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A note on a one-parameter family of non-symmetric number triangles

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Abstract

The recently growing interest in special Clifford Algebra valued polynomial solutions of generalized Cauchy-Riemann systems in $(n+1)$-dimensional Euclidean spaces suggested a detailed study of the arithmetical properties of their coefficients, due to their combinatoric relevance. This concerns, in particular, a generalized Appell sequence of homogeneous polynomials whose coefficient's set can be treated as a one-parameter family of non-symmetric triangles of fractions. The discussion of its properties, similar to those of the ordinary Pascal triangle (which itself does not belong to the family), is carried out in this paper.

1 Introduction

In [11, 18], we have considered for the first time the infinite array of numbers

$$T_s^k(n) = \frac{k!}{n_{(k)}} \frac{\left(\frac{n+1}{2}\right)_{(k-s)}}{(k-s)!} \frac{\left(\frac{n-1}{2}\right)_{(s)}}{s!}, \quad n,k = 1,2,\ldots; s = 0,1,\ldots,k,$$

where $a_{(r)}$ denotes the Pochhammer symbol, given by $a_{(r)} := \frac{\Gamma(a+r)}{\Gamma(a)}$, for any integer $r \geq 1$ and $a_{(0)} := 1$, as well as $0_{(0)} := 1$. These numbers were introduced in the framework of Clifford Analysis (cf. [4]), in order to construct special polynomials in $\mathbb{R}^{n+1}$. Their relation with the elements of the Pascal triangle is obvious, since we can write them also in the form

$$T_s^k(n) = \binom{k}{s} \frac{\left(\frac{n+1}{2}\right)_{(k-s)} \left(\frac{n-1}{2}\right)_{(s)}}{n_{(k)}}.$$

But considered in the form (1), they show their connection with the coefficients of the geometric series and its higher degree relatives, namely with

$$\frac{1}{(1-t)^m} = \sum_{r=0}^{\infty} \frac{m_{(r)}}{r!} t^r, \text{ where } t \in \mathbb{C}, m > 0. \tag{1}$$

Indeed, they are the product of two factors of the form $\frac{m_{(r)}}{r!}$ and of a third factor, which is the reciprocal of such an expression. The series expansion of complex holomorphic functions through the series expansion of the Cauchy kernel in its integral representation is well known and relies on the geometric series $(m = 1)$. Analogously, Clifford Analysis deals with the series expansion of generalized holomorphic functions in $\mathbb{R}^{n+1}$ through the series expansion of the generalized Cauchy kernel in their integral representation. Therefore it seems obvious to expect some similar relation to geometric series (1) of degree $m > 1$. For readers familiar with the basics of Clifford Analysis this connection surely comes not as a surprise, but so far as we know, it has never been explicitly noticed before in this way.

Our main concern will be some arithmetical properties of the family of number triangles composed by fractions $T_s^k(n)$ for different parameter values n in lines of height $k = 0, 1, \ldots,$ and ordered from $s = 0$ up to $s = k$. Both representations (1) and (1) show that they are not symmetric triangles like the ordinary Pascal triangle, because $T_s^k(n) \neq T_{k-s}^k(n)$.

We will try to omit as much as possible details from Clifford Analysis, but due to the particular role in Clifford Analysis we would like to stress in this introduction at least their origin as coefficients in the construction of generalized Appell polynomials in that framework. Those generalized Appell polynomials have recently received a lot of attention from several authors ([3, 8, 15, 16, 20]) due to their important role in theory and applications ([6, 7, 9, 13]), specially in elasticity [3], PDE and Special Functions ([5, 12]), or 3D-quasiconformal mapping problems ([10, 11]).

The mentioned polynomial sequences in $(n+1)$ real variables take their values in the real vector space of paravectors of the corresponding Clifford algebra $\mathcal{C}\ell_{0,n}$. To understand what this means let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ with a non-commutative product according to the multiplication rules

$$e_k e_l + e_l e_k = -2\delta_{kl}, \; k, l = 1, \cdots, n,$$

where δ_{kl} is the Kronecker symbol. The set $\{e_A : A \subseteq \{1, \cdots, n\}\}$ with

$$e_A = e_{h_1} e_{h_2} \cdots e_{h_r}, \; 1 \leq h_1 < \cdots < h_r \leq n, \; e_\emptyset = e_0 = 1,$$

forms a basis of the 2^n-dimensional Clifford algebra $\mathcal{C}\ell_{0,n}$ over $\mathbb{R}$.

Let now $\mathbb{R}^{n+1}$ be embedded in $\mathcal{C}\ell_{0,n}$ by identifying $(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}$ with the algebra's element $x = x_0 + \underline{x} \in \mathcal{A}_n := \mathrm{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset \mathcal{C}\ell_{0,n}$. Here

$$x_0 = \mathrm{Sc}(x)$$

and

$$\underline{x} = \mathrm{Vec}(x) = e_1 x_1 + \cdots + e_n x_n$$

are the so-called *scalar part* resp. *vector part* of the *paravector* $x \in \mathcal{A}_n$. The *conjugate* of x is given by

$$\bar{x} = x_0 - \underline{x}$$

and the *norm* $|x|$ of x is defined by

$$|x|^2 = x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + \cdots + x_n^2.$$

It follows that $\mathcal{C}\ell_{0,n}$-valued functions defined in some open subset $\Omega \subset \mathbb{R}^{n+1}$, in general, are of the form $f(z) = \sum_A f_A(z) e_A$, where $f_A(z)$ are real valued.

The *generalized Cauchy-Riemann operator* in $\mathbb{R}^{n+1}$, $n \geq 1$, is defined by

$$\overline{\partial} := \tfrac{1}{2}(\partial_0 + \partial_{\underline{x}}), \quad \partial_0 := \frac{\partial}{\partial x_0}, \quad \partial_{\underline{x}} := e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}. \tag{2}$$

$\mathscr{C}^1$-functions f satisfying the equation $\overline{\partial} f = 0$ (resp. $f\overline{\partial} = 0$) are generalized holomorphic functions, usually called *left monogenic* (resp. *right monogenic*).

A monogenic function f is hypercomplex differentiable in Ω in the sense of [14], i.e. it has a uniquely defined areolar derivative f' in the sense of Pompeiu in each point of Ω (for more details see also [17]). The hypercomplex (areolar) derivative f' of a monogenic function is given by $f' = \frac{1}{2}(\partial_0 - \partial_{\underline{x}})f$ where

$$\partial := \frac{1}{2}(\partial_0 - \partial_{\underline{x}})$$

is just the *conjugate generalized Cauchy-Riemann operator.* If we recall the complex partial derivatives (also called Wirtinger derivatives)

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}) \text{ and } \frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y})$$

then it is clear that the hypercomplex derivative f' is also a generalized hypercomplex Wirtinger derivative $f' = \partial f = \frac{1}{2}(\partial_0 - \partial_{\underline{x}})f$. There use is vital for the definition of a basic polynomial sequence. Since a hypercomplex differentiable function belongs to the kernel of $\overline{\partial}$, it follows that in fact $f' = \partial_0 f = -\partial_{\underline{x}} f$ corresponding to the complex case of a holomorphic function where

$$\frac{df}{dz} = \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

After this excursion on the fundamentals of Clifford Analysis we can now recall the definition of a generalized Appell sequence (cf. [2, 18]) of monogenic polynomials associated to ∂.

A sequence of monogenic polynomials $(\mathcal{F}_k)_{k\geq 0}$ is called a generalized Appell sequence with respect to ∂ if

1. $\mathcal{F}_0(x) \equiv 1.$
2. $\mathcal{F}_n(0) = 0.$
3. $\partial\, \mathcal{F}_k = k\, \mathcal{F}_{k-1},\ k = 1, 2, \ldots.$

In [11, 18], we have shown for the first time that for all $n \geq 1$ and $T_s^k(n)$ given by (1) the polynomials

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} T_s^k(n)\, x^{k-s}\, \bar{x}^s, \tag{3}$$

form such a set of generalized Appell sequences. Moreover, we showed in those papers also how these Appell sequences can be expressed in terms of several hypercomplex variables of the form $z_k = x_k - x_0 e_k; x_0, x_k \in \mathbb{R}; k = 1, 2, \ldots, n$.

But in the form (3) these polynomials are *special* monogenic polynomials in the sense of [1], where a monogenic polynomial P is said to be special if there exist constants $a_{ij} \in \mathcal{A}_n$ for which

$$P(x) = \sum_{i,j}{}' \bar{x}^i x^j a_{i,j}$$

(the primed sigma stands for a finite sum). This paper [1] is concerned with the extension of the theory of basic sets of polynomials in one complex variable, as introduced by J. M. Whittaker and B. Cannon, and was published 10 years before the introduction of the hypercomplex derivative in [14]. Hence, it has nothing to do with Appell sequences.

In the following we prove several properties of the triangle numbers (1). In particular, we present results that provide different constructive methods for obtaining the aforementioned fractional number triangles in arbitrary dimensions $n \geq 2$. We also derive results concerned with the sum and alternating sum of the rows of the triangle (1) which play an important role in the context of Clifford Analysis.

2 Pascal-like fractional number triangles

We start by first recalling some well known properties of the Pochhammer symbol, namely

$$a_{(r)} = (a+r-1)a_{(r-1)} \quad \text{and} \quad a(a+1)_{(r)} = (a+r)a_{(r)}. \tag{4}$$

These properties can be used to derive straightway the following relations:

$$\left(\frac{n+1}{2}\right)_{(k+1)} = \frac{n+2k+1}{2}\left(\frac{n+1}{2}\right)_{(k)}, \tag{5}$$

$$\left(\frac{n-1}{2}\right)_{(s+1)} = \frac{n+2s-1}{2}\left(\frac{n-1}{2}\right)_{(s)}, \tag{6}$$

and

$$(n-1)\left(\frac{n+1}{2}\right)_{(r)} = (n+2r-1)\left(\frac{n-1}{2}\right)_{(r)}. \tag{7}$$

It is also easy to conclude that, as already mentioned in formula (1)

$$T_s^k(n) = \binom{k}{s}\frac{\prod_{i=1}^{k-s}(n+2i-1)\prod_{i=0}^{s}(n+2i-1)}{2^k n_{(k)}},$$

i.e.

$$2^k T_s^k(n) = \tbinom{k}{s} R_s^k(n),$$

where $R_s^k(n)$ is a rational function which is the quotient of two monic polynomials in n, both of degree k.

For $n=1$ the only possible value of s is $s=0$ and we have $T_0^k(1) \equiv 1$ for all $k=0,1,\ldots$. On the other end of the range of the parameter n we have $T_s^k(\infty) = 2^{-k}\binom{k}{s}$ as consequence of (1).

The last result reveals a connection between the infinite triangular table $2^k T_s^k(n)$, $k=0,1,\ldots,$ $s=0,\ldots,k$ and the well known Pascal triangle which becomes more clear on Table 1, where we present the first 5 rows of the table and highlight (see the boldface numbers) the aforementioned relationship.

Table 1: The first 5 rows of $2^k T_s^k(n)$

$\mathbf{1}$				
$\frac{\mathbf{1}(n+1)}{n}$	$\frac{\mathbf{1}(n-1)}{n}$			
$\frac{\mathbf{1}(n+3)}{n}$	$\frac{\mathbf{2}(n-1)}{n}$	$\frac{\mathbf{1}(n-1)}{n}$		
$\frac{\mathbf{1}(n+5)(n+3)}{n(n+2)}$	$\frac{\mathbf{3}(n+3)(n-1)}{n(n+2)}$	$\frac{\mathbf{3}(n^2-1)}{n(n+2)}$	$\frac{\mathbf{1}(n+3)(n-1)}{n(n+2)}$	
$\frac{\mathbf{1}(n+5)(n+7)}{n(n+2)}$	$\frac{\mathbf{4}(n+5)(n-1)}{n(n+2)}$	$\frac{\mathbf{6}(n^2-1)}{n(n+2)}$	$\frac{\mathbf{4}(n^2-1)}{n(n+2)}$	$\frac{\mathbf{1}(n+5)(n-1)}{n(n+2)}$

Moreover, computing the ratio of each number in (1) with its left-hand neighbor (as Pascal himself did in [21]) it is not difficult to accept either the designation of Pascal-like triangle for the values (1) of $T_s^k(n)$ or to guess a general law for the numbers in Table 2 (see formula (11)).

3 Properties

In this section we prove several properties of the number triangle (1). First of all, we deduce the relationships of a given entry with its immediate neighbors.

Table 2: A Pascal-like triangle

$\frac{\mathbf{1}}{\mathbf{1}}\,\frac{n-1}{n+1}$				
$\frac{\mathbf{2}}{\mathbf{1}}\,\frac{n-1}{n+3}$	$\frac{\mathbf{1}}{\mathbf{2}}\,\frac{n+1}{n+1}$			
$\frac{\mathbf{3}}{\mathbf{1}}\,\frac{n-1}{n+5}$	$\frac{\mathbf{2}}{\mathbf{2}}\,\frac{n+1}{n+3}$	$\frac{\mathbf{1}}{\mathbf{3}}\,\frac{n+3}{n+1}$		
$\frac{\mathbf{4}}{\mathbf{1}}\,\frac{n-1}{n+7}$	$\frac{\mathbf{3}}{\mathbf{2}}\,\frac{n+1}{n+5}$	$\frac{\mathbf{2}}{\mathbf{3}}\,\frac{n+3}{n+3}$	$\frac{\mathbf{1}}{\mathbf{4}}\,\frac{n+5}{n+1}$	
$\frac{\mathbf{5}}{\mathbf{1}}\,\frac{n-1}{n+9}$	$\frac{\mathbf{4}}{\mathbf{2}}\,\frac{n+1}{n+7}$	$\frac{\mathbf{3}}{\mathbf{3}}\,\frac{n+3}{n+5}$	$\frac{\mathbf{2}}{\mathbf{4}}\,\frac{n+5}{n+3}$	$\frac{\mathbf{1}}{\mathbf{5}}\,\frac{n+7}{n+3}$

$$
\begin{tikzcd}
\boxed{T_s^k} \arrow[r, "\text{Th 3}"] \arrow[d, "\text{Th 1}"] \arrow[dr, "\text{Th 2}"] & \boxed{T_{s+1}^k} \\
\boxed{T_s^{k+1}} & \boxed{T_{s+1}^{k+1}}
\end{tikzcd}
$$

Figure 1: The starting point T_s^k

Theorem 1 *For* $k = 0, 1, \ldots$ *and* $s = 0, \ldots, k$

$$T_s^{k+1}(n) = \frac{(k+1)(n+2k-2s+1)}{2(k-s+1)(n+k)}\, T_s^k(n). \tag{8}$$

Proof. From (1) we get

$$T_s^{k+1}(n) = \frac{(k+1)!}{n_{(k+1)}}\, \frac{(\frac{n+1}{2})_{(k-s+1)}(\frac{n-1}{2})_{(s)}}{(k-s+1)!s!}$$

and from relations (4) and (5) we have

$$\begin{aligned} T_s^{k+1}(n) &= \frac{(k+1)!}{(n+k)n_{(k)}}\, \frac{n+2(k-s)+1}{2}\, \frac{(\frac{n+1}{2})_{(k-s)}(\frac{n-1}{2})_{(s)}}{(k-s+1)!s!} \\ &= \frac{(k+1)(n+2k+1)}{2(k-s+1)(n+k)}\, T_s^k(n). \end{aligned}$$

□

Theorem 2 *For* $k = 0, 1, \ldots$ *and* $s = 0, \ldots, k$

$$T_{s+1}^{k+1}(n) = \frac{(k+1)(n+2s-1)}{2(s+1)(n+k)}\, T_s^k(n). \tag{9}$$

Proof. From (1) we get

$$T_{s+1}^{k+1}(n) = \frac{(k+1)!}{n_{(k+1)}}\, \frac{(\frac{n+1}{2})_{(k-s)}(\frac{n-1}{2})_{(s+1)}}{(k-s)!(s+1)!}$$

and from relations (4) and (6) we have

$$
\begin{aligned}
T_{s+1}^{k+1}(n) &= \frac{(k+1)!}{(n+k)n_{(k)}} \frac{n+2s-1}{2} \frac{(\frac{n+1}{2})_{(k-s)}(\frac{n-1}{2})_{(s)}}{(k-s)!(s+1)!} \\
&= \frac{(k+1)(n+2s-1)}{2(s+1)(n+k)} T_s^k(n).
\end{aligned}
$$

□

Theorem 3 *For $k = 1, 2, \dots$ and $s = 0, \dots, k-1$*

$$T_{s+1}^k(n) = \frac{(k-s)(n+2s-1)}{(s+1)(n+2k-2s-1)} T_s^k(n). \tag{10}$$

Proof. We make use of (5) in the equivalent form

$$\left(\frac{n+1}{2}\right)_{(k-s-1)} = \frac{2}{n+2k-2s-1} \left(\frac{n+1}{2}\right)_{(k-s)}$$

in order to obtain

$$
\begin{aligned}
T_{s+1}^k(n) &= \frac{k!}{n_{(k)}} \frac{(\frac{n+1}{2})_{(k-s-1)}(\frac{n-1}{2})_{(s+1)}}{(k-s-1)!(s+1)!} \\
&= \frac{n+2s-1}{n+2k-2s-1} \frac{k-s}{s+1} T_s^k(n).
\end{aligned}
$$

□

Corollary 1 *If $Q_{s+1}^k(n)$ denote the numbers presented in Table 2, then*

$$Q_{s+1}^k(n) = \frac{(k-s)(n+2s-1)}{(s+1)(n+2k-2s-1)}, \quad k = 1, 2, \dots, \quad s = 0, \dots, k. \tag{11}$$

Proof. The result follows at once from Theorem 3, since

$$Q_{s+1}^k(n) = \frac{T_{s+1}^k(n)}{T_s^k(n)}.$$

□

The recursive use of formulae (8)-(10) provides an easy way of constructing the triangle in Table 1. The scheme in Figure 1 summarizes the above properties and Figure 2 contains an example illustrating the relations between the elements of the first 4 rows of the Pascal-like triangle (1).

The next result shows a n-independent relation between adjacent elements in the row k and an element in the row $k-1$.

Theorem 4 *For $k = 1, 2 \dots$, and $s = 0, \dots, k-1$*

$$(k-s)T_s^k(n) + (s+1)T_{s+1}^k(n) = kT_s^{k-1}(n). \tag{12}$$

Proof. Using Theorem 3 we obtain

$$
\begin{aligned}
(k-s)T_s^k(n) + (s+1)T_{s+1}^k(n) &= (k-s)T_s^k(n)\left(1 + \frac{n+2s-1}{n+2k-2s-1}\right) \\
&= 2(k-s)\frac{n+k-1}{n+2k-2s-1} T_s^k(n).
\end{aligned}
$$

The use of Theorem 1 in the form

$$T_s^k(n) = \frac{k(n+2k-2s-1)}{2(k-s)(n+k-1)} T_s^{k-1}(n)$$

yields the final result. □

Finally, the next relation underlines once more the lack of symmetry of the triangle under consideration. In fact, in each row k of the triangle, we can relate the element in position $(k, k-s)$ with the element in position (k, s) as follows:

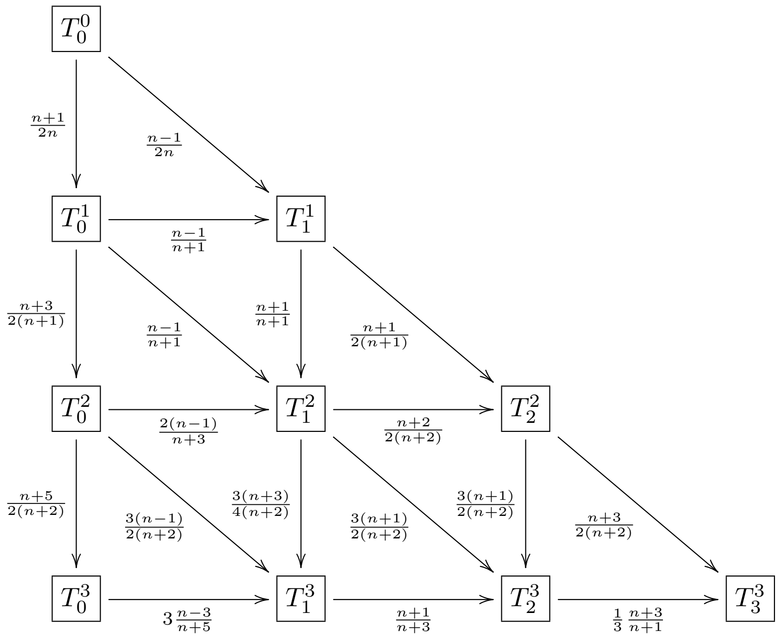

Figure 2: Relations between the first triangle elements

Theorem 5 *For* $k = 0, 1, \ldots$ *and* $s = 0, \ldots, k$

$$T_{k-s}^k(n) = \frac{2s+n-1}{2(k-s)+n-1} T_s^k(n). \tag{13}$$

Proof. From relation (7) we obtain

$$\begin{aligned} T_{k-s}^k(n) &= \frac{k!}{(n)_{(k)}} \frac{(\frac{n-1}{2})_{(s)}(\frac{n+1}{2})_{(k-s)}}{s!(k-s)!} \frac{(2s+n-1)(n-1)}{(n-1)(2(k-s)+n-1)} \\ &= \frac{2s+n-1}{2(k-s)+n-1} T_s^k(n). \end{aligned}$$

□

Figure 3 contains an illustration of Theorems 4 and 5.

Theorem 6 *For* $k = 0, 1, \ldots$

$$\sum_{s=0}^{k} T_s^k(n) = 1. \tag{14}$$

Proof. Denote by $\sigma_k(n)$ the sum $\sigma_k(n) := \sum_{s=0}^{k} T_s^k(n)$. By using (12), we get

$$\sum_{s=0}^{k-1} (k-s) T_s^k(n) + \sum_{s=0}^{k-1} (s+1) T_{s+1}^k(n) = k \sum_{s=0}^{k-1} T_s^{k-1}(n),$$

i.e.

$$k \sum_{s=0}^{k-1} T_s^k(n) + \sum_{s=0}^{k-1} \left[(s+1) T_{s+1}^k(n) - s T_s^k(n)\right] = k \sigma_{k-1}(n)$$

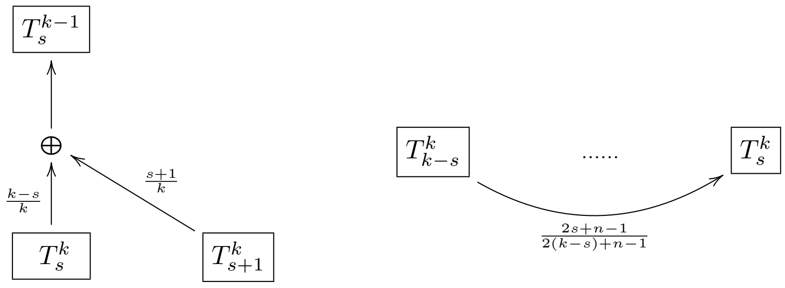

Figure 3: Relations between consecutive and distant neighbors

or

$$k\sum_{s=0}^{k-1} T_s^k(n) + kT_k^k(n) = k\sigma_k(n) = k\sigma_{k-1}(n).$$

We have just proved that $\sigma_k(n) = \sigma_{k-1}(n)$, which means that

$$\sigma_k(n) = \sigma_{k-1}(n) = \sigma_{k-2}(n) = \cdots = \sigma_0(n) = 1. \qquad \square$$

Remark 1 *Theorem 6 can also be obtained as a particular case of the well known Vandermonde Convolution Identity for Pochhammer symbols*

$$(a+b)_{(k)} = \sum_{s=0}^{k} \binom{k}{s} a_{(k-s)} b_{(s)}.$$

In fact, using $a = \frac{n+1}{2}$ *and* $b = \frac{n-1}{2}$ *as well as writing* $\binom{k}{s}$ *as* $\frac{k}{(k-s)!k!}$ *we get automatically by division of the left side by* $(a+b)_{(k)} = n_{(k)}$ *that*

$$1 = \sum_{s=0}^{k} \frac{k!}{n_{(k)}} \frac{(\frac{n+1}{2})_{(k-s)}(\frac{n-1}{2})_{(s)}}{(k-s)!s!} = \sum_{s=0}^{k} T_s^k(n), \tag{15}$$

for $n = 1, 2, \ldots; k = 0, 1, \ldots; s = 0, \ldots, k.$

Finally, we present a property concerned with the alternating sum of the elements of a row of the Pascal-like table (1).

Theorem 7 $\sum_{s=0}^{k} (-1)^s T_s^k(n) = c_k(n)$, *where*

$$c_k(n) = \begin{cases} \frac{k!!(n-2)!!}{(n+k-1)!!}, \text{ if } k \text{ is odd} \\ \\ c_{k-1}(n), \text{ if } k \text{ is even} \end{cases} \tag{16}$$

Proof. From Theorem 4, we conclude that

$$\sum_{s=0}^{k-1} (-1)^s (k-s) T_s^k(n) + \sum_{s=0}^{k-1} (-1)^s (s+1) T_{s+1}^k(n) = kc_{k-1}.$$

Reordering the sums, we obtain

$$kc_k(n)+2\sum_{s=0}^{k-1}(-1)^s(s+1)T_{s+1}^k(n)=kc_{k-1}(n). \tag{17}$$

Denoting by $\vartheta_k(n)$ the alternating sum $\vartheta_k(n):=\sum_{s=0}^{k-1}(-1)^s(s+1)T_{s+1}^k(n)$, we prove now that

$$\vartheta_k(n)=\begin{cases} 0, \text{ if } k \text{ is even} \\ \\ \frac{1}{2}\frac{k(n-1)}{n+k-1}\sum_{s=0}^{k-1}(-1)^s T_s^{k-1}(n), \text{ if } k \text{ is odd} \end{cases} \tag{18}$$

In fact, supposing first that $k=2m,\ m\in\mathbb{N}$, we get

$$\begin{aligned}\vartheta_{2m}(n) &= \sum_{s=0}^{m-1}(-1)^s(s+1)T_{s+1}^{2m}(n)+\sum_{s=m}^{2m-1}(-1)^s(s+1)T_{s+1}^{2m}(n)\\ &= \sum_{s=0}^{m-1}(-1)^s(s+1)T_{s+1}^{2m}(n)+\sum_{s=0}^{m-1}(-1)^{2m-1-s}(2m-s)T_{2m-s}^{2m}(n)\\ &= \sum_{s=1}^{m}(-1)^{s-1}\left(sT_s^{2m}(n)-(2m-s+1)T_{2m-s+1}^{2m}(n)\right).\end{aligned}$$

Applying (10) and (13) we can write

$$\begin{aligned}\vartheta_{2m}(n)=&\sum_{s=1}^{m}(-1)^{s-1}\frac{(2m-s+1)(n+2s+1)}{n+4m-2s+1}T_{s-1}^{2m}(n)\\ &-\sum_{s=1}^{m}(-1)^{s-1}(2m-s+1)\frac{n+2s+1}{n+4m-2s+1}T_{s-1}^{2m}(n)=0.\end{aligned}$$

On the other hand, if $k=2m+1,\ m\in\mathbb{N}$, by the use of Theorem 2, we can conclude that

$$\vartheta_{2m+1}(n)=\sum_{s=0}^{2m}(-1)^s\frac{(2m+1)(n+2s-1)}{2(n+2m)}T_s^{2m}(n),$$

which means that

$$\begin{aligned}\frac{n+2m}{2m+1}\vartheta_{2m+1}(n) &= \frac{1}{2}\sum_{s=0}^{2m}(-1)^s(n-1)T_s^{2m}(n)+\sum_{s=1}^{2m}(-1)^s sT_s^{2m}(n)\\ &= \frac{1}{2}\sum_{s=0}^{2m}(-1)^s(n-1)T_s^{2m}(n)-\sum_{s=0}^{2m-1}(-1)^s(s+1)T_{s+1}^{2m}\\ &= \frac{n-1}{2}\sum_{s=0}^{2m}(-1)^sT_s^{2m}(n)-\vartheta_{2m}(n).\end{aligned}$$

Since $\vartheta_{2m}(n)=0$, result (18) is proved and can be used in (17) in order to obtain

$$k(c_{k-1}(n)-c_k(n))=\begin{cases} 0, \text{ if } k \text{ is even} \\ \\ \frac{k(n-1)}{n+k-1}c_{k-1}(n), \text{ if } k \text{ is odd}\end{cases}$$

or equivalently,

$$c_k(n) = c_{k-1}(n), \text{ if } k \text{ is even and } c_k(n) = \frac{k}{n+k-1}c_{k-1}(n), \text{ if } k \text{ is odd.}$$

The last relations can be used to obtain

$$\begin{aligned} c_{2m-1}(n) &= \frac{2m-1}{n+2m-2}c_{2m-2}(n) = \frac{2m-1}{n+2m-2}c_{2m-3}(n) \\ &= \frac{(2m-1)(2m-3)}{(n+2m-2)(n+2m-4)}c_{2m-4}(n) = \cdots \\ &= \frac{(2m-1)(2m-3)\cdots 3}{(n+2m-2)(n+2m-4)\cdots(n+2)}c_1(n). \end{aligned}$$

But $c_1(n) = T_0^1(n) - T_1^1(n) = \frac{n+1}{2n} - \frac{n-1}{2n} = \frac{1}{n}$ and hence

$$c_{2m-1}(n) = \frac{k!!}{(n+k-1)!!}(n-2)!!.$$

□

Remark 2 *At the end we would like to mention that the case $n = 2$ leads to*

$$c_{2m}(2) = \frac{1}{2^{2m}}\binom{2m}{m} = \frac{1}{2^{2m-1}}\binom{2m-1}{m-1} = c_{2m-1}(2),$$

calling the attention to the special role of the central binomial coefficient. It is also worth to underline the similarity of the sequence $(c_{2m}(2))_{m\geq 0}$ and the Catalan numbers

$$\mathcal{C}_m = \frac{1}{m+1}\binom{2m}{m}.$$

Whereas the Catalan numbers are the ratio of the central binomial coefficient $\binom{2m}{m}$ in the $2m-th$ row of the Pascal triangle and the total number of binomial coefficients in the $m-th$ row, the $c_{2m}(2)$ are the ratio of the same central binomial coefficient and the $sum\ of\ all\ binomial\ coefficients$ in the $2m-th$ row.

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