The (2,2,0) Drazin Inverse Problem

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Abstract

We consider the additive Drazin problem and we study the existence of the Drazin inverse of a two by two matrix with zero (2,2) entry.

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1 Introduction

Unless otherwise stated, all elements are in a ring R with unity 1.

The *Drazin inverse* (D-inverse, for short) of a, denoted by a^d, is the unique solution to the equations $a^{k+1}x = a^k$, $xax = x$, $ax = xa$, for some $k \geq 0$, if any. The minimal such k is called the *index*, denoted $in(a)$, of a. If the Drazin inverse exists, we shall call the element D-invertible, or strongly-pi-regular. When $in(a) \leq 1$, we say a has a *group inverse*, denoted by $a^\#$.

We say $a \in R$ is regular if $a \in aRa$. We shall need the concept of regularity, which guarantees solutions to $aa^-a = a$ and $aa^+a = a$, $a^+ = a^+aa^+$. a^- is called an inner inverse of a, and a^+ is called a reflexive inverse of a.

Two elements x and y are said to be left(right) orthogonal (LO/RO), if $xy = 0$ (resp. $yx = 0$), and orthogonal, denoted by $x \perp y$, if $xy = yx = 0$. Semi-orthogonality means either LO or RO.

If a is D-invertible, then $a = (a^2a^d) + a(1 - aa^d) = c_a + n_a$ is referred as the *core-nilpotent* decomposition of a. Note that $c_a \perp n_a$, n_a is nilpotent, and $a^d = c_a^\#$.

R. Cline showed in [7] how to relate $(ab)^d$ with $(ba)^d$, namely $(ab)^d = a[(ba)^d]^2b$. This equality is known as *Cline's formula*.

In this paper, we shall examine the representation of the Drazin inverses of the block matrix $M = \begin{bmatrix} a & c \\ b & 0 \end{bmatrix}$, in which the (2,2) entry is zero. We aim for results in terms of "words" in a, b

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and c, and their g-inverses, such as inner or Drazin inverses. Needless to say, the search for a formula for this D-inverse is closely related to the "additive problem" of finding the D-inverse of a sum $(a+b)^d$ in terms of words in a and b, and their g-inverses. We cannot expect a single "good formula" without additional assumptions on a, b and c, as seen from the example of $M = \begin{bmatrix} a & c+1 \\ b & 0 \end{bmatrix} = A + \mathbf{e}_1\mathbf{e}_2^T$, in which the D-inverse depends on the "invertibility" of A (being a unit, group member or neither) as well as the interaction of A with $\mathbf{e}_1$ and $\mathbf{e}_2$. Numerous recent papers have dealt with special cases ([8]) and special approaches, such as the use of Catalan Numbers and their recurrence relations ([3], [4]).

Finally, we shall use $rk(\cdot)$ to denote rank.

2 Factorizations and splittings

There are three main paths for research in the study of D-inverses, and these are (semi) orthogonality, nilpotency and commutativity. The former is used in conjunction with suitable factorization or splitting (that is, writing an element as a sum of others with special form), and is combined with Cline's formula. The latter two have the effect of keeping things finite and limiting the number of cases that can occur. Rank may be useful but does not really show us what is going on.

There is *no* formula for general $(a+b)^{-1}$ and hence without one of the three conditions, applied to certain integer combinations of a, b, a^d, b^d, there is no hope in finding a "formula" for $(a+b)^d$.

The factorizations

$$\begin{bmatrix} a & c \\ b & 0 \end{bmatrix} = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix} = \begin{bmatrix} a & 1 \\ cb & 0 \end{bmatrix}$$

show that, as far as D-inverse computation goes, we may without loss of generality take $c = 1$ and replace b by cb (or set $b = 1$ and replace b by bc), i.e

$$\begin{bmatrix} a & c \\ b & 0 \end{bmatrix}^d = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}\left(\begin{bmatrix} a & 1 \\ cb & 0 \end{bmatrix}^d\right)^2\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}.$$

We shall then attempt to turn the vertical factorizations into the corresponding horizontal factorizations. More generally (cf. [5], [16]), we may factor

$$M = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & a & 1 \\ 1 & b & 0 \end{bmatrix}\begin{bmatrix} 0 & d \\ 1 & 0 \\ 0 & c \end{bmatrix} = FG,$$

$$GF = \begin{bmatrix} 0 & d \\ 1 & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} 0 & a & 1 \\ 1 & b & 0 \end{bmatrix} = \left[\begin{array}{c|cc} d & db & 0 \\ \hline 0 & a & 1 \\ c & cb & 0 \end{array}\right]$$

Now, if $db = 0$, then finding M^d reduces to finding $\begin{bmatrix} a & 1 \\ cb & 0 \end{bmatrix}^d$, and we are back to the (2,2,0) case $\begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}^d$. By symmetry, we have analogous results when $dc = 0$ or when $ac = 0$ or $ab = 0$.

The key results used in semi-orthogonal splitting were given in [14] and [15].

Theorem 2.1. Given D-invertible a and b, with $k = in(a)$ and $\ell = in(b)$, then

$$\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}^d = \begin{bmatrix} a^d & x \\ 0 & b^d \end{bmatrix}^d,$$

with $x = (a^d)^2 \sum_{i=0}^{\ell-1} (a^d)^i cb^i(1 - bb^d) + (1 - aa^d) \sum_{i=0}^{k-1} a^i c(b^d)^i (b^d)^2 - a^d cb^d$.

This gives the standard form for left-orthogonal (LO) splittings.

Theorem 2.2. If $pq = 0$ then

$$(p+q)^d = (1 - qq^d)\Big[\sum_{i=0}^{k-1} q^i (p^d)^i\Big]p^d + q^d\Big[\sum_{i=0}^{k-1} (q^d)^i p^i\Big](1 - pp^d),$$

and

$$(p+q)(p+q)^d = (1 - qq^d)\Big[\sum_{i=0}^{k-1} q^i (p^d)^i\Big]pp^d + qq^d\Big[[\sum_{i=0}^{k-1} (q^d)^i p^i\Big](1 - pp^d) + qq^d pp^d,$$

where $\max\{in(p), in(q)\} \le k \le in(p) + in(q)$.

When u is a unit and n is nilpotent, then $u + n$ can be a unit, a group member or even nilpotent. As such, little can be said about its D-inverse without assuming stronger conditions such as semi-orthogonality or commutativity. The simplest semi-orthogonal case occurs when one element is nilpotent.

Six special cases are useful:

Corollary 2.1. (i-a) If $pq = 0$ and p is nilpotent, then

$$(p+q)^d = q^d[\sum_{i=0}^{k-1} (q^d)^i p^i] \text{ and } (p+q)(p+q)^d = qq^d[\sum_{i=0}^{k-1} (q^d)^i p^i].$$

(i-b) If $pq = 0$ and q is nilpotent, then

$$(p+q)^d = [\sum_{i=0}^{k-1} q^i (p^d)^i]p^d \text{ and } (p+q)(p+q)^d = [\sum_{i=0}^{k-1} q^i (p^d)^i]pp^d.$$

(ii-a) If $pq = 0$ and $p^2 = 0$ then

$$(p+q)^d = q^d(1 + q^d p) \text{ and } (p+q)(p+q)^d = qq^d(1 + q^d p).$$

(ii-b) If $pq = 0$ and $q^2 = 0$ then

$$(p+q)^d = (1+qp^d)p^d \text{ and } (p+q)(p+q)^d = (1+qp^d)pp^d.$$

(iii-a) If $pq = 0$ and $p^\#$ exists then

$$(p+q)^d = (1-qq^d)[\sum_{i=0}^{k-1} q^i (p^\#)^i]p^\# + q^d(1-pp^\#)$$

and

$$(p+q)(p+q)^d = (1-qq^d)[\sum_{i=0}^{k-1} q^i (p^\#)^i]pp^\# + qq^d.$$

(iii-b) If $pq = 0$ and $q^\#$ exists then

$$(p+q)^d = (1-qq^\#)p^d + q^\#[\sum_{i=0}^{k-1} (q^\#)^i p^i](1-pp^d)$$

and

$$(p+q)(p+q)^d = pp^d + qq^\#[\sum_{i=0}^{k-1} (q^\#)^i p^i (1-pp^d).$$

For the commutative case, we can obtain the $(a+b)^d$ from the nilpotent case [14].

Lemma 2.1. 1. If $ab = ba$ and b is nilpotent then

$$(a+b)^d = a^d(1+a^d b)^{-1} \text{ and } (a+b)(a+b)^d = aa^d.$$

2. If $ab = ba$ then

$$(a+b)^d = a^d(1+a^d b)^d + (1-aa^d)b^d[1+b^d n_a]^{-1}.$$

Proof. (1). Let $e = aa^d$ and use the right-splitting to split $a+b = (c_a+be)+(n_a+b(1-e)) = p+q$. Then clearly $pq = 0 = qp$ and $(a+b)^d = p^d + q^d$ as well as $(a+b)(a+b)^d = pp^d + qq^d$. Now q is nilpotent and $p = c_a(1+a^d b)$. Since the factors in p commute, we see that $p^d = a^d(1+a^d b)^{-1}$ and $(a+b)(a+b)^d = pp^d = aa^d$.

(2). Taking p and q as in the previous item, from the left-splitting we see that $pq = 0 = qp$ so that $(a+b)^d = p^d + q^d$ and $(a+b)(a+b)^d = pp^q + qq^d$. Now $p^d = [c_a(1+a^d b)]^d = (c_a)^d(1+a^d b)^d = a^d(1+a^d b)^d$ and $q = r+s$, where $r = (1-e)b$ and $s = n_a$. Clearly $rs = sr$ and s is nilpotent. By item (1), we see that $q^d = r^d(1+r^d n_a)^{-1}$ and $qq^d = rr^d = (1-e)bb^d$. Hence $(a+b)(a+b)^d = pp^d + rr^d = e(1+a^d b)](1+a^d b)]^d + (1-e)bb^d$. □

Remarks

If $ab = ba$ then the computation of $(a+b)^d$ has been reduced to computing $(1+a^d b)^d$, which is generally *just as difficult*, even in perturbation theory. If b is "small" then we can use norms to guarantee that $(1+a^d b)$ is invertible.

The following triplet result reduces to a LO splitting when $r = 1$.

Theorem 2.3. If $rpq = 0 = p^2q$, then

$$(p+qr)^d = (qr)^d + pq[(rq)^d]^3 r + q(rq)^d xp + pq[(rq)^d]^2 xp + qxpp^d + pq(rq)^d xpp^d + (1+pqx)p^d,$$

and

$$(p+q)(p+qr)^d = (qr)^d r + qxp + p(qr)^d r + pq[(rq)^d]xp + pp^d + pqxpp^d$$

where

$$x = [(rq)^d]^2 \sum_{i=0}^{\ell-1} [(rq)^d]^i rp^i(1-pp^d) + [1-(rq)(rq)^d] \sum_{i=0}^{k-1} (rq)^i r (p^d)^i (p^d)^2 - (rq)^d rp^d. \quad (1)$$

Proof. We factor $p+qr$ as $\begin{bmatrix} q & 1 \end{bmatrix} \begin{bmatrix} r \\ p \end{bmatrix} = BC$ and hence, by Cline's formula,

$$(p+qr)^d = B[(CB)^d]^2 C.$$

As such, we need

$$CB = \begin{bmatrix} r \\ p \end{bmatrix} \begin{bmatrix} q & 1 \end{bmatrix} = \begin{bmatrix} rq & r \\ pq & p \end{bmatrix} = \begin{bmatrix} rq & r \\ 0 & p \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ pq & 0 \end{bmatrix} = R + S.$$

Because $rpq = 0 = p^2q$, we see that $RS = 0 = S^2$, and hence by Corollary (2.1), $(CB)^d = (I+SR^d)R^d$. Moreover as $R^dS = 0$, we then get $[(CB)^d]^2 = (I+SR^d)[R^d]^2$. This gives

$$(p+rq)^d = \begin{bmatrix} q & 1 \end{bmatrix} (I+SR^d)[R^d]^2 \begin{bmatrix} r \\ p \end{bmatrix}$$

in which $R^d = \begin{bmatrix} (rq)^d & x \\ 0 & p^d \end{bmatrix}$, where x is as in equation (1).

It should be noted that $xpq = 0 = (rq)^d xpp^d$ and that $I + SR^d = \begin{bmatrix} 1 & 0 \\ pq(rq)^d & 1+pqx \end{bmatrix}$, is a unit.

Lastly,

$$\begin{aligned} B[(CB)^d]^2 C &= \begin{bmatrix} q & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ pq(rq)^d & 1+pqx \end{bmatrix} \begin{bmatrix} (rq)^d & x \\ 0 & p^d \end{bmatrix} \begin{bmatrix} (rq)^d & x \\ 0 & p^d \end{bmatrix} \begin{bmatrix} r \\ p \end{bmatrix} \\ &= [q(rq)^d + pq[(rq)^d]^2, qx + pq(rq)^d x + (1+pqx)p^d] \begin{bmatrix} (rq)^d r + xp \\ pp^d \end{bmatrix} \\ &= (qr)^d + pq[(rq)^d]^3 r + q(rq)^d xp + pq[(rq)^d]^2 xp + qxpp^d + pq(rq)^d xpp^d + (1+pqx)p^d. \end{aligned}$$

We may simplify this by using

$$\begin{aligned}
pq[(rq)^d]^3 r &= p[(qr)^d]^2, \\
q(rq)^d xp &= q[(rq)^d]^3[\sum_{i=0}^{\ell-1}[(rq)^d]^i rp^i(1-pp^d)p - [(qr)^d]^3 rpp^d, \\
pq[(rq)^d]^2 xp &= pq[(rq)^d]^4[\sum_{i=0}^{\ell-1}[(rq)^d]^i rp^i(1-pp^d)p - p[(qr)^d]^2 pp^d, \\
qxpp^d &= q[1-(rq)(rq)^d)]\sum_{i=0}^{k-1}(rq)^i r(p^d)^i(p^d)^2 - q(rq)^d rp^d, \\
pq(rq)^d xpp^d &= -pq(rq)^d rp^d \text{ and} \\
pqxp^d &= pq[1-(rq)(rq)^d)]\sum_{i=0}^{k-1}(rq)^i r(p^d)^i(p^d)^3 - pq(rq)^d r[p^d]^2.
\end{aligned}$$

For the remaining result, we simply multiply out $B(CB)^dC = \begin{bmatrix} q & 1 \end{bmatrix}(I+SR^d)R^d\begin{bmatrix} r \\ p \end{bmatrix}$.

□

One source of LO splitting is obtained by creating idempotents and applying the Pierce decomposition

$$x = exe + (1-e)xe + ex(1-e) + (1-e)x(1-e), \text{ with } e^2 = e,$$

to a and b. The obvious idempotents that present themselves are $e = aa^d$ and $f = bb^d$. Selecting the former, we know that $c_a = ae = ea$ and $n_a = a(1-e)$. This gives

$$a + b = ae + a(1-e) + ebe + (1-e)be + eb(1-e) + (1-e)b(1-e).$$

The idea now is to group the terms together as $p + q$ and then assume enough conditions to force $pq = 0$. For example,

$$\begin{aligned}
a+b &= [ea + ebe + eb(1-e)] + [(1-e)a + (1-e)be + (1-e)b(1-e)] \\
&= [ea + eb] + [(1-e)a + (1-e)b] \\
&= c_a(1 + a^d b) + [n_a + (1-aa^d)b].
\end{aligned}$$

In addition, we shall need enough conditions to provide a second LO splitting of $p = r + s$ and $q = t + u$, to ensure that we can get back to a and b. The simplest cases are

(I). **Left-Splitting**: $a + b = aa^d(a+b) + (1-aa^d)(a+b) = c_a(1+a^d b) + [n_a + (1-aa^d)b] = p + q = p + (r + s)$.

(II). **Right-Splitting**: $a + b = (a+b)e + (a+b)(1-e) = (1+ba^d)c_a + [n_a + b(1-e)] = p' + q' = p' + (r' + s)$.

For the left-splitting, it easily follows that

1. $pq = qp$ if and only if $pq = 0 = qp$.

2. $pq = 0$ if and only if $eb(1-e)(a+b) = 0$, in which case $eb(1-e)be = 0$.

3. $qp = 0$ if and only if $(1-e)be(a+b) = 0$, in which case $(1-e)beb(1-e) = 0$.

Likewise, for the right-splitting we have

1. $p'q' = q'p'$ if and only if $p'q' = 0 = q'p'$.

2. $q'p' = 0$ if and only if $(a+b)(1-e)be = 0$, in which case $eb(1-e)be = 0$.

3. $p'q' = 0$ if and only if $(a+b)eb(1-e) = 0$, in which case $(1-e)beb(1-e = 0$.

Needless to say we may switch a and b to give a second formula.

Three especially simple cases occur when $be = 0$ or $eb = 0$. We begin with

Proposition 2.1. Let $e = aa^d$ and $f = bb^d$. If $be = 0$ then

$$(a+b)^d = (1-aa^d)p^d + a^d \sum_{i=0}^{k-1} (a^d)^i p^i (1-pp^d) \tag{2}$$

where $p = a(1-e) + b$.

Proof. We use the right-splitting,

$$a + b = (a+b)(1-e) + (a+b)e = [a(1-e) + b] + c_a = p + q,$$

were $pq = 0$ and $q^\# = a^d$. As such we have a LO splitting and, by Theorem (2.2),

$$(a+b)^d = (1-aa^d)p^d + a^d \sum_{i=0}^{k-1} (a^d)^i p^i (1-pp^d).$$

□

It is clear that $a^d p = a^d b$, but $(a^d)^2 p^2$ cannot be simplified. We now must impose sufficient conditions on a and b so that we can split p as well.

We require the following preliminary fact:

Lemma 2.2. 1. If $be = 0$ then $(1-e)ab = (1-e)ba$ if and only if n_a commutes with $(1-e)b$.

2. If $eb = 0$ then $ab(1-e) = ba(1-e)$ if and only if n_a commutes with $b(1-e)$.

Proof. (1). $(1-e)bn_a = (1-e)b(1-e)a = (1-e)ba$ and $n_a(1-e)b = n_a b = (1-e)ab$.
(2). By symmetry $b(1-e)n_a = bn_a = ba(1-e)$ and $n_a b(1-e) = a(1-e)b(1-e) = ab(1-e)$. □

We are now ready for

Proposition 2.2. Let $e = aa^d$ and $f = bb^d$.

(I). If $be=0$ and $(1-e)ab=0$ then

$$(a+b)^d = (1-aa^d)b^d\sum_{i=0}^{k-1}(b^d)^ia^i + a^d\sum_{i=0}^{k-1}(a^d)^i(n_a+b)^i[1-bb^d\sum_{i=0}^{k-1}(b^d)^ia^i]. \tag{3}$$

(II). If $be=0$ and $(1-e)ab=(1-e)ba$ then

$$(a+b)^d = (1-aa^d)b^du^{-1} + \sum(a^d)^{i+1}\left((1-e)a+b\right)^i bb^d - ebb^d + ebb^du^{-1}bb^d.$$

(III). If $be=0$ and $(1-e)ba=0$ then

$$\begin{aligned}(a+b)^d &= (1-aa^d-a^db)\sum_{j=0}a^j(b^d)^{j+1} + \\ &\quad +a^d(1+a^db)\left(\sum_{i=0}\left(a^d(1+a^db)\right)^i\left((1-e)b+(1-e)a\right)^i\right)\left(1-(1-e)\sum_{i=0}a^i(b^d)^i\right).\end{aligned}$$

Proof. CASE (I): $be=0$ and $(1-e)ab=0$.
$p=n_a+b=r+s$, in which $rs=0$ and r is nilpotent. Hence, by Corollary (2.1),

$$p^d = s^d\sum_{i=0}^{k-1}(s^d)^ir^i) = b^d\sum_{i=0}^{k-1}(b^d)^i(n_a)^i) = b^d\sum_{i=0}^{k-1}(b^d)^ia^i.$$

In addition, $pp^d = bb^d\sum_{i=0}^{k-1}(b^d)^ia^i$. Substituting these in equation (2) gives the desired result. Slight simplification occur when we use the facts that $(n_a+b)^ibb^d = b^{i+1}b^d$ and $a^d(n_a+b)^i = a^db(a+b)^i$.

CASE (II): $be=0$ and $(1-e)ab=(1-e)ba$.

We observe that, by Lemma (2.2), n_a commutes with $(1-e)b$. We now split p in (2) further as $p=[n_a+(1-e)b]+eb=r+s$. It is clear that $rs=0$, so that we have a LO splitting and shall need r^d and s^d.

Now, $s^d=0$ since $(eb)^2=0$, while $r^d=[n_a+(1-e)b]^d$ can be computed from Lemma (2.1)-1, since n_a is nilpotent and n_a and $(1-e)b$ commute.

Lastly, we also need $[(1-e)b]^d = (1-e)[b(1-e)]^{d^2}b = (1-e)b^d$ and thus

$$r^d = [n_a+(1-e)b]^d = (1-e)b^d[1+(1-e)b^dn_a]^{-1} = (1-e)b^du^{-1}, \tag{4}$$

where $u=1+(1-e)b^da$ and $(1-e)b^dn_a = (1-e)b^d(1-e)a = (1-e)b^da$.

Note that $(1-e)u=u(1-e)$ and that $(1-e)b^d$ and $(1-e)bb^d$ commute with a. Also $rr^d=(1-e)bb^d$. Substituting gives

$$\begin{aligned}p^d &= (r+s)^d = (1+sr^d)r^d \\ &= (1+eb(1-e)b^du^{-1})(1-e)b^du^{-1} \\ &= (1+ebb^du^{-1})(1-e)b^du^{-1}\end{aligned}$$

and

$$\begin{aligned} pp^d &= (1+sr^d)rr^d \\ &= (1+eb(1-e)b^du^{-1})(1-e)bb^d \\ &= (1+ebb^du^{-1})(1-e)bb^d \\ &= bb^d - ebb^d + ebb^du^{-1}bb^d = bb^d + eX. \end{aligned}$$

We then arrive at

$$(a+b)^d = (1-aa^d)[1+ebb^du^{-1}](1-e)b^du^{-1} + a^d[\sum_{i=0}^{k-1}(a^d)^i(n_a+b)^i(1-bb^d)]. \tag{5}$$

Case (III): $be = 0$ and $(1-e)ba = 0$.

We now use a different splitting

$$a+b = (1-e)(a+b) + e(a+b) = p+q,$$

where $pq = 0$. Now $p = r+s$ where $r = (1-e)b$ and $s = a(1-e)$. Thus $rs = 0$, s is nilpotent, $r^d = [(1-e)b]^d = (1-e)b^d$ and $rr^d = (1-e)bb^d$. Hence by Corollary (2.1), we see that

$$p^d = (r+s)^d = \sum_{i=0}^{k-1} s^i(r^d)^ir^d = \sum_{i=0}^{k-1}(n_a)^i[(1-e)b)]^d)^{i+1} = (1-e)Wb^d$$

where $W = \sum\limits_{i=0}^{k-1} a^i(b^d)^i$ and

$$pp^d = \sum_{i=0}^{k-1} s^i(r^d)^irr^d = (1-e)\sum_{i=0}^{k-1} a^i(b^d)^{i+1}(1-e)b = (1-e)W.$$

Next we observe that $q = eb + ae = t + u$, where $tu = 0 = t^2$. Again by Corollary (2.1) we get $q^d = u^d(1+u^dt) = a^d(1+a^db)$ and $qq^d = uu^d(1+u^dt) = aa^d(1+a^db)$

Lastly, we substitute the expressions for p^d, pp^d, q^d and qq^d in Theorem (2.2) and obtain

$$\begin{aligned} (a+b)^d &= (1-qq^d)\left(\sum_{i=0} q^i(p^d)^i\right)p^d + q^d\left(\sum_{i=0}(q^d)^ip^i\right)(1-pp^d) \\ &= (1-aa^d-a^db)\sum_{i=0}[e(a+b)]^i[(1-e)Wb^d]^{i+1} + \\ &\quad + a^d(1+a^db)\sum_{i=0}[a^d(1+a^db)]^i[(1-e)(a+b)]^i[1-(1-e)W]. \end{aligned}$$

Since $(e(a+b))^i = ea^{i-1}(b+a)$, for all $i \geq 1$, and also $(1-aa^d-a^db)e = 0$, we see that

$$(1-aa^d-a^db)\left(\sum_{i=1}[e(a+b)]^i\left((1-e)Wb^d\right)^{i+1}\right) = 0$$

and hence we finally arrive at

$$\begin{aligned}(a+b)^d &= (1-aa^d-a^db)\sum_{j=0} a^j(b^d)^{j+1} + \\ &+ a^d(1+a^db)\sum_{i=0}[a^d(1+a^db)]^i[(1-e)(a+b)]^i[1-(1-e)W].\end{aligned}$$

We may alternatively note that $b(1-e)ba = b^2a = 0$, so that we can use the triplet splitting of $p = b + a(1-e) = [a, 1]\begin{bmatrix} 1-e \\ b \end{bmatrix} = BC$. Indeed, observe that

$$CB = \begin{bmatrix} n_a & 1-e \\ ba & b \end{bmatrix} = \begin{bmatrix} n_a & 1-e \\ 0 & b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ ba & 0 \end{bmatrix} = R + S,$$

where $RS = 0$. It is now clear that

$$R^d = \begin{bmatrix} 0 & x \\ 0 & b^d \end{bmatrix} \text{ and } RR^d = \begin{bmatrix} 0 & n_ax + (1-e)b^d \\ 0 & bb^d \end{bmatrix},$$

where $x = \sum_{i=0}^{k-1}(n_a)^i(1-e)(b^d)^i[(b^d)]^2 = (1-e)\sum_{i=0}^{k-1} a^i(b^d)^{i+2}$. It then follows that

$$\begin{aligned}(a+b)^d &= B[(CB)^d]^2C = [a,1]\begin{bmatrix} 0 & x \\ 0 & b^d \end{bmatrix}^2 \begin{bmatrix} 1-e \\ b \end{bmatrix} \\ &= b^d + axbb^d = b^d + a(1-e)\sum_{i=0}^{k-1} a^i(b^d)^{i+2}.\end{aligned}$$

Likewise, $(a+b)(a+b)^d = B(CB)C = axb + bb^d$. □

Remarks

1. In neither case did we assume that b was nilpotent.

2. Dual expressions of Proposition 2.2 appear when we assume $eb = 0$ instead of $be = 0$. For example:

   (a) If $eb = 0 = (1-e)ab$, then we may split $a + b = c_a + (n_a + b)$ as a LO sum, whose second summand is itself LO since $n_ab = (1-e)ab = 0$;

   (b) If $eb = 0 = ab(1-e)$, then we can write $a + b = p + q$ with $p = c_a + be$ and $q = n_a + b(1-e)$, as hence we have a LO splitting since $pq = 0$. Indeed, $c_a \perp n_a$, $ben_a = 0$, $c_ab(1-e) = eab(1-e) = 0$ and $beb(1-e) = 0$. We note that $p = r + s$ and $q = t+u$ are themselves LO sums, as $c_abe = aebe = 0 = n_ab(1-e) = (1-e)ab(1-e)$, and we are able to compute p^d and q^d. Now, $r^\# = a^d$, $s^2 = 0 = s^d$, and we may apply Corollary 2.1(ii-b) to obtain $p^d = (1 + sr^\#)r^\# = (1 + ba^d)a^d$ and $pp^d = (1 + ba^d)aa^d$. We now focus on $q = t + u$, where $tu = 0$, t is nilpotent and $u^d = b^d(1-e)$. Applying

Corollary 2.1(i-a), and since $(b^d(1-e))^i = (b^d)^i(1-e)$, we obtain $q^d = u^d[\sum(u^d)^i t^i] = b^d(1-e)\sum(b^d)^i[(1-e)a]^i$ and $qq^d = bb^d(1-e)\sum(b^d)^i[(1-e)a]^i$. Theorem 2.2 allows us, now, to compute the D-inverse of $p+q = a+b$.

(c) If $eb = 0$ and $(1-e)ab = (1-e)ba$ then n_a and $b(1-e)$ commute. Taking $p = c_a$ and $q = n_a + b$, then $a + b = p + q$, with $pq = 0$ and $p^\#$ exists. We split q further as $q = be + (n_a + b(1-e)) = r + s$, with $rs = 0$, $s^d = b^d(1-e)w^{-1}$, $ss^d = bb^d(1-e)$, $r^2 = 0 = r^d$, where $w = 1 + b^d(1-e)a$. Therefore, $p^\# = a^d$, $q^d = s^d(1+s^d r) = b^d(1-e)w^{-1}(1+b^d(1-e)w^{-1}be)$, $qq^d = ss^d(1+s^d r) = bb^d(1-e)w^{-1}$, which gives

$$(a+b)^d = (1 - bb^d(1-e)w^{-1})[\sum(n_a+b)^i(a^d)^i]a^d + b^d(1-e)w^{-1}(1-e).$$

3 Inclusions

Consider the matrix $M = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}$, where a^d and b^d exist, and set $f = bb^d$ and $n_b = b(1-bb^d)$. We shall show below that in the following list each of the special cases implies the next. Moreover we shall show that finding the D-inverse for the "most general" case (7) uses the computation of the D-inverse for the "simplest" case (1). As such we can say that all cases are really "equivalent".

1. $a = 1$, $b = n$ is nilpotent
2. $a = 1$
3. $ab = ba$ and a is a unit
4. $a^d b = ba^d$ and $baa^d = b$
5. $ab = ba$
6. $af = fa$ and $an_b = n_b a$
7. $af = faf$ and $an_b = n_b a$

It is also clear that the last conditions automatically hold if b is invertible, when M is invertible, or when b is nilpotent and $ab = ba$. We shall compute M^d for the latter case, which in turn however, makes use of the first case. The chain of implications is supported by the following result.

Lemma 3.1. (i) If $ab = bc$ then $a^d b = bc^d$ (but not conversely).

(ii) If $a^d b = bc^d$, then $aa^d b = bcc^d$ (but not conversely).

(iii) If A is a square matrix over a field then A^d is a polynomial in A.

Proof. (i) Let $s = \max\{in(a), in(c)\}$. If $ab = bc$ then $a^k b = bc^k$ for all k. Hence

$$a^d bc^{s+1} = a^d a^{s+1} b = a^s b = bc^s = bc^{s+1} c^d = a^{s+1} bc^d.$$

We claim by induction that $(a^d)^r ba^{s+1} = a^{s+1} b(a^d)^r$ for all r. Indeed, assuming it for r we get $a^d((a^d)^r ba^{s+1} = a^d a^{s+1} b(c^d)^r = a^s b(c^d)^r = bc^s(c^d)^r = bc^{s+1} c^d (c^d)^r = a^{s+1} b(c^d)^{s+1}$. In particular taking $r = s+2$ shows that

$$a^d b = (a^d)^{s+2} a^{s+1} b = (a^d)^{s+2} bc^{s+1} = a^{s+1} b(c^d)^{s+2} = bc^{s+1}(c^d)^{s+2} = bc^d.$$

(ii) $a^2 a^d b = (a^d)^d b = b(c^d)^d = bc^2 c^d$. Now pre-multiply by a^d.
(iii) This is well known (see, for instance, [1, Theorem 7, page 164]). □

Corollary 3.1.

$$\begin{array}{ll} \text{(i)} & a^d b = ba^d, baa^d = b \\ & \Downarrow \\ \text{(ii)} & ab = ba \\ & \Downarrow \\ \text{(iii)} & abb^d = bb^d a, ab(1-bb^d) = b(1-bb^d)a \end{array}$$

Proof. (i)⇒(ii). By the above $a^d b = ba^d \Rightarrow aa^d b = baa^d$. As such, $ab = abaa^d = aa^d ba = baa^d a = ba$.
(ii)⇒(iii) Clear. □

Remark

We cannot simplify the pair of conditions of (iii) . They *do not* imply that $a^2 a^d b = ba^2 a^d$ nor that $ab = ba$.

We now come to our main result.

Let $M = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}$ and $f = bb^d$, and suppose that $af = faf$ and $an_b = n_b a$.

Stage (I). We split $M = \begin{bmatrix} a(1-f) & 1-f \\ b(1-f) & 0 \end{bmatrix} + \begin{bmatrix} af & f \\ bf & 0 \end{bmatrix} = P + Q$ in which $PQ = 0$, so all we need is P^d and Q^d. It can be verified directly, using the identity $af = faf$, that $Q^{\#} = \begin{bmatrix} 0 & fb^d \\ f & -ab^d \end{bmatrix}$, and $QQ^{\#} = \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix}$. As such we can apply Corollary (2.1) -(iii-b) giving

$$M^d = (P+Q)^d = \begin{bmatrix} 1-f & 0 \\ 0 & 1-f \end{bmatrix} P^d + Q^{\#}[\sum_{i=0}^{k-1}(Q^{\#})^i P^i](1 - PP^d). \tag{6}$$

Next, we split P as $P = \begin{bmatrix} 0 & -f \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a(1-f) & 1 \\ n_b & 0 \end{bmatrix} = R + S$. Clearly $RS = 0 = SR$, $R^2 = 0$ and $P^d = S^d$, so that $PP^d = SS^d$.

We thus have

$$(P+Q)^d = (1-f)S^d + (Q^\#)^2 R + Q^\#[\sum_{i=0}^{k-1}(Q^\#)^i(S^i)](I - SS^d), \tag{7}$$

where $S = \begin{bmatrix} a(1-f) & 1 \\ n_b & 0 \end{bmatrix}$. Now, since $a(1-f)$ commutes with n_b, the computation of S^d has been reduced to the "commutative nilpotent" case.

Stage(II). Consider $S = \begin{bmatrix} \alpha & 1 \\ n & 0 \end{bmatrix}$, where $\alpha = a(1-f)$ and $n = n_b$. In addition $\alpha n = n\alpha$ and n is nilpotent. Also set $e = \alpha\alpha^d$.
We split S as $S = \begin{bmatrix} \alpha e & e \\ en & 0 \end{bmatrix} + \begin{bmatrix} \alpha(1-e) & 1-e \\ (1-e)n & 0 \end{bmatrix} = G + H$, where $GH = HG = 0$. Thus $S^d = G^d + H^d$ and $SS^d = GG^d$. Moreover $G = \begin{bmatrix} \alpha e & 0 \\ 0 & \alpha e \end{bmatrix}\begin{bmatrix} 1 & \alpha^d \\ \alpha^d n & 0 \end{bmatrix} = (\alpha e I)T$ and $H = (1-e)I\begin{bmatrix} \alpha(1-e) & 1 \\ n & 0 \end{bmatrix} = [(1-e)I]U$, in which $(\alpha e I)T = T(\alpha e I)$ and $(1-e)IU = U(1-e)I$.

We now recall

Lemma 3.2. Let $M = \begin{bmatrix} a & c \\ b & 0 \end{bmatrix}$ where a, b and c commute and a and bc are nilpotent. Then M is also nilpotent.

Proof. $M^2 = (bc)I + aM$, in which $(bc)I$ and aM commute and are both nilpotent. This forces M^2 to be nilpotent. □

Applying this to the above case we see that U and H are nilpotent. As such, because $\alpha e I$ and T commute, we have $S^d = G^d = (\alpha^d I)T^d$ in which $T = \begin{bmatrix} 1 & \alpha^d \\ \alpha^d n & 0 \end{bmatrix}$.

We next factor $T = \begin{bmatrix} 1 & 1 \\ \alpha^d n & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \alpha^d \end{bmatrix} = BC$ so that we can use Cline's formula, to obtain

$$T^d = (BC)^d = B(K^d)^2 C \text{ and } TT^d = BK^dC, \tag{8}$$

where $K = CB = \begin{bmatrix} 1 & 1 \\ \eta & 0 \end{bmatrix}$ with $\eta = (\alpha^d)^2 n_b$.

As such we now come to our final stage.

Stage(III). Let $K = \begin{bmatrix} 1 & 1 \\ \eta & 0 \end{bmatrix}$, where $\eta = (a^d)^2 n_b$ is nilpotent. The computation of K^d was given in [3]. We shall derive the formula by considering the quadratic equation $m^2 = ma + b$, in a ring R with 1, where a and b have Drazin inverses and a commutes with b.

This in turn will depend on the master recurrence relation $y_{k+1} = y_k a + y_{k-1} b$, with $y_0 = 1$ and $y_1 = a$ [12].

A key role is played by the **reduced** quadratic $m^2 = m+n$, with $n^r = 0 \neq n^{r-1}$. Indeed, we shall show that in this case m has a Drazin inverse (is strongly-pi-regular) and we shall compute its D-inverse in the form $m^d = m\mu(n) + n\nu(n)$, where μ and ν are suitable polynomials in n of degree at most $in(n) - 1$. Some interesting combinatorial identities will be developed along the way.

4 Difference Equations

We begin by considering the recurrence relation

$$y_{k+1} = y_k a + y_{k-1} b, \tag{9}$$

where $ab = ba$ and with $y_{-1} = 0$, $y_0 = 1$ [18]. From [12] we know that the exact solution is given by

$$y_s = \sum_{t=0}^{[s/2]} \binom{s-1-t}{t} a^{s-1-2t} b^t$$

in which the upper limit can be replaced by any $L \geq [s/2]$, such as $L = s-1$. This gives the solution chain $\mathbf{y} = (0, 1, a, a^2 + b, a^3 + ab + ba, \ldots)$.

Now consider $M = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}$. Then $M^2 = Ma + bI$, and it is convenient to consider the ring quadratic $m^2 = ma + b$ with a, b and m commuting.

Repeated application shows that $m^k = m\alpha_k + \beta_k$ for $k = 0, 1, \ldots$, where α_i and β_i are polynomials in a and b, and $\alpha_0 = 0, \alpha_1 = 1$ and $\beta_0 = 1, \beta_1 = 0$. Multiplying by m gives

$$m^{k+1} = m(a\alpha_k + \beta_k) + b\alpha_k = m\alpha_{k+1} + \beta_{k+1}.$$

This gives the recurrence relations

$$\text{(i) } \alpha_{k+1} = a\alpha_k + b\alpha_{k-1} \quad \text{and (ii)} \quad \beta_{k+1} = b\alpha_k \tag{10}$$

with $\alpha_0 = 0$ and $\alpha_1 = 1$. This shows that the terms of $\{\alpha_i\}$ precisely satisfy the basic recurrence (9), and are thus known.

We note in passing that if a and b commute and are nilpotent, say $a^r = 0 = b^r$, then $\alpha_s = 0$ for $s \geq 3r$. Indeed, if $s - 1 - 2t$ and t are both smaller than r, then $t \leq 3r + 1$ which cannot be. This means that M is also nilpotent, as observed earlier.

5 The nilpotent case

Next we examine the special case where $a = 1$ and $b = n$ is nilpotent, i.e. we consider $m^2 = m + n$, with $n^r = 0 \neq n^{r-1}$ (and $r = in(n)$).

We first note that $m(m-1) = n$, and hence that $m^r(m-1)^r = n^r = 0$. We now set $c = 1-m$. Then $m^r c^r = 0$ or $m^r(1 - c^r) = m^r$. Hence if $(1 - c^r) = (1 - c)g$, then $m^r(1 - c)g = m^r$ i.e. $m^{r+1}g = m^r = gm^{r+1}$. This guarantees that m has a Drazin inverse of the form $m^d = m^r g^{r+1}$ and $mm^d = m^r g^r$ [9].

To simplify the expression for m^d, we shall need several results involving the infinite geometric sum $G = G(c) = 1 + c + c^2 + \ldots$ and its r-th partial sum $g = g_r(c) = 1 + c + .. + c^{r-1}$. Since $m^r c^r = 0$ it is clear that $m^r G = m^r g$. Consequently $m^r G^{r+1} = m^r g^{r+1}$. Powering G corresponds to powering the semi-infinite matrix F of all ones above or on the main diagonal. This shows that $G^{r+1} = \binom{r}{0} + \binom{r+1}{1}c + \binom{r+2}{2}c^2 + \ldots + \binom{2r-1}{r-1}c^{r-1} + \ldots$ and it now follows that

$$m^d = \sum_{k=0}^{r-1} \binom{r+k}{k}(-1)^k m^{r-k} n^k \qquad (r = in(n)). \tag{11}$$

For example,

$$\begin{aligned} r = 2 &: \quad m^d = m^2 - 3nm \\ r = 3 &: \quad m^d = m^3 - 4nm^2 + 10n^2m \\ r = 4 &: \quad m^d = m^4 - 5nm^3 + 15n^2n^2 - 35n^3m \end{aligned}$$

Substituting the power form $m^k = m\alpha_k + \beta_k$ into the expression for m^d now gives

$$m^d = \sum_{k=0}^{r-1}(-1)^k \binom{r+k}{k} n^k (m\alpha_{r-k} + n\alpha_{r-k-1}) = \mu m + \nu n$$

where

$$\mu = \sum_{k=0}^{r-1}(-1)^k \binom{r+k}{k} n^k \alpha_{r-k} \text{ and } \nu = \sum_{k=0}^{r-2}(-1)^k \binom{r+k}{k} n^k \alpha_{r-k-1}.$$

Next we substitute the exact solution $\alpha_s = \sum_{t=0}^{[s-1} \binom{s-1-t}{t} n^t$ into μ and ν and obtain

$$\mu = \sum_{k=0}^{r-1}(-1)^k \binom{r+k}{k} n^k \sum_{t=0}^{r-k-1} \binom{r-k-t-1}{t} n^t$$

and

$$\nu = \sum_{k=0}^{r-2}(-1)^k \binom{r+k}{k} n^k \sum_{t=0}^{r-k-2} \binom{r-k-t-2}{t} n^t.$$

Now, setting $k + s = t$ and interchanging summations

$$\sum_{k=0}^{r-1}\sum_{s=k}^{r-1}(.) = \sum_{s=0}^{r-1}\sum_{k=0}^{s}(.)$$

and

$$\sum_{k=0}^{r-2}\sum_{s=k}^{r-2}(.) = \sum_{s=0}^{r-2}\sum_{k=0}^{s}(.),$$

we arrive at

$$\mu(r) = \sum_{s=0}^{r-1} P(s) n^s \text{ and } \nu(r) = \sum_{s=0}^{r-1} Q(s) n^s,$$

where, for $s \le [(r-1)/2]$,

$$P(s) = \sum_{k=0}^{s} (-1)^k \binom{r+k}{k}\binom{r-s-1}{s-k} \quad \text{and} \quad Q(s) = \sum_{k=0}^{s} (-1)^s \binom{r+k}{k}\binom{r-s-2}{s-k}.$$

To evaluate these polynomials, we need the following combinatorial identity:

Lemma 5.1. For all natural r, s and t,

$$\sum_{k=0}^{s} (-1)^k \binom{r+k}{k}\binom{r-s-t}{s-k} = (-1)^s \binom{2s+t}{s} \tag{12}$$

Proof. For all rational α, β, we have

$$(1+x)^\alpha (1+x)^\beta = (1+x)^{\alpha+\beta}.$$

Thus on equating powers of x^s we get

$$\sum_{k=0}^{s} \binom{\alpha}{k}\binom{\beta}{s-k} = \binom{\alpha+\beta}{k}.$$

It should be noted that the upper limit β may depend on s.

Now recall that $\binom{-r-1}{k} = (-1)^k \binom{r+k}{k}$ and set $\alpha = -r-1$ and $\beta = -r-t-s$. Then $\alpha + \beta = -1-t-s$, and we arrive at $\binom{-1-t-s}{s} = (-1)^s \binom{t+2s}{s}$. $\square$

Taking $t = 1$, gives $P(s) = (-1)^s \binom{2s+1}{s}$ while $t = 2$ yields $Q(s) = (-1)^s \binom{2s+2}{s}$, as desired.

6 Combinations

Let us now combine all the above steps and compute the Drazin inverse of $M = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}$, when a^d and b^d both exist, $af = faf$ and $an_b = n_b a$, where $f = bb^d$ and $n_b = b(1-f)$.

We may apply the results of the previous section to the matrix $K = \begin{bmatrix} 1 & 1 \\ \eta & 0 \end{bmatrix}$, and thus have $K^d = \begin{bmatrix} \mu + \eta\nu & \mu \\ \mu\eta & \eta\nu \end{bmatrix}$, where

$$\eta = (\alpha^d)^2 n_b,\ \mu = \sum_{s=0}^{in(b)} P(s)[(a(1-f))^d]^2 n_b \text{ and } \nu = \sum_{s=0}^{in(b)} Q(s)[(a(1-f))^d]^2 n_b, \tag{13}$$

with $P(s) = (-1)^s \binom{2s+1}{s}$, $Q(s) = (-1)^s \binom{2s+2}{s}$ and $\eta = (\alpha^d)^2 n_b$.

From equation (8), $TT^d = \begin{bmatrix} 1 & 1 \\ \alpha^d n_b & 0 \end{bmatrix} K^d \begin{bmatrix} 1 & 0 \\ 0 & \alpha^d \end{bmatrix}$ and $T^d = \begin{bmatrix} 1 & 1 \\ \alpha^d n_b & 0 \end{bmatrix} \left(K^d\right)^2 \begin{bmatrix} 1 & 0 \\ 0 & \alpha^d \end{bmatrix}$, where $\alpha = a(1-f)$, and $G = \alpha e T, GG^d = \alpha\alpha^d TT^d, N_G = \alpha e N_T$. Substituting in equation (7),

$$\begin{aligned} M^d &= (1-f)(\alpha)^d T^d + \begin{bmatrix} 0 & -b^d \\ 0 & -ab^d \end{bmatrix} + \begin{bmatrix} 0 & fb^d \\ f & -ab^d \end{bmatrix} (I - \alpha\alpha^d TT^d) + \\ &\quad + \sum_{i=1}^{in(b)} \begin{bmatrix} 0 & fb^d \\ f & -ab^d \end{bmatrix}^{i+1} \left((\alpha e N_T)^i + \begin{bmatrix} \alpha(1-e) & 1-e \\ (1-e)n_b & 0 \end{bmatrix}^i \right) \end{aligned}$$

7 Comments and Conclusions

1. The (2,2,0) problem appears naturally in the *additive* D-inverse problem. For example when $b^2 a = 0$ then the computation of $(a+b)^d$ reduces to the computation of $\begin{bmatrix} a & 1 \\ ba & 0 \end{bmatrix}^d$.
   Indeed, if $a + b = [a,1]\begin{bmatrix} 1 \\ b \end{bmatrix} = BC$, then $CB = \begin{bmatrix} a & 1 \\ ba & b \end{bmatrix}$ which we can split as $CB = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} a & 1 \\ ba & 0 \end{bmatrix} = P + Q$, with $PQ = 0$. Hence a knowledge of $\begin{bmatrix} a & 1 \\ ba & 0 \end{bmatrix}^d$ would suffice for the computation of $(a+b)^d$.
   Conversely, if $(a+b)^d$ is known and $b^2 = 0$ then $\begin{bmatrix} a & 1 \\ ba & 0 \end{bmatrix}^d$ can be computed, via the splitting $\begin{bmatrix} a & 1 \\ ba & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -b \end{bmatrix} + \begin{bmatrix} a & 1 \\ ba & b \end{bmatrix} = R + S$, in which $RS = 0$. We may without loss of generality assume that b is nilpotent.

2. If $ba = 0$, then we may also compute $M = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}^d$, but this condition is independent from the condition (7) : $af = faf$ and $an_b = n_b a$. Indeed, we may split $M = \begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = R + S$, in which $RS = 0$. We can immediately compute $R^d = \begin{bmatrix} 0 & b^d \\ 1 & 0 \end{bmatrix}$, $RR^d = \begin{bmatrix} 1 & 0 \\ 0 & bb^d \end{bmatrix}$, $S^d = \begin{bmatrix} a^d & 0 \\ 0 & 0 \end{bmatrix}$ and $SS^d = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$.
   From item (1), we obtain $(R+S)^d = (I - SS^d)\left[\sum_{i=0}^{k-1} (SR^d)^i\right] R^d + S^d \left[\left[\sum_{i=0}^{k-1} (S^d R)^i\right] (1 - RR^d)\right.$, in which
   $$SR^d = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & (b)^d \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ab^d \\ 0 & 0 \end{bmatrix}.$$
   Likewise,
   $$S^d R = \begin{bmatrix} a^d & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ ba & 0 \end{bmatrix} = \begin{bmatrix} 0 & a^d \\ 0 & 0 \end{bmatrix}.$$
   Clearly both are nilpotent. Only two terms survive and we get
   $$(R+S)^d = (I - SS^d)[I + SR^d]R^d + S^d[I + S^d R](I - RR^d), \tag{14}$$

which reduces to

$$M^d = \begin{bmatrix} 1-e & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & ab^d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & b^d \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} a^d & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & a^d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1-bb^d \end{bmatrix}$$

or

$$M^d = \begin{bmatrix} (1-e)ab^d & (1-e)b^d + (a^d)^2(1-bb^d) \\ 1 & 0 \end{bmatrix} \tag{15}$$

3. If we replace b by ba in the previous result, then we can compute $Q^d = \begin{bmatrix} a & 1 \\ ba & 0 \end{bmatrix}^d$, under assumption that $ba^2 = 0$. This gives

$$Q^d = \begin{bmatrix} (1-e)a(ba)^d & (1-e)(ba)^d + (a^d)^2(1-(ba)(ba)^d) \\ 1 & 0 \end{bmatrix} \tag{16}$$

4. Combining the above we see that if $ba^2 = 0$ as well as $b^2a = 0$ then we can compute $(a+b)^d$ via the computation of Q^d.

5. We could try to use condition (7) to compute $Q^d = \begin{bmatrix} a & 1 \\ ba & 0 \end{bmatrix}^d$, and hence find $(a+b)^d$ under the assumptions that $b^2a = 0$ and (7) holds. This, however, will not give anything new because

   Lemma 7.1. $ba = 0$ if and only if (i) $b^2a = 0$ (ii) $AF = FaF$ and (iii) $a(ba)[1-F] = (ba)[1-F]a$, where $F = (ba)(ba)^d$.

6. The identity $M^2 = \begin{bmatrix} 1 & c \\ b & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & c \\ b & 0 \end{bmatrix} + \begin{bmatrix} cb & 0 \\ 0 & bc \end{bmatrix} = M + X$ shows that the recurrence relation cannot be used to compute M^d when b and c do not commute.

7. We cannot expect $\begin{bmatrix} A & I \\ B & 0 \end{bmatrix}^d$ to be expressible in terms of standard functions of A and B, even in the case where A is invertible and B is nilpotent. We can however, perform a sequence of row-column perturbations to compute the desired D-inverse.

8. If $M = \begin{bmatrix} A & I \\ B & 0 \end{bmatrix}$, then $rk(M) = rk(I) + rk(B)$. However, the character of the matrix M can vary greatly! Indeed, when $A = I$, then $M^2 = \begin{bmatrix} I+B & 1 \\ B & B \end{bmatrix}$, and $rk(M^2) = rk(I) + rk(B^2)$. If B is nilpotent, $rk(M) \neq rk(M^2)$ and hence M can *never* have a group inverse. For other invertible A, the matrix M may have a group inverse, as seen from the example

$$M = \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} P & \mathbf{q} \\ 0 & 0 \end{bmatrix}$$

where $P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is invertible. This means that $M^\#$ exists.

9. Taking determinants throughout the identity

$$\begin{bmatrix} xI_n & C \\ 0 & I_n \end{bmatrix} \begin{bmatrix} xI - A & -C \\ -B & xI_n \end{bmatrix} = \begin{bmatrix} x(xI - A) - CB & 0 \\ -B & xI \end{bmatrix}$$

shows that $x^n \Delta_M(x) = x^n |x(x-1)A - CB|$. Cancelling gives

$$\Delta_M(x) = |x(xI - A) - CB| = |x^2 I - xA - CB|.$$

This shows that the D-inverse of M depends on the determinant of the quadratic "stencil" $x^2 I - xA - CB$, which also appears in the study of the Riccati equation!

Even when A is invertible this does not admit simplification! As such we cannot expect to find a splitting of M that will only use $A^{-1}CB$!

10. The (2,2,0) problem frequently appears when we use semi-orthogonal splittings. For example when $M = \begin{bmatrix} I & P \\ Q & R \end{bmatrix}$ and $RQ = 0$, then we can obtain a semi-orthogonal splitting via $M = \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} I & P \\ Q & 0 \end{bmatrix} = S + T$ where $ST = 0$. Likewise when $PR = 0$ we have the semi-orthogonal splitting $M = \begin{bmatrix} I & P \\ Q & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} = T + S$ with $TS = 0$. In either case we can reduce the computation of the D-inverse to the (2,2,0) case, by using the formula given in [15].

11. Suppose that $m^2 = mp + q$, in a ring R with 1 where p and q do not commute. If we set $M = \begin{bmatrix} p & 1 \\ q & 0 \end{bmatrix}$ and $U = [m, 1]$, then $UM = mU$. It is clear that now $UM^k = m^k U$ for all k. If M has a right index and $M^{r+1}X = M^r$ then $m^r[m, 1] = m^r U = UM^r = UM^{r+1}X = M^{r+1}(UX)$. Setting $UX = [f, g]$, then shows that $m^{r+1}g = m^r$. That is, m has the same right index.

    It is not clear what happens to the left index.

8 Questions

We close with some pertinent questions.

- Can we find $(n + b)^d$ where n is nilpotent and b is idempotent? Or where n is nilpotent and $n^2 b = 0 = bn^2$?

- Are there any other sufficient conditions allowing the computation of $\begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}^d$?

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