sequence of functions $\{\iota_\varepsilon^k\}_{k\in\mathbb{N}}$ in the space of test functions $S_W(\mathbb{T})$ converging to ι_ε in the $L^2(\mathbb{T})$-norm, as k tends to infinity.*

Proof. In fact, we are going to approximate the function ι_ε by a sequence of functions on the space $\mathfrak{D}_W$, which is a subset of $S_W(\mathbb{T})$, as defined in Subsection 3.3.

Define

$$\iota_\varepsilon^k(u) := \int_{(0,u]}\Big(\int_0^y g_\varepsilon^k(z)\,dz\Big)W(dy), \quad \forall\, u \in \mathbb{T}, \tag{4.4}$$

where the function $g_\varepsilon^k(z) \in L^2(\mathbb{T})$ is given by

$$g_\varepsilon^k(z) := c_\varepsilon^{k,+}g_\varepsilon^{k,+}(z) - c_\varepsilon^{k,-}g_\varepsilon^{k,-}(z), \quad \forall\, z \in \mathbb{T},$$

with

$$
\begin{aligned}
g_\varepsilon^{k,+}(z) &:= k\Big[\mathbf{1}_{(0,1/k]}(z) - \mathbf{1}_{(1/k,2/k]}(z)\Big], \quad \forall\, z \in \mathbb{T},\\
g_\varepsilon^{k,-}(z) &:= k\Big[\mathbf{1}_{(\varepsilon-1/k,\varepsilon]}(z) - \mathbf{1}_{(\varepsilon,\varepsilon+1/k]}(z)\Big], \quad \forall\, z \in \mathbb{T},\\
c_\varepsilon^{k,+} &:= \frac{1}{\varepsilon}\left(\int_{(0,1]} \Big(\int_0^y g_\varepsilon^{k,+}(z)\, dz\Big) W(dy)\right)^{-1},\\
c_\varepsilon^{k,-} &:= \frac{1}{\varepsilon}\left(\int_{(0,1]} \Big(\int_0^y g_\varepsilon^{k,-}(z)\, dz\Big) W(dy)\right)^{-1}.
\end{aligned}
$$

We consider $k \in \mathbb{N}$ such that $k > 1/\varepsilon$, in order that the formulas above make sense. We claim that

$$
\int_0^1 g_\varepsilon^k(z)\, dz = 0 \qquad \text{and} \qquad \int_{(0,1]} \Big(\int_0^y g_\varepsilon^k(z)\, dz\Big) W(dy) = 0.
$$

The first equality above follows from the fact that $\int_0^1 g_\varepsilon^{k,+}(z)dz = \int_0^1 g_\varepsilon^{k,-}(z)dz = 0$, which can be easily checked. The second equality follows from a simple computation.

Under the choice of g_ε^k, the function ι_ε^k defined in (4.4) has the following behavior: for $u \in (2/k, \varepsilon - 1/k]$, $\iota_\varepsilon^k(u)$ is equal to ε^{-1} and for $u \in (\varepsilon + 1/k, 1]$, $\iota_\varepsilon^k(u)$ vanishes. Therefore, for each $k \in \mathbb{N}$, the function ι_ε^k differs from ι_ε only on the set $(0, 2/k] \cup (\varepsilon - 1/k, \varepsilon + 1/k]$.

Since $|\iota_\varepsilon^k - \iota_\varepsilon|$ is bounded by a constant that does not depend on k, the Dominated Convergence Theorem implies that ι_ε^k converges to ι_ε in $L^2(\mathbb{T})$, as k goes to infinity, concluding the proof. □

4.3. **The Gaussian limit.** At this point we have all the needed ingredients in order to prove our main result, namely, Theorem 2.2. In this subsection, we follow the ideas from the proof of the Theorem 2.9 of [8].

We know that the occupation time is close to the additive functional of the density of particles evaluated on ι_ε, which in turn can be very well approximated by the additive functional of the density of particles evaluated on a function in the space of test functions $S_W(\mathbb{T})$. At this point, we can take the limit as n tends to infinity, because, by Theorem 3.5, the convergence of $\mathscr{Y}_t^n(H)$ to $\mathscr{Y}_t(H)$ holds for any $H \in S_W(\mathbb{T})$.

Next, we prove that the additive functional associated to $\mathscr{Y}_t(\iota_\varepsilon)$ converges, as ε tends to 0, to a Gaussian process. For that purpose, define

$$
\tilde{\Gamma}_\varepsilon(t) = \int_0^t \mathscr{Y}_s(\iota_\varepsilon) ds, \tag{4.5}
$$

where $\mathscr{Y}_t$ is the Ornstein-Uhlenbeck process given in (3.7).

Remark 4.7. We point out that definition above is, in principle, not well defined since ι_ε does not belong to the space $S_W(\mathbb{T})$. To handle that, it is necessary to look at the limit of Cauchy sequences $\{\mathscr{Y}_t^n(\iota_\varepsilon^k)\}_{k\in\mathbb{N}}$, where $\{\iota_\varepsilon^k\}_{k\in\mathbb{N}}$ is given in Lemma 4.6. By the convergence of $\mathscr{Y}_t^n(\iota_\varepsilon^k)$ towards $\mathscr{Y}_t(\iota_\varepsilon^k)$ as n goes to infinity, and the fact that $\{\mathscr{Y}_t^n(\iota_\varepsilon^k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in k (uniformly in n), a diagonal argument leads to a precise definition of $\tilde{\Gamma}_\varepsilon(t)$. This was very well detailed in [5] or [8] and to keep the present text short, we ask the reader to accept (4.5) or to go into the details in these references.

The next lemma characterizes, for fixed t, the dependency of $\tilde{\Gamma}_\varepsilon(t)$ on $\varepsilon > 0$.

Lemma 4.8. *For any fixed $t \in [0, T]$ and any $\varepsilon > \delta > 0$,*

$$\mathbb{E}\big[\big(\tilde{\Gamma}_\varepsilon(t) - \tilde{\Gamma}_\delta(t)\big)^2\big] \le C\varepsilon^\theta t, \tag{4.6}$$

where $C > 0$ is some constant that does not depend on ε nor δ.

Proof. Fix $\varepsilon > \delta > 0$. Repeatedly applying the inequality $(x + y)^2 \le 2(x^2 + y^2)$, we bound the expectation in (4.6) by four times the sum of

$$\mathbb{E}\Big[\Big(\tilde{\Gamma}_\varepsilon(t) - \int_0^t \mathscr{Y}^n_s(\iota_\varepsilon)ds\Big)^2\Big], \tag{4.7}$$

$$\mathbb{E}\Big[\Big(\Gamma_n(t) - \int_0^t \mathscr{Y}^n_s(\iota_\varepsilon)ds\Big)^2\Big], \tag{4.8}$$

$$\mathbb{E}\Big[\Big(\Gamma_n(t) - \int_0^t \mathscr{Y}^n_s(\iota_\delta)ds\Big)^2\Big], \tag{4.9}$$

and

$$\mathbb{E}\Big[\Big(\tilde{\Gamma}_\delta(t) - \int_0^t \mathscr{Y}^n_s(\iota_\delta)ds\Big)^2\Big]. \tag{4.10}$$

The term in (4.8) can be estimated by using Proposition 4.5, from where we get that

$$\mathbb{E}\Big[\Big(\Gamma_n(t) - \int_0^t \mathscr{Y}^n_s(\iota_\varepsilon)ds\Big)^2\Big] \le C\varepsilon^\theta t.$$

Analogously, for (4.9), we have

$$\mathbb{E}\Big[\Big(\Gamma_n(t) - \int_0^t \mathscr{Y}^n_s(\iota_\delta)ds\Big)^2\Big] \le C\delta^\theta t < C\varepsilon^\theta t.$$

The next step is to guarantee that (4.7) is arbitrarily small for large n. We do the following. By Lemma 4.6 there exists a sequence of functions $\{\iota^k_\varepsilon\}_{k\in\mathbb{N}}$ in the space of test functions $S_W(\mathbb{T})$ approximating the function ι_ε in the $L^2(\mathbb{T})$-norm, as k tends to infinity. By adding and subtracting terms, we bound (4.7) by four times the sum of the terms below:

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\tilde{\Gamma}_\varepsilon(t) - \int_0^t \mathscr{Y}_s(\iota^k_\varepsilon)ds\Big)^2\Big],\\
&\mathbb{E}\Big[\Big(\int_0^t \mathscr{Y}_s(\iota^k_\varepsilon)ds - \int_0^t \mathscr{Y}^n_s(\iota^k_\varepsilon)ds\Big)^2\Big],\\
&\mathbb{E}\Big[\Big(\int_0^t \mathscr{Y}^n_s(\iota^k_\varepsilon)ds - \int_0^t \mathscr{Y}^n_s(\iota_\varepsilon)ds\Big)^2\Big].
\end{aligned} \tag{4.11}$$

The first expectation in (4.11) can be estimated by using the linearity of $\mathscr{Y}_t$ together with Lemma 6.1 (postponed to the appendix), from where we get that

$$\mathbb{E}\Big[\Big(\int_0^t [\mathscr{Y}_s(\iota_\varepsilon) - \mathscr{Y}_s(\iota^k_\varepsilon)]ds\Big)^2\Big] = t^2\chi(\rho)\int_{\mathbb{T}} (\iota_\varepsilon(u) - \iota^k_\varepsilon(u))^2 du.$$

By Lemma 4.6, the right hand-side of the previous equality goes to zero, as k goes to infinity.

The second expectation in (4.11) goes to zero, as n tends to infinity, as a consequence of the Theorem 3.5 and of Lemma 4.6.

To bound the third expectation in (4.11) we apply the Cauchy-Schwarz inequality, leading to

$$\mathbb{E}\Big[\Big(\int_0^t [\mathscr{Y}_s^n(\iota_\varepsilon^k) - \mathscr{Y}_s^n(\iota_\varepsilon)]ds\Big)^2\Big] \leq t^2 \frac{1}{n}\sum_{x\in\mathbb{T}_n}(\iota_\varepsilon^k - \iota_\varepsilon)^2\Big(\frac{x}{n}\Big)\chi(\rho).$$

Taking n sufficiently large, the right hand-side of the previous expression is close to

$$t^2\int_{\mathbb{T}}(\iota_\varepsilon^k(u) - \iota_\varepsilon(u))^2du\,\chi(\rho)$$

and again by Lemma 4.6, this expression is small for k sufficiently big.

Expression (4.10) can be treated in same way as (4.7), finishing the proof of the lemma. □

Proposition 4.9. *As ε goes to zero, the sequence of processes $\{\tilde{\Gamma}_\varepsilon(t) : t \in [0, T]\}_{\varepsilon>0}$ converges in distribution, with respect to the uniform topology of $\mathscr{C}([0, T], \mathbb{R})$, to a Gaussian process $\{\tilde{\Gamma}(t) : t \in [0, T]\}$.*

Proof. We begin by claiming that

$$\mathbb{E}\Big[\Big(\tilde{\Gamma}_\varepsilon(t)\Big)^2\Big] \leq t^2\frac{\chi(\rho)}{\varepsilon}. \tag{4.12}$$

By the Cauchy-Schwarz inequality,

$$\mathbb{E}\Big[\Big(\tilde{\Gamma}_\varepsilon(t)\Big)^2\Big] \leq\ t\,\mathbb{E}\Big[\int_0^t(\mathscr{Y}_s(\iota_\varepsilon))^2ds\Big].$$

By Fubini's Theorem and Lemma 6.1 we get that

$$\mathbb{E}\Big[\Big(\tilde{\Gamma}_\varepsilon(t)\Big)^2\Big] \leq t\int_0^t \mathbb{E}\Big[(\mathscr{Y}_s(\iota_\varepsilon))^2\Big]ds = t^2\chi(\rho)\int_{\mathbb{T}}(\iota_\varepsilon(u))^2du = t^2\frac{\chi(\rho)}{\varepsilon},$$

proving the claim. We observe that Lemma 6.1 is stated only for functions in the space $S_W(\mathbb{T})$. Nevertheless, an aproximation procedure in L^2 as described in the Remark 4.7 extends the statement of the Lemma 6.1 for ι_ε as well.

Fix $\varepsilon > 0$. For $\delta < \varepsilon$, applying (4.6) and (4.12) we have that

$$\begin{aligned}\mathbb{E}\Big[\Big(\tilde{\Gamma}_\delta(t)\Big)^2\Big] &= \mathbb{E}\Big[\Big(\tilde{\Gamma}_\delta(t) - \tilde{\Gamma}_\varepsilon(t) + \tilde{\Gamma}_\varepsilon(t)\Big)^2\Big]\\ &\leq 2\mathbb{E}\Big[\Big(\tilde{\Gamma}_\delta(t) - \tilde{\Gamma}_\varepsilon(t)\Big)^2\Big] + 2\mathbb{E}\Big[\Big(\tilde{\Gamma}_\varepsilon(t)\Big)^2\Big]\\ &\leq 2C\varepsilon^\theta t + 2t^2\frac{\chi(\rho)}{\varepsilon}.\end{aligned} \tag{4.13}$$

If $t \geq \delta^{1+\theta}$, taking $\varepsilon = t^{1/1+\theta}$ we conclude that

$$\mathbb{E}\Big[\Big(\tilde{\Gamma}_\delta(t)\Big)^2\Big] \leq Ct^{\frac{1+2\theta}{1+\theta}}, \tag{4.14}$$

where C does not depend on ε nor t. On the other hand, if $t < \delta^{1+\theta}$, then $t^{\frac{1}{1+\theta}} < \delta$ and using (4.12), the previous inequality is also true. Therefore, by the stationarity of $\mathscr{Y}_t$ and since $\tilde{\Gamma}_\varepsilon(0) = 0$, we get that

$$\begin{aligned}\mathbb{E}\big[\big(\tilde{\Gamma}_\varepsilon(t) - \tilde{\Gamma}_\varepsilon(s)\big)^2\big] &= \mathbb{E}\big[\big(\tilde{\Gamma}_\varepsilon(t-s) - \tilde{\Gamma}_\varepsilon(0)\big)^2\big]\\ &= \mathbb{E}\big[\big(\tilde{\Gamma}_\varepsilon(t-s)\big)^2\big] \leq C|t-s|^{\frac{1+2\theta}{1+\theta}},\end{aligned} \tag{4.15}$$

for all $t, s \in [0, T]$. Estimate (4.15) allows to invoke Kolmogorov-Centsov's compactness criterion (see problem 2.4.11 in [10]), assuring that the sequence of processes $\{\tilde{\Gamma}_\varepsilon(t) : t \in [0, T]\}_{\varepsilon>0}$ is tight. Besides that, for fixed t, (4.6) implies that $\{\tilde{\Gamma}_\varepsilon(t)\}_{\varepsilon>0}$ is a Cauchy sequence in L^2, implying the uniqueness of limit points. This concludes the proof.

□

A final observation: since we have, in general, no manageable formula for the semigroup $\{P_t : t \geq 0\}$ associated to $\tilde{c}'(\rho)\mathfrak{L}_W$, we are not able to explicitly characterize the covariance of the Gaussian process $\{\tilde{\Gamma}(t) : t \in [0, T]\}$ obtained above (and hence we can not characterize the process itself beyond of proving its existence). In next subsection we detail two cases where the covariances can be computed explicitly.

5. Further extensions and examples

In this section we present the extension of the previous results for two models evolving in the one-dimensional lattice $\mathbb{Z}$. For that purpose we need to introduce some notation. From now on, we take the state space $\Omega := \{0, 1\}^{\mathbb{Z}}$ and the Markov process $\{\eta_t : t \geq 0\}$ with infinitesimal generator given on local functions $f : \Omega \to \mathbb{R}$ by

$$(\mathscr{L}_n f)(\eta) = \sum_{x \in \mathbb{Z}} a^n_{x,x+1}(\eta)[f(\eta^{x,x+1}) - f(\eta)],$$

where $a^n_{x,x+1}$ will be defined later accordingly to model we consider. Above, $\eta^{x,x+1}$ denotes the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(x+1)$, as in (2.2). For this process we define the occupation time of the origin $\Gamma_n(t)$ as in Definition 2.1. Below we present two examples for which we are able to characterize completely the limiting Gaussian process appearing in the statement of Theorem 2.2.

5.1. **Porous media models.** In this section we consider a collection of models whose scaling limits were studied in [9]. First, we consider the Markov process $\{\eta_t : t \geq 0\}$ with generator given by $\mathscr{L}_n$ as above with

$$a^n_{x,x+1}(\eta) := a_{x,x+1}(\eta) = 1 + b(\eta(x-1) + \eta(x+2))),$$

where $b > -1/2$. In the particular case where $b = 0$, the process becomes the well known symmetric simple exclusion process.

We notice that all the results of Section 4 are true for these models simply by rewriting the proofs adapted to the infinite volume context. First, the results in Subsection 4.1 are true for this choice of the jump rates: Lemma 4.1 holds in this case by making the simple choice of W equal to the identity function which corresponds to conductances given by $\xi_{x,x+1} = 1$ for all $x \in \mathbb{Z}$, therefore Proposition 4.5 is also true for this model. Second, to prove the results of Subsection 4.2 we define the density fluctuation field $\mathscr{Y}^n_t$ (see (3.6)) with the sum taken with $x \in \mathbb{Z}$ and on the Schwarz space of test functions, that we denote by $\mathscr{S}(\mathbb{R})$. Therefore, by the chosen space of test functions the approximation arguments of Subsection 4.2 are standard and are left to the reader. Finally, the results of Section 4.3 are also simple to check in this setting, for details we refer to [8]. We omitted the details of these proofs since they are basically a modification of notation to fit the infinite volume setting. Moreover, since the equilibrium fluctuations were proved for these models in [9], then Theorem 2.2 holds in this case with the limiting process $\{\Gamma(t) : t \geq 0\}$ being a fractional Brownian motion of Hurst exponent $H = 3/4$.

We remark that we can even take more general rates $a^n_{x,x+1}(\eta) := a_{x,x+1}(\eta)$ equal to

$$1 + b\Big(\prod_{j=-(m-1)}^{-1} \eta(x+j) + \prod_{\substack{j=-(m-2)\\ j\neq 0,1}}^{2} \eta(x+j) + \cdots + \prod_{\substack{j=-1\\ j\neq 0,1}}^{m-1} \eta(x+j) + \prod_{j=2}^{m} \eta(x+j)\Big),$$

with $b > -1/2$ and $m \in \mathbb{N}\backslash\{1\}$. Under these rates, particles more likely hop to unoccupied nearest-neighbor sites when there are at least $m-1 \geq 1$ other neighboring sites fully occupied.

Summarizing, since in [9] the equilibrium fluctuations were obtained for these models, the limit being a generalized Ornstein-Uhlenbeck process and since all the results of Section 4 are also true for these models, then we are able to show the following result.

Theorem 5.1. *As n goes to infinity, the sequence of processes $\{\Gamma_n(t) : t \in [0,T]\}_{n\in\mathbb{N}}$ converges in distribution with respect to the uniform topology of $\mathscr{C}([0,T],\mathbb{R})$ to a fractional Brownian motion of Hurst exponent $H = 3/4$.*

5.2. **Symmetric Exclusion with a slow bond.** In this section, we consider the Markov process $\{\eta_t : t \geq 0\}$ with generator given by $\mathscr{L}_n$ as above with

$$a^n_{x,x+1}(\eta) \;=\; \begin{cases} \frac{\alpha}{n^\beta}, & \text{if } x = -1, \\ 1, & \text{otherwise}, \end{cases} \tag{5.1}$$

for $\alpha > 0$ and $\beta \in [0,\infty]$.

These models correspond to the symmetric exclusion process with a slow bond, which was extensively studied in [3, 4, 5]. We first notice that if we take the process evolving on $\mathbb{T}$, the case $\beta = 1$ and $\alpha = 1$ corresponds to a particular case of the ones described above by simply taking $W : \mathbb{R} \to \mathbb{R}$ as $W(x) = x + \lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the biggest integer number smaller or equal to x. It is simple to check that this function W satisfies the conditions imposed in Section 2, and for $b = 0$ the conductances are given by $\xi_{-1,0} = \frac{1}{n+1}$, while for $x \neq -1$, $\xi_{x,x+1} = 1$. Therefore, asymptotically the behavior of this model is the same as for the slow bond introduced above. We remark that when we take other values of β or α we can only write the conductances in terms of a function W that depends on n and this is not covered by the results we presented above, since there the function W is fixed. For these models we are also able to prove the Theorem 2.2 for all the ranges of the parameters $\alpha > 0$ and $\beta \in [0,\infty]$.

Now we sketch the proof of this result following the steps of Section 4. First, we notice that all the results of Subsection 4.1 are true for these models by replacing there the jump rates $\xi_{x,x+1}c_{x,x+1}$ by our choice of $a^n_{x,x+1}$ given above. In order to prove the results of Subsection 4.2 we define the density fluctuation field $\mathscr{Y}^n_t$ (see (3.6)) with the sum taken with $x \in \mathbb{Z}$ on the space of test functions that we denote by $\mathscr{S}_\beta(\mathbb{R})$ which is defined as follows.

First of all, we define first $\mathscr{S}(\mathbb{R}\backslash\{0\})$ as the space of functions $H : \mathbb{R} \to \mathbb{R}$, such that $H \in C^\infty(\mathbb{R}\backslash\{0\})$ and H is continuous from the right at $x = 0$, with

$$\|H\|_{k,\ell} \;:=\; \sup_{x\in\mathbb{R}\backslash\{0\}} |(1+|x|^\ell)\, H^{(k)}(x)| \;<\; \infty\,,$$

for all integers $k, \ell \geq 0$ and $H^{(k)}(0^-) = H^{(k)}(0^+)$, for all k integer, $k \geq 1$, where

$$H(0^+) := \lim_{\substack{u\to 0,\\ u>0}} H(u) \quad \text{and} \quad H(0^-) := \lim_{\substack{u\to 0,\\ u<0}} H(u)\,,$$

when the above limits exist.

Now, let $\mathscr{S}_\beta(\mathbb{R})$ be the subset of $\mathscr{S}(\mathbb{R}\backslash\{0\})$ composed of functions H satisfying:

- for $\beta \in [0,1)$, $H(0^-) = H(0^+)\,,$
- for $\beta = 1$, $H^{(1)}(0^+) \;=\; H^{(1)}(0^-) \;=\; \alpha\Big(H(0^+) - H(0^-)\Big)\,,$
- for $\beta \in (1,\infty]$, $H^{(1)}(0^+) \;=\; H^{(1)}(0^-) \;=\; 0\,.$

We remark that for $\beta < 1$, $\mathscr{S}_\beta(\mathbb{R})$ coincides with the Schwarz space $\mathscr{S}(\mathbb{R})$. Since we are working with different spaces for the test functions, we need to show that we are able approximate the function ι_ε by a suitable sequence of test functions $\mathscr{S}_\beta(\mathbb{R})$. This is the content of the next lemma.

Lemma 5.2. *For fixed $\varepsilon \in (0,1)$, there exists a sequence of functions $\{\iota_\varepsilon^k\}_{k\in\mathbb{N}}$ in the space of test functions $\mathscr{S}_\beta(\mathbb{R})$ converging to ι_ε in the $L^2(\mathbb{T})$-norm, as k tends to infinity.*

Proof. This proof is the same proof as in Lemma 4.6, if we consider

$$\iota_\varepsilon^k(u) := \int_{-\infty}^{u} \Big(\int_{-\infty}^{y} h_\varepsilon^k(z)\,dz\Big)dy, \quad \forall\, u \in \mathbb{R},$$

where h_ε^k is an approximation of the function g_ε^k, defined above, in the space $\mathscr{S}(\mathbb{R}\backslash\{0\})$.

Then the function ι_ε^k belongs to space of test functions $\mathscr{S}_\beta(\mathbb{R})$, and converges to ι_ε, as k tends to infinity in the $L^2(\mathbb{T})$-norm. □

Moreover, all the results of Subsection 4.3 are of straight verification for these models, since the equilibrium fluctuations for these models were proved in Theorem 2.6 of [5], the limit being a generalized Ornstein-Uhlenbeck process. As a consequence we have the following result.

Theorem 5.3. *As n goes to infinity, the sequence of processes $\{\Gamma_n(t) : t \in [0,T]\}_{n\in\mathbb{N}}$ converges in distribution with respect to the uniform topology of $\mathscr{C}([0,T],\mathbb{R})$ to:*

• *For $\beta \in [0,1)$, a mean-zero Gaussian process $\{\Gamma_\infty(t) : t \in [0,T]\}$ with variance given by*

$$\mathbb{E}\big[\big(\Gamma_\infty(t)\big)^2\big] = \frac{4}{3}\frac{\chi(\rho)}{\sqrt{\pi}}t^{3/2}. \tag{5.2}$$

Or else, $\{\Gamma_\infty(t) : t \in [0,T]\}$ is a fractional Brownian motion of Hurst exponent $3/4$.

• *For $\beta = 1$, a mean-zero Gaussian process $\{\Gamma_\alpha(t) : t \in [0,T]\}$ with variance given by*

$$\mathbb{E}\big[\big(\Gamma_\alpha(t)\big)^2\big] = \frac{4}{3}\frac{\chi(\rho)}{\sqrt{\pi}}t^{3/2} + 2\chi(\rho)\int_0^t\int_0^s \frac{F_\alpha(s-r)}{\sqrt{4\pi(s-r)}}drds, \tag{5.3}$$

where

$$F_\alpha(t) = \frac{1}{2t}\int_0^{+\infty} z\, e^{-z^2/4t-2\alpha z}\,dz. \tag{5.4}$$

Moreover, this process $\{\Gamma_\alpha(t) : t \in [0,T]\}$ is not self-similar, hence it is not a fractional Brownian motion.

• *For $\beta \in (1, \infty]$, a mean-zero Gaussian process $\{\Gamma_0(t) : t \in [0, T]\}$ with variance given by*

$$\mathbb{E}\big[\big(\Gamma_0(t)\big)^2\big] = \frac{8}{3}\frac{\chi(\rho)}{\sqrt{\pi}}t^{3/2}. \tag{5.5}$$

Or else, $\{\Gamma_0(t) : t \in [0, T]\}$ is a fractional Brownian motion of Hurst exponent $3/4$ with twice the variance of $\{\Gamma_\infty(t) : t \in [0, T]\}$.

Proof. As mentioned above, the previous results can be obtained by following the proof in Section 4 together with Theorem 2.6 of [5]. In order to characterize the limiting processes, by stationarity, since they are mean-zero and equal to 0 for $t = 0$, it is enough to compute their variances. For notational convenience let $\Gamma(t)$ be the limiting process for all the ranges of β. Then, by symmetry, we get that

$$\mathbb{E}\big[\big(\Gamma(t)\big)^2\big] = \lim_{\varepsilon\to 0} 2\int_0^t \int_0^s \mathbb{E}[\mathscr{Y}_s(\iota_\varepsilon)\mathscr{Y}_r(\iota_\varepsilon)]\, dr\, ds,$$

where $\mathscr{Y}_t$ is the stationary solution of

$$d\mathscr{Y}_t = \Delta_\beta \mathscr{Y}_t dt + \sqrt{2\chi(\rho)}\nabla_\beta d\mathscr{W}_t, \tag{5.6}$$

$\mathscr{W}_t$ being a space-time white noise of unit variance and the characteristic operators Δ_β and ∇_β were defined in [5]. By equation (33) in the proof of Theorem 2.7 of [5],

$$\mathbb{E}[\mathscr{Y}_s(\iota_\varepsilon)\mathscr{Y}_r(\iota_\varepsilon)] = \chi(\rho)\int_{\mathbb{R}} (T^\beta_{s-r}\iota_\varepsilon)(u)\iota_\varepsilon(u)du,$$

where T^β_t is the semigroup associated to Δ_β. It remains only to take the limit of expression above as ε goes to zero. Performing a simple but long computation we get the result. For the sake of completeness we present this computation in the Lemma 6.2 of the Appendix.

Finally, the fact that $\{\Gamma_\alpha(t) : t \in [0, T]\}$ is not self-similar it is a consequence of the fact that its variance is not invariant under a time-transformation of a power type, see [11]. □

It is a folklore conjecture that the fluctuations of the current and of the occupation time should be of same order. By means of the previous theorem, we offer a counter-example for such idea. In [5], it was proved that the fluctuations for the current at the origin in the regime $\beta > 1$ are null. Opposed to that, in the theorem above, we get that the fluctuations for the occupation time at the origin are not null. Of course, this does not eliminate the possibility the conjecture to be true under some additional hypothesis on the particle system. Anyway, the particle system we have used here to present the counter-example has some strong properties as reversibility and the order preservation of particles.

As a consequence of the Theorem 5.3 we discover also that the three processes obtained as the limit of the occupation time are continuously related through the parameter α presented in (5.1). This result is stated in the following corollary.

Corollary 5.4. *The sequence of processes $\{\Gamma_\alpha(t) : t \in [0, T]\}_{\alpha>0}$ converges, as α tends to infinity, to the mean-zero Gaussian process $\{\Gamma_\infty(t) : t \in [0, T]\}$ with variance given by* (5.2). *On the other hand, as α tends to zero, the sequence of processes $\{\Gamma_\alpha(t) : t \in [0, T]\}_{\alpha>0}$ converges to the mean zero Gaussian process $\{\Gamma_0(t) : t \in [0, T]\}$ with variance given by* (5.5). *The convergence above is in the sense of finite dimensional distributions.*

Proof. Gaussian processes are characterized by their covariance. Reversibility in all cases allows to characterize the covariance in terms of the variance. Therefore, it is enough to show the convergence of the variances in each case which is a consequence of the Dominated Convergence Theorem and the fact that

$$\forall\, t \geq 0, \ \lim_{\alpha\to\infty} F_\alpha(t) = 0 \ \text{ and } \ \lim_{\alpha\to 0} F_\alpha(t) = 1,$$

where $F_\alpha(t)$ was defined in (5.4). Both limits above are of straightforward verification and are left to the reader. □

Remark 5.5. We notice that the result of Theorem 2.2 can be extended for particle systems for which Corollary 4.4, Lemma 4.6 and Theorem 3.5 hold. The characterization of the Gaussian process would depend on the knowledge of the semigroup associated to the corresponding operator $\mathfrak{L}_W$ in (3.7).

6. Appendix

We present in this appendix the proof of the following lemma, which is a standard one in the area. Because we were not able to find it written anywhere in the literature, we include it here for the sake of completeness.

Lemma 6.1. *If* $\{\mathscr{Y}_t : t \geq 0\}$ *is a solution of* (3.7)*, then for all* $H \in S_W(\mathbb{T})$*, it holds that*

$$\mathbb{E}\big[\mathscr{Y}_t(H)\mathscr{Y}_s(H)\big] = \chi(\rho)\int_{\mathbb{T}} (P_{t-s}H)(u)H(u)du, \tag{6.1}$$

where $\{P_t : t \geq 0\}$ *is the semigroup associated to* $\tilde{c}'(\rho)\mathfrak{L}_W$*.*

Proof. From [2], since $\mathscr{Y}_t$ solves (3.7), $\mathscr{Y}_t$ solves the following martingale problem: for every $H \in S_W(\mathbb{T})$,

$$\mathscr{M}_t(H) = \mathscr{Y}_t(H) - \mathscr{Y}_0(H) - \tilde{c}'(\rho)\int_0^t \mathscr{Y}_s(\mathscr{L}_W H)ds \tag{6.2}$$

is a martingale with respect to the natural filtration $\mathscr{F}_t := \sigma(\eta_s : 0 \leq s \leq t)$. At first, we claim that

$$\mathbb{E}\big[\mathscr{Y}_t(H)\mathscr{Y}_0(H)\big] = \chi(\rho)\int_{\mathbb{T}} (P_t H)(u)H(u)du. \tag{6.3}$$

For this purpose, notice that

$$\begin{aligned} \mathbb{E}\big[\mathscr{Y}_t(H)\mathscr{Y}_0(H)\big] &= \mathbb{E}\Big[\Big(\mathscr{M}_t(H) + \mathscr{Y}_0(H) + \tilde{c}'(\rho)\int_0^t \mathscr{Y}_s(\mathscr{L}_W H)ds\Big)\,\mathscr{Y}_0(H)\Big] \\ &= \mathbb{E}\big[\mathscr{M}_t(H)\mathscr{Y}_0(H)\big] + \mathbb{E}\big[\mathscr{Y}_0(H)\mathscr{Y}_0(H)\big] + \mathbb{E}\Big[\mathscr{Y}_0(H)\ \tilde{c}'(\rho)\int_0^t \mathscr{Y}_s(\mathscr{L}_W H)ds\Big]. \end{aligned} \tag{6.4}$$

The first expectation above vanishes because

$$\begin{aligned} \mathbb{E}\big[\mathscr{M}_t(H)\mathscr{Y}_0(H)\big] = \mathbb{E}\big[\mathbb{E}\big[\mathscr{M}_t(H)\mathscr{Y}_0(H)\big|\mathscr{F}_0\big]\big] &= \mathbb{E}\big[\mathscr{Y}_0(H)\mathbb{E}\big[\mathscr{M}_t(H)\big|\mathscr{F}_0\big]\big] \\ &= \mathbb{E}\big[\mathscr{Y}_0(H)\mathscr{M}_0(H)\big] = 0, \end{aligned}$$

where last equality above is due to $\mathscr{M}_0(H) = 0$.

The second term can be handled as follows. By computing the characteristic function of $\mathscr{Y}_0^n(H)$ and by the Theorem 2.1 of [2], we get that

$$\mathbb{E}\big[\mathscr{Y}_0(H)\mathscr{Y}_0(H)\big] = \chi(\rho)\int_{\mathbb{T}} (H(u))^2 du. \tag{6.5}$$

Now we develop the last expectation of (6.4) by using again (6.2), that is:

$$\mathbb{E}\Big[\mathscr{Y}_0(H)\tilde{c}'(\rho)\int_0^t \mathscr{Y}_s(\mathscr{L}_W H)ds\Big] = \int_0^t \mathbb{E}\Big[\mathscr{Y}_0(H)\mathscr{Y}_s\Big(\tilde{c}'(\rho)\mathscr{L}_W H\Big)\Big]\, ds$$

$$= \int_0^t \mathbb{E}\Bigg[\mathscr{Y}_0(H)\mathscr{M}_s\Big(\tilde{c}'(\rho)\mathscr{L}_W H\Big) + \mathscr{Y}_0(H)\mathscr{Y}_0\Big(\tilde{c}'(\rho)\mathscr{L}_W H\Big)$$

$$+ \mathscr{Y}_0(H)\int_0^s \mathscr{Y}_r\Big((\tilde{c}'(\rho))^2\mathscr{L}_W^2 H\Big)dr\Bigg]ds.$$

Repeating the same argument as above, last expression can be rewritten as

$$\chi(\rho)\int_{\mathbb{T}} t\tilde{c}'(\rho)(\mathscr{L}_W H)(u)H(u)du + \int_0^t\int_0^s \mathbb{E}\Big[\mathscr{Y}_0(H)\mathscr{Y}_r\Big((\tilde{c}'(\rho))^2\mathscr{L}_W^2 H\Big)\Big]\; dr\; ds.$$

Let us introduce the temporary notation $G := (\tilde{c}'(\rho))^2\mathscr{L}_W^2 H$ and rewrite expression above simply as

$$\chi(\rho)\int_{\mathbb{T}} t\tilde{c}'(\rho)(\mathscr{L}_W H)(u)H(u)du + \int_0^t\int_0^s \mathbb{E}\big[\mathscr{Y}_r(G)\mathscr{Y}_0(H)\big]\; dr\; ds. \tag{6.6}$$

We want to characterize the expectation in the second parcel above. Invoking (6.2) again we have that

$$\mathscr{M}_r(G) = \mathscr{Y}_r(G) - \mathscr{Y}_0(G) - \tilde{c}'(\rho)\int_0^r \mathscr{Y}_l(\mathscr{L}_W G)dl$$

is a martingale. Provided by this fact and repeating the previous arguments, we are lead to

$$\mathbb{E}\big[\mathscr{Y}_r(G)\mathscr{Y}_0(H)\big] = \chi(\rho)\int_{\mathbb{T}} G(u)H(u)du + \mathbb{E}\Big[\mathscr{Y}_0(H)\;\tilde{c}'(\rho)\int_0^r \mathscr{Y}_l(\mathscr{L}_W G)dl\Big] \tag{6.7}$$

Putting together (6.5), (6.7) and (6.6), we obtain:

$$\mathbb{E}\big[\mathscr{Y}_t(H)\mathscr{Y}_0(H)\big] = \chi(\rho)\int_{\mathbb{T}}(H(u))^2du + \chi(\rho)t\int_{\mathbb{T}}\tilde{c}'(\rho)(\mathscr{L}_W H)(u)H(u)du$$

$$+\chi(\rho)\frac{t^2}{2}\int_{\mathbb{T}}(\tilde{c}'(\rho))^2(\mathscr{L}_W^2 H)(u)H(u)du + M_t(H),$$

where

$$M_t(H) := \int_0^t\int_0^s \mathbb{E}\Big[\mathscr{Y}_0(H)\;\tilde{c}'(\rho)\int_0^r \mathscr{Y}_l(\mathscr{L}_W G)dl\Big]dr\,ds.$$

From the Lemma 3.5 of [2], we have that $\mathscr{L}_W : S_W(\mathbb{T}) \to S_W(\mathbb{T})$ is a bounded operator with respect to the norm associated to the inner product defined in (3.5). Therefore, it makes sense to define the exponential of this operator. A long, but elementary induction argument over the previous formula leads to

$$\mathbb{E}\big[\mathscr{Y}_t(H)\mathscr{Y}_0(H)\big] = \chi(\rho)\int_{\mathbb{T}}(e^{t\tilde{c}'(\rho)\mathscr{L}_W}H)(u)H(u)du$$

$$= \chi(\rho)\int_{\mathbb{T}}(P_t H)(u)H(u)du,$$

where $\{P_t : t \geq 0\}$ is the semigroup associated to $\tilde{c}'(\rho)\mathfrak{L}_W$. Finally, since $\{\mathscr{Y}_t : t \geq 0\}$ is a stationary process, we get that

$$\mathbb{E}\big[\mathscr{Y}_t(H)\mathscr{Y}_s(H)\big] = \mathbb{E}\big[\mathscr{Y}_{t-s}(H)\mathscr{Y}_0(H)\big] = \chi(\rho)\int_{\mathbb{T}}(P_{t-s}H)(u)H(u)du,$$

as desired. □

We finish this Appendix fulfilling some details in the proof of Theorem 5.3. Let T_t^β be the semigroup associated to the operator Δ_β. For $\beta \in [0,1)$ it is the semigroup related to the heat equation in the line. Quite classical, it acts on $g \in \mathscr{S}_\beta(\mathbb{R})$ as

$$T_t g(x) \;=\; \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}}\,g(y)\,dy\,,\quad \text{for } x\in\mathbb{R}. \tag{6.8}$$

For $\beta \in (1,\infty]$, the semigroup T_t^β is also known and it acts on $g \in \mathscr{S}_\beta(\mathbb{R})$ as

$$T_t^{\text{Neu}} g(x) \;=\; \begin{cases} \dfrac{1}{\sqrt{4\pi t}}\displaystyle\int_0^{+\infty}\Big[e^{-\frac{(x-y)^2}{4t}}+e^{-\frac{(x+y)^2}{4t}}\Big]g(y)\,dy\,, & \text{for } x>0\,,\\ \dfrac{1}{\sqrt{4\pi t}}\displaystyle\int_0^{+\infty}\Big[e^{-\frac{(x-y)^2}{4t}}+e^{-\frac{(x+y)^2}{4t}}\Big]g(-y)\,dy\,, & \text{for } x<0. \end{cases} \tag{6.9}$$

Denote by g_{even} and g_{odd} the even and odd parts of a function $g:\mathbb{R}\to\mathbb{R}$, respectively, or else, for $x\in\mathbb{R}$,

$$g_{\text{even}}(x) = \frac{g(x)+g(-x)}{2} \quad\text{and}\quad g_{\text{odd}}(x) = \frac{g(x)-g(-x)}{2}. \tag{6.10}$$

As proved in [5], for $\beta = 1$, the semigroup T_t^β acts on $g \in \mathscr{S}_\beta(\mathbb{R})$ as

$$\begin{aligned} T_t^\alpha g(x) = \frac{1}{\sqrt{4\pi t}}\Bigg\{ &\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} g_{\text{even}}(y)\,dy \\ &+ e^{2\alpha x}\int_x^{+\infty} e^{-2\alpha z}\int_0^{+\infty}\Big[\big(\tfrac{z-y+4\alpha t}{2t}\big)e^{-\frac{(z-y)^2}{4t}} + \big(\tfrac{z+y-4\alpha t}{2t}\big)e^{-\frac{(z+y)^2}{4t}}\Big]\, g_{\text{odd}}(y)\,dy\,dz\Bigg\}, \end{aligned} \tag{6.11}$$

for $x>0$ and

$$\begin{aligned} T_t^\alpha g(x) = \frac{1}{\sqrt{4\pi t}}\Bigg\{ &\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} g_{\text{even}}(y)\,dy \\ &- e^{-2\alpha x}\int_{-x}^{+\infty} e^{-2\alpha z}\int_0^{+\infty}\Big[\big(\tfrac{z-y+4\alpha t}{2t}\big)e^{-\frac{(z-y)^2}{4t}} + \big(\tfrac{z+y-4\alpha t}{2t}\big)e^{-\frac{(z+y)^2}{4t}}\Big]\, g_{\text{odd}}(y)\,dy\,dz\Bigg\}, \end{aligned}$$

for $x<0$.

Below, we state and prove a lemma required in the proof of the Theorem 5.3.

Lemma 6.2. *For $\beta \in [0,1)$,*

$$\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}}(T_t^\beta \iota_\varepsilon)(u)\iota_\varepsilon(u)du \;=\; \frac{1}{\sqrt{4\pi t}}.$$

For $\beta = 1$,

$$\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}}(T_t^\beta \iota_\varepsilon)(u)\iota_\varepsilon(u)du \;=\; \frac{1}{\sqrt{4\pi t}}\Big(1+\frac{1}{2t}\int_0^{+\infty} z e^{-\frac{z^2}{4t}-2\alpha z}dz\Big). \tag{6.12}$$

Finally, for $\beta \in (1,\infty]$,

$$\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}}(T_t^\beta \iota_\varepsilon)(u)\iota_\varepsilon(u)du \;=\; \frac{2}{\sqrt{4\pi t}}.$$

Proof. Consider $\beta \in [0,1)$. In this case,

$$\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}}(T_t^\beta \iota_\varepsilon)(u)\iota_\varepsilon(u)du \;=\; \lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon^2}\int_0^\varepsilon\int_0^\varepsilon \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}}dx\,dy \;=\; \frac{1}{\sqrt{4\pi t}}, \tag{6.13}$$

because the gaussian kernel is a continuous function. The case $\beta \in (1,\infty]$ is quite similar. Indeed, for this regime of β,

$$\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}}(T_t^\beta \iota_\varepsilon)(u)\iota_\varepsilon(u)du \;=\; \lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon^2}\int_0^\varepsilon\int_0^\varepsilon \frac{e^{-\frac{(x-y)^2}{4t}}+e^{-\frac{(x+y)^2}{4t}}}{\sqrt{4\pi t}}dx\,dy \;=\; \frac{2}{\sqrt{4\pi t}},$$

The case $\beta = 1$ deserves more attention. For $g(u) = \iota_\varepsilon(u)$, we have that

$$g_{\text{even}}(u) = \tfrac{1}{2\varepsilon}\mathbf{1}_{[-\varepsilon,\varepsilon]} \quad\text{and}\quad g_{\text{odd}}(u) = \tfrac{1}{2\varepsilon}\Big(\mathbf{1}_{(0,\varepsilon]} - \mathbf{1}_{[-\varepsilon,0)]}\Big),$$

according to (6.10). Recalling formula (6.11), we obtain

$$\int_{\mathbb{R}}(T_t^\beta \iota_\varepsilon)(u)\iota_\varepsilon(u)du = \frac{1}{\sqrt{4\pi t}}\Big(\frac{1}{\varepsilon^2}\int_0^\varepsilon\int_0^\varepsilon e^{-\frac{(x-y)^2}{4t}}\,dy\,dx + \tfrac{S(\varepsilon)}{\varepsilon^2}\Big), \tag{6.14}$$

where $S(\varepsilon)$ is

$$\int_0^\varepsilon\Big(\tfrac{e^{2\alpha x}}{2}\int_x^{+\infty} e^{-2\alpha z}\int_0^\varepsilon\Big[\tfrac{(z-y+4\alpha t)}{2t}e^{\frac{-(z-y)^2}{4t}} + \tfrac{(z+y+4\alpha t)}{2t}e^{\frac{-(z+y)^2}{4t}}\Big]\,dy\,dz\Big)dx.$$

We want to precise the limit of (6.14) as $\varepsilon \searrow 0$. By (6.13), it only remains to evaluate the limit of $S(\varepsilon)/\varepsilon^2$ as ε goes to zero. A direct verification shows that

$$\lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon^2}S(\varepsilon) = \frac{1}{2t}\int_0^\infty z e^{-\frac{z^2}{4t}-2\alpha z}dz,$$

leading to (6.12) and hence finishing the proof. □

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