On a Hyperbolic Augmented Lagrangian Artificial Fish Swarm Based Method: Convergence Issues

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Extended Abstract

1 Introduction

The problem to be addressed in this paper has the general form:

$$\min_{x\in\Omega} \quad f(x) \quad \text{subject to} \quad g(x) \leq 0 \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are nonlinear continuous functions and $\Omega = \{x \in \mathbb{R}^n : -\infty < l \leq x \leq u < \infty\}$. Problems with equality constraints, $h(x) = 0$, can be reformulated into the above form by converting into a couple of inequality constraints $h(x) - \beta \leq 0$ and $-h(x) - \beta \leq 0$, where β is a small positive relaxation parameter. Since we do not assume that the objective function f is convex, the problem may have multiple optimal solutions in the feasible region. A global optimal solution is to be computed. For this class of optimization problems, methods based on penalty functions are quite common in the literature. In this type of methods, the constraint violation is combined with the objective function to define a penalty function, which aims at penalizing infeasible solutions by increasing their fitness values proportionally to their level of constraint violation. We are interested in a particular class of penalty functions known as augmented Lagrangian functions to handle the equality and inequality constraints of the problem (1). An augmented Lagrangian is a more sophisticated penalty function for which a finite penalty parameter value is sufficient to yield convergence to the solution of the constrained problem.

We aim at analyzing the theoretical and practical behavior of an augmented Lagrangian algorithm that is constructed following the most common augmented Lagrangian paradigm [1, 2, 4]. The penalty function basis of our proposal is the exterior penalty function presented in [5], the so-called 2-parameter hyperbolic penalty function. This is a continuously differentiable function that depends on two penalty parameters $\tau \geq 0$ and $\rho \geq 0$, whose penalty term is:

$$\mathcal{P}(x,\tau,\rho) = \sum_{i=1}^{p} \left(\tau g_i(x) + \sqrt{\tau^2 \left(g_i(x)\right)^2 + \rho^2} \right). \tag{2}$$

2 Hyperbolic Augmented Lagrangian methodology

An augmented Lagrangian technique solves a sequence of very simple subproblems where the objective function penalizes all or some of the constraint violation. Our implementation of the hyperbolic augmented Lagrangian framework aims to penalize the inequality constraints of the nonlinear optimization problem (1). The penalty term of the hyperbolic (2) is herein reformulated so that a new penalty parameter μ emerges as $\mu = \dfrac{\rho}{\tau}$ and $\tau > 0$. The simple bounds are not included in the augmented Lagrangian although the solution method to solve the subproblem (see (3) below) guarantees that they are satisfied. Thus, each subproblem of the sequence that is solved for fixed values of the penalties τ and μ is a bound constrained optimization problem of the form:

$$\min_{x\in\Omega} \quad \phi(x,\delta,\tau,\mu) \equiv f(x) + \sum_{i=1}^{p} \delta_i g_i(x) + \tau \sum_{i=1}^{p} \left(g_i(x) + \sqrt{\left(g_i(x)\right)^2 + \mu^2} \right) \tag{3}$$

where $\delta = (\delta_1, \ldots, \delta_p)^T$ is the Lagrange multiplier vector associated with the inequality constraints. Another important issue in the augmented Lagrangian context is concerned with the selection of the method to compute an approximate solution to the subproblem (3), within a specified 'optimal' tolerance. Since a global solution of problem (1) is to be computed, a global optimization solution method for the subproblem (3) is to be provided.

A formal description of the hyperbolic augmented Lagrangian algorithm for solving problem (1) is presented in Algorithm 1.

We note that in the presented algorithm, the penalty parameter μ is reduced at all iterations aiming to converge to 0. Although the penalty τ is to be increased as $k \to \infty$, to avoid an ill-conditioned subproblem, its value is not updated whenever complementarity, as well as feasibility, improve. Another important issue addressed in this paper is the estimation of the multiplier vector. A safeguarded first-order Lagrange multiplier formula is used, so that multiplier vector boundedness is guaranteed.

2.1 Artificial Fish Swarm Algorithm

To solve subproblem (3) the artificial fish swarm (AFS) algorithm is used. This is a stochastic population based method. At each iteration, a population of NP solutions, herein denoted by x^i, $i = 1, \ldots, NP$, are used to generate a trial population

Algorithm 1 Hyperbolic augmented Lagrangian algorithm

```
Given initial values and parameters;
Set $k = 0$;
while feasibility and complementarity are not achieved do
    Find an approximate global minimizer $x^{(k)}$ of subproblem (3), within an 'optimal' tolerance, using Algorithm 2;
    Estimate the multiplier vector $\delta$;
    Reduce the penalty $\mu$;
    if feasibility and complementarity have been improved then
        Maintain the value of penalty $\tau$;
    else
        Increase the value of penalty $\tau$;
    end if
    Reduce 'optimal' tolerance;
    $k = k + 1$;
end while
```

y^i, $i = 1 \ldots, NP$. Initially, the population is randomly generated in the entire search space Ω. Each fish/point movement is defined according to the number of points inside its visual. The visual is the closed neighborhood centered at x^i with a radius $v > 0$. The three possible situations that may occur are:

i) the visual is empty;

ii) the visual is crowded;

iii) the visual is not crowded.

When the visual is empty, a random behavior is performed, in which the trial y^i is randomly generated inside the visual of x^i. When the visual is crowded, and a point randomly selected from the visual has a better fitness than x^i, the searching behavior is implemented, i.e., y^i is randomly generated along the direction from x^i to that selected point. Otherwise, the random behavior is performed. When the visual is not crowded, and the best point inside the visual has a better fitness than x^i, the chasing behavior is performed. This means that y^i is randomly generated along the direction from x^i to that best point. However, if that best point is not better than x^i, the swarming behavior is tried instead. Here, the central point of the visual is computed and if it has better fitness than x^i, y^i is computed randomly along the direction from x^i to the central; otherwise, the searching behavior is implemented instead [3].

When selecting between the current x^i and the trial y^i, the solution with best objective function value (fitness) wins. The pseudo code of this algorithm is shown below in Algorithm 2.

Algorithm 2 Artificial fish swarm algorithm

```
Generate a population of $NP$ points: $x^i$, $i = 1, \dots, NP$;
while an approximate minimizer of subproblem (3) is not reached do
    for all $x^i$ do
        Compute visual;
        Perform accordingly chasing, swarming, searching or random behavior to
        find $y^i$;
        Select between $y^i$ and $x^i$;
    end for
end while
```

2.2 Convergence to Global Minimizers

The global convergence of the hyperbolic augmented Lagrangian algorithm has been analyzed. The analysis has some similarities with that of the global augmented Lagrangian method presented in [2] and relies on the fact that, at each iteration k, an approximate global minimizer of the subproblem (3) is obtained, $x^{(k)}$, within a certain 'optimal' tolerance. Based on the following assumptions

- a global minimizer z of the problem (1) exists;
- the sequence $\{x^{(k)}\}$ generated by the Algorithm 1 is well defined and there exists a set of indices $\mathcal{N} \subseteq \mathbb{N}$ so that $\lim_{k \in \mathcal{N}} x^{(k)} = x^*$;
- The functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are continuous in a neighborhood of x^*;
- for all $i = 1, \dots, p$, the optimal multipliers $\delta_i^* \in [0, \delta_{\max}]$;
- the sequence of penalty parameters $\{\tau^{(k)}\}$ tends to ∞;
- the sequence of penalty parameters $\{\mu^{(k)}\}$ tends to zero;

we prove that every limit point x^* of the sequence $\{x^{(k)}\}$ generated by the Algorithm 1 is feasible and is an approximate global minimizer of the problem (1), within a sufficiently small 'optimal' tolerance.

A stochastic algorithm for solving an optimization problem of the type (3) is said to exhibit global convergence when the sequence of solutions generated by the algorithm converges in some stochastic manner to the global optimum. Since AFS algorithm generates random sequences of solutions, convergence here means stochastic convergence that relies on a probability measure and is different from the convergence concept of classical analysis. There are different types of stochastic convergence for a sequence of random solutions. Convergence with probability one (or almost sure convergence), convergence in probability, convergence in mean square, and convergence in distribution are the most known. Until now, convergence of the fish swarm iterative process in mean square has been guaranteed [4].

3 Discussion of Preliminary Numerical Results

We consider the below example to analyze the behavior of the algorithm when dealing with both equality and inequality constraints:

$$\begin{array}{lcl} \min f(x) & \equiv & \frac{\sin^3(2\pi x_1)\sin(2\pi x_2)}{x_1^3(x_1+x_2)} \\ \text{subject to} & & x_1^2 - x_2 + 2.272 \leq 0 \\ & & 1 - x_1 + (x_2 - 4)^2 = 0 \\ & & 0 \leq x_i \leq 10, i = 1, 2. \end{array}$$

The global optimum is $f^* = -0.047764467932642$ located at (1.1550767073652, 3.6062021998978), where both constraints are active. This example was solved 10 times by the presented algorithm with $NP = 50$, and the best obtained solution ('$f_{\text{best point}}$') is -0.047762. During the best of the 10 runs, the hyperbolic augmented Lagrangian algorithm completed 10 iterations, after 9 166 function evaluations ('n.func.eval.'). Algorithm 2 was allowed to run for 20 iterations, at each outer iteration. For the augmented Lagrangian shown in (3), the equality was transformed into an inequality using $\beta = 0.00001$.

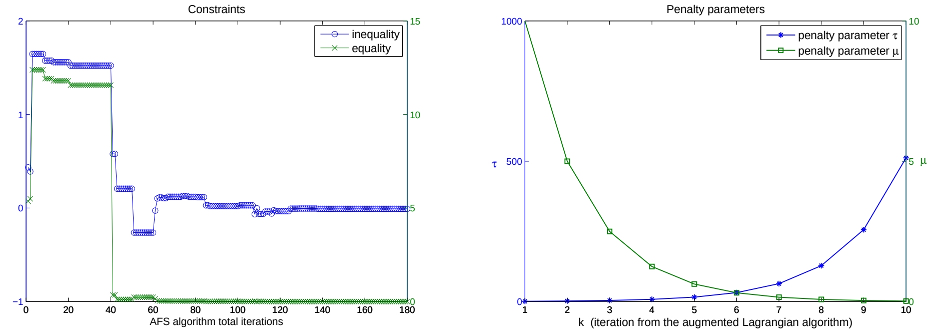

Figure 1: Constraints behavior (left) and penalty parameters behavior (right).

From Figure 1 - plot on the left - we may observe that the feasible region is reached within $k = 7$ outer iterations (equivalent to 140 AFS algorithm total iterations), with a sum of violations ('violation') equal to 2.545298E-05. The plot on the right shows the values of the penalty parameters τ and μ at every outer iteration. Their convergence behavior is as expected. Figure 2 aims to depict the initial population at the beginning of the iterative process and the population after completing six iterations. In this figure, we show the values of 'n.func.eval.', '$f_{\text{best point}}$', 'violation', τ and μ attained at iterations $k = 0$ and $k = 6$. It is also shown that the initial population is nicely spread over the set $[0, 10]^2$ and the minimizer is attracting the points of the population.

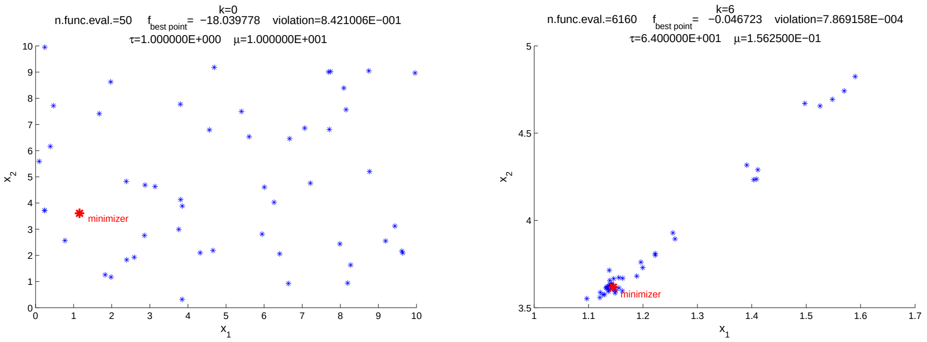

Figure 2: Minimizer and population at $k = 0$ (left) and $k = 6$ (right).

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