Exclusion and zero-range in the rarefaction fan

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Abstract In these notes we briefly review asymptotic results for the totally asymmetric simple exclusion process and the totally asymmetric constant-rate zero-range process, in the presence of particles with different priorities. We review the Law of Large Numbers for a second class particle added to those systems and we present the proof of crossing probabilities for a second and a third class particles. This is done, for the exclusion process, by means of a particle-hole symmetry argument, while for the zero-range process it is a consequence of a coupling argument.

1 Introduction

In these notes we review some asymptotic results on two classical interacting particle systems: the totally asymmetric simple exclusion process and the totally asymmetric constant-rate zero-range process, in the presence of particles with different priorities. These processes are taken on $\mathbb{Z}$ and at each site $x \in \mathbb{Z}$ we place a random clock T_x, which is distributed according to an exponential law with parameter 1. The collection of clocks $\{T_x\}_{x\in\mathbb{Z}}$ forms a sequence of independent and identically distributed random variables. Initially we randomly distribute particles along the lattice and each time a clock rings, if there is a particle at the corresponding site, then it decides to jump to one of its nearest-neighbors. If there is no particle at that site, then nothing happens and the clocks restart. We consider two types of jumps in these notes. The first type of jump is realized under an exclusion rule, therefore the particle system coined the name simple exclusion process. In this process a jump from a site x to $x+1$ occurs at rate 1, but the jump is performed if and only if the

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destination site is empty. In the figure below we represent particles by $\bullet$, holes by $\circ$ and the jumping particle by $\underline{\bullet}$. A possible jump is

$$\bullet\bullet\circ\bullet\overset{\curvearrowright}{\underline{\bullet}}\circ\circ\bullet\bullet \longrightarrow \bullet\bullet\circ\bullet\circ\bullet\circ\bullet\bullet.$$

A forbidden jump is

$$\bullet\bullet\circ\overset{\curvearrowright}{\underline{\bullet}}\bullet\circ\circ\bullet.$$

This Markov process is denoted by $\{\eta_t : t \geq 0\}$ and has state space $\Omega_{EP} := \{0,1\}^{\mathbb{Z}}$. For this model there is at most a particle per site, so its configurations, denoted by η, consist in vectors whose components are either 0 or 1. Physically the interpretation $\eta(x) = 1$ means that the site x is occupied.

The second type of jump that we consider is described as follows. There is no restriction on the number of particles at each site and if the clock at the site x rings and if there is at least one particle at that site, then it jumps from x to $x+1$ at rate $1/\xi_x$, where ξ_x denotes the number of particles at the site x. In this case, the jump occurs independently from the number of particles at the destination site.

A possible jump is

$$\begin{array}{ccccccccccc}
\bullet & \bullet & & & & & & & & & \\
\bullet & \bullet & \bullet & \bullet & \curvearrowright & & & & \bullet & & \\
\bullet & \bullet & \bullet & \bullet & \underline{\bullet} & \bullet & \circ & \circ & \bullet & \circ & \circ
\end{array}$$

This Markov process is denoted by $\{\xi_t : t \geq 0\}$ and has state space $\Omega_{ZR} := \mathbb{N}_0^{\mathbb{Z}}$. The configurations of this model are denoted by ξ and consist in vectors whose components contain one number of $\mathbb{N}_0$. Physically, the interpretation $\xi(x) = k$, for $k \in \mathbb{N}_0$ means that the site x is occupied with k particles.

We will add to these particle systems a ”special” particle, which is seen by the remaining particles as a hole and it is seen by the holes as a particle, therefore this particle is called a *second class particle*. We will first present the LLN for this particle starting both processes from initial conditions in the rarefaction fan. Then, we will consider both processes in the presence of a second class particle and a third class particle at its right site. The first and second class particles see the third class particle as a hole, but the third class particle does not distinguish the second class particle from the first class particles. We will prove, by a symmetry argument, that for the exclusion, the probability of the second class particle swapping order with the third class particle is equal to $2/3$. As a consequence, by coupling the exclusion with the zero-range, the probability of the second class particle being at the right hand side or at the same site of the third class particle, in the zero-range, equals $2/3$.

The outline of these notes is described as follows. In Section 2, we define the processes, their invariant measures and a set of measures which are not invariant but lead in the hydrodynamics to the rarefaction fan of the associated hydrodynamic equation. In Section 3, we describe the hydrodynamics for these processes and in Section 4, we state a LLN for a second class particle in a rarefaction setting. In Section 5 we present a coupling between both processes and in Section 6 we discuss crossing probabilities for second and third class particles.

2 The models

Let $\{\eta_t; t \geq 0\}$ be the one-dimensional totally asymmetric simple exclusion process (TASEP), a continuous time Markov process with state space Ω_{EP} whose infinitesimal generator is defined on local functions $f : \Omega_{EP} \to \mathbb{R}$ as

$$\mathcal{L}_{EP} f(\eta) = \sum_{x\in\mathbb{Z}} \eta(x)(1-\eta(x+1))[f(\sigma_{EP}^{x,x+1}\eta) - f(\eta)].$$

Above $(\sigma_{EP}^{x,x+1}\eta)(x) = \eta(x+1)$, $(\sigma_{EP}^{x,x+1}\eta)(x+1) = \eta(x)$ and on other sites $\sigma_{EP}^{x,x+1}\eta$ coincides with η. As an example see the figure below in which the particle underlined is at the site x.

$$\bullet\bullet\circ\bullet\underline{\bullet}\circ\circ\bullet \quad \eta \longrightarrow \bullet\bullet\circ\bullet\underline{\circ}\bullet\circ\bullet \quad \sigma_{EP}^{x,x+1}\eta$$

Now, let $\{\xi_t; t \geq 0\}$ be the one-dimensional constant-rate totally asymmetric zero-range process (TAZRP), a continuous time Markov process with state space Ω_{ZR} whose infinitesimal generator is defined on local functions $f : \Omega_{ZR} \to \mathbb{R}$ as

$$\mathcal{L}_{ZR} f(\xi) = \sum_{x\in\mathbb{Z}} \mathbf{1}\{\xi(x) \geq 1\}[f(\sigma_{ZR}^{x,x+1}\xi) - f(\xi)],$$

where $(\sigma_{ZR}^{x,x+1}\xi)(x) = \xi(x) - 1$, $(\sigma_{ZR}^{x,x+1}\xi)(x+1) = \xi(x+1) + 1$ and on other sites $\sigma_{ZR}^{x,x+1}\xi$ coincides with ξ. As an example see the figure below in which the particle underlined is at the site x.

$$\begin{array}{l}
\bullet\bullet \\
\bullet\bullet\bullet\bullet\quad\;\;\;\bullet \\
\bullet\bullet\bullet\bullet\underline{\bullet}\bullet\circ\circ\bullet\circ\circ \qquad \xi \\
\qquad\quad\downarrow \\
\bullet\bullet \\
\bullet\bullet\bullet\bullet\;\;\bullet\;\;\;\bullet \\
\bullet\bullet\bullet\bullet\underline{\circ}\bullet\circ\circ\bullet\circ\circ \qquad \sigma_{ZR}^{x,x+1}\xi
\end{array}$$

Fore more details on the construction of these models we refer to [9, 2].

Now, we describe briefly the invariant measures for these processes. We start with the TASEP. It is well known that the Bernoulli product measure of parameter $\alpha \in [0,1]$, that we denote by ν_α, is invariant for the TASEP. This measure is defined on Ω_{EP}, is translation invariant and parameterized by the density α, namely: $E_{\nu_\alpha}[\eta(x)] = \alpha$ for any $x \in \mathbb{Z}$. For $x \in \mathbb{Z}$, $k \in \{0,1\}$ and $\alpha \in [0,1]$, its marginal is given by

$$\nu_\alpha(\eta : \eta(x) = k) = \alpha^k(1-\alpha)^{1-k}.$$

For the TAZRP, it is known that the Geometric product measure of parameter $\frac{1}{1+\rho}$ with $\rho \in (0,+\infty)$, that we denote by μ_ρ, is invariant. That is, μ_ρ is defined on Ω_{ZR} and for $x \in \mathbb{Z}$ and $k \in \mathbb{N}_0$, μ_ρ has marginal given by

$$\mu_\rho(\xi : \xi(x) = k) = \Big(\frac{\rho}{1+\rho}\Big)^k \frac{1}{1+\rho}.$$

Since we are interested in analyzing the processes in the rarefaction fan, we will make use of the following measures. For $\alpha, \beta \in [0,1]$ let $\nu_{\alpha,\beta}$ be the product measure, such that for $x \in \mathbb{Z}$ and $k \in \{0,1\}$

$$\nu_{\alpha,\beta}(\eta : \eta(x) = k) = \begin{cases} \alpha^k(1-\alpha)^{1-k}, \text{ if } x < 0 \\ \beta^k(1-\beta)^{1-k}, \text{ if } x \geq 0 \end{cases}. \tag{1}$$

Analogously, for $\rho, \lambda \in (0, +\infty)$ let $\mu_{\rho,\lambda}$ be the product measure such that for $k \in \mathbb{N}_0$ and $x \in \mathbb{Z}$

$$\mu_{\rho,\lambda}(\xi : \xi(x) = k) = \begin{cases} \Big(\frac{\rho}{1+\rho}\Big)^k \frac{1}{1+\rho}, \text{ if } x < 0 \\ \Big(\frac{\lambda}{1+\lambda}\Big)^k \frac{1}{1+\lambda}, \text{ if } x \geq 0 \end{cases}. \tag{2}$$

Moreover, below we also consider the zero-range process starting from the measure $\mu_{\infty,\lambda}$, with $\lambda \geq 0$. This means that, if a configuration $\xi \in \Omega_{ZR}$ is distributed according to $\mu_{\infty,\lambda}$, then $\xi(x) = \infty$ for $x < 0$ and $\xi(x)$ is distributed according to μ_λ for $x \geq 0$. When $\lambda = 0$, $\mu_{\infty,0}$ gives weight one to the configuration $\tilde{\xi}$, such that $\tilde{\xi}(x) = \infty$ for $x < 0$ and $\tilde{\xi}(x) = 0$ for $x \geq 0$.

3 Hydrodynamics

The hydrodynamic limit consist in a LLN for the empirical measure process associated to a particle system [10]. For that purpose, given a process ζ_t, let $\pi^n(\zeta_t, du)$ be the empirical measure given by

$$\pi^n(\zeta_t, du) = \frac{1}{n}\sum_{x\in\mathbb{Z}} \zeta_t(x)\delta_{\frac{x}{n}}(du).$$

Here δ_u denotes the Dirac measure at u.

Now, fix a measure μ_n associated to a profile $\rho_0 : \mathbb{R} \to \mathbb{R}$ defined by

$$\rho_0(u) = \theta_1 \mathbf{1}\{u < 0\} + \theta_2 \mathbf{1}\{u \geq 0\},$$

with $\theta_1 > \theta_2$. Since the work of [12], it is known that starting the TASEP or TAZRP from such μ_n, if $\pi^n_0(\xi, du)$ converges to $\rho_0(u)du$ in probability, as $n \to +\infty$, then $\pi^n_{tn}(\zeta, du)$ converges to $\rho(t,u)du$ in probability, as $n \to +\infty$, where $\rho(t,u)$ is the unique entropy solution of the corresponding hydrodynamic equation. For both processes the hydrodynamic equation is given by

$$\partial_t \rho(t,u) + \partial_u j(\rho(t,u)) = 0,$$

with initial condition $\rho(0,u) := \rho_0(u)$ for all $u \in \mathbb{R}$. In fact, the aforementioned result is more general [12], but as it is stated it is sufficient for our purposes. The function $j(\rho)$ corresponds to the mean (with respect to the invariant measure of the process, that we represent generically by m_r, indexed in r) of what is called the instantaneous current at the bond $\{0,1\}$. Since jumps are totally asymmetric, this current is simply the jump rate to the right neighboring site. For the TASEP, the instantaneous current is $\eta(0)(1-\eta(1))$ and since $m_r = \nu_\alpha$, we get

$$j(\alpha) := E_{\nu_\alpha}[\eta(0)(1-\eta(1))] = \alpha(1-\alpha)$$

and the hydrodynamic equation becomes

$$\partial_t \rho(t,u) + \partial_u \Big(\rho(t,u)(1-\rho(t,u))\Big) = 0, \tag{3}$$

which is known as the *inviscid Burgers equation*. For the TAZRP, the instantaneous current is $\mathbf{1}\{\xi(0) \geq 1\}$ and since $m_r = \mu_\rho$, we get

$$j(\rho) := E_{\mu_\rho}[\mathbf{1}\{\xi(0) \geq 1\}] = \frac{\rho}{1+\rho}$$

and the hydrodynamic equation becomes

$$\partial_t \rho(t,u) + \partial_u \Big(\frac{\rho(t,u)}{1+\rho(t,u)}\Big) = 0. \tag{4}$$

Now, we notice that the solution of (3) under the initial condition $\rho(0,u) = \alpha \mathbf{1}\{u<0\} + \beta \mathbf{1}\{u \geq 0\}$, with $\alpha > \beta$, is given by

$$\rho(t,u) = \begin{cases} \alpha, \text{ if } u < (1-2\alpha)t \\ \beta, \text{ if } u > (1-2\beta)t \\ \frac{t-u}{2t}, \text{ if } (1-2\alpha)t \leq u \leq (1-2\beta)t \end{cases} \tag{5}$$

and the solution of (4) under the initial condition $\rho(0,u) = \rho \mathbf{1}\{u<0\} + \lambda \mathbf{1}\{u \geq 0\}$, with $\rho > \lambda$, is given by

$$\rho(t,u) = \begin{cases} \rho, \text{ if } u < \frac{t}{(1+\rho)^2} \\ \lambda, \text{ if } u > \frac{t}{(1+\lambda)^2} \\ \frac{\sqrt{t}-\sqrt{u}}{\sqrt{u}}, \text{ if } \frac{t}{(1+\rho)^2} \leq u \leq \frac{t}{(1+\lambda)^2}. \end{cases} \tag{6}$$

We will see that these solutions, under a proper renormalization, are the probability density functions of a ”special” particle whose dynamics we define below.

4 Law of Large Numbers for a second class particle

In this section we describe the LLN for a second class particle added to the TASEP and to the TAZRP. Since the dynamics of this particle is completely different in these processes, we start by describing its motion in the TASEP. Suppose to start the TASEP from a configuration η as for example

$$\bullet\bullet\circ\bullet\underline{\circledast}\circ\circ\bullet \qquad \eta$$

In this case $\circledast$ represents the second class particle and a particle underlined means it stands at the origin. Suppose now, that the clock T_{-1} rings and the particle at $x=-1$ jumps to the origin. In spite of the exclusion rule and the fact that the origin being occupied with a second class particle, the jump is performed and the particles exchange positions.

$$\bullet\bullet\circ\bullet\underline{\circledast}\circ\circ\bullet \qquad \eta \longrightarrow \bullet\bullet\circ\circledast\underline{\bullet}\circ\circ\bullet \qquad \eta^{-1,0}$$

For this reason the particle represented by $\circledast$ is a second class particle and the particle represented by $\bullet$ is a first class particle, since it has priority to jump.

On the other hand, if on η the second class particle jumps to its right, then the jump is performed and it exchanges positions with the hole to its right.

$$\bullet\bullet\circ\bullet\underline{\circledast}\circ\circ\bullet \qquad \eta \longrightarrow \bullet\bullet\circ\bullet\underline{\circ}\circledast\circ\bullet \qquad \eta^{0,1}$$

Now, if initially the second class particle attempts to jump to its right neighboring site which is occupied by a first class particle, then nothing happens.

$$\bullet\bullet\circ\bullet\underline{\circledast}\bullet\circ\bullet \qquad \zeta \longrightarrow \bullet\bullet\circ\bullet\underline{\circledast}\bullet\circ\bullet \qquad \zeta$$

Concluding, in the TASEP, a second class particle can jump backwards and this happens if and only if a first class particle at its left attempts to jump to the right.

In the TAZRP the dynamics of a second class particle is substantially different from the dynamics described above. Consider the TAZRP starting from a configuration ξ as for example the one given below.

$$\begin{array}{ccccccccccc}
\bullet & \bullet & & & & & & & & & \\
\bullet & \bullet & \bullet & \bullet & & & & & & & \\
\bullet & \bullet & \bullet & \underline{\circledast} & \bullet & \circ & \circ & \bullet & \circ & \qquad & \xi
\end{array}$$

Suppose now, that the clock at the origin rings. Then, the first class particle at the origin jumps to the right and the second class particle remains at the origin.

$$\begin{array}{ccccccccccc}
\bullet & \bullet & & & & & & & & & \\
\bullet & \bullet & \bullet & & \bullet & & & & & & \\
\bullet & \bullet & \bullet & \underline{\circledast} & \bullet & \circ & \circ & \bullet & \circ & \qquad & \xi^{0,1}
\end{array}$$

Now, if the clock at the origin rings again, then the second class particle can jump to the right.

$$\begin{array}{ccccccccc}
\bullet & \bullet & & & \bullet & & & & \\
\bullet & \bullet & \bullet & & \bullet & & & & \\
\bullet & \bullet & \bullet & \underline{\circ} & \circledast & \circ & \circ & \bullet & \circ
\end{array}$$

Concluding, in the TAZRP, a second class particle can never jump backwards and it only jumps from a site x to $x+1$, if there is no other first class particle at x, and the jump occurs independently of the number of particles at $x+1$.

A second class particle in the TASEP or TAZRP can be obtained considering the 'basic coupling' for those processes. The idea is the following. Consider two TAZRP ξ_t^0 and ξ_t^1 starting from initial configurations ξ_0^0 and ξ_0^1 such that $\xi_0^0(x) \leq \xi_0^1(x)$ for all $x \in \mathbb{Z}$. We couple the two processes so that whenever a particle in the ξ^0 configuration moves, a corresponding ξ^1 particle makes the same jump. That is, a particle at x in the ξ^0 and ξ^1 processes jumps to $x+1$ with rate $\mathbf{1}\{\xi^0(x) \geq 1\}$ and also one of the particles at x in the ξ^1 process displaces by 1 with rate $\mathbf{1}\{\xi^1(x) \geq 1\} - \mathbf{1}\{\xi^0(x) \geq 1\}$. Then, we can write $\xi_t^1 = \xi_t^0 + Z(t)$, where, $Z(t)(x)$ counts the second-class particles. For the TASEP it is analogous.

Now we describe the asymptotic limit for a second class particle in TASEP.

Theorem 1. *([4, 7, 8, 11])*
Consider the TASEP starting from $\nu_{\alpha,\beta}$ with $0 \leq \beta < \alpha \leq 1$. At time $t = 0$ put a second class particle at the origin regardless the value of the configuration at this point and let $X_2^{EP}(t)$ denote the position of this particle at time t. Then

$$\lim_{t\to+\infty} \frac{X_2^{EP}(t)}{t} = \mathcal{U}, \quad \textit{almost} \quad \textit{surely},$$

where $\mathcal{U}$ is uniformly distributed on $[1-2\alpha, 1-2\beta]$. That is

$$F_{\mathcal{U}}(u) := P(\mathcal{U} \leq u) = \frac{\beta - \rho(1,u)}{\beta - \alpha},$$

where $\rho(t,u)$ is given in (5).

The proof of last result for convergence in distribution was given in [7] and it was generalized to partial asymmetric jumps in [4]. The almost sure convergence was derived in [8] and in [11]. In TAZRP the asymptotic limit for a second class particle is given in the next theorem.

Theorem 2. *([3])*
Consider the TAZRP starting from $\mu_{\rho,\lambda}$, with $0 \leq \lambda < \rho \leq \infty$. At time $t = 0$ add a second class particle at the origin and let $X_2^{ZR}(t)$ denote its position at time t. Then

$$\lim_{t\to+\infty} \frac{X_2^{ZR}(t)}{t} = \mathcal{V} = \Big(\frac{1+\mathcal{U}}{2}\Big)^2, \quad \textit{almost surely},$$

where $\mathcal{U}$ is uniformly distributed on $\Big[\dfrac{1-\rho}{1+\rho}, \dfrac{1-\lambda}{1+\lambda}\Big]$. That is,

$$F_{\mathcal{V}}(u) := P(\mathcal{V} \leq u) = \frac{1+\lambda}{\rho-\lambda}\Big((1+\rho)(1-j(\rho(1,u))) - 1\Big),$$

where $\rho(t,u)$ is given in (6) and $j(\cdot)$ is given above (4).

5 Coupling TASEP and TAZRP with a second class particle

In this section we present a coupling between the TASEP and the TAZEP in the presence of one second class particle. It uses the particle to particle coupling introduced in [3] and it relates the TAZRP and TASEP in such a way that the position of the second class particle in the TAZRP corresponds to the flux of holes that crossover the second class particle in the TASEP. Now we explain the relation between the configurations of the two processes. To make easier the exposition we give an example of a initial configuration for TASEP as below.

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Let $X_2^{EP}(t)$ denote the position at time t of the second class particle in TASEP. Initially, we label the holes by denoting the position of the i-th hole at time 0 by $x_i(0)$. To simplify notation, we label the leftmost (resp. rightmost) hole at the right (resp. left) hand side of the second class particle at time $t = 0$ by 1 (resp. -1). Both processes are related in such a way that basically on the TASEP the distance between two consecutive holes minus one is the number of particles at a site in the TAZRP, but near the second class particle one has to be more careful. At time 0, we define:

- for $i = X_2^{EP}(0) - 1$: $\xi(i)$ is the number of particles between $X_2^{EP}(0)$ and the first hole to its left, therefore, $\xi(i) = x_1(0) - X_2^{EP}(0) - 1$;
- for $i = X_2^{EP}(0)$: $\xi(i)$ has a second class particle plus a number of first class particles that coincides with the number of first class particles between $X_2^{EP}(0)$ and the first hole to its right, therefore, $\xi(i)$ has $X_2^{EP}(0) - x_{-1}(0) - 1$ first class particles and a second class particle;
- for $i \in \mathbb{Z} \setminus \{X_2^{EP}(0) - 1, X_2^{EP}(0)\}$: $\xi(i)$ corresponds to the number of particles between consecutive holes, therefore, for $\kappa > 0$ and for $i = X_2^{EP}(0) + \kappa$, $\xi(i) = x_{\kappa+1}(0) - x_\kappa(0) - 1$, similarly for $\kappa < 0$;

For example in the configuration above we have $x_{-3}(0) = -11$, $x_{-2}(0) = -7$, $x_{-1}(0) = -3$, $X_2^{EP}(0) = 0$, $x_1(0) = 2$, $x_2(0) = 4$, $x_3(0) = 5$, $x_4(0) = 6$, $x_5(0) = 8$, $x_6(0) = 9$, which corresponds in TAZRP to

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With the established relations we notice that for a positive site (resp. negative site) if in the TAZRP there are k particles at a given site, then for the TASEP there are k particles plus a hole to their right (resp. left). For positive (resp. negative) sites there are k particles at that site with probability $\alpha^k(1-\alpha)$ (resp. $\beta^k(1-\beta)$). For the TAZRP at the site $X_2^{ZR}(t)$ there are k particles, if in the TASEP there are k particles plus a hole to the right of the second class particle. By the definition of the invariant measures for the TAZRP we have that $\alpha = \rho/(1+\rho)$ and $\beta = \lambda/(1+\lambda)$.

On the figure below, we represent a possible initial configuration in the TAZRP and its corresponding configuration in the TASEP.

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●●●$\underline{\circledast}$●○○●○   ○●●●○●●●○●●$\underline{\circledast}$●○●○○○●○○

Now, if for the TAZRP the clock rings at the origin we get

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●●● ●
●●●$\underline{\circledast}$●○○●○   ○●●●○●●●○●●$\underline{\circledast}$○●●○○○●○○

Now, if for the TAZRP the clock at the origin rings again we get

●●   ●
●●● ●
●●●$\underline{\circ}$ ⊛○○●○   ○●●●○●●●○●●○$\underline{\circledast}$●●○○○●○○

Now, in the TAZRP the second class particle cannot jump since there are two first class particles at its site, neither the second class particle in TASEP. Therefore, from the mapping described above and for any initial configuration with a single second class particle, we have that $J_2^{EP}(t) = J_2^{ZR}(t)$ and $H_2^{EP}(t) = X_2^{ZR}(t)$, where $J_2^{EP}(t)$ (resp. $J_2^{ZR}(t)$) is the process that counts the number of first class particles that jump over the second class particle in the time interval $[0,t]$ in the TASEP (resp. in the TAZRP) and $H_2^{EP}(t)$ is the process that counts the number of holes that the second class particles jumps over in the time interval $[0,t]$ in the TASEP.

Since these processes have been very well studied in the TASEP, see for example [5] and references there in, from there one can get information on a second class particle in the TAZRP.

6 Second and Third class particles

In this section we present a simple proof of the following theorem:

Theorem 3. *([4, 1])*
Consider the TASEP, starting from the configuration η, such that all the sites $x \in \mathbb{Z}_-$ are occupied by first class particles, the origin is occupied by a second class particle, while the site $x = 1$ is occupied by a third class particle and the rest is empty. See the figure below, where the second class particle is represented by ⊛ and the third class particle is represented by ⊖.

●●●●$\underline{\circledast}$⊖○○○○ η

Let $X_2^{EP}(t)$ and $X_3^{EP}(t)$ denote the position of the second class particle and the position of the third class particle, respectively, at time t. Then

$$\lim_{t\to+\infty} P\Big(X_2^{EP}(t) > X_3^{EP}(t)\Big) = \frac{2}{3}.$$

Proof. Denote by $\tilde{\eta}$, the configuration that has a second class particle at the origin, while the negative sites are occupied by first class particles and the rest is empty.

$$\bullet\bullet\bullet\bullet\underline{\circledast}\circ\circ\circ\circ \quad \tilde{\eta}$$

Let Ξ denote the space of configurations of $\{0,1\}^{\mathbb{Z}}$ that have exactly one second class particle. For a configuration $\eta \in \Xi$, let $X_2^{EP}(t,\eta)$ denote the position of the second class particle at time t in the configuration η. The process $(\eta_t, X_2^{EP}(t,\eta))$ has generator given on local functions $f : \{0,1\}^{\mathbb{Z}} \times \mathbb{Z} \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{L}^2 f(\eta,z) = &\sum_{x,x+1\neq z} \eta(x)(1-\eta(x+1)\{f(\eta^{x,x+1},z) - f(\eta,z)\} \\ &+ \eta(z-1)\{f(\eta^{z-1,z},z-1) - f(\eta,z)\} \\ &+ (1-\eta(z+1))\{f(\eta^{z,z+1},z+1) - f(\eta,z)\}.\end{aligned} \tag{7}$$

This generator translates the dynamics of the second class particle in the TASEP that we defined above: the second class particle has the same jump rate as the first class particles, but whenever a first class particle attempts to jump to a site occupied by a second class particle they exchange positions and when a second class particle attempts to jump to a site occupied by a first class particle, the jump is forbidden.

For a configuration $\eta \in \Xi$, denote by $J_t^2(\eta)$ the process that counts the number of first class particles that jump from $X_2^{EP}(s,\eta)-1$ to $X_2^{EP}(s,\eta)$, for $s \in [0,t]$. This current can be formally defined by:

$$J_t^2(\eta) = \sum_{x\geq 0} \eta_t(x + X_2^{EP}(t,\eta)) - \eta_0(x),$$

so that

$$J_t^2(\tilde{\eta}) = \sum_{x \geq X_2^{EP}(t,\tilde{\eta})} \tilde{\eta}_t(x).$$

Then, applying the Kolmogorov backwards equation, we have that

$$\frac{d}{dt}E\Big(J_t^2(\tilde{\eta})\Big) = E\Big(\mathcal{L}^2(J_t^2(\tilde{\eta}))\Big) = E\Big(J_t^2(\tilde{\eta}^{-1,0})\Big) + E\Big(J_t^2(\tilde{\eta}^{0,1})\Big) - 2E\Big(J_t^2(\tilde{\eta})\Big), \tag{8}$$

where $\tilde{\eta}^{-1,0}$ corresponds to a jump of the rightmost first class particle in η from the site -1 to 0, which is the site occupied by the second class particle and $\tilde{\eta}^{0,1}$ corresponds to a jump of the second class particle from the site 0 to the site 1 which is occupied by the leftmost hole.

$$\bullet\bullet\bullet\circledast\underline{\bullet}\circ\circ\circ\circ \quad \tilde{\eta}^{-1,0}$$

$$\bullet\bullet\bullet\bullet\underline{\circ}\circledast\circ\circ\circ \quad \tilde{\eta}^{0,1}$$

Analogously, for a configuration η in Ξ, we denote by $H_t^2(\eta)$ the process that counts the number of holes that jump from $X_2^{EP}(s,\eta)+1$ to $X_2^{EP}(s,\eta)$, for $s \in [0,t]$, formally defined by

$$H_t^2(\eta) = \sum_{x \le 0} \Big\{ (1 - \eta_t(x - X_2^{EP}(t,\eta))) - (1 - \eta_0(x)) \Big\}.$$

Notice that,

$$H_t^2(\tilde\eta) = \sum_{x \le X_2^{EP}(t,\tilde\eta)} \Big(1 - \tilde\eta_t(x) \Big).$$

Now, the processes $J_t^2(\eta)$ and $H_t^2(\eta)$ behave symmetrically when starting them from the configurations $\tilde\eta^{-1,0}$ and $\tilde\eta^{0,1}$, respectively, see Lemma 1. Therefore, by Lemma 1, we can write (8) as

$$\frac{d}{dt} E(J_t^2(\tilde\eta)) = E(H_t^2(\tilde\eta^{0,1})) + E(J_t^2(\tilde\eta^{0,1})) - 2E(J_t^2(\tilde\eta)). \tag{9}$$

On the other hand we also have that $H_t^2(\tilde\eta) = J_t^2(\tilde\eta)$ in distribution, see Lemma 2.

Now, we are in a good position to compute (9) by coupling the TASEP starting from $\tilde\eta^{0,1}$ and $\tilde\eta$. Initially we have two discrepancies between the configurations $\tilde\eta^{0,1}$ and $\tilde\eta$ at sites 0 and 1 as can be seen in the figure below:

$$\bullet\bullet\bullet\bullet\underline{\circledast}\circ\circ\circ \qquad \tilde\eta$$

$$\bullet\bullet\bullet\bullet\underline{\circ}\circledast\circ\circ \qquad \tilde\eta^{0,1}$$

Let $Y_0(t)$ and $Y_1(t)$ denote the position at time t of the discrepancies initially at site 0 and 1, respectively. These discrepancies behave as a second class particle and as a third class particle in the coupled process, until the time they meet. The coupled process starts from η. Then, until this meeting time, we have that

$$X_2^{EP}(t) = X_2^{EP}(t,\eta) = Y_0(t),$$

$$X_3^{EP}(t) = X_3^{EP}(t,\eta) = Y_1(t).$$

Now, let $A_t = \Big\{ Y_0(t) < Y_1(t) \Big\}$. If A_t happens, then

$$H_t^2(\tilde\eta^{0,1}) = H_t^2(\tilde\eta) + 1 + \sum_{x=Y_0(t)+1}^{Y_1(t)} \Big(1 - \tilde\eta_t(x) \Big)$$

and

$$J_t^2(\tilde\eta) = J_t^2(\tilde\eta^{0,1}) + \sum_{x=Y_0(t)+1}^{Y_1(t)} \tilde\eta_t(x),$$

see the figure below.

$$\bullet\bullet\bullet\circ\bullet\circledast\bullet\bullet\circ\circ\circ\circ\circ$$

$$\bullet\bullet\bullet\circ\bullet\circ\bullet\bullet\circ\circledast\circ\circ\circ$$

Otherwise $H_t^2(\tilde\eta^{0,1}) = H_t^2(\tilde\eta)$ and $J_t^2(\tilde\eta) = J_t^2(\tilde\eta^{0,1})$, since the configurations at time t are equal. See the figure below.

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Then we can partition the space to rewrite (9) as

$$\begin{aligned}\frac{d}{dt}E(J_t^2(\tilde{\eta})) =& E\Big(1_{A_t}(H_t^2(\tilde{\eta}^{0,1})+J_t^2(\tilde{\eta}^{0,1})-2J_t^2(\tilde{\eta}))\Big)\\ &+E\Big(1_{A_t^c}(H_t^2(\tilde{\eta}^{0,1})+J_t^2(\tilde{\eta}^{0,1})-2J_t^2(\tilde{\eta}))\Big).\end{aligned}$$

Using the relations established above, we have that:

$$\frac{d}{dt}E(J_t^2(\tilde{\eta})) = P(A_t)+E\Big(1_{A_t}\Big\{\sum_{x=Y_0(t)+1}^{Y_1(t)}\Big(1-\tilde{\eta}_t(x)\Big)-\sum_{x=Y_0(t)+1}^{Y_1(t)}\tilde{\eta}_t(x)\Big\}\Big).$$

Now, by symmetry it holds that

$$\sum_{x=Y_0(t)+1}^{Y_1(t)}\Big(1-\tilde{\eta}_t(x)\Big) =_{law} \sum_{x=Y_0(t)+1}^{Y_1(t)}\tilde{\eta}_t(x).$$

Then, we obtain

$$\frac{d}{dt}E(J_t^2(\tilde{\eta})) = P(A_t) = P\Big(X_2^{EP}(t)<X_3^{EP}(t)\Big). \tag{10}$$

It remains to compute the left hand side of last expression. For the configuration $\tilde{\eta}$ we can label the first class particles from the left to the right, in such a way that $P_i(0,\tilde{\eta})$ denotes the position of the i-th first class particle at time 0. Clearly one has $P_i(0,\tilde{\eta})=-i$. Let $P_i(t,\tilde{\eta})$ denote the position of this particle at time t.

Since first class particles preserve their order, it is easy to see that the current through the second class particle $J_t^2(\tilde{\eta})$, can be written as

$$J_t^2(\tilde{\eta}) = \sum_{x=X_2^{EP}(t,\tilde{\eta})}^{P_1(t,\tilde{\eta})}\tilde{\eta}_t(x),$$

see the figure below where the rightmost particle is at $P_1(t,\tilde{\eta})=6$, $X_2^{EP}(t,\tilde{\eta})=-1$ and $J_t^2(\tilde{\eta})=3$.

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It was shown in [5, 6], that

$$\frac{J_t^2(\tilde{\eta})}{t} \xrightarrow[t\to+\infty]{} \Big(\frac{1-\mathcal{U}}{2}\Big)^2, \quad \text{almost} \quad \text{surely},$$

where $\mathcal{U}$ is the random variable with Uniform distribution on $[-1,1]$ given in Theorem 1 with $\alpha = 1$ and $\beta = 0$. In particular the convergence in distribution also holds.

Using the martingale decomposition of the current it is easy to show, that for any $\varepsilon > 0$,

$$\Big(\frac{J_t^2(\tilde{\eta})}{t}\Big)^{2-\varepsilon}$$

is uniformly integrable since its L^2-norm is finite. As a consequence, by a well know result on weak convergence of random variables, it holds that

$$\lim_{t\to+\infty} E\Big(\frac{J_t^2(\tilde{\eta})}{t}\Big) = E\Big(\frac{1-\mathcal{U}}{2}\Big)^2 = \frac{1}{3}. \tag{11}$$

Moreover,

$$\frac{1}{t}\int_0^t \frac{d}{ds}E(J_s^2(\tilde{\eta}))ds = E\Big(\frac{J_t^2(\tilde{\eta})}{t}\Big), \tag{12}$$

and by (10), the left hand side of last expression is equal to

$$\frac{1}{t}\int_0^t P(A_s)ds.$$

Now, since A_s are decreasing sets, then $P(A_s)$ decreases and as a consequence the limit, as $s \to +\infty$ exists. By the Césaro theorem

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t P(A_s)ds = \lim_{t\to+\infty} P(A_t).$$

Putting together last result, (11) and (12), we obtain that $\lim_{t\to+\infty} P(A_t) = \frac{1}{3}$, which concludes the proof.

Lemma 1. *The process $J_t^2(\tilde{\eta}^{-1,0})$ has the same distribution as the process $H_t^2(\tilde{\eta}^{0,1})$.*

Proof. In other words, we have to show that if L^J and L^H represent the generators of the processes $J_t^2(\tilde{\eta}^{-1,0})$ and $H_t^2(\tilde{\eta}^{0,1})$, respectively, then for every local function $f:\{0,1\}^{\mathbb{Z}}\times\mathbb{Z}\to\mathbb{R}$, $L^J f(\tilde{\eta}^{-1,0},z) = L^H f(\tilde{\eta}^{0,1},z)$. The easiest way of showing this is to consider the process seen from the position of the second class particle.

For a configuration $\eta \in \Xi$, let $\eta_t' = \tau_{X_2^{EP}(t,\eta)}\eta_t$ be such that for a site $x \in \mathbb{Z}$, $\eta_t'(x) = \eta_t(x + X_2^{EP}(t,\eta))$ be the process whose generator is given on local functions $f:\{0,1\}^{\mathbb{Z}}\to\mathbb{R}$ by

$$\begin{aligned} L'f(\eta') = &\sum_{x,x+1\neq 0}\eta'(x)(1-\eta'(x+1))\{f(\sigma_{EP}^{x,x+1}\eta')) - f(\eta')\} \\ &+\eta'(-1)\{f(\tau_{-1}\sigma_{EP}^{-1,0}\eta') - f(\eta')\} \\ &+(1-\eta'(1))\{f(\tau_1\sigma_{EP}^{0,1}\eta') - f(\eta')\}. \end{aligned}$$

Above $\tau_x\eta$ is the shift in η that places the second class particle at the origin. In this process the position of $X_2^{EP}(t,\eta)$ corresponds to the number of shifts of the system, of size -1, during the time interval $[0,t]$ and as a consequence, in this process the site 0 is always occupied by a second class particle.

Denote by $N_1(t,\eta')$ the number of particles that jump from the site -1 to 0 during the time interval $[0,t]$:

$$N_1(t,\eta') = \sum_{x\geq 0}\Big(\eta'_t(x) - \eta'_0(x)\Big).$$

Note that $N_1(t,\eta')$ corresponds to the number of particles at the right hand side of $X_2^{EP}(t,\eta)$ at time t, and as a consequence one has that $J_t^2(\eta) = N_1(t,\eta')$.

Consider now the process $(\eta'_t, N_1(t,\eta'))$ with generator given on local functions $f:\{0,1\}^{\mathbb{Z}}\times\mathbb{Z}\to\mathbb{R}$ by

$$\begin{aligned}L_1 f(\eta',N) = &\sum_{x,x+1\neq 0}\eta'(x)(1-\eta'(x+1))\{f(\sigma_{EP}^{x,x+1}\eta',N)-f(\eta',N)\}\\ &+\eta'(-1)\{f(\tau_{-1}\sigma_{EP}^{-1,0}\eta',N+1)-f(\eta',N)\}\\ &+(1-\eta'(1))\{f(\tau_1\sigma_{EP}^{0,1}\eta',N)-f(\eta',N)\}.\end{aligned} \tag{13}$$

Analogously, we can consider $N_{-1}(t,\eta')$ as the number of jumps, of size 1, of the second class particle, that is, the number of shifts of the system of size 1. Whenever the second class particle jumps one unit ahead, the hole placed before the jump at site 1 jumps to the site -1, then we can write:

$$N_{-1}(t,\eta') = \sum_{x\leq 0}\Big((1-\eta'_t(x)) - (1-\eta'_0(x))\Big).$$

In this case we also have that $H_t^2(\eta) = N_{-1}(t,\eta')$.

The process $(\eta'_t, N_{-1}(t,\eta'))$ has generator given on local functions $f:\{0,1\}^{\mathbb{Z}}\times\mathbb{Z}\to\mathbb{R}$ by

$$\begin{aligned}L_{-1} f(\eta',N) = &\sum_{x,x+1\neq 0}\eta'(x)(1-\eta'(x+1))\{f(\sigma_{EP}^{x,x+1}\eta',N)-f(\eta',N)\}\\ &+\eta'(-1)\{f(\tau_{-1}\sigma_{EP}^{-1,0}\eta'^{-1,0},N)-f(\eta',N)\}\\ &+(1-\eta'(1))\{f(\tau_1\sigma_{EP}^{0,1}\eta',N+1)-f(\eta',N)\}.\end{aligned}$$

To fix notation, let $\zeta = \tilde{\eta}^{-1,0}$ and $\varsigma = \tilde{\eta}^{0,1}$, as shown below.

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As before denote by ς' and ζ' the configurations ς and ζ seen from the second class particle, respectively. We couple the processes starting from ζ and ς under the basic coupling, so that clocks are attached to sites. By the symmetry of the configurations, it is easy to see that $\forall x \neq 0$, $\zeta(x) = 1 - \varsigma(-x)$ and both have a second class particle at the origin. Now simple computations show that

$$L^J f(\tilde{\eta}^{-1,0},N) = L_1 f(\varsigma',z) = L_{-1} f(\zeta',N) = L^H(\tilde{\eta}^{0,1},N),$$

which concludes the proof.

We give a sketch of last equality. Let f be a local function and ζ' and ς' as defined above, then:

$$\begin{aligned} L_1 f(\varsigma',N) = \sum_{x,x+1\neq 0} &(1-\zeta'(-x))(1-(1-\zeta'(-(x+1))) \\ &\qquad\qquad\qquad \times \{f(\sigma_{EP}^{-x-(x+1)}\zeta',N) - f(\zeta',N)\} \\ &+ (1-\zeta'(1))\{f(\tau_1\sigma_{EP}^{0,1}\zeta',N+1) - f(\zeta',N)\} \\ &+ \zeta'(-1)\{f(\tau_{-1}\sigma_{EP}^{-1,0}\zeta',N) - f(\zeta',N)\}. \end{aligned}$$

In the first equality we used the fact that $\forall x \neq 0$, $\zeta'(x) = 1 - \varsigma'(-x)$ and notice that last expression is precisely $L_{-1} f(\zeta',N)$.

Lemma 2. *The process $J_t^2(\tilde{\eta})$ has the same distribution as the process $H_t^2(\tilde{\eta})$.*

Proof. The proof follows the same computations as the ones performed in the proof of last lemma since what we have to show is that for every local function $f : \{0,1\}^{\mathbb{Z}} \times \mathbb{Z} \to \mathbb{R}$, $L^J f(\tilde{\eta},z) = L^H f(\tilde{\eta},z)$. This is a consequence of the particle-hole symmetry of the processes for the configuration $\tilde{\eta}$:

$$\bullet\bullet\bullet\bullet\,\underline{\circledast}\circ\circ\circ\circ$$

$$\bullet\bullet\bullet\bullet\underline{\circledast}\,\circ\circ\circ\circ$$

As a consequence of Theorem 3 and a simple modification of the coupling described in the previous section (see [3] for details) the following result holds.

Corollary 1. *Consider the TAZRP starting from the configuration ξ, such that all the sites $x \in \mathbb{Z}_-$ are occupied by infinitely many first class particles, the origin is occupied by a second class particle, the site $x = 1$ is occupied by a third class particle and the rest is empty. See the figure below, where the second class particle is represented by $\circledast$ and the third class particle is represented by $\ominus$.*

$$\begin{array}{l} \vdots \\ \bullet \\ \bullet \\ \bullet\underline{\circledast}\ominus\circ\circ\circ \end{array}$$

Let $X_2^{ZR}(t)$ and $X_3^{ZR}(t)$ denote the position of the second class particle and the position of the third class particle, respectively, at time t. Then

$$\lim_{t\to+\infty} P(X_2^{ZR}(t) \geq X_3^{ZR}(t)) = \frac{2}{3}.$$

To finish I would like to mention that it would be an interesting problem to derive the previous result without going to the coupling argument. It would also be a very interesting problem to extend the results presented here for more general zero-range processes with a rate function given by $g(\cdot)$ and with partially asymmetric jumps. In each case the coupling with TASEP presented in Section 5 fails dramatically.

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