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PARAMETER OPTIMIZATION FOR STABILIZERS

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The aim of this work is to develop an effective numerical tool oriented to optimization of stabilizer parameters according to different criteria that appear in engineering practice. We formulate a special optimization problem that allows us to determine optimal parameters of a stabilizer. The obtained results are applied to choose parameters of a spacecraft stabilization system. We discuss the choice of optimization criteria comparing the degree of stability objective function, the H_∞ norm of the system transfer matrix, and the minimal "peak" criterion.

INTRODUCTION

The increase dynamic complexity of space missions motivates research on engineering tools and methodologies capable to improve the performance of space systems. Development of effective high-precision control systems for large variety of applications, including flexible spacecraft and large space structures, requires parameter optimization of the system transient processes.

Assume that the differential equation describing the behavior of the system has zero as its asymptotically stable equilibrium position, and depends on a vector parameter. During the system stabilization to the zero equilibrium position, the properties of transition process would depend on the choice of this parameter. That choice should be done to optimize, in some sense, the behavior of the trajectories. In practice, this parameter can be selected from a number of alternatives, based on various criteria, and it is impossible, of course, to construct a stabilizer optimal in all aspects.

For example, for a linear controllable system, the pole assignment theorem guarantees the existence of a linear feedback yielding a linear differential equation with any given set of eigenvalues. One can choose a stabilizer with a very high damping speed. However, such a stabilizer is practically useless because of so-called peak-effect.[2,6,9] Namely, there exists a large deviation of the solutions from the equilibrium position at the beginning of the stabilization process, whenever the module of an eigenvalues real part is big enough.

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The same phenomenon is observed if one chooses a robust stabilizer constructed using the degree of stability concept.[10]

Recently, the H_∞ approach has been developed as a tool for analyzing the robustness and performance of stabilization systems.[3] Usually an available model describes the plant only approximately. H_∞- optimal control is a frequency-domain synthesis method developed to solve the problem of influence of modeling errors. The main idea is to treat the worst case scenario: it is necessary to design so-called robust controller capable to stabilize all systems from some class. A robust controller should stabilize the system even in the case of the largest possible modeling error. Due to the Nyquist criterion this can be formalized in terms of the minimization of H_∞-norm of the transfer matrix. If a linear system is asymptotically stable, then the maximum of the modulus of its transfer function over the closed right-half of the complex plane will always occur on the imaginary axis. On the other hand, the smaller H_∞- norm, the bigger the distance between the eigenvalues of the asymptotically stable matrix and the imaginary axis. Therefore, the result of H_∞- norm minimization is rather close to that of maximization of the degree of stability.

In Reference 5 we develop a new tool to optimize the parameters of stabilizers, according to different criteria that appear in the engineering practice. We formulate a special mathematical programming problem that allows us to determine optimal parameters of a stabilizer. The objective function and the functions describing the constraints of the problem are defined by the maximum norm of solutions to the system. The functions have strongly nonlinear nature and numerical methods have been developed to solve the respective optimization problems. In this work we apply the above techniques to analyze the properties of H_∞-norm as an objective function and compare them with other optimality criteria for an example of parameter optimization of a spacecraft stabilization system.

Throughout this paper, we denote the set of real numbers by R, and the usual n-dimensional space of vectors with components in R by R^n. By $|\cdot|$ we denote the Euclidean norm. By B we denote the set of vectors in R^n satisfying $|x| \leq 1$. The conjugate transpose of a matrix A is denoted by A^*.

MATHEMATICAL BACKGROUND

Consider a linear sistem of differential equations

$$\dot{x} = A(u)x,\ x \in R^n,\ t \in [0,T], \tag{1}$$

where u is a parameter belonging to a compact set $U \subset R^k$. Denote by $\lambda_i(u), i = \overline{1,n}$ the eigenvalues of matrix $A(u)$. Assume that

$$\operatorname{Re} \lambda_i(u) < 0,\ i = \overline{1,n},$$

i.e., all eigenvalues of matrix $A(u)$ have negative real parts, when $u \in U$. Let us briefly describe some well known criteria used to find parameters of a linear system.

The Degree of Stability

One of the most used optimality criteria is the degree of stability. It is defined by

$$\delta(u) = -\max_{i=\overline{1,n}} \operatorname{Re} \lambda_i(u).$$

The respective optimization problem takes the form

$$\begin{aligned} \delta(u) \;&\rightarrow\; \max, \\ u \;&\in\; U. \end{aligned} \tag{2}$$

The problem can also include some additional constraints involving the trajectories of the system.

The H_∞-norm

Another widely used objective function is the H_∞-norm of the system transfer matrix. Denote by

$$G(s,u) = (sI - A(u))^{-1},$$

the transfer matrix of Equation (1). Here s is a complex number and I is the identity matrix. The largest singular value of $G(s,u)$ is defined as the square root of the maximum eigenvalue of the matrix $G^*(s,u)G(s,u)$ and is denoted by $\sigma(G(s,u))$. The H_∞-norm of the system transfer matrix is defined by

$$\|G(\cdot,u)\|_\infty = \sup_{\omega \in R} \sigma(G(i\,\omega,u)).$$

The parameters of System (1) can be choosen as solutions to the following optimization problem

$$\begin{aligned} \|G(\cdot,u)\|_\infty &\rightarrow \min, \\ u &\in U. \end{aligned} \tag{3}$$

As in the previous case the problem can also include some additional constraints involving the trajectories of the system.

The above two criteria do not take into account a very important property of solutions to linear asymptotically stable systems: a possible large deviation of trajectories during the stabilization process. This phenomenon, known as peak-effect, is a ”pay-off” for a fast or robust stabilization.[2,6,9] The minimal overshooting can also be an objective functions. Below we show how to formulate and solve such problems.

Problem of Optimal Parameters Determination

Let $x(t,x_0,u)$ be the solution to the Cauchy problem

$$\begin{aligned} &\dot{x} = A(u)x,\ x \in R^n,\ t \in [0,T], \\ &x(0) = x_0. \end{aligned}$$

Define the functions

$$\varphi_i(u) = \max_{t\in\Delta_i}\ \max_{x_0\in B_i} |x(t, x_0, u)|_i,\ \ i = \overline{0, m}. \tag{4}$$

Here $B_i = \{x \in R^n \mid\ |x|_i \leq 1\}$, $|\cdot|_i$ are some norms in R^n, and $\Delta_i \subseteq [0, T]$ are a closed intervals. Consider the following mathematical programming problem

$$\begin{aligned} &\varphi_0(u) \longrightarrow \min, \\ &\varphi_i(u) \leq \bar{\varphi}_i,\ \ i = \overline{1, m}, \\ &\quad u \in U. \end{aligned} \tag{5}$$

Many problems of stabilization systems parameters optimization can be written in this form. The involved functions depend only on the parameters of the system. Consider two examples.

Minimization of the final deviation The problem is to determine optimal values for the system parameters guaranteeing minimal deviation of the system state from the zero equilibrium position at the final moment of time. This problem can be formalized as follows:

$$\begin{aligned} \max_{x_0\in B} &|x(T, x_0, u)| \longrightarrow \min, \\ &u \in U. \end{aligned}$$

For $T \gg 1$, this is an approximation to the problem of stability degree maximization.

Minimization of the maximal deviation This problem consists in determination the parameters corresponding to minimal deviation of a family of trajectories satisfying certain restrictions at the final moment of time. This problem can be formalized as follows:

$$\begin{aligned} &\max_{t\in\,[0,T]}\ \max_{|x_0|=1}\ |x(t, x_0, u)|\ \to \min, \\ &\max_{|x_0|=1}\ \ |x(T, x_0, u)| \leq \delta, \\ &\quad u \in\ U. \end{aligned}$$

The above problems are of interest for stabilization theory; they both have Form (5). The study of these type of problems can hardly be performed analytically for more or less complex systems. For this reason, we focus on the numerical aspects of this problem.

Discrete Optimization Problem

Let $t \in \Delta_i$ and $x_0 \in B_i$, $i = \overline{0, m}$. Consider the Euler approximation to solutions $x(t, x_0, u)$,

$$\tilde{x}(t^i_{k+1}, x^i_j, u) = \tilde{x}(t^i_k, x^i_j, u) + \tau_i\, A(u)\tilde{x}(t^i_k, x^i_j, u),\ k = \overline{0, K_i}, \tag{6}$$

where $\tau_i = t^i_{k+1} - t^i_k$ are small enough constants, $t^i_0 = 0$, $t^i_k \in \Delta_i$, and $\tilde{x}(0, x^i_j, u) = x^i_j$. Let $\varepsilon > 0$ and $\delta > 0$ be small enough. Let $K_i(\delta)$ and $J_i(\delta)$ be sets of indices such that the points $t_k^i \in \Delta_i$, $k \in K_i(\delta)$, and $x_j^i \in B_i$, $j \in J_i(\delta)$ form a δ-net in Δ_i and B_i, $i = \overline{1,m}$, respectively. Define the functions

$$\varphi_i^\delta(u) = \max_{k \in K_i(\delta)} \max_{j \in J_i(\delta)} \left| \tilde{x}(t_k^i, x_j^i, u) \right|, \; i = \overline{0,m}$$

and consider the problem

$$\begin{aligned} &\varphi_0^\delta(u) \longrightarrow \min, \\ &\varphi_i^\delta(u) \le \bar{\varphi}_i + \varepsilon, \;\; i = \overline{1,m}, \\ &\quad u \in U. \end{aligned} \tag{7}$$

Denote by $\hat{u}$ and u^δ the optimal parameters for problems (5) and (7), respectively. Recall the following result obtained in Reference 5.

Theorem 1 *For any $\epsilon > 0$ there exists $\delta > 0$ such that u^δ is an admissible solution to the following problem*

$$\begin{aligned} &\varphi_0(u) \longrightarrow \min, \\ &\varphi_i(u) \le \bar{\varphi}_i + 2\varepsilon, \;\; i = \overline{1,m}, \\ &\quad u \in U, \end{aligned}$$

and

$$\varphi_0(u^\delta) \le \varphi_0(\hat{u}) + 2\varepsilon.$$

This theorem allows one to choose the parameters of discretization in order to obtain optimal stabilizer parameters with a necessary precision (see Reference 5 for details).

Note that the use of techniques developed in Reference 5 allows one to obtain more precise estimates for the number of points in the meshes needed to achieve a given discretization accuracy in the case of asymptotically stable systems. Optimization problem (7) is a hard nonlinear nonsmooth problem and can be solved only using numerical methods. Our computational experience shows that the multi-start Nelder - Mead method (see Reference 4) is the most adequate one. Note that the problem of optimal choice of parameters is solved only once, at the stage of the control system's development. Therefore it is worthwhile to dedicate more resources to its solution. The methods developed in Reference 5 allow one to solve the problem within a reasonable time.

SIMULATION RESULTS

Consider motion of a connected two-body system in a circular orbit around the Earth. Body 1 is a satellite with the center of mass O_1, body 2 is a stabilizer with the center of mass O_2. These two bodies are linked to each other at the point P through a dissipative hinge mechanism. Let O be the center of mass of the system. We use three reference frames: $OXYZ$ is the orbital coordinate frame, its axis OZ is directed along the radius vector of the point O with respect to the center of the Earth, OX is directed along the velocity of the point O, and OY is normal to the orbit plane. The axes of referential frames $O_1x_1y_1z_1$ and $O_2x_2y_2z_2$ are the central principal axes of inertia for bodies 1 and 2 respectively. Consider motion of the system in the orbit plane supposing that the bodies are connected in their centres of mass, i.e., the points O_1, O_2, O, and P coincide. Let α_1 and α_2 be the angles between the axis OX and the axes O_1x_1 and O_2x_2 respectively. The equations of motion for this system can be written as

$$\begin{aligned} &\ddot{\alpha}_1 + 3p_1 \sin\alpha_1 \cos\alpha_1 + k_1(\dot{\alpha}_1 - \dot{\alpha}_2) = 0, \\ &\ddot{\alpha}_2 + 3p_2 \sin\alpha_2 \cos\alpha_2 - \tfrac{k_1}{\mu}(\dot{\alpha}_1 - \dot{\alpha}_2) = 0. \end{aligned} \tag{8}$$

Here the dot stands for the derivative with respect to the orbital time $\tau = \omega_0\, t$, where t is the time, and ω_0 is is the angular velocity of the system center of mass motion along the circular orbit. Parameters $(p_1,\, p_2,\, k_1,\, \mu)$ depend on the principal moments of inertia of the bodies and on the damping coefficient of the stabilizer.[1,7,8]

Linearizing the equations at the equilibrium position $\alpha_{10} = 0,\ \ \alpha_{20} = 0$, we get the following linear system

$$\dot{x} = A(u)\, x, \tag{9}$$

where $u = (p_1,\, p_2,\, k_1,\, \mu)$, and the matrix $A(u)$ has the form

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3p_1 & 0 & -k_1 & k_1 \\ 0 & -3p_2 & \frac{k_1}{\mu} & -\frac{k_1}{\mu} \end{pmatrix}. \tag{10}$$

The set of admissible parameters is given by

$$U = \{(p_1,\, p_2,\, k_1,\, \mu)\ :\ k_1 > 0,\, \mu > 0,\, 0 < p_1 \le 1,\, 0 < p_2 \le 1, p_1 \ne p_2\} \tag{11}$$

(see Reference 7).

Below we formalize the optimal parameter determination problem for this system in different manners and compare the results.

Consider the following problem

$$\max_{|x_0|=1}\ |\, x(T, x_0, u)\,|\ \to \min, \qquad u \in U. \tag{12}$$

Set $T = 3\,\pi$. The are two quartets of optimal parameters

$$p_1 = 0.06928,\, p_2 = 1.00757,\, k_1 = 0.59209,\, \mu = 0.33161, \tag{13}$$

and

$$p_1 = 1.00521,\, p_2 = 0.06920,\, k_1 = 1.78178,\, \mu = 3.01152. \tag{14}$$

The value of the global minimum is $m = 0.00378$.

Recall that the maximum degree of stability for this system is achieved if the parameters take the values

$$p_1 = 0.0294,\ p_2 = 1,\ k_1 = 0.4203,\ \mu = 0.1716, \tag{15}$$

or

$$p_1 = 1,\ p_2 = 0.0294,\ k_1 = 2.4495,\ \mu = 5.8284, \tag{16}$$

(see Reference 8). The corresponding eigenvalues are -0.59, $-0.72 \pm 0.12i$, -0.83. In this case we have

$$\max_{|x_0|=1} |\, x(3\pi, x_0, u)| = 0.47. \tag{17}$$

The minimization of the H_∞-norm of the transfer matrix leads to two sets of parameters

$$p_1 = 0.12254,\ p_2 = 1.00014,\ k_1 = 0.66944,\ \mu = 0.60089, \tag{18}$$

and

$$p_1 = 1.00011,\ p_2 = 0.12279,\ k_1 = 1.11284,\ \mu = 1.66412. \tag{19}$$

These sets of parameters correspond to the eigenvalues $-0.48 \pm 0.55i$, $-0.41 \pm 1.39i$. For this choice of parameters we find

$$\max_{|x_0|=1} |\, x(3\pi, x_0, u)| = 0.056. \tag{20}$$

Therefore the value of the problem

$$\begin{gathered} \|G(\cdot, u)\|_\infty \to \min, \\ \max_{|x_0|=1} |x(3\pi, x_0, u)| \le \delta, \\ u \in U, \end{gathered} \tag{21}$$

does not depend on δ, whenever $\delta > 0.0559$.

A different behaviour we observe if the maximal deviations of trajectories from the equilibrium position is considered as the objective function. For example, consider the problem

$$\begin{gathered} \max_{t\in[0,3\pi]} \max_{|x_0|=1} |x(t, x_0, u)| \to \min, \\ \max_{|x_0|=1} |x(3\pi, x_0, u)| \le \delta, \\ u \in U. \end{gathered} \tag{22}$$

Numerical solution of this problem leads to the following results:

1. $\delta = 0.47$. The optimal parameters for Problem (22) are

$$p_1 = 0.22359,\ p_2 = 0.65771,\ k_1 = 0.44747,\ \mu = 0.61902, \tag{23}$$

and

$$p_1 = 0.57767,\ p_2 = 0.16774,\ k_1 = 0.74318,\ \mu = 2.10926. \tag{24}$$

The corresponding eigenvalues of the linear system and the value of the problem are

$$-0.34 \pm 0.97i,\ -0.25 \pm 1.20i$$

and

$$m = 1.20, \tag{25}$$

respectively.

2. $\delta = 0.06$. The optimal parameters for Problem (22) are

$$p_1 = 0.14827,\ p_2 = 0.82299,\ k_1 = 0.52301,\ \mu = 0.61786, \tag{26}$$

and

$$p_1 = 0.82390,\ p_2 = 0.14867,\ k_1 = 0.85127,\ \mu = 1.60847. \tag{27}$$

The corresponding eigenvalues of the linear system and the value of the problem are

$$-0.34 \pm 0.69i,\ -0.35 \pm 1.24i$$

and

$$m = 1.34, \tag{28}$$

respectively.

3. $\delta = 0.005$. The optimal parameters for Problem (22) are

$$p_1 = 0.07048,\ p_2 = 1.00220,\ k_1 = 0.59611,\ \mu = 0.33961, \tag{29}$$

and

$$p_1 = 1.00175,\ p_2 = 0.0743,\ k_1 = 1.75718,\ \mu = 2.94874. \tag{30}$$

The corresponding eigenvalues of the linear system and the value of the problem are

$$-0.59 \pm 0.34i,\ -0.59 \pm 1.02i$$

$$m = 1.59, \tag{31}$$

respectively.

It is interesting that the optimal parameters for Problem (21) with $\delta = 0.005$ are also

$$p_1 = 0.07048,\ p_2 = 1.00220,\ k_1 = 0.59611,\ \mu = 0.33961,$$

and

$$p_1 = 1.00175,\ p_2 = 0.0743,\ k_1 = 1.75718,\ \mu = 2.94874.$$

Note that for Parameters (15) and (16) we have

$$\max_{t\in[0,3\pi]} \max_{|x_0|=1} \left|x(t, x_0, u_{(15)})\right| = 1.92,$$

i.e, the overshooting is rather significant.

The analysis of the obtained results shows that the H_∞-norm is a good choice for the objective function. It combines the robustness guaranteed by the maximal degree of stability criterion with a reasonably small overshooting. However, the H_∞ tool can only be applied to linear systems. The direct approach based on solution of Problem (5) can be used also for nonlinear systems with well succeed implementation (see Reference 5).

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