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Solving Systems of Nonlinear Equations by Harmony Search

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Abstract

In this paper, we aim to analyze the performance of some variants of the harmony search (HS) metaheuristic when solving systems of nonlinear equations through the global optimization of an appropriate merit function. The HS metaheuristic draws its inspiration from an artistic process, the improvisation process of musicians seeking a wonderful harmony. A new differential best HS algorithm, based on an improvisation operator that mimics the best harmony and uses a differential variation, is proposed. Computational experiments involving a well-known set of small-dimensional problems are presented.

Key words: nonlinear equations, metaheuristic, harmony search

1 Introduction

The present study aims to investigate the performance of the harmony search (HS) metaheuristic to locate a solution of a nonlinear system of equations of the form

$$f(x) = 0, \quad f(x) = (f_1(x), f_2(x), \dots, f_n(x))^T \quad (1)$$

where each $f_i : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous function in the search space, and Ω is a closed convex set, herein defined as $[l, u] = \{x : -\infty < l_i \leq x_i \leq u_i < \infty, i = 1, \dots, n\}$. We do not assume that the functions $f_i(x)$, $i = 1, \dots, n$ are differentiable. Thus, the search for an efficient derivative-free technique that does not assume smoothness, convexity and differentiability is of great importance in mathematics and engineering. The basic HS algorithm that emerged in 2001 relies on a population of points and is inspired by natural

phenomena [5]. It draws its inspiration not from a biological or physical process like most metaheuristic optimization techniques, but from an artistic one – the improvisation process of musicians seeking a wonderful harmony.

Some problems in engineering, chemistry, physics, medicine and even economic areas, aim at determining the roots of a system of equations. In general, these problems are nonlinear and difficult to solve. The most famous techniques to solve nonlinear equations are based on the Newton's method [3, 4, 6, 13, 16]. They require analytical or numerical first derivative information. Newton's method is the most widely used algorithm for solving nonlinear systems of equations. It is computationally expensive, in particular if n is large, since the Jacobian matrix and the solution of a system of linear equations are required at each iteration. On the other hand, Quasi-Newton methods avoid either the necessity of computing derivatives, or the necessity of solving a full linear system per iteration or both tasks [14]. Thus, Quasi-Newton methods have less expensive iterations than Newton, and their convergence properties are not very different from the Newton one.

The problem of solving a nonlinear system of equations can be naturally formulated as a global optimization problem. Problem (1) is equivalent to

$$\min_{x \in \Omega \subset \mathbb{R}^n} \mathcal{M}(x) \equiv \sum_{i=1}^n f_i(x)^2, \quad (2)$$

in the sense that they have the same solutions. These required solutions are the global minima, and not just the local minima, of the function $\mathcal{M}(x)$, known as merit function, in the set Ω . Problem (2) is similar to the usual least squares problem for which many iterative methods have been proposed. They basically assume that the objective function is twice continuously differentiable. However, the objective \mathcal{M} in (2) is only once differentiable if some, or just one, of the f_i , ($i = 1, \dots, n$) are not differentiable. Thus, methods for solving the least squares problem cannot be directly applied to solve (2).

When a global solution of a nonlinear optimization problem is required, Newton-type methods have some disadvantages, when compared with global search methods, because they rely on searching locally for the solution. The final solution is heavily dependent on the initial approximation of the iterative process and they can be trapped in a local minimum. Local optimization techniques guarantee globally only under certain convexity assumptions. Furthermore, most of the methods require differentiable properties of all the equations in the nonlinear system, such as, for example, the trust-region Gauss-Newton method presented in [15].

Preventing premature convergence to a local while trying to locate a global solution of problem (2) is the goal of the present study. Here, we aim to investigate the performance of a metaheuristic to solve globally optimization problems without the use of derivative information. Metaheuristics are general heuristic methods which can be applied to a wide variety of optimization problems.

HS has efficient strategies for exploring the entire search space, as well as techniques to exploit locally a promising region to yield a high quality solution in a reasonable time. The dynamic updating of two important parameters in the HS algorithm has improved the efficiency and robustness of the metaheuristic [12]. Here, we combine this improved HS algorithm with ideas from the global best variant of HS [17], and from the mutation strategy present in the differential evolution method introduced in [21], to propose the differential best HS algorithm.

Although other metaheuristics have been proposed to solve systems of nonlinear equations, the computational effort to achieve a solution is meaningful. Further, the quality of the solution is not in general satisfactory. A genetic algorithm (GA) is proposed in [2]. In [7], a new technique for solving systems of nonlinear equations reshaping the system as a multiobjective optimization problem is proposed. A technique of evolutionary computation is applied to solve the multiobjective problem. A modified version of the classical particle swarm optimization (PSO) algorithm is presented in [10] and a tabu search based framework has been implemented together with a local search strategy to enhance the search about a promising region and improve the quality of the solution in [18, 19, 20]. Multiple solutions in nonlinear systems have been addressed in the literature. In [8], the authors propose techniques for computing all the multiple solutions in nonlinear systems and in [9], a continuous global optimization heuristic, known as C-GRASP, with areas of repulsion around already detected solutions, is described.

The structure of the remainder of the paper is the following. Section 2 describes the classic HS algorithm, the improved HS, the global best HS, and the proposed differential best HS. Section 3 reports the preliminary numerical experiments carried out with the presented metaheuristic when solving a set of small-dimensional problems. Section 4 contains the conclusions of this study and some ideas for future work.

2 Harmony search algorithms

The HS algorithm was developed to solve global optimization problems in an analogy with the music improvisation process where music players improvise the pitches of their instruments to obtain better harmony (see [5, 11]).

The algorithm parameters are:

- i) the harmony memory size (HMS), which gives the number of solution vectors in the harmony memory (HM), and has an equivalent meaning of the population size in GA and PSO algorithms;
- ii) the harmony memory considering rate (HMCR);
- iii) the pitch adjusting rate (PAR);

- iv) the number of allowed improvisations (NI), which is similar to the number of generations/iterations allowed in GA and PSO algorithms.

The HM matrix is a memory location to store all the solution vectors. The HMCR and PAR parameters are used to find globally and locally improved solutions, respectively.

In the remaining part of this section we present the main steps of the classic HS algorithm, as well as an improved variant of HS containing dynamic updating of parameters, the global-best version of the HS and the proposed differential best HS. An overview of the existing variants of the HS is presented by Alia and Mandava in [1].

2.1 Classical HS

The main steps of the classical HS algorithm are represented in Algorithm 1. After genera-

Algorithm 1 HS algorithm

- Step 0. Set values to the parameters. Set $k = 0$.
 - Step 1. Initialize the HM randomly in Ω : $x^j, j = 1, \dots, \text{HMS}$.
 - Step 2. Evaluate the HM, select the best, x^{best} , and the worst harmony.
 - Step 3. Improvise a new harmony y and evaluate.
 - Step 4. Update the HM and select x^{best} and the worst harmony.
 - Step 5. If x^{best} is sufficiently accurate then STOP else increase k and go to Step 3.
-

ting the HM randomly in the search space $\Omega, x^j, j = 1, \dots, \text{HMS}$, the vectors are evaluated and the best harmony, herein denoted by x^{best} , and the worst in terms of merit function value are selected (see Step 2). Thereafter, a new harmony is improvised (in Step 3) meaning that a new vector y is generated using three improvisation operators:

- o_1 . HM operator;
- o_2 . random selection operator;
- o_3 . pitch adjustment operator.

The HMCR parameter varies between 0 and 1 and gives the probability of choosing the component of the new harmony/vector from the HM (operator o_1). Otherwise, the component is randomly generated in Ω (operator o_2):

$$y_i = \begin{cases} x_i^j, j \text{ randomly chosen from } \{1, \dots, \text{HMS}\} & \text{if } rand() < \text{HMCR} \\ l_i + rand()(u_i - l_i) & \text{otherwise} \end{cases} \quad (3)$$

for $i = 1, \dots, n$, where $rand()$ represents a random number in the range $[0, 1]$. The operator o_3 is subsequently applied with probability PAR to refine only the components produced

by o_1 , as follows:

$$y_i = \begin{cases} \min\{\max\{y_i \pm \text{rand}()BW, l_i\}, u_i\} & \text{if } \text{rand}() < \text{PAR} \\ y_i & \text{otherwise} \end{cases} \quad (4)$$

where BW is an arbitrary distance bandwidth. In Step 4 of the algorithm, the HM is updated. The new harmony is compared with the worst harmony in the HM, in terms of \mathcal{M} values. The new harmony is included in the HM, replacing the worst one if it is better than the worst harmony.

2.2 Improved HS

The classical HS algorithm uses fixed value for both PAR and BW . Small PAR values with large BW values can considerably increase the number of iterations required to converge to the optimal solution of (2). Experience has shown that BW must take large values at the beginning of the iterative process to enforce the algorithm to increase the diversity of solution vectors. However, small BW values in the final iterations increase the fine-tuning of solution vectors. Furthermore, large PAR values with small BW values usually cause the improvement of best solutions in the final stage of the process. To eliminate the drawbacks due to fixed values of PAR and BW , the improved HS (I-HS) algorithm proposed in [12] uses parameter values dynamically dependent on the improvisation/iteration number k , as shown:

$$\text{PAR}(k) = \text{PAR}_{\min} + k \frac{(\text{PAR}_{\max} - \text{PAR}_{\min})}{NI} \quad (5)$$

where PAR_{\min} and PAR_{\max} are the minimum and maximum pitch adjusting rate respectively, k denotes the iteration number, and

$$\text{BW}(k) = \text{BW}_{\max} e^{ck}, \quad \text{for } c = \frac{\ln(\frac{\text{BW}_{\min}}{\text{BW}_{\max}})}{NI} \quad (6)$$

where BW_{\min} and BW_{\max} are the minimum and maximum bandwidth respectively.

2.3 Global-best HS

In [17], a new variant of harmony search, called the global-best harmony search (G-bHS), is proposed. The pitch-adjustment step of the HS is modified in a way that the new harmony can mimic the best harmony in the HM, adding a social dimension to the HS and replacing the BW related parameters altogether. Thus, the new pitch adjustment operator, o_3 , is applied with probability $\text{PAR}(k)$, computed from (5), to refine only the components produced by o_1 , in the following way:

$$y_i = \begin{cases} \min\{\max\{x_t^{\text{best}}, l_i\}, u_i\}, t \text{ randomly chosen from } \{1, \dots, n\} & \text{if } \text{rand}() < \text{PAR}(k) \\ y_i & \text{otherwise} \end{cases} \quad (7)$$

where *best* is the index of the best harmony in the HM.

2.4 Proposed differential best HS

We borrow the ideas from the global best HS algorithm and increase the explorative power of the classical HS to improvise a new harmony, for the Step 3 in Algorithm 1. Our proposal then replaces the improvisation operator o_1 by another one that mimics the best harmony and uses a mutation strategy present in the differential evolution method [21]. The differential mutation scheme aims to explore the search space for a promising region where the global solution of problem (2) lies. Furthermore, the pitch adjustment operator is maintained and the parameters PAR and BW are dynamically updated according to (5) and (6) respectively.

Thus, the parameter HMCR sets the probability of choosing the component of the new harmony from the best harmony in HM adding a differential variation, i.e., for each $i = 1, \dots, n$, (3) is replaced by the following equation:

$$y_i = \begin{cases} \min\{\max\{x_i^{best} + F(x_i^{j_1} - x_i^{j_2}), l_i\}, u_i\} & \text{if } rand() < \text{HMCR} \\ l_i + rand()(u_i - l_i) & \text{otherwise} \end{cases} \quad (8)$$

where F is a real parameter from $[0, 2]$ which controls the amplification of the differential variation, $x^{j_1} - x^{j_2}$. The indices j_1, j_2 are uniformly chosen random values from the set $\{1, \dots, \text{HMS}\}$, mutually different. Then, the operator o_3 , is applied with probability $\text{PAR}(k)$ to refine only the components produced by o_1 , as previously defined in equation (4), where k denotes the iteration counter. This variant will be denoted by differential best HS (D-bHS).

3 Computational experiments

In this section, we aim to compare the performance of the above described variants of the HS algorithm for global optimization when solving systems of nonlinear equations. The results of these experiments are obtained in a personal computer with an AMD Turion 2.20 GHz processor and 3 GB of memory, and all program codes were written in MATLAB R2010b. During the experiments, we set $\text{HMS} = \min\{2n, 10\}$ and $\text{HMCR} = 0.95$. When running I-HS, we tested two sets for $(\text{BW}_{\min}, \text{BW}_{\max})$, respectively $(1e-4, 1)$ and $(1e-6, 5)$ and set $\text{PAR}_{\min} = 0.35$ and $\text{PAR}_{\max} = 0.99$. In G-bHS we set $\text{PAR}_{\min} = 0.01$ and $\text{PAR}_{\max} = 0.99$ as proposed in [17]. The experiments carried out with D-bHS use both pairs of $(\text{BW}_{\min}, \text{BW}_{\max})$: $(1e-4, 1)$ and $(1e-6, 5)$, and the parameter F was set to 0.9.

The algorithms in comparison were terminated when

$$\|f(x^{best})\| \leq \tau \quad (9)$$

is satisfied for the best point in the HM, for $\tau = 10^{-6}$. However, if this condition is not satisfied after nfe_{\max} function evaluations then the algorithm will stop. The nfe_{\max} is an adequate alternative to the parameter NI to measure algorithms efficiency.

First, we note that in the subsequence tables, the subscript in the notation ‘I-HS $_{1e-4}^1$ ’, corresponds to BW_{\min} and the superscript corresponds to BW_{\max} . The tables contain the results, in terms of number of function evaluations of the best run, nfe_{best} , among the 30 runs. The values in parentheses correspond to $\mathcal{M}(x^{best})$ when nfe_{\max} function evaluations are attained.

3.1 An illustrative non-smooth problem

Here, we analyze the performance of the described I-HS, G-bHS and D-bHS algorithms on a non-smooth nonlinear system with two variables [7]:

$$\begin{cases} x_1^2 - x_2^2 = 0 \\ 1 - |x_1 - x_2| = 0 \end{cases}$$

which has more than one solution. The evolutionary approach in [7] is applied to a population of 200 points and converges to a Pareto curve with a set of nine nondominated solutions after 150 generations and at least 30 000 function evaluations. The least fitness (sum of the absolute values of the objectives) is around 0.01. We also run `fsolve` from MATLABTM and after 20 iterations and 23 function evaluations, the point (1, 1), with merit function value equals to one, is obtained. (The output parameter ‘exitflag’ is ‘-2’. The solver stopped due to a very small direction vector.) After trying several initial points, the solution (-0.5, 0.5) was finally reached (in three iterations and 12 function evaluations). Considering $\Omega = [-10, 10]^2$, the HS variants behave as shown in Table 1. We observe that the variants I-HS and D-bHS converge to the solution (-0.500000, 0.500000), with merit function values of order 10^{-13} , and the pair (1e-6, 5) for (BW_{\min} , BW_{\max}) gives a better performance than the other in comparison. G-bHS stops after 100 000 function evaluations with a merit function value of 2.8e-6.

Table 1: Results of nfe_{best} for $nfe_{\max}=100\ 000$.

	n	I-HS $_{1e-4}^1$	I-HS $_{1e-6}^5$	G-bHS	D-bHS $_{1e-4}^1$	D-bHS $_{1e-6}^5$
non-smooth problem	2	425	278	(2.8e-6)	664	492

3.2 Other small-dimensional problems

In this comparative study, six well-known small-dimensional problems are used, where two of them are tested with three different search space sizes [22]. Table 2 contains the number of

function evaluations of the best run when solving the problems **Reactor**, **Steering**, **Merlet** and **Floudas**. The algorithms were allowed to run for a maximum of 10 000 function evaluations, when solving the problems **Reactor** and **Steering**, and for 1 000 when solving **Merlet** and **Floudas**. First, we note that the problem **Steering** has two solutions inside the feasible set $[0.06, 1]^3$ and the algorithms are not able to decrease the merit function value below $6.1e-8$. Further, although I-HS $_{1e-6}^5$ requires fewer function evaluations to converge to the solutions of **Merlet** and **Floudas**, with the desired accuracy (see (9)), the variant D-bHS $_{1e-6}^5$ outperforms the others since it converges to a solution of **Reactor**, as well as to the solutions of **Merlet** and **Floudas**, with an acceptable computational effort.

Results concerned with the problems **Effati-Grosan₁** and **Effati-Grosan₂**, with different search spaces, are reported in Table 3. Here, the algorithms in comparison are allowed to run for a maximum of 100 000 function evaluations. We observe that the proposed variant D-bHS with the pair of minimum and maximum bandwidth set to (1e-6, 5) is the most effective.

Table 2: Results of nfe_{best} for the small problems.

Prob.	n	Ω	I-HS $_{1e-4}^1$	I-HS $_{1e-6}^5$	G-bHS	D-bHS $_{1e-4}^1$	D-bHS $_{1e-6}^5$
Reactor	2	$[0, 1]^2$	(2.1e-4)	(2.0e-3)	(2.5e-3)	776	550
Steering	3	$[0.06, 1]^3$	(6.1e-8)	(6.1e-8)	(8.4e-8)	(6.1e-8)	(6.1e-8)
Merlet	2	$[0, 2\pi]^2$	437	292	(1.3e-5)	659	504
Floudas	2	$[0.25, 1] \times [1.5, 2\pi]$	398	278	(7.2e-5)	698	438

Table 3: Results of nfe_{best} for $nfe_{max}=100\ 000$.

Prob.	Ω	I-HS $_{1e-4}^1$	I-HS $_{1e-6}^5$	G-bHS	D-bHS $_{1e-4}^1$	D-bHS $_{1e-6}^5$
Effati-Grosan₁	$[-2, 2]^2$	704	517	(9.8e-6)	662	536
	$[-10, 10]^2$	696	501	(1.1e-5)	710	523
	$[-100, 100]^2$	752	445	(1.9e-3)	726	508
Effati-Grosan₂	$[-2, 2]^2$	706	484	(3.1e-7)	755	509
	$[-10, 10]^2$	673	463	(1.3e-5)	656	473
	$[-100, 100]^2$	(2.0e-4)	(4.9e-4)	(6.8e-4)	642	482

4 Conclusions

We analyzed the performance of different variants of the harmony search metaheuristic when computing a solution of a system of nonlinear equations. A new variant, denoted by differential best harmony search, and based on an improvisation operator that mimics the best

harmony of the harmony memory and uses a differential variation has been presented. Preliminary computational experiments have been carried out with a set of small-dimensional problems. We observed that the differential mutation strategy has enforced the algorithm to increase the diversity. Furthermore, a large range between the minimum and maximum bandwidth during the pitch adjustment step has increased the effectiveness of the algorithm.

In the coming months, the proposed D-bHS algorithm will be embedded into a multi-start algorithm based on clustering and regions of attractions for computing multiple roots of systems of nonlinear equations.

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