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Research Paper

Analysis of a Prestressed Sleeper for Railway Turnout under the Effect Static Load with a Unilateral Contact Model and Support Stiffness of Existing Railway Bed of a Railway Section in Vietnam

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ABSTRACT

In the process of analyzing prestressed sleepers for turnout on elastic foundations with large lengths, authors around the world often use the one-stiffness model of Winkler or the two-stiffness model of the Filonenko-Borodich and Pasternak. Railway bed stiffness is assumed to be a constant value from the design to the end of the operating process. In this study, the authors use a one-stiffness model to simulate the interaction between the prestressed sleepers for turnout and the railway bed, which helps to describe the contact and non-contact between the prestressed sleepers of turnout and the railway bed, the phenomenon that the many traditional models do not implement yet. In addition, the railway bed stiffness included in the analysis is obtained by testing at the field of a railway section in operation in Vietnam. The authors use the finite element method and combine it with the Newton iteration method. Accordingly, the analysis results of prestressed sleepers for turnout will be more consistent with actual behavior. We need to check the actual stiffness of the foundation before it is included in the analysis because the actual stiffness of the foundation gives more accurate results during analysis.

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1 Introduction

Currently, prestressed sleepers for turnout are often considered as a beam on an elastic foundation, the models for calculating beams on an elastic foundation in research as well as in construction design calculations do not consider the non-contact between the beam and the foundation. In the calculating process, authors often usefoudation models with one-stiffness or two-stiffness. Vu Dinh Lai et al. [1], Luong Tho Trinh et al. [2], and Anil K. Chopa [3] introduced the method of calculating beams on elastic foundation according to the model of Winkler with one-stiffness. Vu ThiBich Quyen [4] calculates beams

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on an elastic foundation with the Winkler model by boundary element method. Pham Hoang Anh [5] calculates beams on elastic foundations with complex boundary conditions with the Winkler model by analytical method.

In order to closely simulate the actual behavior of beams on an elastic foundation, authors around the world considered the contact and non-contact between the beam and the foundation. Z. Celep et al. [6] investigated the dynamic effects of finite-length beams on a one-dimensional foundation. Diego Froio et al. [7] analyzed beams on a nonlinear foundation under the effect of load changing with moving time.P. Castro Jorge et al. [8] studied the effect of the beam with articulated joint double-end on an elastic foundation subjected to a moving constant load with Winkler, one-dimensional, and order nonlinear foudation models of degree three. Cristiano Viei Rodrigues [9] analyzes beams on a nonlinear foundation subjected to moving oscillators by the finite element method. D. Froio et al. [10] used the numerical method to calculate simple beams on degree three nonlinear foundation under the effect of load changing with moving time. S. M. Abdelghany et al. [11] investigated the behavior of beams on a nonlinear foundation subjected to moving loads using Galerkin and Runge-Kutta methods. Salih N Akour [12] analyses the dynamics of the beam on a nonlinear foundation under the influence of harmonic loads distributed on the beam surface using the Runge-Kutta method to solve. Do Xuan Quy et al. [13] studied the mechanical behavior of a bar with an anisotropic connection under dynamic load. Do Xuan Quy et al. [14] empirically studied the behavior of anisotropic connections under the effect of dynamic loads.

Due to the large length of the turnout sleepers, and under the effect of load, there will be sleeper segments in contact and segments not in contact with the foundation, making the problem more complicated. In order to find the solution to the problem, the authors do the following: For the calculating model, the authors use the unilateral contact model to describe the behavior between the sleepers and the foundation, and the applied unilateral foundation stiffness has been determined experimentally. For the method, the authors use the finite element method and combine it with the Newton iteration method to find the nonlinear root of the problem.

2 Calculation Model of Turnout Sleepers on Elastic Foundation

Under the action of the wheel load of the train, the sleepers have a segment displacement downward and another segment displacement upward. In segment displacement downward, ballast acts force on the bottom of the sleeper. In segment displacement upward, the bottom of the sleepers is separated from the surface of the ballast, there is no force of the ballast to the bottom of the sleeper.

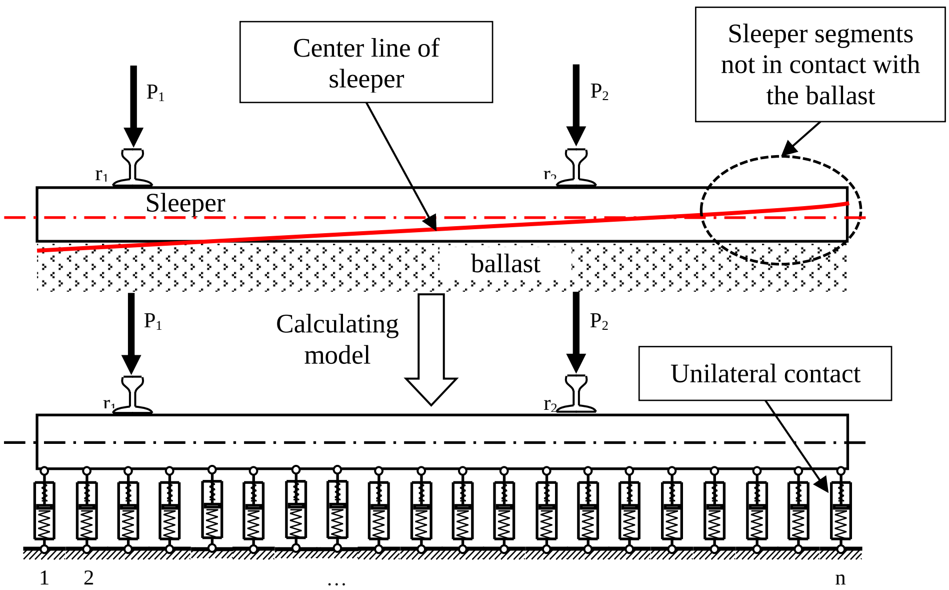


Fig. 1 - Calculation model of turnout sleepers on elastic foundation

In order to be able to describe the interaction between the ballast and the sleepers as closely as possible, the authors model the ballast foundation into a system of unilateral contact. At this time, the calculation model of turnout sleeper on the ballast foundation is replaced by a beam system placed on system with unilateral contact n (Fig. 1.).

The relationship between unilateral contact reaction and contact support displacement is used in this study as a formula (1).

$$N = \frac{k}{2}(\Delta) + \frac{k}{2}|\Delta| \quad (1)$$

where

N: reaction of the unilateral contact with the sleepers.

k: the stiffness of the unilateral contact when the point of contact between the bottom of the sleepers and the ballast moves downwards.

Δ: displacement of the contact point between the bottom of the sleeper and the ballast.

The foundation stiffness*k* is calculated as follows:

For links i (i from 2nd to n-1): $k_i = l_{pt} \times c \times b$ (2)

For the rest of the links: $k_1 = k_n = 0.5 \times k_i$. (3)

where

l_{pt}: length of sleeper element bounded by link i and i+1

b: width of sleeper bottom

k: foundation stiffness reference to the literature and determined by field experiments.

3 Theoretical Basis for Calculating Beams on a Unilateral Contact System under the Effect of Static Loads by Finite Element Method

3.1 The Basic Equation of the Finite Element Method for the Beam System on a Unilateral Contact System under the Effect of Static Loads

- For beam system with normal contact:

The basic equation of the finite element method for the normally contacted beam system, after processing the boundary conditions, is written as follows:

$$[K]\{\Delta\} = \{P\} \quad (4)$$

where

K: the overall stiffness matrix of the structure

$\{\Delta\}$: node displacement vector of the structure

$\{P\}$: node load vector

- For beam systems with anisotropic contact:

Considering anisotropic contact reactions as a type of load, transferring these reactions to the left side obtains the basic equation for an anisotropic contact beam system as follows:

$$[K]\{\Delta\} + \{N(\Delta)\} = \{P\} \quad (5)$$

where

$\{N(\Delta)\}$: reaction vector of the anisotropic contact of the structure is calculated according to formula (1).

In this study, the elements used are flat beam elements with stiffness matrix $[k^e]$, as follows:

$$[k^e] = \begin{bmatrix} \frac{12EJ}{l^3} & \frac{6EJ}{l^2} & -\frac{12EJ}{l^3} & \frac{6EJ}{l^2} \\ \frac{6EJ}{l^2} & \frac{4EJ}{l} & -\frac{6EJ}{l^2} & \frac{2EJ}{l} \\ -\frac{12EJ}{l^3} & -\frac{6EJ}{l^2} & \frac{12EJ}{l^3} & -\frac{6EJ}{l^2} \\ \frac{6EJ}{l^2} & \frac{2EJ}{l} & -\frac{6EJ}{l^2} & \frac{4EJ}{l} \end{bmatrix} \quad (6)$$

where

E: elastic modulus of the material

l: element length

F: cross-section area of element

J: axial inertia moment of the element cross section

Since $\{N(\Delta)\}$is a function that depends on the variable Δ, equation (5) is a nonlinear equation, which is solved by the authors using Newton's iterative method.

3.2 *The Basic Equation of Beam System with a Unilateral Contact System by Newton Iterative Method*

3.2.1 *Set up an iterative formula of Newton's method to solve the system of basic equations of a one-way connected beam system*

Equation (5) is written in the expanded form as follows:

$$\begin{gathered} K_{11}\Delta_1 + K_{12}\Delta_2 + K_{13}\Delta_3 + \cdots + N_1 = P_1 \\ K_{21}\Delta_1 + K_{22}\Delta_2 + K_{23}\Delta_3 + \cdots + N_2 = P_2 \\ K_{31}\Delta_1 + K_{32}\Delta_2 + K_{33}\Delta_3 + \cdots + N_3 = P_3 \\ \cdots \\ K_{41}\Delta_1 + K_{42}\Delta_2 + K_{43}\Delta_3 + \cdots + N_4 = P_4 \end{gathered} \quad (7)$$

Moving all the right sides of the equation to the left side resets the functions

$$f_i = \sum_j K_{ij}\Delta_j + N_i - P_i = 0 \quad (8)$$

Transform, get Newton's iterative formula to solve the system of equations (5) as follows:

$$\begin{gathered} \frac{\partial f_1(\Delta^0)}{\partial \Delta_1}\delta\Delta_1 + \frac{\partial f_1(\Delta^0)}{\partial \Delta_2}\delta\Delta_2 + \cdots + \frac{\partial f_1(\Delta^0)}{\partial \Delta_n}\delta\Delta_n = -f_1(\Delta^0) \\ \frac{\partial f_2(\Delta^0)}{\partial \Delta_1}\delta\Delta_1 + \frac{\partial f_2(\Delta^0)}{\partial \Delta_2}\delta\Delta_2 + \cdots + \frac{\partial f_2(\Delta^0)}{\partial \Delta_n}\delta\Delta_n = -f_2(\Delta^0) \\ \frac{\partial f_3(\Delta^0)}{\partial \Delta_1}\delta\Delta_1 + \frac{\partial f_3(\Delta^0)}{\partial \Delta_2}\delta\Delta_2 + \cdots + \frac{\partial f_3(\Delta^0)}{\partial \Delta_n}\delta\Delta_n = -f_3(\Delta^0) \\ \cdots \\ \frac{\partial f_n(\Delta^0)}{\partial \Delta_1}\delta\Delta_1 + \frac{\partial f_n(\Delta^0)}{\partial \Delta_2}\delta\Delta_2 + \cdots + \frac{\partial f_n(\Delta^0)}{\partial \Delta_n}\delta\Delta_n = -f_n(\Delta^0) \end{gathered} \quad (9)$$

3.2.2 *Algorithm*

- *Step 1.* Giving the node displacement vector an initial value

$$\Delta_i = \Delta_i^0 \text{(can be given: } \Delta_i\text{=0)}$$

- *Step 2.* Calculating the value of functions $f_i(\Delta^0)$and partial derivatives $\frac{\partial f_i(\Delta^0)}{\partial \Delta_j}$.Solve the system of equations (9). The obtained root is the node displacement increment vector $\delta\Delta_i$.

- *Step 3.* Recalculating the node displacement vector$\Delta_i^1 = \Delta_i^0 + \delta\Delta_i$

- *Step 4.* Checking program stop condition

Calculating deviation: $\varepsilon = \sum f_i^2(\Delta_1)$

If $\varepsilon \leq [\varepsilon]$, then stop the program to take the root$\Delta_i = \Delta_i^1$

If $\varepsilon > [\varepsilon]$ then continue to loop with $\Delta_i^0 = \Delta_i^1$

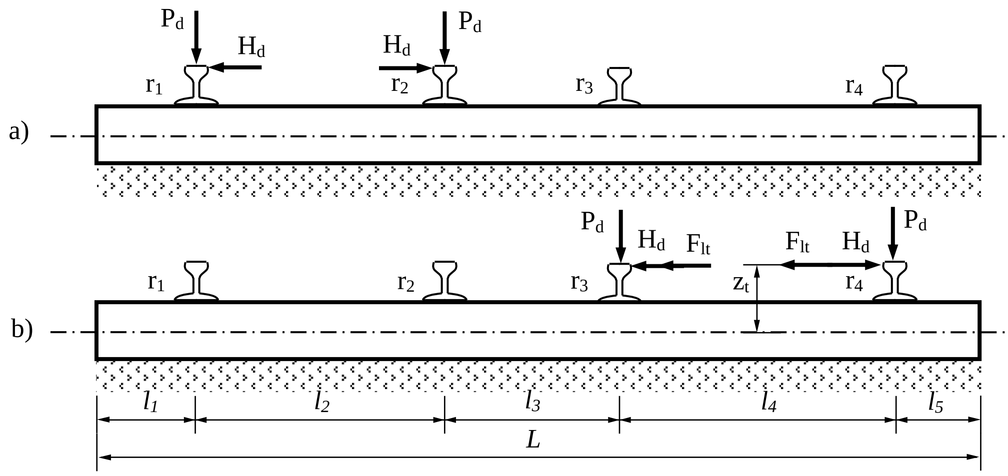
4 Research problem

A turnout sleeper made of prestressed concrete. Train runs on the sleeper with a straight line and a turning route. There is a bearing diagram as shown in Fig. 2. Dynamic force acting on the top of the rail P_d = 70.560 kN. Ballast with stiffness c is referenced in the literature [15-17] and experimentally. In this problem, the authors will calculate, compare, and analyze the results of calculating sleepers of turnout with Winkler and unilateral contact models as follows:

Comparing the results of the sleeper analysis calculated according to the Winkler model and the unilateral contact model.

Comparing the results of the analysis of the sleepers according to the traditional method with railway bed stiffness the reference in the literature [15-17] and according to the unilateral contact model with the experimental railway bed stiffness.

4.1.1 The Parameters Included in the Analysis



a - in the case of a train running on a straight line; b - in the case the train running on the turn line

Fig. 2 – Bearing diagram of sleepers at turnout

Load: P_d = 70,560 N; H_d = 28,224 N; F_{lt} = 2885 N; the distance from the point of application of loads H_d, F_{lt} to the axis of the sleepers is z_t = 0.2305 m. Assume that the rails are closely linked to the sleepers. The load acting on the rails when the locomotive runs on a straight line includes the dynamic load P_d, the horizontal shaking force H_d. When the locomotive runs on the line turning outside P_d and H_d, the rails are subjected to additional centrifugal force F_{lt} (Fig. 2).

Sleepers made of prestressed concrete: $E = 36 \times 10^9$ N/m^2; $F = 0.0513$ m^2; $J = 138.4958 \times 10^{-6}$ m^4;

bottom width B_t = 0.2900 m;

length L = 3.9000 m; l_1 = 0.4650 m; l_2 = 1.0700 m; l_3 = 0.7082 m; l_4 = 1.0750 m; l_5 = 0.5818 m.

Railway bed stiffness according to the document [17], c_{tl}=15 ×10^7 N/m^3, according to the experiment c_{tn}=143.35×10^7 N/m^3 (Fig. 3).

Fig. 3 – Measuring railway bed stiffness at the site of turnout

4.1.2 Selecting the Number of Elements to Discretely Sleepers While Ensuring Accuracy

Discreting the sleepers into n elements of length lpt, the ballast system below the sleepers is also modeled as unilateral contact placed at the end of the elements. Unilateral contacts between sleepers have stiffness $k = l_{pt} \times c \times$ Bt. For two unilateral contacts at the end of sleepers, stiffness: k= $0.5 \times l_{pt} \times c \times B_t$(Fig. 1).

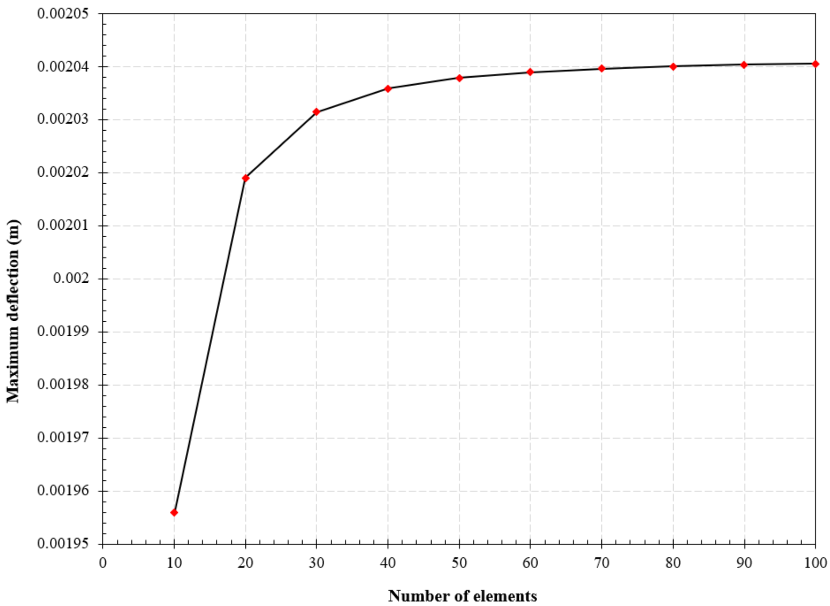


Fig. 4 – The maximum deflection in sleepers with different number of elements

Before analyzing and comparing with the results of the sleepers with the Winkler model and the unilateral foundation model, the authorsanalyze the sleepers with the discrete options of the foundation into the system 11, 21,…,101 links, corresponding to discretizing sleepers into 10, 20,..., 100 elements. Figures 4 and Table 1 are histogram and table of values showing the maximum deflection change in sleepers according to the number of elements used to discretely separate the sleepers. The analysis results show that as the number of elements used to discretize sleepers increases, the maximum deflection in sleepers gradually approaches a constant value. As the number of elements gets closer to 100, the difference in maximum deflection in the sleepers decreases between the split options with the other number of elements. Since the difference in maximum deflection between the discrete option of sleepers into 90 and 100 elements is only 0.0101%, it can be considered that the option of discrete sleepers into 100 elements will give results close to the behavior its reality. The next analysis of the problem the authors will perform with the sleepers are discrete into 100 elements and the foundation system is replaced by 101 unilateral contacts.

Table 1 – Maximum deflection value in sleepers with different number of elements

Number of elements	$V_{max} \times 10^{-3}$	Deviation (%)
10	1.9558	
20	2.0190	3.2269
30	2.0314	0.6163
40	2.0358	0.2155
50	2.0378	0.0991
60	2.0389	0.0536
70	2.0395	0.0325
80	2.0400	0.0210
90	2.0403	0.0143
100	2.0405	0.0101

4.1.3 Comparing the Results of Sleeper Analysis with Winkler Foundation Model and Unilateral Foundation Model When the Foundation Stiffness Is Referenced in the Literature [17]

Discreting sleepers into 100 elements, set up input data files, analyze internal forces, beam displacement using ANISOL program, export analysis results in *.csv format and charts. These analyzes account for two load cases running on the straight line and running on the turn line. The analysis results of sleepers on ballast with unilateral foundation model and Winkler foundation model are shown in Figs.5, 6 and Table 2.

Table 2 – Comparing the results of calculating the deflection, and moment when the locomotive runs straight and turn line according to the unilateral foundation model and the Winkler model

Parameter	Straight line			Turn line		
	Unilateral	Winkler	Deviation (%)	Unilateral	Winkler	Deviation (%)
Minimum deflection (upward displacement), (m)	0.0015	0.0002	534.0962	0.0013	0.0003	349.9982
Maximum deflection (downward displacement), (m)	-0.0020	-0.0020	0.0426	-0.0015	-0.0015	1.1222
Maximum moment (N.m)	15142.4012	14078.4329	7.5574	15065.2808	14063.7210	7.1216
Minimum moment (N.m)	-4847.0153	-5092.7152	4.8245	-1811.6215	-2805.4637	35.4252

For conventional sleepers, when the train is running on them, sleepers always move downwards, so the traditional calculation with the Winkler model and the unilateral foundation model will give the same results. But with sleepers for turnout, it is different. when there is a train running on one side of the sleepers, the displacement is downward and the other side is moving up. Fig. 5 is a graph of the deflection of sleepers calculated according to the Winkler foundation model and the unilateral foundation model in the case of locomotives running on turn and straight lines. The deflection graph shows that the sleeper segment has an upward displacement (the sleeper and the ballast do not contact each other) up to 27% of the sleeper length with the locomotive running on the turn line and 31% of the sleeper length in the case of locomotive running on a straight line. Such a large percentage of the sleeper lengths that are not in contact with the ballast leads to very different calculation schemes obtained by the traditional Winkler base model and the unilateral foundation base model. Regarding the deflection shown in Fig. 5 and Table 2, the deflection chart analyzed according to the Winkler model is very different from the unilateral foundation model, the smallest difference in deflection calculated according to the two foundation models is up to 349.9982% in the case of trains running on a turn line, and 534.0962% in the case of trains running on a straight line. Regarding the bending moment, the analysis results according to the Winkler model and the unilateral foundation model are also very different (Fig. 6.). The difference in the value of the bending moment according to the two models is also up to 35.4252% for the case train running the turn line.

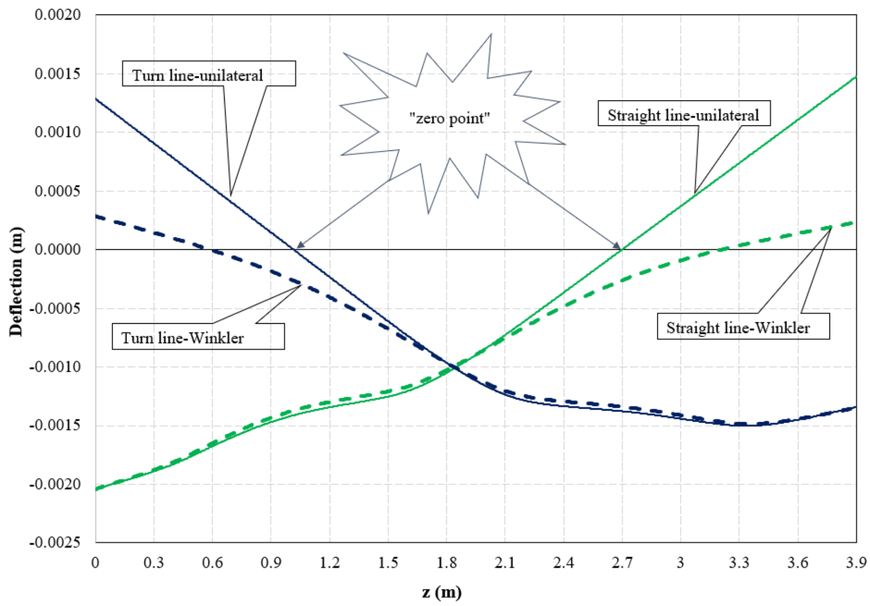


Fig. 5 – The deflections of the sleepers when the locomotive runs on the straight and the turn line are calculated according to the unilateral and Winkler model

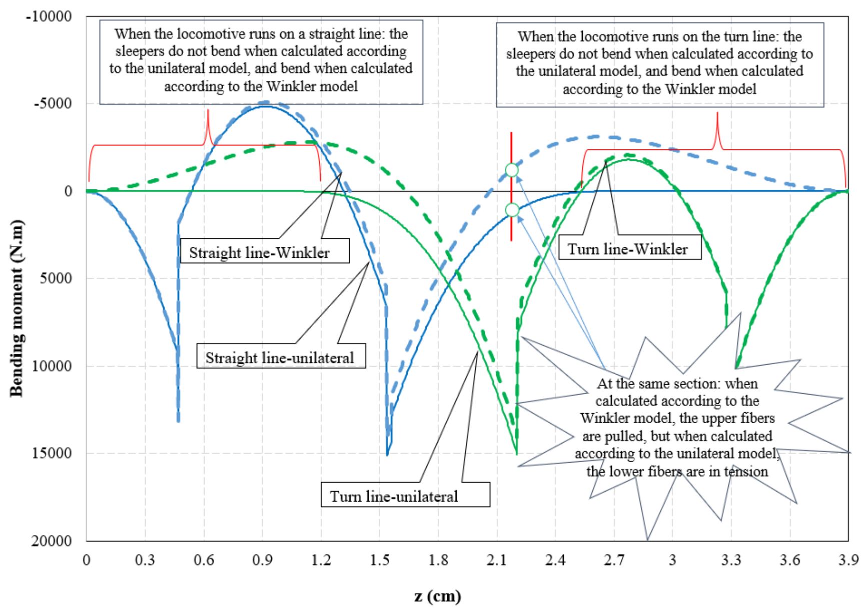


Fig. 6 – The moments of sleepers when the locomotive runs on straight and turn line are calculated according to the unilateral and Winkler model

More specifically, when analyzing sleepers according to the Winkler model, all sleepers are bent, but when analyzing sleepers according to a unilateral foundation model, in both cases of train running, the end of segments of sleepers no appear bending (Fig.6.). There are some sleeper sections, when calculated according to the Winkler model, the upper fibers are pulled, but when calculated according to the unilateral model, the lower fibers are in tension. These differences will cause designers to have a wrong orientation in the arrangement of load-bearing materials for sleepers if they continue to use the Winkler base model in computational analysis.

"Zero point" is the boundary point lying on the axis of the sleepers to distinguish between the downward and upward displacement of the beam. The exact determination of the "zero point" is a long-standing wish of railway technicians, but there are no tools to implement it. With the theoretical basis and the ANISOL calculation program, the authors hope to be able to help railway technicians determine this "zero point".

4.1.4 Comparing the Results of Turnout Sleeper Analysis between the Winkler Foundation Model with the Reference Foundation Stiffness in the Document [17] and the Unilateral Foundation Model with the Experimental Foundation stiffness.

Usually, the design of cross-sections and materials is based on the results of analysis of internal forces and displacements with the Winkler foundation model with a constant foundation stiffness as the reference value in the document [17]. However, in the railway section studied, the foundation stiffness has a great change after a period of use. This makes the results of the re-analysis of the sleepers with the new foundation stiffness are very different from the results of the original analysis with the Winkler foundation model and the original assumed foundation stiffness (Figs. 7, 8).

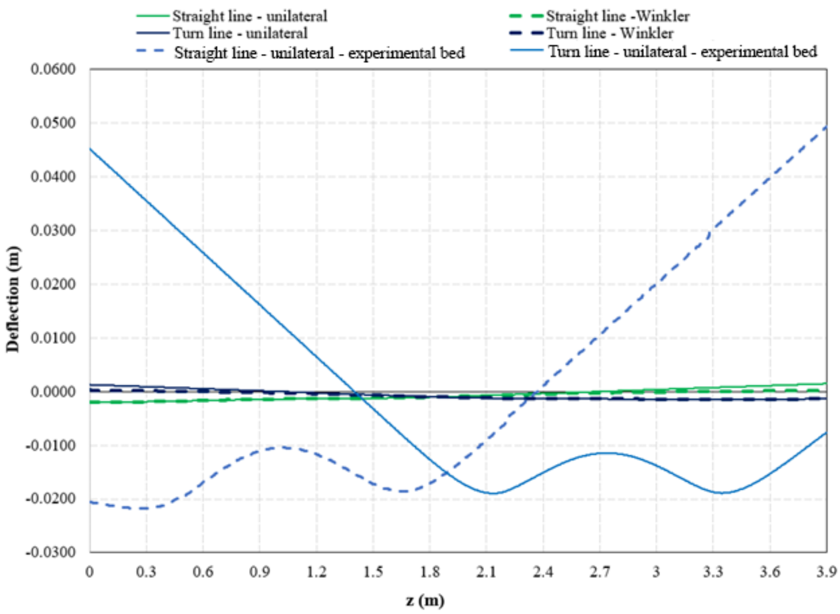


Fig. 7 – Deflections of sleepers when the locomotive runs on straight and turn line in the following cases: Winkler model with hypothetical foundation stiffness, unilateral foundation model with hypothetical foundation stiffness, and unilateral foundation model with experimental foundation stiffness

This difference value is up to units of thousands (Table 3). In this survey, although the bending moment in the sleepers in the case of using the experimental foundation stiffness is smaller than the bending moment obtained when calculating the calculation. according to the Winkler model with the assumed foundation stiffness, but this assumption is completely changed when paying attention to the deflection graph. This also means that over time, the scale of the sleepers being separated from the ballast during the load-bearing process increases gradually, accordingly the fatigue effect due to the dynamic load transmitted to the sleepers also increases, causing the sleepers to fall. The turnout sleeper tends to suffer fatigue damage after a period of use.

From the above analysis results, it is shown that, using the Winkler model with a constant foundation stiffness not only cannot describe the sleeper's performance closely, but also cannot predict the change in foundation stiffness in the future, leading to analysis results recorded after a period of use are very different from the results of the original analysis. The authors believe that it is necessary to add new calculations in the design of turnout sleepers such as analysis of turnout sleepers with a one-unilateral foundation model, analysis of behavior of turnout sleepers over time of use, etc.

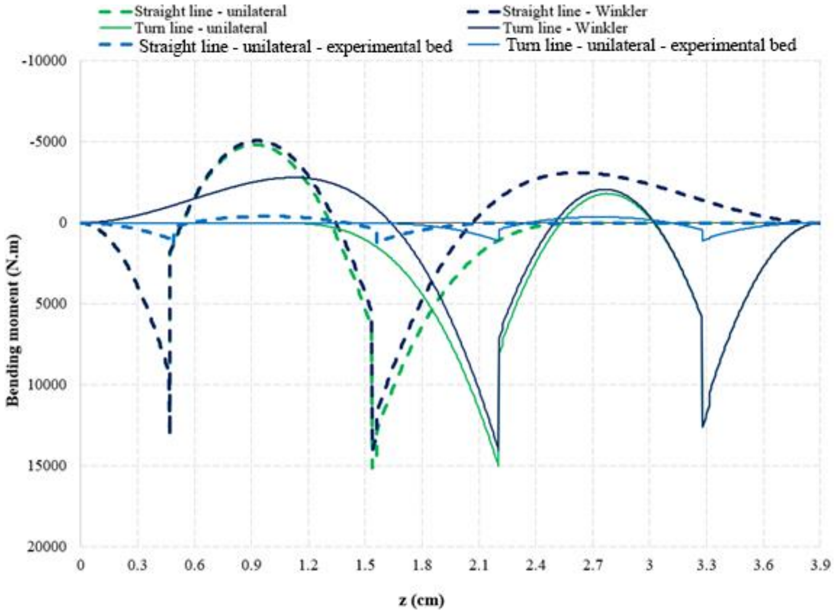


Fig. 8 – Bending moment of sleepers when locomotive runs on straight and turn line in the following cases: Winkler model with hypothetical foundation stiffness, unilateral foundation model with hypothetical foundation stiffness, and unilateral foundation model with the experimental foundation stiffness

Table 3 – Comparing the results of calculating according to unilateral foundation model - experimental foundation stiffness and Winkler model - reference foundation stiffness [17]

Parameter	Straight line			Turn line		
	Unilateral	Winkler	Deviation (%)	Unilateral	Winkler	Deviation (%)
Minimum deflection (upward displacement), (m)	0.0493	0.0002	21110.1385	0.0452	0.0003	15725.7899
Maximum deflection (downward displacement), (m)	-0.0218	-0.0020	966.2919	-0.0190	-0.0015	1175.8275
Maximum moment, (N.m)	1358.7240	14078.4329	90.3489	1110.0081	14063.7210	92.1073
Minimum moment, (N.m)	-410.9525	-5092.7152	91.9306	-366.0275	-2805.4637	86.9530

5 Conclusions

The authors have set up a calculation model, theoretical basis, and an algorithm to analyze sleepers under the effect of static loads.

The analysis shows that the analysis of traditional turnout sleepers with an assumed foundation stiffness is not close to the actual behavior.

It is recommended to include the unilateral contact model in the analysis of turnout sleepers, as well as the need to check the actual stiffness of the foundation before it is included in the analysis.

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