YOUNG CHILDREN SOLVING ADDITVE STRUCTURE PROBLEMS

Ema Mamede — CIEC, University of Minho

Florbela Soutinho — CIEC, University of Minho

This paper describes a study to analyse how 4-6-year-olds (N=45) children solve different types of additive reasoning problems. Individual interviews were conducted on kindergarten children when solving the problems. Their performance as well as their explanations were analysed when solving additive reasoning problems. The additive reasoning problems comprised simple, inverse and comparative problems. Results suggested that Portuguese kindergarten children have some informal knowledge that allowed them to solve additive structure problems with understanding. Children performed better in the simple additive problems and found the comparative problems more difficult.

INTRODUCTION

In mathematics children are expected to be able to attribute a number to a quantity, which is measuring (Nunes & Bryant, 2010a), but they also are expected to be able to quantify relations. When quantities are measured, they have a numerical value, but it is possible to reason about the quantities without measure them. In agreement with Nunes, Bryant and Watson (2010), it is crucial for children to learn to make both connections and distinctions between number and quantity. Quantitative reasoning results from a quantifying relations and manipulate them (Nunes & Bryant, 2010a), making relationships between quantities valuable (Thompson, 1994). For Nunes and Bryant (2010a), quantifying relations can be done by additive or multiplicative reasoning. Quoting the authors "[…] Additive reasoning tells us about the difference between quantities; multiplicative reasoning tells us about the ratio between quantities." (p.8). In the literature additive reasoning is associated to addition and subtraction (see Vergnaud, 1983) and multiplicative reasoning is associated to multiplication and division problems (see Steffe, 1994; Vergnaud, 1983).

Children can use their informal knowledge to analyse and solve simple addition and subtraction problems before they receive any formal instruction on addition and subtraction operations (Nunes & Bryant, 1996).

ABOUT THE ADDITVE REASONING

Piaget (1952) argued that children's understanding of arithmetical operations arises from their *schema*. A 'schema' is a representation of an action in which only the essential aspects of the action are evident. He identified three schemas related to additive reasoning: joint, separate and one-to-one correspondence. The author pointed out that children are able to master addition and subtraction only when they understand the inverse relation between these operations, which is achieved by the 7-year-olds. More recently, Nunes and Bryant (1996) referred that kindergarten children of 5-6-year-olds can relate their understanding of number as a measure of set size to their conception of addition / subtraction as an increase / decrease in quantities. This can help children to begin to understand that one operation is the inverse of the other. The schema from which children begin to understand addition and subtraction are representations of the act of joint and separate, respectively (Nunes, Campos, Magina & Bryant, 2005). These schemas allow 5-year-olds children to solve a problem such as: ”Anna has 3 candies. Her mother gave her 2 more candies. How many candies does Anna have now?”.

Additive reasoning problems involve one variable and they tell us about the difference between quantities. The part-whole relation is the invariant of the additive reasoning. The whole equals the sum of the parts. Nunes, Bryant and Watson (2010) argue that additive relations are used in one variable problems when quantities of the same kind are put together, separated or compared.

Carpenter and Moser (1982, 1984) presented a classification of addition and subtraction problem that does not characterize all the types of word problems involving additive reasoning, but those who are appropriate for primary age children. They distinguished four categories of addition and subtraction problems: change, combine, compare and equalize (see Carpenter & Moser, 1982, 1984).

Carpenter and Moser (1984) conducted a research on primary school children to analyse their solution strategies according to the type of problem presented. The authors argue that the processes that children use to solve addition and subtraction problems are intrinsically related to the structure of the problem. This idea that addition and subtraction word problems differ both in semantic relations used to describe a particular problem situation and in the identity of the quantity that is left unknown is also supported by other researchers (see De Corte & Verschaffel, 1987; Carpenter & Moser, 1982; Riley, Greeno & Heller, 1983; Fuson & Willis, 1986), who argue that addition and subtraction problem types are related to fairly systematic differences in children’s performance at various grade levels.

According to Nunes et al. (2005), children’s ability to solve problems involving an additive structure develops in three phases: first children can solve simple problems; then they can solve the inverse problems; and finally they can solve static problems. The addition and subtractions simple problems are those in which children are asked to transform one quantity by adding to it or subtracting from it (e.g., Joe had 5 marbles. Then he gave 3 to Tom. How many marbles does he have now?). These types of problems involve relations between the whole and its parts. The inverse problems are those in which the situation presented in the problem relates to a schema, but the correct resolution demands the inverse schema. For example, in the problem “Joe had some marbles. Then he won 2 more marbles in a game. Now Joe has 6 marbles. How many marbles did Joe have in the beginning?” (Nunes & Bryant, 2010a), subtraction appears as the inverse of addition; the quantity increased and the final one are given, and the initial quantity is unknown. The addition and subtraction static problems are those in which children are asked to quantify comparisons. For example, “Joe has 8 marbles and Tom has 5. Who has more marbles? (an easy question) How many more marbles does Joe have than Tom?” (a difficult question) (Nunes & Bryant, 1996; Nunes et al., 2005).

For Nunes and Bryant (1996) the difficulty of the problem is determined not only by the situation but also by the invariants of addition and subtraction that have to be understood by the children in order to solve a particular problem, and these invariants change according to the unknown parts of the problem. Nunes and Bryant (1996) also point out that the success in addition and subtraction tasks for young children is also determined by the resources that children are using to implement computational procedures, the system of signs. For the authors problems that involve relations are more difficult than those that involve quantities. The literature about additive reasoning has been giving evidence that compare problems, which involve relations between quantities, are more difficult than those that involve combining sets or transformations. Carpenter and Moser (1984) refer that many children do not seem to know what to do when asked to solve a compare problem.

Nunes et al. (2005) conducted a research with primary school Brazilian children, from grades 1 to 4, to analyse their performance when solving problems of additive reasoning. Their results indicate levels of success above 70% for the children of all grades when solving simple problems of part-whole relations involving addition and subtraction. When children were asked to solve inverse problems only 60% of the first graders and more than 80% of the 4th-graders succeeded in a problem such as: ”Kate had some candies. She won 2 more in a game. Now she has 12 candies. How many candies did Kate have in the beginning?”. Their study also analysed comparative problems, such as: “In a classroom there are 9 pupils and 6 chairs. Are there more chairs or pupils? How many pupils are there more?”. The authors reported around 50% of success for the second question, and almost 90% among the 4th-graders. These results support the idea that the development of children’s additive reasoning is progressive, but also suggest that children are able to solve many of these problems before they receive any formal instruction on addition and subtraction.

Literature gives evidence that kindergarten children are able to solve some addition and subtraction problems (see Fuson, 1992; Nunes & Bryant, 1996), but that does not mean that they understand all the relations in the context of additive reasoning problems. The children's understanding of addition a subtraction is progressive and develops over a long period of time.

To understand more about the children's additive reasoning, it becomes relevant to analyze younger children's ideas of addition and subtraction. Following previous research of Nunes et al. (2005), it was conducted a study with young children, from 4 to 6 years of age, concerning these issues. The study was developed to examine children's understanding of additive reasoning problems. For that two questions were addressed: a) how do children perform when solving additive reasoning problems?; and b) what explanations do they present when solving these problems?

METHODS

Individual interviews were conducted to 45 kindergarten children (4- to 6-year-olds), from Viseu, Portugal. There were 15 children from each age level. In these interviews children were challenged to solve 12 additive reasoning problems (4 direct problems, 4 inverse problems, 4 comparative problems). The interviews were conducted always by the same researcher.

The problems presented to the children were an adaptation of the problems previously documented in the literature by Nunes et al. (2005). Table 1 gives some examples of additive problems presented to children.

Type of problem	Example
Direct	Kate's mum gave her 4 pencils. Later she gave her 2 more. How many pencils does she have now?
	Ben had 7 candies and he gave 5 to his sister. How many candies does he have now?
Inverse	Anna had some candies. She gave 3 to her sister. Anna has 2 candies now. How many candies did she have in the beginning?
	Mark had 5 chocolate candies, he ate some and now he has 3 candies. How many chocolate drops did he eat?
Comparative	In a classroom there are 6 pupils and 4 chairs. Are there more pupils or chairs? How many more?
	Mary has 3 flowers. She has 2 more flowers than Betty. How many flowers does Betty have?

Table 1: Examples of additive reasoning problems.

All the problems were presented to the children by the means of a story problem and material was available to represent the problems.

No feedback was given to any child when solving the problems. All the children were asked “Why do you think so?” after his/her resolution in order to know children’s arguments. In the comparative problems, it was expected that some children could requested help to understand the problem. In some cases the interviewer had to repeat the problem to the child or to put a second question, transforming a static question into a dynamic one, in order to facilitate their understanding of the problem. For example, instead of “how many cars are there more than planes?” – a static question – the child would then be asked “How many planes should we give to Mark for him to have as many toys has Ben?” – a dynamic question.

For all these problems, the assessment of children’s performance was 0 for an incorrect response, and 1 for a correct one.

Data collection took place by means of video record and interviewer’s field notes.

Results

A descriptive analysis of children’s performance when solving additive reasoning problems was conducted. Table 2 summarizes this information for each type of additive structure problem according to the age level.

Additive reasoning problems			
		Mean (s.d.)	
Type of problem	4-year-olds (n=15)	5-year-olds (n=15)	6-year-olds (n=15)
Direct	2.13 (1.25)	3.75 (1.36)	3.53 (0.83)
Inverse	1.47 (1.30)	1.80 (1.27)	2.53 (1.25)
Comparative	0.80 (0.78)	2.33 (1.23)	2.33 (1.29)

Table 2: Mean and (standard deviation) of correct responses when solving the additive structure problems by age level.

It is remarkable the children’s success levels when solving additive reasoning problems. Even the 4-year-olds were able to solve successfully some of these problems. The inverse problems and the comparative problems seemed to be more difficult for children than the direct ones, but even in those 5- and 6-year-olds children presented a correct resolution. The comparative problems were the most difficult for the children. Very often the interviewer had to repeat the problem to the child or to ask a second question in the same problem in order to facilitate children's understanding of the problem, moving from a static question to a dynamic one, as referred before. Thus, the number of cases in which the interviewer had to transform a static problem into a dynamic one was registered producing two categories: without transformation, in which the child solved the problem with no changes; and with transformation in which the child need the interviewer to transform the problem. In any of these cases, the assessment was 0/1 for incorrect/correct responses.

Table 3 summarizes the number of correct responses given by the children when solving the comparative problems according to the need of changes in the presentation of the problem. As each child solved 4 comparative problems, 60 resolutions for each age group were produced.

	Correct responses in comparative problems		
Difficulty level	4-year-olds (n=15)	5-year-olds (n=15)	6-year-olds (n=15)
Without Transformation	2	14	19
With Transformation	10	21	16
Total correct responses	12	35	35

Table 3: Number of correct resolutions in the comparative problems, with the transformation and without it, according to the age.

Figures 1 to 3 present the distributions of the total of correct responses for the three types of additive reasoning problems, according to the age level.

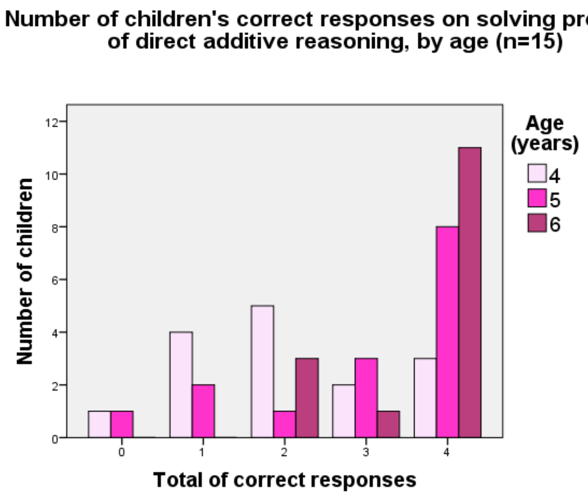

Figure 1: Number of correct responses for direct problems by age level.

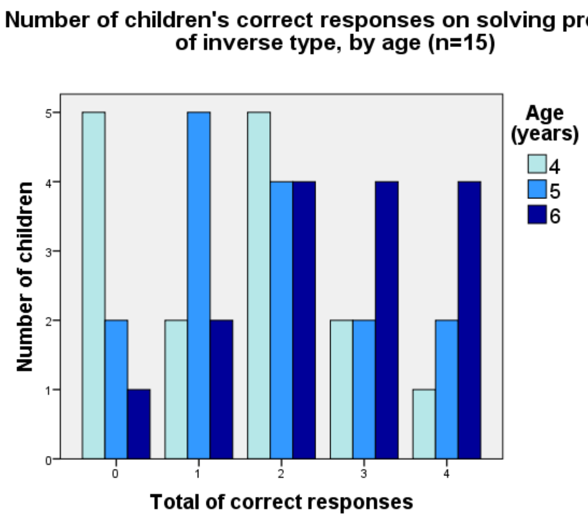

Figure 2: Number of correct responses for inverse problems by age level.

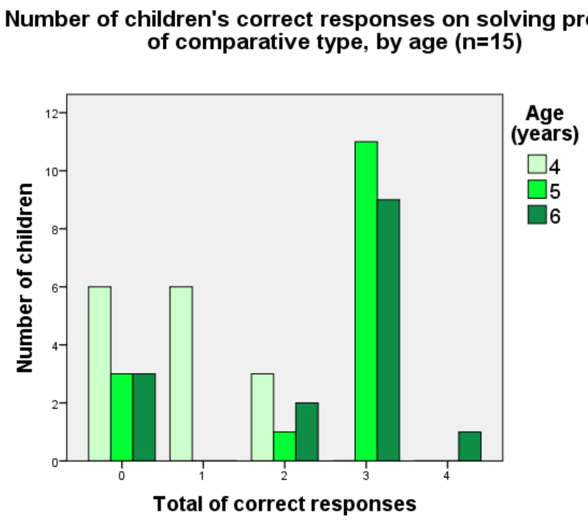

Figure 3: Number of correct responses for comparative problems by age level.

In order to analyse the effect of the age on children’s performance solving the different types of additive problems a one-way Analysis of Variance (ANOVA) was conducted with performance in the type of problem (direct, inverse, comparative) as dependent list and age (4-, 5- and 6-year-olds) as a factor. There were no significant effects of the age on the direct problems neither on the inverse problems, but there is a significant effect of age on comparative problems ($F(2,42)=9.3$, $p< .001$) indicating that older children performed on this problems than the 4-year-olds. Bonferroni post-hoc tests indicate that children of 5- and 6-year-olds performed better than the 4-year-olds, but no significant differences were found on children’s performance of 5- and 6-year-olds. Thus, in direct and inverse type of problems there was no age effect; the comparative problems were easier for older children than for the younger ones.

To know more about children's reasoning when solving these problems, their arguments were analysed for each type of problem. Four categories of children's arguments were considered in this analysis. The valid arguments comprise the justifications in which children consider all the quantities involved in the problem correctly; the incomplete category comprises children's arguments that refers only to one part of the quantities involved in the problem; the invalid arguments are those in which children do not articulate the quantities involved in the problems; and the no argument category that comprises all the cases of absence of argument.

Table 4 presents the number of arguments of each type that were used by children when solving additive reasoning problems correctly, according to the age.

Additive reasoning problems									
	Type of problem								
	direct			inverse			comparative		
Type of argument	4yrs	5yrs	6yrs	4 yrs	5yrs	6yrs	4 yrs	5yrs	6yrs
Valid	17	19	38	12	17	28	8	22	22
Incomplete	1	9	-	-	2	1	-	-	6
Invalid	3	8	4	7	2	7	3	9	4
No argument	11	9	11	3	6	2	1	4	3
Total correct resp.	32	45	53	22	27	38	12	35	35

Table 4: Number of arguments of each type given when solving the additive structure problems by age level.

Four categories of children's arguments were considered in this analysis. The valid arguments comprise the justifications in which children consider all the quantities involved in the problem correctly; the incomplete category comprises children's arguments that refers only to one part of the quantities involved in the problem; the invalid arguments are those in which children do not articulate the quantities involved in the problems; and the no argument category that comprises all the cases of absence of argument. Table 4 presents the number of arguments of each type that were used by children when solving additive reasoning problems correctly, according to the age.

Children of all age levels presented valid arguments were associated to correct resolutions. This suggests that the results obtained from children's performance are associated to an understanding of the problems presented to them. Around 53% of the 4-year-olds could solve correctly the simple problems presenting valid justifications; these percentage increases to almost 72% for the group of 6-year-olds children. Valid arguments were also presented in 54.5% of the correct answers given by the 4-year-olds children when solving the inverse problems, and in 66.7% of the correct resolutions of the comparative problems. In all type of problems there were children who were able to solve them correctly, but were unable to present a valid argument.

The use of an incomplete argument can be understood as child difficulty to articulate verbally a logic explanation that was carried on. Also children who solved correctly the problems presented no argument, as it happen with 34.4% of the 4-year-olds that solved correctly the simple problems.

DISCUSSION AND CONCLUSION

Children's informal knowledge is supposed to be the starting point for the formal instruction. Thus, it makes sense to know better what do children can and cannot do before being taught about arithmetic operations in primary school. The results presented here suggest that Portuguese kindergarten children are able to solve some problems involving additive structures with understanding, in particular conditions.

These results converge with those presented by Nunes et al. (2005) who analysed 5-8-year-olds children's performance when solving additive reasoning problems. These authors also reported that additive comparative problems were more difficult to young children than the direct and inverse ones. Our study extended these findings about children's additive reasoning as it gives evidence that 4-year-olds children can succeed in solving direct, inverse and also comparative problems. Their procedures do not vary from those used by the 5- and 6-year-olds relying on the schema of the act of join and separate for the direct and inverse problems previously identified in the literature (see Nunes & Bryant, 1996; Nunes et al., 2005).

The children's arguments were also analysed in order to get an insight on their reasoning when solving the additive structure problems. These arguments give evidence that children as young as 4 years of age can establish a correct reasoning and solve this type of problems. This suggests that their correct answers were not achieved by chance. If there are children of 4-year-olds able to solve some additive structure problems with understanding, relying in their informal knowledge, perhaps kindergarten could stimulate their early ideas about addition and subtraction. More research is needed to analyse these issues and to find out what sort of problems, if there are any, should be presented to kindergarten children in order to help them to develop their reasoning.

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