FOCUSING ON YOUNG CHINDREN’S ADDITIVE AND MULTIPLICATIVE REASONING

Ema Mamede
CIEC, University of Minho
Braga, Portugal

Florbela Soutinho
CIEC, University of Minho
Braga, Portugal

This paper describes a brief study to analyse how 4-6-years-old children solve different types of additive and multiplicative reasoning problems. Individual interviews were conducted on kindergarten children when solving the problems. Their performance as well as their explanations were analysed when solving additive and multiplicative reasoning problems. The additive reasoning problems comprised simple, inverse and comparative problems; the multiplicative ones comprised simples and inverse problems. Results suggested that Portuguese kindergarten children have some informal knowledge that allowed them to solve additive and multiplicative reasoning problems with understanding.

Key-words: additive reasoning, multiplicative reasoning, kindergarten children.

Framework

In their process of understanding numbers children need to make connection between quantities and numbers. Numbers are used to represent quantities and to represent relations. Nunes and Bryant (2010a) refer that when numbers are used to represent quantities they are the result of a measurement operation from which a quantity can be represented by a number of conventional units (e.g., 3 children, 4 chairs). When a number is used to represent relations, the number does not refer to a quantity but to a relation between two quantities, expressing how many more or fewer (e.g., there is 1 more chair than children).

In mathematics children are expected to be able to attribute a number to a quantity, which is measuring (Nunes & Bryant, 2010a), but they also are expected to be able to quantify relations. When quantities are measured, they have a numerical value, but it is possible to reason about the quantities without measure them. In agreement with Nunes, Bryant and Watson (2010), it is crucial for children to learn to make both connections and distinctions between number and quantity. Quantitative reasoning results from a quantifying relations and manipulate them (Nunes & Bryant, 2010a), making relationships between quantities valuable (Thompson, 1993). For Thompson (1993) “Quantitative reasoning is the analysis of a situation into a quantitative structure – a network of quantities and quantitative relationships […] What it important is relationships among quantities.” (p.165). Quoting Nunes and Bryant (2010a), “[…] quantifying relations can be done by additive or multiplicative reasoning. Additive reasoning tell us about the difference between quantities; multiplicative reasoning tell us about the ratio between quantities.” (p.8). In literature, additive reasoning is associated to addition and subtraction (see Vergnaud, 1983) and multiplicative reasoning is associated to multiplication and division problems (see Steffe, 1994; Vergnaud, 1983).

Children can use their informal knowledge to analyse and solve simple addition and subtraction problems before they receive any formal instruction on addition and subtraction operations (Nunes & Bryant, 1996). But they also can know quite a lot about multiplicative reasoning when they start school (Nunes & Bryant, 2010b). To have an opportunity to solve addition and subtractions problems can help children to construct a more complete understanding of these arithmetic operations.

Additive reasoning

Piaget (1952) argued that children’s understanding of arithmetical operations arises from their *schema*. A ‘schema’ is a representation of an action in which only the essential aspects of the action are evident. He identified three schemas related to additive reasoning: joint, separate and one-to-one correspondence. The author pointed out that children are able to master addition and subtraction only when they understand the inverse relation between these operations, which is achieved by the 7-years-old. More recently, Nunes and Bryant (1996) referred that kindergarten children of 5-6-years-old can relate their understanding of number as a measure of set size to their conception of addition / subtraction as an increase / decrease in quantities. This can help children to begin to understand that one operation is the inverse of the other. The schema from which children begin to understand addition and subtraction are representations of the act of joint and separate, respectively (Nunes, Campos, Magina & Bryant, 2005). These schemas allow 5-years-old children to solve a problem such as: ”Anna has 3 candies. Her mother gave her 2 more candies. How many candies does Anna have now?”.

Additive reasoning problems involve one variable and they tell us about the difference between quantities. The part-whole relation is the invariant of the additive reasoning. The whole equals the sum of the parts. Nunes, Bryant and Watson (2010) argue that additive relations are used in one variable problems when quantities of the same kind are put together, separated or compared.

Carpenter and Moser (1982, 1984) presented a classification of addition and subtraction problem that does not characterize all the types of word problems involving additive reasoning, but those who are appropriate for primary age children. They distinguished four categories of addition and subtraction problems: change, combine, compare and equalize (see Carpenter & Moser, 1982, 1984).

Carpenter and Moser (1984) conducted a research on primary school children to analyse their solution strategies according to the type of problem presented. The authors argue that the processes that children use to solve addition and subtraction problems are intrinsically related to the structure of the problem. This idea that addition and subtraction word problems differ both in semantic relations used to describe a particular problem situation and in the identity of the quantity that is left unknown is also supported by other researchers (see De Corte & Verschafefel, 1987; Carpenter & Moser, 1982; Riley, Greeno & Heller, 1983; Fuson & Willis, 1986), who argue that addition and subtraction problem types are related to fairly systematic differences in children's performance at various grade levels. Even though slightly different classifications of problems were used by different authors, their empirical research consistently found that different class of problems vary in their level of difficulty.

According to Nunes et al. (2005), children's ability to solve problems involving an additive structure develops in three phases: first children can solve simple problems; then they can solve the inverse problems; and finally they can solve static problems. The addition and subtractions simple problems are those in which children are asked to transform one quantity by adding to it or subtracting from it (e.g., Joe had 5 marbles. Then he gave 3 to Tom. How many marbles does he have now?). These types of problems involve relations between the whole and its parts. The inverse problems are those in which the situation presented in the problem relates to a schema, but the correct resolution demands the inverse schema. For example, in the problem "Joe had some marbles. Then he won 2 more marbles in a game. Now Joe has 6 marbles. How many marbles did Joe have in the beginning?" (Nunes & Bryant, 2010a), subtraction appears as the inverse of addition; the quantity increased and the final one are given, and the initial quantity is unknown. The addition and subtraction static problems are those in which children are asked to quantify comparisons. For example, "Joe has 8 marbles and Tom has 5. Who has more marbles? (an easy question) How many more marbles does Joe have than Tom?" (a difficult question) (Nunes & Bryant, 1996; Nunes et al., 2005).

For Nunes and Bryant (1996) the difficulty of the problem is determined not only by the situation but also by the invariants of addition and subtraction that have to be understood by the children in order to solve a particular problem, and these invariants change according to the unknown parts of the problem. Nunes and Bryant (1996) also point out that the success in addition and subtraction tasks for young children is also determined by the resources that children are using to implement computational procedures, the system of signs. For the authors problems that involve relations are more difficult than those that involve quantities. The literature about additive reasoning has been giving evidence that compare problems, which involve relations between quantities, are more difficult than those that involve combining sets or transformations. Carpenter and Moser (1984) refer that many children do not seem to know what to do when asked to solve a compare problem.

Nunes et al. (2005) conducted a research with primary school Brazilian children, from grades 1 to 4, to analyse their performance when solving problems of additive reasoning. Their results indicate levels of success above 70% for the children of all grades when solving simple problems of part-whole relations involving addition and subtraction. When children were asked to solve inverse problems only 60% of the first graders and more than 80% of the 4^{th}-graders succeeded in a problem such as: "Kate had some candies. She won 2 more in a game. Now she has 12 candies. How many candies did Kate have in the beginning?". Their study also analysed comparative problems, such as: "In a classroom there are 9 pupils and 6 chairs. Are there more chairs or pupils? How many pupils are there more?". The authors reported around 50% of success for the second question, and almost 90% among the 4^{th}-graders. These results support the idea that the development of children's additive reasoning is progressive, but also suggest that children are able to solve many of these problems before they receive any formal instruction on addition and subtraction.

Literature gives evidence that kindergarten children are able to solve some addition and subtraction problems (see Fuson, 1992; Nunes & Bryant, 1996), but that does not mean that they understand all the relations in the context of additive reasoning problems. The children's understanding of addition a subtraction is progressive and develops over a long period of time.

Multiplicative reasoning

Piagetian theory supports the idea that children first quantify additive relations and can only quantify multiplicative relations much later (see Piaget, 1960). In spite of his undoubted contribution to research, more recently research has been giving evidence of a different position. Thompson (1994), Vergnaud (1983) and Nunes and Bryant (2010a) support the idea that additive and multiplicative reasoning have different origins. Thompson (1994) considers quantitative operations as a mental operation by which one conceives a new quantity in relation to one or more already conceived quantities. He argues that "Quantitative operations originate in action: The quantitative operation of combining two quantities additively originates in the actions of putting together to make a whole and separating a whole to make parts; operation of comparing two quantities additively originates in the action of matching two quantities with the goal of determining the excess or deficit; the quantitative operation of comparing two quantities multiplicatively originates in matching and subdividing with the goal of sharing." (Thompson, 1994, pp. 185-186). Also Vergnaud (1983) in his theory of conceptual fields distinguishes the field of additive structures and the field of multiplicative structures, considering them as sets of problems involving operations of the additive or the multiplicative type. Vergnaud (1983) argues that "multiplicative structures rely partly on additive structures; but they also have their own intrinsic organization which is not reducible to additive aspects" (p.128). Nunes and Bryant (2010a) also consider that additive and multiplicative reasoning have different origins, arguing that "Additive reasoning stems from the actions of joining, separating and placing sets in one-to-one correspondence. Multiplicative reasoning stems from the action of putting two variables in one-to-many correspondence (one-to-one is just a particular case), an action that keeps the ratio between the variables constant." (p.11).

Multiplicative reasoning involves two (or more) variables in a fixed ratio. Thus, problems such as: "Joe bought 5 sweets. Each sweet costs 3p. How much did he spent?" Or "Joe bought some sweets; each sweet costs 3p. He spent 30p. How many sweets did he buy?" are examples of problems involving multiplicative reasoning. The former can be solved by a multiplication to determine the unknown total cost; the later would be solved by means of a division to determine an unknown quantity, the number of sweets (Nunes & Bryant, 2010a). Research has been giving evidence that children can solve multiplication and division problems of these kinds even before receiving formal instruction about multiplication and division in school. For that they use the schema of one-to-many correspondence. Carpenter, Ansell, Franke, Fennema and Weisbeck, referred by Nunes and Bryant (2010a), reported high percentages of success when observing kindergarten children solving multiplicative reasoning problems involving correspondence 2:1, 3:1 and 4:1. Nunes et al. (2005) analysed primary Brazilian school children performance when solving multiplicative reasoning problems. When children were shown a picture with 4 houses and then were asked to solve the problem: "In each house are living 3 puppies. How many puppies are living in the 4 houses altogether?", 60% of the 1st-graders and above 80% of the children of the other grades succeeded. When children were asked to solve a division problem, such as: "There are 27 sweets to share among three children. The children want to get all the same amount of sweets. How many sweets will each one get?", the levels of success for 1st-graders was 80% and above that for the other graders (2nd to 4th-graders).

Kornilaki, refereed by Nunes et al. (2005) analysed 5- to 8-years-old children performance when solving multiplicative reasoning problems, presented to them using only pictures. She presented multiplication and division problems of two types, direct and inverse problems. In the direct problems children can reach the solution using directly correspondence and distribution to solve multiplication and division problems, respectively. In the inverse problems this cannot be done immediately. In an inverse multiplication problem such as "It's Charles birthday. Each friend that is coming to his party will get 3 balloons. He bought 18 balloons. How many friends are there in the party?". Kornilaki's results showed that 30% of the 5-year-olds and 50% of the 6-year-olds children succeeded in this problem. In the inverse division problem "It's Ana's birthday and she is going to share cookies among her friends. She prepared small bags with 3 cookies each to share between her friends. She used 18 cookies to prepare the bags. How many bags did she make?", 40% of the 5-year-old and almost 68% of the 6-year-olds children succeeded. Again, research is giving evidence that children can solve multiplicative reasoning problems before being taught in school about it and before achieving all the additive reasoning development.

In this scenario some questions arise that still have no answer in the literature. How do 4-6-year-olds children master the different types of additive and multiplicative reasoning problems? How much the development of the children’s additive reasoning affects the development of their multiplicative reasoning? To what extent do children’s additive and multiplicative reasoning can be improved in the kindergarten? This paper explored the first question focusing on children’s informal knowledge when solving some additive and multiplicative reasoning problems.

Methods

Individual interviews were conducted to 6 kindergarten children (4-6-year-olds), from Viseu, Portugal. These interviews were conducted in two different sessions. There was one week between these two sessions. Each session last approximately 25 minutes. In the first session, children were challenged to solve 9 additive reasoning problems (3 direct problems, 3 inverse problems, 3 comparative problems); in the second session, the children were challenged to solve 6 multiplicative reasoning problems (4 direct problems, 2 inverse problems). The problems presented to the children were an adaptation of the problems previously documented in the literature by Nunes et al. (2005). Tables 1 and 2 give some examples of additive and multiplicative problems presented to children, respectively.

Table 1: Examples of additive reasoning problems

Type of problem	Example
Direct	Kate’s mum gave her 4 pencils. Later she gave her 2 more. How many pencils does she have now?
Inverse	Anna had some candies. She gave 3 to her sister. Anna has 2 candies now. How many candies did she have in the beginning?
Comparative	In a classroom there are 6 pupils and 4 chairs. Are there more pupils or chairs? How many more?

All the problems were presented to the children by the means of a story problem and material was available to represent the problems.

Table 2: Examples of multiplicative reasoning problems presented to the children.

Type of problem	Example
Direct	In this street there are 3 houses. In each house are living 2 rabbits. How many rabbits are living in the houses altogether?
Inverse	It’s Bill’s birthday. He is going to offer 3 balloons to each friend in his party. He bought all these balloons to offer (Showing a bowl with 15 balloons). How many friends are in the party?

The interviewer was the same in all the interviews. No feedback was given to any child when solving the problems. All the children were asked “Why do you think so?” after his/her resolution in order to know children’s arguments. Data collection took place by means of digital video record and interviewer’s field notes.

Results

A descriptive analysis of children’s performance when solving additive and multiplicative reasoning problems was conducted. Table 3 summarizes this information for each type of problem, when solving problems.

It is remarkable children's success levels when solving additive and multiplicative reasoning problems. The inverse problems are more difficult for children than the direct ones, but even in those children presented a correct resolution in approximately 78% of the additive inverse problems and in 75% of the multiplicative inverse problems presented to them.

Table 3: Number of correct/incorrect resolutions presented by children when solving additive and multiplicative reasoning problems.

	Additive reasoning problems			Multiplicative reasoning problems	
Resolution	Direct (18 resol.)	Inverse (18 resol.)	Comparative (18 resol.)	Direct (24 resol.)	Inverse (12 resol.)
Correct	17	14	10	18	9
Incorrect	1	4	8	6	3

In the additive reasoning problems, the comparative ones were the most difficult for the children, in which children presented around 56% of correct resolutions.

A bit more about the additive reasoning problems

The number of correct responses on this type of problems gives evidence that children possess some informal knowledge about addition and subtraction that allow then to successfully solve additive reasoning problems. Nevertheless, some of these problems seem to be more difficult for them than others.

Because the comparative problems were the most difficult ones, in some cases the interviewer had to repeat or even reformulate the problem presented to the child. In the cases in which the reformulation was needed, the interviewer had to present a new question in the problem, transforming a static question (e.g. – "how many cars are there more than planes?") into a dynamic question (e.g. – "how many cars more does Tom need to have as many as Ben?"). This fact made us reconsider the analysis developed and introduce a new to point, the level of performance. Thus, when a child solved easily an additive problem, the level of performance was "easy"; when a reformulation of the problem was required, the level of performance was "difficult". Table 4 summarizes the levels of performance observed among those who solved correctly the additive problems.

Table 4: Number of children who correctly solved the additive problems by level of performance.

	Additive reasoning problems		
Performance	Direct (18 resol.)	Inverse (18 resol.)	Comparative (18 resol.)
Easy	17	7	7
Difficult	0	7	3

An analysis of children's strategies when solving these tasks was conducted. For this, four categories were distinguished: join, separate, one-to-one correspondence, and invalid. These categories were previously presented in the literature (see Nunes et al., 2005; Nunes & Bryant, 2010). The *join* category comprises all the cases in which a child join two quantities to produce the result (e.g., "4 plus 2, it's 6"); the category of separate comprises the cases in which a child separates an amount to produce the result (e.g., "there are 6, I took 2 and now there are 4"); the one-to-one correspondence comprises the cases in which a child establishes this type of correspondence to share items and produce a final amount. Table 5 presents the number of strategies of each type observed on the children who correctly solved the additive reasoning problems.

Most of the additive reasoning problems were correctly solved using join and separate strategies. The correspondence seemed to play an important role only on the comparative problems. Figures 1 and 2 give examples of children strategies using join and one-to-one correspondence to solve the problems.

Table 5: Number of strategies of each type used by the children who correctly solved the additive problems according.

	Additive reasoning problems		
	Direct	Inverse	Comparative
Strategy	(18 resol.)	(18 resol.)	(18 resol.)
Join	12	12	3
Separate	5	2	5
Correspondence	0	0	2

Figure 1 – A child using the strategy of join to solve an additive direct problem.

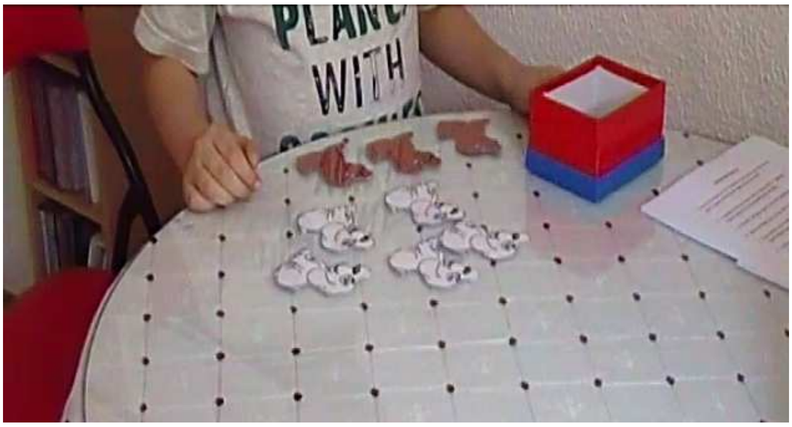

Figure 2 – A child using the strategy of one-to-one correspondence to solve an additive comparative problem.

To know more about children's reasoning when solving these problems, their arguments were analysed for each problem. Four categories of children's arguments were considered in this analysis. The valid arguments comprise the justifications in which children consider all the quantities involved in the problem correctly (e.g., after solving a problem "4 plus 2, it's 6" explains using his/her fingers "four…1, 2, 3, 4 plus two, is 5, 6"; or "because there were 3 and then plus 2 is 5"); the incomplete category comprises children's arguments that refers only to one part of the quantities involved in the problem; and the invalid arguments are those in which children do not articulate the quantities involved in the problems. Table 6 presents the number of arguments of each type that were used by children when solving additive reasoning problems correctly.

Children presented valid arguments in most of the cases of correct resolutions observed. This suggests that the results obtained from children's performance are associated to their understanding of the additive reasoning problems.

Table 6: Number of arguments of each type presented by the children when solving additive reasoning problems.

	Additive reasoning problems		
Type of argument	Direct	Inverse	Comparative
Valid	17	14	8
Incomplete	0	0	1
Invalid	0	0	1

In the additive–comparative reasoning problems there was a child that solved the problems correctly, but who presented an incomplete argument; another child in these conditions presented an invalid argument. The use of an incomplete argument can be understood as a child difficulty to articulate verbally a logic explanation for the procedure that was carried out.

A bit more about the multiplicative reasoning problems

The number of correct responses on this type of problems gives evidence that children possess some informal ideas of multiplicative relations. These ideas allowed them to successfully solve the direct and inverse problems. Nevertheless, the inverse problems seem to be more difficult for them than the inverse ones.

Children's performance was analysed and two levels of performance were considered: the easy one, comprising the resolutions in which a child solved the problem immediately after its presentation, with no additional intervention of the interviewer; and the difficult one, comprising the cases in which the interviewer had to repeat. Table 7 summarizes the levels of performance observed among those who solved correctly the multiplicative reasoning problems.

Table 7 – Number of children who correctly solved the multiplicative problems by level of performance.

	Multiplicative reasoning problems	
Performance	Direct (18 correct resol.)	Inverse (9 correct resol.)
Easy	14	8
Difficult	4	1

There were 24 direct problems presented to the children who solved correctly 18 of these problems. Most of the children who succeed were able to solve easily the direct problems and almost half of the inverse problems. This fact suggests that children can understand multiplicative relations much earlier than they receive instruction on multiplication.

An analysis of children's strategies when solving these tasks was conducted. For this, three categories were distinguished: equal share, one-to-many correspondence, and trial and adjust. The first two categories were previously presented in the literature (see Nunes et al., 2005; Nunes & Bryant, 2010). The equal share category comprises all the cases in which a child shares a quantity among recipients to obtain the result; the category of one-to-many correspondence comprises the cases in which a child establishes a correspondence between an element of a set and another set with more than one element; the trial and adjust comprises the cases in which a child uses the trial and error strategy but refines and approximates each attempt to produce equal subsets of the initial set. Table 8 presents the number of strategies of each type observed on the children who correctly solved the multiplicative reasoning problems.

Most of the multiplicative reasoning problems were correctly solved using strategies that rely on correspondence. Figure 3 gives an example of child using one-to-many correspondence to solve a multiplicative reasoning problem.

Table 8: Number of strategies of each type used by the children who correctly solved the multiplicative problems according.

	Multiplicative reasoning problems	
	Direct	Inverse
Strategy	(18 correct resol.)	(18 correct resol.)
Correspondence	14	0
Equal share	1	6
Trial & adjust	3	3

Most of the multiplicative reasoning problems were correctly solved using strategies that rely on correspondence. Figure 3 gives an example of child using one-to-many correspondence to solve a multiplicative reasoning problem.

Figure 3 – A child using the strategy of one-to-many correspondence to solve a multiplicative direct problem.

When analysing children’s arguments for each problem, three categories were considered: the valid arguments, comprising the justifications in which children consider all the quantities involved in the problem correctly; the incomplete arguments, comprising arguments that refers only to one part of the quantities involved in the problem; and the invalid arguments, in which children do not articulate the quantities involved in the problems or answer “I don’t know!”. Table 9 presents the number of arguments of each type that were used by children when solving multiplicative reasoning problems correctly.

Table 9: Number of arguments of each type presented by the children when solving multiplicative reasoning problems.

	Multiplicative reasoning problems		
Type of argument	Direct	Inverse	Comparative
Valid	17	19	7
Incomplete	1	0	0
Invalid	0	0	2

Children presented valid arguments in most of the cases of correct resolutions observed, suggesting some understanding of the multiplicative reasoning problems.

Similarly to the additive reasoning solving problems, in the multiplicative-simple reasoning problems there was a child who solved correctly the problem, but could not articulate a complete explanation for it. The use of an incomplete argument can be understood as a child difficulty to articulate verbally a logic explanation that was carried on.

Final remarks

The results of this study give evidence that kindergarten children possess some type of informal knowledge that allow them to successfully solve some problems of additive and multiplicative structure. Children's informal knowledge is supposed to be the starting point for the formal instruction. Thus, it makes sense to know better what do children can and cannot do before being taught about arithmetic operations in primary school. The results presented here suggest that Portuguese kindergarten children are able to solve some problems involving additive and multiplicative structures in particular conditions.

Our findings suggest that direct and inverse additive problems can be solved by children from 4- to 6-years-old, in particular conditions. Children's strategies as well as their arguments support the idea that these levels of success were not obtained by chance. The comparative problems seem to be more difficult for these children. These ideas converge with those presented by Nunes et al. (2005) who analysed 5-8-years-old children's performance when solving additive reasoning problems. These authors also reported that additive comparative problems were more difficult to young children than the simple and inverse ones.

Also the multiplicative structure problems presented to the children of this study were correctly solved by many young children. The solution to direct and inverse multiplicative structure problems was reached by many children, and arguments to support their procedures were presented by them, revealing an understanding of the situation. These findings converge with the idea presented previously by Nunes et al. (2005), and Nunes and Bryant (2010) when they argue that children possess some informal knowledge that allow them to solve multiplication and division problems much earlier than they receive any formal instruction about these operations at school.

Nevertheless, this study involved a very small sample which makes impossible the idea of establishing any type of generalization of these findings. Research refers that additive and multiplicative reasoning involve different schema of action. This suggests that possibly these two types of reasoning develop differently, and one can be more difficult than the other. This study was not designed with the appropriate controls in the research in order to provide comparative information about these two types of problems. More research is needed to analyse the children's understanding of these issues and to find out what sort of problems, if there are any, could be presented to kindergarten children in order to help them to develop their reasoning.

References

Carpenter, T. & Moser, J. (1982). The development of Addition and Subtraction Problem-solving Skills. In T. Carpenter, J. Moser & T. Romberg (Eds.), *Addition and Subtraction: A cognitive perspective* (pp.9-24). Hillsdale, NJ: LEA.

Carpenter, T. & Moser, J. (1984). The Acquisition of Addition and Subtraction Concepts in Grades One through Three. *Journal for Research in Mathematics Education*, 15 (3), 179-202.

De Corte, E. & Verschaffel, L. (1987). The Effect of Semantic Structure on First Graders' Strategies for Solving Addition and Subtraction Problems. *Journal for Research in Mathematics Education*, 18(5), 363-381.

Fuson, K. (1992). Research on while number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*, (pp.243-275). New York: Macmilla Publishing Company.

Fuson, K. & Willis, G. (1986). *First and Second Graders' Performance on Compare and Equalize Word Problems*. In L. Burton & C. Hoyles (Eds.), *Proceedings of the Tenth International Conference of Psychology of Mathematics Education* (pp.19-24). London: University of London - Institute of Education.

Nunes, T. & Bryant, P. (2010a). *Understanding relations and their graphical representation.* In T. Nunes, P. Bryant & A. Watson (Eds), Key understanding in mathematics learning. (Acedido em 20 de Abril, 2011, de http://www.nuffieldfoundation.org/sites/defaukt/files/P4.pdf.

Nunes, T. & Bryant, P. (2010b). *Understanding whole numbers.* In T. Nunes, P. Bryant & A. Watson (Eds), Key understanding in mathematics learning. (Acedido em 20 de Abril, 2011, de http://www.nuffieldfoundation.org/sites/defaukt/files/P2.pdf

Nunes, T. & Bryant, P. (1996). *Children Doing Mathematics*. Oxford: Blackwell Publishers.

Nunes, T.; Campos, T; Magina, S. & Bryant, P. (2005). *Educação matemática – Números e operações numéricas*. São Paulo: Cortez Editora.

Piaget, J. (1952). *The Child's Conception of Number*. London: Routledge & Kegan Paul Limited

Riley, M, Greeno, J. & Heller, J. (1983). Development of children's problem-solving ability in arithmetic. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp.153-196). New York: Academic Press.

Steffe, L. (1994). Children's Multiplying Schemes. In G. Harel & J. Confrey (Eds.), *The Development of multiplicative reasoning in the learning of mathematics*, (pp. 3-40). Albany, NY: SUNY Press.

Thompson, P. (1994). The Development of the Concept of Speed and Its Relationship to Concepts of Rate.. In G. Harel & J. Confrey (Eds.), *The Development of multiplicative reasoning in the learning of mathematics*, (pp. 181-236). Albany, NY: SUNY Press.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes*, (pp. 128-175). New York: Academic Press.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Berh (Eds.), *Number Concepts and operations in middle-grades*, (pp. 141-161). Reston VA: NCTM.

The authors:

Ema Mamede, PhD, Professor in Elementary Mathematics, Institute of Education, University of Minho, Portugal. Develops research at the Research Centre on Child Studies (CIEC); main focus of research are on the development of number sense at elementary school level and on mathematics for kindergarten children.

Florbela Soutinho, PhD student, Institute of Education, University of Minho, Portugal. Develops research at the Research Centre on Child Studies (CIEC); main focuses of research is on mathematics for kindergarten children.