A Pascal-like Triangle with Quaternionic Entries*

M. Irene Falcão[1 [0000−0001−7948−8341]] and Helmuth R. Malonek[2 [0000−0003−0830−3316]]

[1] Centro de Matemática, Universidade do Minho, Portugal mif@math.uminho.pt
[2] CIDMA, Universidade de Aveiro, Portugal hrmalon@ua.pt

Abstract. In this paper we consider a Pascal-like triangle as result of the expansion of a binomial in terms of the generators e_1, e_2 of the non-commutative Clifford algebra $\mathcal{C}\ell_{0,2}$ over $\mathbb{R}$. The study of various patterns in such structure and the discussion of its properties are carried out.

Keywords: Pascal triangle · Quaternions · Clifford algebras.

1 Introduction

Over the years several authors have constructed arithmetic triangles by choosing as their elements numbers which satisfy a recurrence relation of the form

$$\mathcal{E}_{k,s} = A(k,s)\mathcal{E}_{k-1,s} + B(k,s)\mathcal{E}_{k-1,s-1}, \tag{1}$$

with appropriate coefficients $A(k,s)$ and $B(k,s)$ and initial conditions. The triangle corresponding to the choice $A(k,s) = B(k,s) = 1$, with the initial conditions

$$\mathcal{E}_{k,0} = 1,\ k = 0,1,\ldots \quad \mathcal{E}_{0,s} = 0,\ s = 1,2,\ldots \tag{2}$$

reduces to the well known Pascal triangle whereas the initial conditions $\mathcal{E}_{1,0} = 1$, $\mathcal{E}_{1,1} = 2$ and $\mathcal{E}_{k,s} = 0$, for $s < 0$ or $s > k$ leads to Lucas triangle. The Stirling triangle of the second kind corresponds to a choice of $A(k,s) = s$ and $B(k,s) = 1$ with the initial values (2). The book [2] contains several results and detailed references concerning generalized Pascal triangles and other arithmetic triangles.

In this work we consider the arithmetic triangle obtained by choosing the generators of the non-commutative Clifford Algebra $\mathcal{C}\ell_{0,2}$ as the coefficients in (1) together with the initial values (2). We study various patterns in its structure and discuss its main properties. They reveal, in a very particular form, similarities with the classic properties of the Pascal triangle with real entries and at the same time illustrate the consequences of the non-commutativity.

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2 A Pascal-like Triangle

Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal base of the Euclidean vector space $\mathbb{R}^n$ with a product according to the multiplication rules

$$e_k e_l + e_l e_k = -2\delta_{kl}, \ k, l = 1, \cdots, n, \tag{3}$$

where δ_{kl} is the Kronecker symbol. This non-commutative product generates the 2^n-dimensional Clifford algebra $\mathcal{C}\ell_{0,n}$ over $\mathbb{R}$.

The vector space $\mathbb{R}^{n+1}$ is embedded in $\mathcal{C}\ell_{0,n}$ by the identification of the element $(x_0, x_1, \cdots, x_n)$ in $\mathbb{R}^{n+1}$ with the element

$$x = x_0 + x_1 e_1 + \cdots + x_n e_n \tag{4}$$

in $\mathcal{A}_n := \text{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset \mathcal{C}\ell_{0,n}$; such an element $x \in \mathcal{A}_n$ is usually called paravector. For a paravector x of the form (4) we define

- *scalar part* resp. *vector part* of x:

$$\text{Sc}(x) = x_0 \quad \text{and} \ \text{Vec}(x) = e_1 x_1 + \cdots + e_n x_n;$$

  if $\text{Sc}(x) = 0$, then x is called a *pure paravector*;

- *conjugate of* x:

$$\bar{x} = x_0 - x_1 e_1 - \cdots - x_n e_n;$$

- *norm of* x:

$$|x| = (x\bar{x})^{\frac{1}{2}} = \sqrt{x_0^2 + x_1^2 + \cdots + x_n^2}.$$

Usually $\{e_1, e_2, \cdots, e_n\}$ are called the imaginary units or *generators* of the Clifford algebra $\mathcal{C}\ell_{0,n}$. Obviously, we can identify the case $n = 1$ with the complex algebra $\mathbb{C}$ by choosing $i := e_1$. The quaternion algebra $\mathbb{H}$ can be obtained through the identification $\mathbf{i} := e_1$, $\mathbf{j} := e_2$ and $\mathbf{k} := e_1 e_2$. We refer the readers to the books [3,10] for details on Clifford algebras.

In this paper we consider the arithmetical triangle obtained by choosing as its elements the numbers which satisfy the recurrence relation

$$\mathcal{E}_{k,s} = e_1 \mathcal{E}_{k-1,s} + e_2 \mathcal{E}_{k-1,s-1} \tag{5}$$

with the initial conditions

$$\mathcal{E}_{k,0} = e_1^k, \ k = 0, 1, \ldots \quad \mathcal{E}_{0,s} = 0, \ s = 1, 2, \ldots \tag{6}$$

The problem (5)-(6) is equivalent to the problem (1)-(2) for the choice of the generators of the Clifford algebra $\mathcal{C}\ell_{0,2}$ as the coefficients $A(k, s)$ and $B(k, s)$.

The various elements $\mathcal{E}_{k,s}$, $(0 \le s \le k)$, defined by (5)-(6) can be arranged in the form of a triangular array as in the case of Pascal triangle. In Fig. 1 we present the first rows of such triangle and highlight the relationships of a given

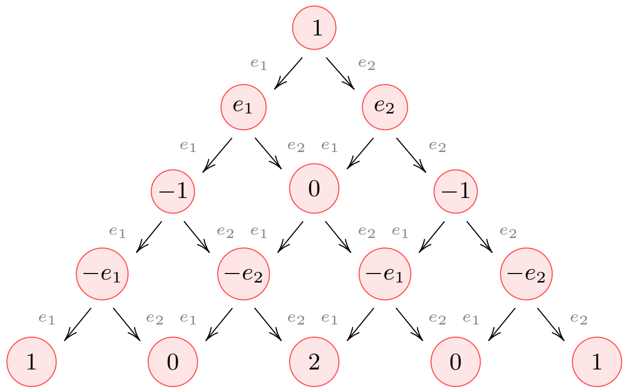

Fig. 1. First rows of the hypercomplex Pascal triangle $\mathcal{E}_{k,s}$

entry with its neighbors. It is also visible that the values of the rows change over alternately between real numbers and paravectors in $\mathcal{A}_2$.

Observe that considering the norm of each element of the triangle of Fig. 1, i.e. considering elements of the form

$$\tilde{\mathcal{E}}_{k,s} = |\mathcal{E}_{k,s}|,$$

we end up with the so-called Pauli Pascal triangle [11], a triangular array made of three copies of the Pascal triangle (see Fig. 2). It is easy to see that the entries of one of the triangles are the non-zero elements of the even rows (in green) and the other two are obtained by considering alternating elements (blue/red) of the odd rows.

Reading the entries of the triangle by rows, we obtain the sequence

$$1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 0, 2, 0, 1, 1, 1, 2, 2, 1, 1, 1, 0, 3, 0, 3, 0, 1, 1, 1, 3, 3, 3, 3, 1, 1 \ldots$$

listed in *The On-Line Encyclopedia of Integer Sequences* [14] (A051159).

For the above reasons the triangle corresponding to (5)-(6) will be called henceforward *quaternionic Pascal triangle* or *Pascal triangle with quaternion entries*. Along the next section we will present several other arguments supporting such designations.

Before we proceed we need to introduce some other tools from the Clifford algebra $\mathcal{C}\ell_{0,2}$, namely the embedding of the non-commutative product into an n-nary symmetric product (see [12]) defined as

$$a_1 \times a_2 \times \cdots \times a_n = \frac{1}{n!} \sum_{\pi(s_1,\ldots,s_n)} a_{s_1} a_{s_2} \cdots a_{s_n}, \tag{7}$$

where the sum runs over all permutations of all $(s_1, \ldots, s_n)$. If the factor a_j occurs μ_j-times in (7), we briefly write $a_1{}^{\mu_1} \times a_2{}^{\mu_2} \times \cdots \times a_n{}^{\mu_n}$ and set parentheses if the powers are understood in the ordinary way.

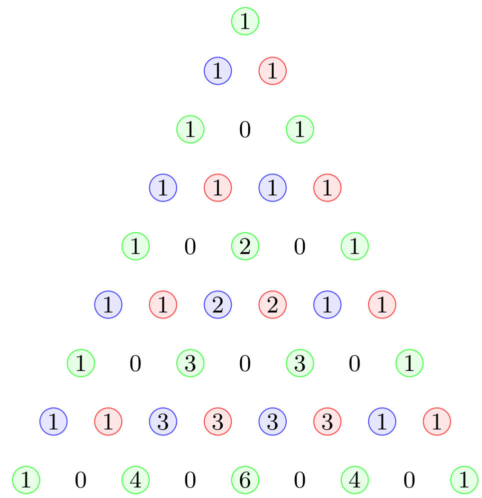

Fig. 2. Three Pascal triangles

For example,

$$e_1 \times e_2 = \tfrac{1}{2}(e_1 e_2 + e_2 e_1) = 0$$

while

$$e_1^2 \times e_2 = e_1 \times e_1 \times e_2 = \tfrac{1}{3!}(2e_1^2 e_2 + 2e_1 e_2 e_1 + 2e_2 e_1^2) = -\tfrac{1}{3} e_2$$

and

$$(e_1)^2 \times e_2 = -e_2.$$

The symmetric product along with the established convention permit to deal with a polynomial formula exactly in the same way as in the case of two commutative variables. More precisely, the following relation holds (see [12,13])

$$(v_1 + v_2)^k = \sum_{m=0}^{k} \tbinom{k}{m} v_1^{k-m} \times v_2^m. \qquad (8)$$

3 Properties of the Quaternionic Pascal Triangle

It is well known that the entries of the Pascal triangle are the coefficients in the expansion of $(x+y)^n$, while the Lucas triangle is generated by the coefficients in the expansion of $(x+y)^{n-1}(x+2y)$ [7] and the Pauli Pascal triangle is generated by the coefficients in expansion of $(x+y)^n$ where x and y anticommute [11].

We prove now that the entries of the quaternionic Pascal triangle are also the coefficients in the expansion of a particular binomial.

Theorem 1. *The entries $\mathcal{E}_{k,s}$ of the quaternionic Pascal triangle are the coefficients in the expansion of $(e_1 + te_2)^n,\ t \in \mathbb{R}$, i.e.*

$$\mathcal{E}_{k,s} = \binom{k}{s} e_1^{k-s} \times e_2^s.$$

Proof. The use of (8) leads to

$$(e_1 + te_2)^k = \sum_{s=0}^{k} \binom{k}{s} e_1^{k-s} \times e_2^s t^s.$$

Denote by $\alpha_{k,s}$ the coefficients of t^s in the right-hand side of last expression, i.e.

$$\alpha_{k,s} := \binom{k}{s} e_1^{k-s} \times e_2^s.$$

Observe that the coefficients $\alpha_{k,s}$ clearly satisfy the initial conditions (6). We are going to prove that they also satisfy the recurrence relation (5), concluding in this way that they are equal to $\mathcal{E}_{k,s}$. To prove the additive property (5) we recall the following recursive formula (see e.g. [12]):

$$v_1^{\mu_1} \times v_2^{\mu_2} = \frac{1}{\mu_1 + \mu_2} [\mu_1 v_1 ({v_1}^{\mu_1 - 1} \times {v_2}^{\mu_2}) + \mu_2 v_2 ({v_1}^{\mu_1} \times {v_2}^{\mu_2 - 1})],$$

which for $v_1 = e_1$, $v_2 = e_2$, $\mu_1 = k - s$, $\mu_2 = s$ allows to write

$$\begin{aligned} \alpha_{k,s} &= \binom{k}{s} \left[\tfrac{k-s}{k} e_1 ({e_1}^{k-s-1} \times {e_2}^s) + \tfrac{s}{k} e_2 ({e_1}^{k-s} \times {e_2}^{s-1}) \right] \\ &= e_1 \binom{k-1}{s} {e_1}^{k-s-1} \times {e_2}^s + e_2 \binom{k-1}{s-1} {e_1}^{k-s} \times {e_2}^{s-1} \\ &= e_1 \alpha_{k-1,s} + e_2 \alpha_{k-1,s-1} \end{aligned}$$

and the proof is completed. □

Theorem 2. *The entries $\mathcal{E}_{k,s}$ of the quaternionic Pascal triangle can be written explicitly as*

$$\mathcal{E}_{k,s} = (-1)^{\lfloor \frac{k}{2} \rfloor} \binom{\lfloor \frac{k}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor} (\epsilon_k \epsilon_s + \epsilon_{k+1} \epsilon_s e_1 + \epsilon_{k+1} \epsilon_{s+1} e_2), \tag{9}$$

where $\epsilon_j = 0$, for odd j and $\epsilon_j = 1$, for even j.

Proof. The result follows by induction on k together with the recursive definition of $\mathcal{E}_{k,s}$. □

A similar result was proved in [4] by using the T. Abadie's formula for the derivative of a composed function. In its present form, (9) can be used to easily identify several properties of the quaternionic Pascal triangle. The first properties can be considered as hypercomplex analogues of well-known properties of the classical Pascal triangle.

Property 1 (Row sum).

$$\sum_{s=0}^{k} \mathcal{E}_{k,s} = \begin{cases} (-2)^{\frac{k}{2}}, & \text{if } k \text{ is even} \\ (-2)^{\frac{k-1}{2}}(e_1+e_2), & \text{if } k \text{ is odd} \end{cases}$$

Proof. If $k = 2m$, $m \in \mathbb{N}$, then (9) together with the well known property $\sum_{s=0}^{k}\binom{k}{s} = 2^k$ give

$$\sum_{s=0}^{2m} \mathcal{E}_{2m,s} = \sum_{s=0}^{2m} (-1)^m \binom{m}{\lfloor \frac{s}{2} \rfloor} \epsilon_s = (-1)^m \sum_{s=0}^{m} \binom{m}{s} = (-1)^m 2^m.$$

On the other hand, for odd values of k, we get

$$\sum_{s=0}^{2m+1} \mathcal{E}_{2m+1,s} = \sum_{s=0}^{m} (-1)^m \binom{m}{\lfloor \frac{s}{2} \rfloor} (\epsilon_{2s} e_1 + \epsilon_{2s+2} e_2) = (-1)^m (e_1+e_2) \sum_{s=0}^{m} \binom{m}{s}$$

and the result is proved. □

Property 2 (Row "alternating" sum).

$$\sum_{s=0}^{k} (-1)^{\lfloor \frac{s}{2} \rfloor} \mathcal{E}_{k,s} = 0, \ k \geq 2.$$

Proof. The property follows at once by combining result (9) with the property $\sum_{s=0}^{k}(-1)^s\binom{k}{s} = 0, k \geq 1$.
□

Property 3 (Central coefficients).

$$\mathcal{E}_{2k,k} = \begin{cases} \binom{k}{\frac{k}{2}}, & \text{if } k \text{ is even} \\ 0, & \text{if } k \text{ is odd} \end{cases}$$

Proof. The proof is immediate. □

Property 4 (Row sum squares).

$$\sum_{s=0}^{k} \mathcal{E}_{k,s}^2 = \begin{cases} \mathcal{E}_{2k,k}, & \text{if } k \text{ is even} \\ -2\mathcal{E}_{2k-2,k-1}, & \text{if } k \text{ is odd} \end{cases}$$

Proof. From (3) and (9) we obtain

$$\mathcal{E}_{k,s}^2 = \binom{\lfloor \frac{k}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor}^2 (\epsilon_k^2 \epsilon_s^2 - \epsilon_{k+1}^2 \epsilon_s^2 - \epsilon_{k+1}^2 \epsilon_{s+1}^2),$$

or equivalently

$$\mathcal{E}_{k,s}^2 = \begin{cases} \binom{m}{\lfloor \frac{s}{2} \rfloor}^2 \epsilon_s^2, & \text{if } k = 2m \\ -\binom{m}{\lfloor \frac{s}{2} \rfloor}^2 (\epsilon_s^2 + \epsilon_{s+1}^2), & \text{if } k = 2m+1 \end{cases} \tag{10}$$

Therefore

$$\sum_{s=0}^{k} \mathcal{E}_{k,s}^2 = \begin{cases} \sum_{s=0}^{m} \binom{m}{s}^2, & \text{if } k = 2m \\ -2 \sum_{s=0}^{m} \binom{m}{s}^2, & \text{if } k = 2m+1 \end{cases}$$

The results follows now from the well known fact that the sum of the squares of each element in the row k of the Pascal triangle equals the central binomial coefficient $\binom{2k}{k}$. □

Table 1 summarizes the properties here deduced, which, as already pointed out, reveal great similarities with some of the most well known properties of the Pascal triangle with real entries (see e.g. [2,9]). We now present properties intrinsic to the quaternionic nature of the structure.

Property 5. $\mathcal{E}_{k,s} = 0$ if and only if k is even and s is odd.

Proof. From (9) we know that $\mathcal{E}_{k,s} = 0$ iff

$$\begin{cases} \epsilon_k \epsilon_s = 0 \\ \epsilon_{k+1} \epsilon_s = 0 \\ \epsilon_{k+1} \epsilon_{s+1} = 0 \end{cases} \Leftrightarrow \begin{cases} k \text{ odd or } s \text{ odd} \\ k \text{ even or } s \text{ odd} \\ k \text{ even or } s \text{ even} \end{cases}$$

and the result follows. □

Property 6.

$$\sum_{s=0}^{k} \binom{k}{s}^{-1} \mathcal{E}_{k,s}^2 = \binom{k}{\lfloor \frac{k}{2} \rfloor}^{-1} (-2)^k.$$

Proof. If $k = 2m$, $m \in \mathbb{N}$, then from (10) we obtain

$$\sum_{s=0}^{k} \binom{k}{s}^{-1} \mathcal{E}_{k,s}^2 = \sum_{s=0}^{m} \binom{2m}{2s}^{-1} \binom{m}{s}^2.$$

Using the identity (cf. [8, Identity (6.11)])

$$\sum_{s=0}^{m} \frac{\binom{m}{s}^2}{\binom{2m}{2s}} = \frac{4^m}{\binom{2m}{m}}, \tag{11}$$

Table 1. Parallels between Pascal triangle with real and quaternionic entries

Properties	**Pascal triangle**	**Quaternion triangle**
coefficients	$\mathcal{C}_{k,s}$	$\mathcal{E}_{k,s}$
recurrence relation	$\mathcal{C}_{k,s} = \mathcal{C}_{k-1,s} + \mathcal{C}_{k-1,s-1}$ $\mathcal{C}_{k,0} = \mathcal{C}_{k,k} = 1$	$\mathcal{E}_{k,s} = e_1\mathcal{E}_{k-1,s} + e_2\mathcal{E}_{k-1,s-1}$ $\mathcal{E}_{0,0} = 1,\ \mathcal{E}_{k,0} = e_1^k,\ \mathcal{E}_{k,k} = e_2^k$
explicit expression	$\mathcal{C}_{k,s} = \binom{k}{s} = \frac{k!}{s!(k-s)!}$	$\mathcal{E}_{k,s} = \begin{cases} (-1)^{\frac{k}{2}} \binom{\frac{k}{2}}{\frac{s}{2}}, & k \text{ even}, s \text{ even} \\ (-1)^{\frac{k-1}{2}} \binom{\frac{k-1}{2}}{\frac{s}{2}} e_1, & k \text{ odd}, s \text{ even} \\ (-1)^{\frac{k-1}{2}} \binom{\frac{k-1}{2}}{\frac{s-1}{2}} e_2, & k \text{ odd}, s \text{ odd} \\ 0, & \text{otherwise} \end{cases}$
sum of values	$\sum_{s=0}^{k} \mathcal{C}_{k,s} = 2^k$	$\sum_{s=0}^{k} \mathcal{E}_{k,s} = \begin{cases} (-2)^{\frac{k}{2}}, & k \text{ even} \\ (-2)^{\frac{k-1}{2}}(e_1+e_2), & k \text{ odd} \end{cases}$
alternating sum	$\sum_{s=0}^{k} (-1)^s \mathcal{C}_{k,s} = 0,\ k \geq 1$	$\sum_{s=0}^{k} (-1)^{\lfloor \frac{s}{2} \rfloor} \mathcal{E}_{k,s} = 0,\ k \geq 2$
central coefficient	$\mathcal{C}_{2k,k}$	$\mathcal{E}_{2k,k} = \begin{cases} \mathcal{C}_{k,\frac{k}{2}}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$
sum of square values	$\sum_{s=0}^{k} \mathcal{C}_{k,s}^2 = \mathcal{C}_{2k,k}$	$\sum_{s=0}^{k} \mathcal{E}_{k,s}^2 = \begin{cases} \mathcal{E}_{2k,k}, & k \text{ even} \\ -2\mathcal{E}_{2k-2,k-1}, & k \text{ odd} \end{cases}$

the results follows.

Consider now the case where $k = 2m+1$. Then

$$\begin{aligned}\sum_{s=0}^{k} \binom{k}{s}^{-1} \mathcal{E}_{k,s}^2 &= \sum_{s=0}^{2m+1} \binom{2m+1}{2s}^{-1} \binom{m}{\lfloor \frac{s}{2} \rfloor}^2 (-\epsilon_s^2 - \epsilon_{s+1}^2) \\ &= -\sum_{s=0}^{m} \binom{m}{s}^2 \left(\binom{2m+1}{2s}^{-1} + \binom{2m+1}{2s+1}^{-1} \right).\end{aligned}$$

From the relation

$$\binom{2m+1}{2s}^{-1} + \binom{2m+1}{2s+1}^{-1} = \frac{2m+2}{2m+1} \binom{2m}{2s}^{-1}$$

and using again (11) it follows that

$$\sum_{s=0}^{k} \binom{k}{s}^{-1} \mathcal{E}_{k,s}^2 = -\frac{2m+2}{2m+1} \frac{4^m}{\binom{2m}{m}} = \frac{m+1}{2m+1} \frac{(-2)^{2m+1}}{\binom{2m}{m}}.$$

and the result is proved, since $\frac{2m+1}{m+1}\binom{2m}{m} = \binom{2m+1}{m}$. □

It is worth to point out that Property 6 hides a sequence of real numbers which combines apparently unrelated subjects in real, complex and hypercomplex analysis [1,5,15,16].

Such sequence was mentioned for the first time in hypercomplex context in [6] and was introduced in the form

$$c_k := \Big[\sum_{s=0}^{k} (-1)^k \binom{k}{s} \left(e_1^{k-s} \times e_2^{s} \right)^2 \Big]^{-1}, \ k = 0, 1, \dots . \tag{12}$$

Using Theorem 1 and Property 6, the explicit expression of c_k can be written as

$$c_k = (-1)^k \Big[\sum_{s=0}^{k} \varepsilon_{k,s}^2 \tbinom{k}{s}^{-1} \Big]^{-1} = \frac{1}{2^k} \binom{k}{\lfloor \frac{k}{2} \rfloor} \tag{13}$$

The first terms of the sequence $(c_k)_k$ are

$$1, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{5}{16}, \frac{5}{16}, \frac{35}{128}, \frac{35}{128}, \frac{63}{256}, \frac{63}{256}, \frac{231}{1024}, \frac{231}{1024}, \frac{429}{2048}, \frac{429}{2048}, \frac{6435}{32768} \dots$$

4 Generalizations

The Pascal triangle obtained in last section by the use of the generators of the Clifford algebra $\mathcal{C}\ell_{0,2}$ can be extended to a n regular polytope structure by considering the generators $e_1, e_2, \dots, e_n$ of the 2^n-dimensional Clifford algebra

$\mathcal{C}\ell_{0,n}$ over $\mathbb{R}$, obtaining in this way a generalization of the Pascal n-simplex. In such case we can construct an hypercomplex Pascal n-simplex by choosing as its elements $\mathcal{E}_{k,s_1,\ldots,s_{n-1}}$, the numbers which satisfy the recurrence relation

$$\mathcal{E}_{k,s_1,\ldots,s_{n-1}} = e_1\mathcal{E}_{k-1,s_1,\ldots,s_{n-1}} + e_2\mathcal{E}_{k-1,s_1-1,\ldots,s_{n-1}} + \cdots + e_n\mathcal{E}_{k-1,s_1-1,\ldots,s_{n-1}-1}$$

with initial conditions $\mathcal{E}_{0,0,\ldots,0} = 1$ and $\mathcal{E}_{s_0,s_1,\ldots,s_{n-1}} = 0$, for $s_i > s_{i-1}$ or $s_i < 0$. Details on this hypercomplex n-simplex will appear in a forthcoming paper.

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