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WEAK EXPONENTIAL STABILITY VIA AVERAGING AND APPLICATIONS

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The averaging method is one of the most powerful methods used to analyze differential equations appearing in the study of nonlinear problems. The averaging method has also been successfully applied in control problems, simplifying the equations and study of the systems involved. In this work, applying the averaging method, we study exponential stability of time varying linear control systems. The obtained results are applied to prove the existence of almost closed relative trajectories for formation flying with single-input control.

INTRODUCTION

In many practical problems, the complexity of dynamics and associated equations imposes the necessity of application of approximations methods to infer important characteristics of the system under consideration. The averaging method is one of the most powerful tools used to analyze differential equations appearing in the study of nonlinear problems. The idea behind the averaging method is to replace the original equation

$$\dot{x} = \epsilon f(t, x), \tag{1}$$

where ϵ is a small parameter, by an averaged one,

$$\dot{\overline{x}} = \epsilon \overline{f}(\overline{x}) = \epsilon \lim_{t\to\infty} \frac{1}{t} \int_0^t f(s, \bar{x}) ds, \tag{2}$$

which is simpler and has similar properties. The rigorous justification of the averaging method application is given by the Bogoliubov first theorem, (Reference 7), for finite time intervals, and by the Samoilenko-Stanzhitskii theorem for infinite time intervals, (Reference 6). The averaging method is an important tool to study asymptotic stability properties of the zero equilibrium position of System (1) (References 3, 5).

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The averaging method has also been successfully applied in control problems. It is known that many controllable nonlinear control systems can be stabilized by means of time-varying feedback laws, (Reference 1). Applying the averaging method to these systems one can simplify the study, as in (Reference 4), where the authors presented an application of the method to the construction of explicit time-varying feedback laws asymptotically stabilizing the attitude of a rigid body spacecraft with two controls.

In this paper we study weak exponential stability of linear control systems via averaging method. The paper is organized as follows. In Section 2, the background mathematical results, used later on, are gathered. In Section 3, we present our main results, concerning the exponential stability of linear time-varying control systems via averaging method. Section 4 contains an application of the obtained results to formation flying problem.

Throughout this paper we denote by $\mathbb{R}^n$ the real n-dimensional space, by $\mathbb{R}_+$ the set of non-negative real numbers, and by $|\cdot|$ the usual Euclidean norm, respectively. We use the notation $B_n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ for the closed unit ball in $\mathbb{R}^n$. The Minkowsky function of a set $A \subset \mathbb{R}^n$ is denoted by $\mu(x) = \inf\{\alpha > 0 \mid x/\alpha \in A\}$. The interior, the convex hull, and the closure of a subset $S \subset \mathbb{R}^n$ are denoted by $\operatorname{int} S$, $\operatorname{co} S$, and $\operatorname{cl} S$, respectively. The transpose of a matrix A is denoted by A'. We denote by $\mathcal{S}_{[0,T]}(F, C)$ the set of solutions to the Cauchy problem $\dot{x} \in F(t, x)$, $t \in [0, T]$, $x(0) \in C \subset \mathbb{R}^n$. The closed unit ball in the space of continuous functions with the uniform norm is denoted by $\mathcal{B}$.

PRELIMINARY NOTES AND RESULTS

When studying control problems, we can write the control systems in term of differential inclusions. For example, the linear control system

$$\dot{x}(t) = Ax(t) + b(t)u(t), \tag{3}$$

is equivalent to the differential inclusion

$$\dot{x}(t) \in Ax(t) + K(t), \tag{4}$$

where $x(t) \in \mathbb{R}^n$ and $K(t) = \{v = b(t)u \mid u \in \mathbb{R}_+\} \subset \mathbb{R}^n$. (The technical assumptions about A, $b(\cdot)$, $u(\cdot)$ and $K(\cdot)$ will be given below). Many proprieties of control systems become almost obvious when we use this approach. For a comprehensive introduction to the theory of differential inclusions the reader is referred to (Reference 8).

In this paper we study weak exponential stability of linear control systems of a special type. Recall the definition of weak exponential stability.

Definition 1. We say that the zero equilibrium position of Differential Inclusion (4) (or, equivalently, of System (3)) is weakly exponentially stable, if there exist positive constants c, γ, and δ such that for any $x_0 \in \delta B_n$ at least one trajectory $\overline{x}(\cdot)$ of (4) (respectively (3)) with $x(0) = x_0$ satisfies

$$|x(t)| \leq c|x_0|e^{-\gamma t}, \;\; t \geq 0.$$

Several results concerning weak exponential stability of System (4) are known. Let $K \subset \mathbb{R}^n$ be a closed convex cone and $x = 0$ be a weakly exponentially stable equilibrium position of (4), then there exist, see (Reference 8, Theorem 9.1), a convex polyhedron $\mathfrak{M}$, with the vertices $\{\overline{x}_1, \ldots, \overline{x}_m\}$ and numbers $h > 0$ and $\delta \in]0, 1[$ such that $0 \in \mathrm{int}\mathfrak{M}$ and for every $k = \overline{1, m}$, there exists a vector $\overline{v}_k \in K$ satisfying

$$\overline{x}_k + h(Ax_k + \overline{v}_k) \in (1 - \delta h)\mathfrak{M}.$$

Let $\overline{x} \in \mathbb{R}^n$. Consider the set-valued maps

$$\Theta(\overline{x}) = \{(\theta_1, \ldots, \theta_m) \mid \sum_{k=1}^{m} \theta_k \overline{x}_k = \overline{x}, \sum_{k=1}^{m} \theta_k = \mu(\overline{x}), \theta_k \geq 0\},$$

and

$$\overline{U}(\overline{x}) = \{\overline{v} = \sum_{k=1}^{m} \theta_k \overline{v}_k \mid (\theta_1, \ldots, \theta_m) \in \Theta(\overline{x})\}, \tag{5}$$

where $\mu(\overline{x})$ is the Minkowski function of $\mathfrak{M}$. We will need the following lemma.

Lemma 1. *The set-valued map $\overline{U}(\cdot)$ has compact convex images and is upper semi-continuous.*

Proof. Let $\overline{x} \in \mathbb{R}^n$. If $\overline{v}^1, \overline{v}^2 \in \overline{U}(\overline{x})$ and $\lambda \in [0, 1]$, then we have

$$\lambda \overline{v}^1 + (1 - \lambda)\overline{v}^2 = \sum_{k=1}^{m} (\lambda \theta_k^1 + (1 - \lambda)\theta_k^2)\overline{v}_k,$$

where $\sum\limits_{k=1}^{m} \theta_k^j \overline{x}_k = \overline{x}$, $\sum\limits_{k=1}^{m} \theta_k^j = \mu(\overline{x})$ and $\theta_k^j \geq 0$, $j = 1, 2$. Since

$$\begin{aligned}
\lambda \theta_k^1 + (1 - \lambda)\theta_k^2 &\geq 0, \quad \lambda \in [0, 1], \\
\sum_{k=1}^{m} (\lambda \theta_k^1 + (1 - \lambda)\theta_k^2)\overline{x}_k &= \lambda \overline{x} + (1 - \lambda)\overline{x} = \overline{x}, \\
\sum_{k=1}^{m} (\lambda \theta_k^1 + (1 - \lambda)\theta_k^2) &= \lambda \mu(\overline{x}) + (1 - \lambda)\mu(\overline{x}) = \mu(\overline{x}),
\end{aligned}$$

we get $\lambda \overline{v}^1 + (1 - \lambda)\overline{v}^2 \in \overline{U}(\overline{x})$, whenever $\lambda \in [0, 1]$, i.e. that the set-valued map has convex images.

If $\overline{v}^i \in \overline{U}(\overline{x}^i)$, $\sum\limits_{k=1}^{m} \theta_k^i \overline{x}_k = \overline{x}^i$, $\sum\limits_{k=1}^{m} \theta_k^i = \mu(\overline{x}^i)$, $\theta_k^i \geq 0$, and $\overline{x}^i \to \overline{x}$, $\overline{v}^i \to \overline{v}$ as $i \to \infty$, then, without loss of generality, $\theta_k^i \to \theta_k$. Since $\mu(\cdot)$ is Lipschitzian, passing to the limit as $i \to \infty$, we obtain

$$\sum_{k=1}^{m} \theta_k \overline{x}_k = \overline{x}, \ \sum_{k=1}^{m} \theta_k = \mu(\overline{x}), \ \theta_k \geq 0.$$

Therefore the graph of $\overline{U}(\cdot)$ is closed. Since $\overline{U}(\cdot)$ is a bounded map this means that $\overline{U}(\cdot)$ is upper semi-continuous. □

Recall a generalization of the Samoilenko-Stanzhitskii Theorem obtained in (Reference 2). Consider a differential inclusion

$$\dot{x} \in \epsilon F(t, x), \tag{6}$$

and the respective averaged inclusion

$$\dot{\overline{x}} \in \epsilon \overline{F}(\overline{x}) = \epsilon \bigcap_{\delta > 0} \bar{F}^{\delta}(x), \tag{7}$$

where $\bar{F}^{\delta}(x)$ is the convex hull of the map

$$\bar{\Phi}^{\delta}(x) = \limsup_{\theta \uparrow 1} \limsup_{T \to \infty} \frac{1}{(1-\theta)T} I(\theta T, T, x, \delta),$$

and

$$I(t_1, t_2, x, \delta) = \left\{ \int_{t_1}^{t_2} v(t) dt \mid v(\cdot) \in L_1^{\text{loc}}([0, \infty[, \mathbb{R}^n), \ v(t) \in F(t, x + \delta B_n) \right\}.$$

(The $\limsup$ stands for the Kuratowski upper limit, i.e., the set of all limit points.) It is assumed that the following conditions are satisfied:

(C1) $\operatorname{cl}\operatorname{co} F(t, x) = F(t, x)$, for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$;

(C2) the set-valued map $F(t, \cdot)$ is upper semi-continuous;

(C3) for any x there exists measurable selection of $F(t, x)$, that is, there exists $f(t, x) \in F(t, x)$ such that $t \to f(t, x)$ is measurable for all x;

(C4) there exists a non-negative $c(\cdot) \in L_1^{\text{loc}}([0, \infty[, \mathbb{R})$ such that $F(t, x) \subset c(t)B$ for all $(t, x) \in [0, +\infty[\times \mathbb{R}^n$;

(C5) there exists the limit

$$c = \lim_{T \to \infty} \frac{1}{T} \int_0^T c(t) dt;$$

Under these assumptions the following theorem holds true.

Theorem 2. *Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a set-valued map satisfying conditions (C1) - (C5). Assume that $\overline{x} = 0$ is an asymptotically stable equilibrium position of the differential inclusion $\dot{x} \in \epsilon \overline{F}(x)$. Then for any $\eta > 0$ there exist $\epsilon_0 > 0$ and $\delta > 0$ such that $\mathcal{S}_{[0,\infty[}(F, \delta B) \subset \eta \mathcal{B}$, whenever $\epsilon \in]0, \epsilon_0[$.*

That is, the asymptotically stability of the equilibrium position of the averaged inclusion, implies the boundedness of the original system solutions.

We are now in a position to state and prove our main result.

EXPONENTIAL STABILITY OF LINEAR TIME-VARYING CONTROL SYSTEM

Consider the following control system

$$\dot{X}(t) = AX(t) + b(t/\epsilon, t)u(t/\epsilon), \tag{8}$$

where $\epsilon > 0$ is a small parameter, A is a $n \times n$ matrix, $b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ is a bounded continuous function, and $u(t/\epsilon)$ is a control subject to the constrain $u(t/\epsilon) \in \mathbb{R}_+$. System (8) has a fast and a slow time scales. Set $\tau = t/\epsilon$ and $x(\tau) = X(\epsilon\tau)$. Then (8) takes the form

$$\frac{dx(\tau)}{d\tau} = \epsilon \frac{dX(\epsilon\tau)}{dt} = \epsilon(Ax(\tau) + b(\tau, \epsilon\tau)u(\tau)), \tag{9}$$

Let K be a closed convex cone, satisfying

$$K \subset \left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T b(\tau, \epsilon\tau)u(\tau)d\tau \mid u(\tau) \geq 0,\ u(\cdot) \in L_\infty([0, \infty[, \mathbb{R}) \right\}.$$

Consider the associated averaged control system

$$\frac{d\overline{x}}{d\tau} = \epsilon(A\overline{x} + \overline{v}), \quad \overline{v} \in K. \tag{10}$$

Assuming that the origin is a weakly exponentially stable equilibrium position of System (10) we can show the weak exponential stability of System (9).

Theorem 3. *Let $\overline{x} = 0$ be a weakly exponentially stable equilibrium position of System (10). Then, there exist $\epsilon_0 > 0$, such that $x = 0$ is a weakly exponentially stable equilibrium position of System (9), whenever $\epsilon \in]0, \epsilon_0[$.*

Proof. Consider the polyhedron $\mathfrak{M} = \mathrm{co}\{\overline{x}_k\}$, introduced in the previous section. Recall that there exist $h > 0$ and $\delta \in [0, 1[$ such that for every k, there is a vector $\overline{v}_k \in K$, satisfying

$$\overline{x}_k + h\epsilon(A\overline{x}_k + \overline{v}_k) \in (1 - \delta h)\mathfrak{M}.$$

If $\gamma > 0$ sufficiently small then we have

$$\overline{x}_k + h\epsilon(A\overline{x}_k + \overline{v}_k + \gamma x_k) \in \left(1 - \frac{\delta h}{2}\right)\mathfrak{M}.$$

for all $k = \overline{1, m}$. Consider the differential inclusion

$$\frac{d\overline{z}}{d\tau} \in \epsilon(A\overline{z} + \gamma\overline{z} + \overline{U}(\overline{z})), \tag{11}$$

where the set valued map $\overline{U}(\cdot)$ is defined by Equality (5). Let $\overline{z}(\cdot)$ be a solution to Inclusion (11). The function $\mu(\overline{z}(\tau))$ is absolutely continuous. Let τ be a point such that the derivatives

$$\frac{d\mu(\overline{z}(\tau))}{d\tau} \quad \text{and} \quad \frac{d\overline{z}(\tau)}{d\tau} \in \epsilon(A\overline{z}(\tau) + \gamma\overline{z}(\tau) + \overline{U}(\overline{z}(\tau))),$$

exist. Then, there exists $\overline{v}(\tau) \in \overline{U}(\overline{z}(\tau))$ such that

$$\frac{d\overline{z}(\tau)}{d\tau} = \epsilon(A\overline{z}(\tau) + \gamma\overline{z}(\tau) + \overline{v}(\tau)). \tag{12}$$

Moreover, we have

$$\overline{z}(\tau) + h\epsilon(A\overline{z}(\tau) + \gamma\overline{z}(\tau) + \overline{v}(\tau)) = \overline{z}(\tau) + h\epsilon\left(A\overline{z}(\tau) + \gamma\overline{z}(\tau) + \sum_{k=1}^{m}\theta_k\overline{v}_k\right),$$

where $(\theta_1, \ldots, \theta_m) \in \Theta(\overline{z}(\tau))$. Therefore we obtain

$$\begin{gathered}\overline{z}(\tau) + h\epsilon(A\overline{z}(\tau) + \gamma\overline{z}(\tau) + \overline{v}(\tau)) = \\ = \sum_{k=1}^{m}\theta_k(\overline{x}_k + h\epsilon(A\overline{x}_k + \gamma x_k + \overline{v}_k)) \in (1 - \tfrac{\delta h}{2})\mu(\overline{x}(\tau))\mathfrak{M}.\end{gathered}$$

From this we get

$$\begin{aligned}\frac{d}{d\tau}\mu(\overline{z}(\tau)) &\leq \frac{\mu(\overline{z}(\tau) + h\epsilon(A\overline{z}(\tau) + \gamma\overline{z}(\tau) + \overline{v}(\tau)) - \mu(\overline{z}(\tau))}{h} \\ &\leq \frac{(1 - \frac{\delta h}{2})\mu(\overline{x}(\tau)) - \mu(\overline{x}(\tau))}{h} \\ &= -\tfrac{\delta}{2}\mu(\overline{z}(\tau)).\end{aligned}$$

Thus we have

$$\mu(\overline{z}(\tau)) \leq \mu(\overline{z}(0))e^{-\frac{\delta\tau}{2}}.$$

This implies that $\overline{z} = 0$ is an asymptotically stable equilibrium position of Inclusion (11). Consider the differential inclusion

$$\frac{dz(\tau)}{d\tau} \in \epsilon(Az(\tau) + \gamma z(\tau) + W(z(\tau), \tau)), \tag{13}$$

where

$$W(z, \tau) = \{w = \sum_{k=1}^{m}\theta_k b(\tau, \epsilon\tau)u_k(\tau) \mid (\theta_1, \ldots, \theta_m) \in \Theta(z)\},$$

and the functions $u_k(\cdot) \in L_\infty([0, \infty[, \mathbb{R}_+)$, $k = \overline{1, m}$, satisfy

$$\overline{v}_k = \lim_{T\to\infty}\frac{1}{T}\int_0^T b(\tau, \epsilon\tau)u_k(\tau)d\tau.$$

As in Lemma 1, one can show that the set $W(z, \tau)$ is compact and convex for all (z, τ) and the set-valued map $W(\cdot, \tau)$ is upper semi-continuous. Moreover, conditions $(C1) - (C5)$ are satisfied. By Theorem 2, there exists a number $\epsilon_0 > 0$ such that the solutions of Inclusion (13) are bounded, whenever $\epsilon \in]0, \epsilon_0[$.

Let $z(\cdot)$ be a solution of Inclusion (13). Set $z(\tau) = e^{\gamma\epsilon\tau}x(\tau)$. Then we get

$$\frac{dz(\tau)}{d\tau} = \gamma\epsilon e^{\gamma\epsilon\tau}x(\tau) + e^{\gamma\epsilon t}\frac{dx(\tau)}{d\tau}.$$

On the other hand, we have

$$\frac{dz(\tau)}{d\tau} = \epsilon(Az + \gamma z(\tau) + \sum_{k=1}^{m} \theta_k(\tau) b(\tau, \epsilon\tau) u_k(\tau)),$$

where $(\theta_1(\tau), \dots, \theta_m(\tau)) \in \Theta(z(\tau))$. Thus we obtain

$$\begin{aligned} \frac{dx(\tau)}{d\tau} &= \epsilon(Ax(\tau) + \sum_{k=1}^{m} e^{-\epsilon\gamma\tau} \theta_k(\tau) b(\tau, \epsilon\tau) u_k(\tau)) \\ &= \epsilon(Ax(\tau) + \sum_{k=1}^{m} \tilde{\theta}_k(\tau) b(\tau, \epsilon\tau) u_k(\tau)) \end{aligned} \tag{14}$$

where $(\tilde{\theta}_1(\tau), \dots, \tilde{\theta}_m(\tau)) \in \Theta(x(\tau))$. Since the solutions of Inclusion (13) are bounded, i.e, there exists a $c > 0$ such that $|z(\tau)| \leq c|z(0)|$ for all $\tau \geq 0$, the solutions of Equation (14) satisfy the inequality

$$|x(\tau)| \leq \tilde{c}|x(0)|e^{-\gamma\epsilon\tau}.$$

Thus, $x = 0$ is a weakly exponentially stable equilibrium position of inclusion

$$\frac{dx}{d\tau} \in \epsilon(Ax + b(\tau, \epsilon\tau)\mathbb{R}_+).$$

□

Now we present our second main result. Consider the system

$$\dot{X}(t) = AX(t) + a(t) + b(t/\epsilon, t)u(t/\epsilon), \tag{15}$$

where $a : \mathbb{R} \to \mathbb{R}^n$ is a continuous function satisfying $|a(t)| \leq a$, $t \geq 0$. Using the previous result, we shown that there exists a admissible control that drives System (15) from a state X_0 to a δ-neighbourhood of state X_1, for every pair $(X_0, X_1) \in \mathbb{R}^n \times \mathbb{R}^n$.

Theorem 4. *Let $X = 0$ be a weakly exponentially stable equilibrium position of System (8) and let $\delta > 0$. If $|e^{\pm At}| \leq Q(1 + t^p)$, with $Q > 0$ and $p \geq 1$, then there exist $\epsilon_0 > 0$ and $T > 0$, such that for any pair $X_0, X_1 \in \mathbb{R}^n$ there is a trajectory $X(\cdot)$ of (15) satisfying*

$$X(0) = X_0 \text{ and } X(T) \in X_1 + \delta B_n,$$

whenever $\epsilon \in]0, \epsilon_0[$.

Proof. Consider a pair $X_0, X_1 \in \mathbb{R}^n$ and $\delta > 0$. Set

$$\tilde{X}_0 = X_0 - e^{-AT}X_1 + \int_0^T e^{-As}a(s)ds.$$

Since $|e^{\pm At}| \leq Q(1+t^p)$, with $p \geq 1$, we have

$$\begin{aligned} |\tilde{X}_0| &\leq |X_0| + Q(1+T^p)|X_1| + aQ\int_0^T (1+s^p)ds \\ &= |X_0| + Q(1+T^p)|X_1| + aQ\left(T + \tfrac{T^{p+1}}{p+1}\right). \end{aligned}$$

Set

$$\phi(T) \equiv |X_0| + Q(1+T^p)|X_1| + aQ\left(T + \frac{T^{p+1}}{p+1}\right).$$

Since zero is a weakly exponentially stable equilibrium position of System (8), we see that there exists a control $\hat{u}(\cdot) \geq 0$ such that the corresponding trajectory of System (8) with the initial condition $\tilde{X}_0$ satisfies the inequality

$$|\hat{X}(T)| \leq ce^{-\gamma T}\phi(T) < \delta,$$

whenever T is sufficiently large. Thus, we have

$$e^{AT}\left(X_0 - e^{-AT}X_1 + \int_0^T e^{-As}a(s)ds\right) + \int_0^T e^{A(T-s)}b(s/\epsilon, s)\hat{u}(s)ds \in \delta B_n.$$

From this we obtain

$$e^{AT}X_0 + \int_0^T e^{A(T-s)}(a(s) + b(s/\epsilon, s)\hat{u}(s))ds \in X_1 + \delta B_n$$

and the end of the proof. □

APPLICATION TO FORMATION FLYING PROBLEM

In this section we apply our results to formation flying problem with a single input control.

Equations of relative motion with single-input control

Consider a modification of the Hill-Clohessy-Wiltshire equations introduced by Schweighart and Sedwick, (Reference 9), to take into account the influence of the J_2-harmonic on relative motion of two satellites with close near-circular orbits:

$$\begin{aligned} &\ddot{x} + 2nc\dot{z} = u(t)e_x(t), \\ &\ddot{y} + q^2y = 2lq\cos(qt+\phi) + u(t)e_y(t), \\ &\ddot{z} - 2nc\dot{x} - (5c^2-2)n^2z = u(t)e_z(t). \end{aligned}$$

The linearization is done with respect to a circular reference orbit with the mean motion n. Here x, y, and z are coordinates in the respective orbital reference frame $Oxyz$. The axes are chosen in the following way: Oz indicates the radial direction outwards from the Earth, Ox is directed along the velocity of the point O, and y is normal to the orbital plane. The coefficients c, q, l, and ϕ are properly defined constants (see Reference 9 for a detailed description). The direction of the control acceleration $u(t)$ is defined by the vector-function

$$e(t) = (e_x(t), e_y(t), e_z(t))'.$$

Unilateral fast rotating control orthogonal to a fixed vector in the inertial space

In (Reference 10) the authors consider a formation of two satellites. The chief satellite is assumed to move passively whereas the deputy satellite possesses a spherically symmetrical mass distribution, is spin-stabilized, and is equipped with a thruster oriented along the spin axis. Here we assume that the thruster is oriented orthogonally to the spin axis. Let Λ be the angle between this axis and the vector pointing to the vernal equinox direction, and ϕ be the inclination of the plane containing these vectors with respect to the Earth's equator. Then in the Earth-centred inertial reference frame the spin axis direction has the components $(\cos\Lambda, \sin\Lambda\cos\phi, \sin\Lambda\sin\phi)^T$. In the $Oxyz$ reference frame the expressions are

$$s_x(t) = \sigma_z \sin\theta(t) - \sigma_x \cos\theta(t),$$

$$s_y = -\sigma_y,$$

$$s_z(t) = -\sigma_x \sin\theta(t) - \sigma_z \cos\theta(t).$$

Here the vector $\sigma = (\sigma_x, \sigma_y, \sigma_z)^T$ defines the direction of spin axis in the ascending node of the orbit via the inclination i and the right ascension Ω:

$$\begin{aligned}
\sigma_x &= \cos\Omega\cos i\sin\Lambda\cos\phi - \sin\Omega\cos i\cos\Lambda + \sin i\sin\Lambda\sin\phi,\\
\sigma_y &= -\cos\Omega\sin i\sin\Lambda\cos\phi + \sin\Omega\sin i\cos\Lambda + \cos i\sin\Lambda\sin\phi,\\
\sigma_z &= \cos\Omega\cos\Lambda + \sin\Omega\sin\Lambda\cos\phi.
\end{aligned}$$

We set $\theta(t) = nct$. Assume that the condition $\sigma_x^2 + \sigma_z^2 \neq 0$ and $\sigma_y \neq 0$ is satisfied. The thruster is oriented along the direction

$$e_x(t) = Q_1(t)s_z(t)\cos(t/\epsilon) + Q_2(t)s_x(t)s_y\sin(t/\epsilon),$$

$$e_y(t) = -Q_2(t)(s_x^2(t) + s_z^2(t))\sin(t/\epsilon),$$

$$e_z(t) = -Q_1(t)s_x(t)\cos(t/\epsilon) + Q_2(t)s_y s_z(t)\sin(t/\epsilon),$$

where ϵ is a small parameter,

$$Q_1(t) = \frac{1}{\sqrt{s_x^2(t) + s_z^2(t)}},$$

and

$$Q_2(t) = \frac{1}{\sqrt{s_x^2(t)s_y^2 + (s_x^2(t) + s_z^2(t))^2 + s_y^2 s_z^2(t)}}.$$

Using the notations

$$H = (x, y, z, \dot{x}, \dot{y}, \dot{z})',$$

$$a(t) = (0, 0, 0, 0, 2lq\cos(qt + \phi), 0)',$$

$$b(t) = (0, 0, 0, e_x(t), e_y(t), e_z(t))',$$

and

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2nc \\ 0 & -q^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & (5c^2-2)n^2 & 2nc & 0 & 0 \end{pmatrix},$$

we obtain the system

$$\dot{H}(t) = AH(t) + a(t) + b(t/\epsilon, t)u(t/\epsilon),$$

i.e., System (15). Introducing the fast time $\tau = t/\epsilon$, we get the system

$$\frac{d\eta(\tau)}{d\tau} = \epsilon(A\eta(\tau) + a(\epsilon\tau) + b(\epsilon\tau, \tau)u(\tau)), \quad u(\tau) \geq 0.$$

The averaged system is given by

$$\frac{d\overline{\eta}}{d\tau} = \epsilon(A\overline{\eta} + \overline{v}), \quad \overline{v} \in K. \tag{16}$$

The structure of the cone K is described by the following

Theorem 5. *If $\sigma_x^2 + \sigma_z^2 \neq 0$ and $\sigma_y \neq 0$, then $K = 0_3 \times \mathbb{R}^3$.*

The proof of this theorem is rather technical and is based on direct calculations. We omit it.

From Theorem 5 we see that averaged System (16) is controllable. Therefore it is weakly exponentially stable. Moreover, $|e^{\pm At}| \leq Q(1+t)$, for some $Q > 0$. Therefore Theorem 4 can be applied. These guarantees the existence of almost closed relative trajectories of the formation.

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