Variational and quasivariational inequalities with first order constraints

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Abstract

We study the existence of solutions of stationary variational and quasivariational inequalities with curl constraint, Neumann type boundary condition and a p-curl type operator. These problems are studied in bounded, not necessarily simply connected domains, with a special geometry, and the functional framework is the space of divergence-free functions with curl in $\boldsymbol{L}^p$ and null tangential or normal traces.
The analogous variational or quasivariational inequalities with a gradient constraint are also studied, considering Neumann or Dirichlet non-homogeneous boundary conditions. The existence of a generalized solution for a Lagrange multiplier problem with homogeneous Dirichlet boundary condition and the equivalence with the variational inequality is proved in the linear case, for an arbitrary gradient constraint.

1 Introduction

The study of variational inequalities had its beginning around 1960. A model problem is the well-known obstacle problem, that we briefly formulate here: to find $u \in \mathbb{K}_\psi$ such that

$$\int_\Omega \nabla u \cdot \nabla(v-u) \geq \int_\Omega f(v-u), \qquad \forall v \in \mathbb{K}_\psi, \tag{1}$$

where f is a given function defined in a bounded open subset Ω of $\mathbb{R}^N$ and, for an obstacle ψ, $\mathbb{K}_\psi = \{v : v \geq \psi\}$.

Under appropriate assumptions the variational inequality (1) is equivalent to the complementary problem

$$\min\{-\Delta u - f, u - \psi\} = 0 \qquad \text{a.e. in } \Omega.$$

The set $\partial I \cap \Omega = \partial\Lambda \cap \Omega$, where $I = \{x \in \Omega : u(x) = \psi(x)\}$ and $\Lambda = \{x \in \Omega : u(x) > \psi(x)\}$, is called the free boundary for the obstacle problem.

Problems where an *a priori* unknown subset of Ω is part of the problem are, in general, called free boundary problems. In the last fifty years many problems arising from other sciences were modeled as free boundary problems. Many of these models can be reduced to variational or quasivariational inequalities, a quasivariational inequality being an implicit problem where the definition of the convex set depends on the solution itself.

Problems with gradient or curl constraint, which we address here, model many different situations, such as the elastoplastic torsion problem ([26], [6] or [7]), sand piles and river networks ([20] or [22]) or electromagnetic problems ([21], [4], [17] or [18]). We remark that, if we consider a longitudinal geometry in electromagnetic problems, i.e., when the magnetic field is of the form $(0,0,h)$, the curl constraint is reduced to a gradient constraint (see [23]). We consider the following particular situation: to find $u \in \mathbb{K}^{\nabla}_{\varphi}$ such that

$$\int_\Omega \nabla u \cdot \nabla(v-u) \geq \int_\Omega f(v-u), \qquad \forall v \in \mathbb{K}^{\nabla}_{\varphi}, \tag{2}$$

where $\mathbb{K}^{\nabla}_{\varphi} = \{v : |\nabla v| \leq \varphi\}$, f and φ being given functions defined in Ω. Decomposing Ω in the sets $\Lambda = \{x \in \Omega : |\nabla u(x)| < \varphi(x)\}$ and $I = \{x \in \Omega : |\nabla u(x)| = \varphi(x)\}$, we also have $-\Delta u = f$ in Λ, but here we do not have a sign for $-\Delta u - f$. This is a difference between the obstacle-type problems and the gradient constraint problems. In fact, the second ones are more difficult to handle, although the constraint in the first derivatives of the solutions has a regularizing effect. The existence of a solution for stationary variational inequalities is immediate, by a theorem due to Lions and Stampacchia (see, for instance, [12]). If we want to study additional regularity, the natural way is to consider a family of penalized equations that approximates the variational inequality. There exists a general way of penalizing any elliptic variational inequality (see [14], p. 370) but, in order to obtain additional regularity of the solutions, we need an explicit definition of the penalization which we can manage to obtain *a priori* estimates for the approximated solutions. And here we point out a difference in the treatment of obstacle problems (zero order constraints) or problems with constraints in the first derivatives. The supposedly natural penalization $\frac{1}{\varepsilon}(|\nabla u^\varepsilon|^2 - \varphi^2 - \varepsilon)^+$ does not penalize the variational inequality (2). In fact, it penalizes a different problem, $\max\{-\Delta u - f, |\nabla u| - \varphi\} = 0$. It was shown by one of the authors that, in the evolutive case, the two problems are not, in general, equivalent (see [24]).

Another possible formulation for the variational inequality (2) consists in finding a pair (u, λ) of functions defined in Ω such that

$$-\nabla \cdot (\lambda \nabla u) = f \quad \text{in } \Omega, \tag{3a}$$

$$|\nabla u| \leq \varphi \quad \text{in } \Omega, \tag{3b}$$

$$\lambda \geq 1, \quad \text{in } \Omega, \tag{3c}$$

$$(\lambda - 1)(|\nabla u| - \varphi) = 0 \quad \text{in } \Omega. \tag{3d}$$

This means that, in the set Λ, the equation $-\Delta u = f$ is satisfied and, in the set I, the Lagrange multiplier λ may take any value greater than or equal to 1, i.e., λ belongs to the maximal monotone graph $k(|\nabla u| - \varphi)$, where $k(s) = 1$ if $s < 0$ and $k(0) = [1, \infty[$. It is easy to show that if (u, λ) solves (3) then u solves (2). Indeed, given $v \in \mathbb{K}^{\nabla}_{\varphi}$, multiplying (3a) by $v - u$ and integrating, we get

$$\int_\Omega \lambda \nabla u \cdot \nabla(v-u) = \int_\Omega f(v-u).$$

But, as

$$\int_\Omega (\lambda-1)\nabla u \cdot \nabla(v-u) \leq \int_\Omega (\lambda-1)|\nabla u|\big(|\nabla v| - |\nabla u|\big) = \int_I (\lambda-1)\varphi\big(|\nabla v| - \varphi\big) \leq 0,$$

we immediately obtain (2).

A very ingenious (although natural) penalization and regularization of problem (2) was introduced by Gerhardt in [11]. He approximated the maximal monotone graph k by a family of smooth monotone convex functions k_ε such that $k_\varepsilon(s) = 1$ if $s \leq 0$ and $k_\varepsilon(s) = e^{\frac{ms}{\varepsilon}}$ if $s \geq \varepsilon$ (m chosen *a posteriori*). This approach gives us the correct penalization for the variational and quasivariational inequalities with gradient or curl constraint, which is also very useful in the treatment of the evolutive problems, not considered in this paper. Nevertheless, this idea is used here to prove the existence of a solution of problem (3) for strictly positive smooth gradient constraint φ, the main result of this work.

Problems with curl constraint for operators of p-curl type were studied by two of the authors in [17] assuming Ω to be simply connected, $p > \frac{6}{5}$ and null normal trace for the test functions. Later advances (see [25] and [2]) allow us to extend our results to a more general case where $p > 1$, Ω is simply or multiply connected with a special geometry and the test functions have null tangential or normal traces. For the sake of completeness, in Section 2 we present these generalizations in the framework of variational or quasivariational inequalities. We prove a continuous dependence result for solutions of the variational inequality with different data and we use this result to prove the existence of a solution for quasivariational inequalities.

In Section 3 we study the case of a gradient constraint. In Subsection 3.1 we consider variational inequalities with non-homogeneous Neumann type boundary condition, for operators of type $-\nabla\cdot(|\nabla u|^{p-2}\nabla u)+|u|^{p-2}u$ and we follow the steps of Section 2. In Subsection 3.2 we consider variational inequalities with non-homogeneous Dirichlet boundary condition, for p-laplacian type operators. We remark that the existence of a solution for the variational inequality is only possible if there exists a compatibility condition between the traces of functions in the convex sets and the constraint on their gradients. It is not easy to obtain the existence of a solution for the quasivariational inequality since we need to guarantee the compatibility condition for all possible solutions. Based on a previous work of two of the authors (see [3]) we were able to prove the existence of a solution for the quasivariational inequality when the boundary data satisfies a compatibility condition that depends on the minimum of the constraint.

In Section 4, we prove the existence of a solution for problem (3) in a weak sense. We approximate the variational inequality using the penalization of Gerhardt and, although the *a priori* estimates for the approximated solutions are not enough to pass to the limit, using the monotonicity of the penalization, we can interpret (3a,3c,3d) in a generalized sense.

The existence of a Lagrange multiplier for the elastoplastic torsion problem (gradient constraint one) with homogeneous Dirichlet boundary conditions was proved by Brezis in [5]. This result was later extended by the third author in [24], in the evolutive case, for nonconstant gradient constraint φ satisfying $\Delta\varphi^2\leq 0$. As this last case is equivalent to a double-obstacle problem, it is easier than the one considered in this paper. Further generalizations of the result of Brezis, considering also the gradient constraint one, have been done, for example, in [8] and [9].

The existence of a Lagrange multiplier remains, to the best of our knowledge, an open problem for $p\neq 2$ in the case of a gradient constraint and for any $p>1$ in the case of a curl constraint.

2 The problem with curl constraint

In this section we study variational and quasivariational inequalities with curl constraint, assuming two different types of boundary conditions: the perfectly conductive boundary and the perfectly permeable boundary.

Spaces of vector-functions will be denoted by boldface symbols, following the standard notations for vector-functions.

Let Ω be an open bounded connected subset of $\mathbb{R}^3$ with a $\mathscr{C}^{1,1}$ boundary Γ. The boundary is not necessarily connected and we denote by Γ_i, $i=0,\dots,I$, the connected components of Γ, being Γ_0 the boundary of the only unbounded connected component of $\mathbb{R}^3\setminus\bar\Omega$.

Following [10], [1] and [2] we assume that the set Ω can be made simply connected by a finite number of regular disjoint cuts, $\Sigma_1,\dots,\Sigma_J$. More precisely, each surface Σ_j is an open subset of a smooth manifold, the boundary of Σ_j is contained in Γ, $\bar\Sigma_i\cap\bar\Sigma_j\neq\emptyset$ for $i\neq j$ and $\Omega^0=\Omega\setminus\bigcup_{j=1}^J\Sigma_j$ is simply connected and pseudo-$\mathscr{C}^{1,1}$.

We denote by $\boldsymbol{n}$ the exterior normal unitary vector to Γ and we consider two types of boundary conditions

$$\boldsymbol{h}\cdot\boldsymbol{n}=0 \text{ on } \Gamma,\qquad \langle\boldsymbol{h}\cdot\boldsymbol{n},1\rangle_{\Sigma_j}=0,\ j=1,\dots,J \tag{4}$$

and

$$\boldsymbol{h}\times\boldsymbol{n}=0 \text{ on } \Gamma,\qquad \langle\boldsymbol{h}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0,\ i=1,\dots,I. \tag{5}$$

The meaning of the notation $\langle\,\cdot\,,\,\cdot\,\rangle_{\Sigma_j}$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{\Gamma_i}$ will be precised later.

Given $1<p<\infty$, we introduce the functional framework necessary to formulate and solve the variational and quasivariational inequalities with curl constraint. For details see [17], [25] and [18]. We consider

$$\boldsymbol{W}^p(\nabla\cdot,\Omega)=\big\{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega):\ \nabla\cdot\boldsymbol{v}\in L^p(\Omega)\big\},$$

endowed with the norm $\|\boldsymbol{v}\|_{\boldsymbol{W}^p(\nabla\cdot,\Omega)}=\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\nabla\cdot\boldsymbol{v}\|_{L^p(\Omega)}$.

Given $\boldsymbol{v}\in\boldsymbol{\mathscr{D}}(\bar\Omega)$ and $\varphi\in W^{1,p'}(\Omega)$, we have the following formula of integration by parts

$$\int_\Omega\nabla\cdot\boldsymbol{v}\,\varphi+\int_\Omega\boldsymbol{v}\cdot\nabla\varphi=\int_\Gamma\boldsymbol{v}\cdot\boldsymbol{n}\,\varphi,$$

that can be extended, by density, to

$$\int_\Omega \nabla\cdot\boldsymbol{v}\,\varphi + \int_\Omega \boldsymbol{v}\cdot\nabla\varphi = \langle\gamma_n(\boldsymbol{v}),\varphi\rangle_{W^{-\frac1p,p}(\Gamma)\times W^{\frac1p,p'}(\Gamma)},\ \forall\boldsymbol{v}\in\boldsymbol{W}^p(\nabla\cdot,\Omega),\ \forall\varphi\in W^{1,p'}(\Omega),$$

where $\langle\cdot,\cdot\rangle_{W^{-\frac1p,p}(\Gamma)\times\ W^{\frac1p,p'}(\Gamma)}$ is the duality bracket between $W^{-\frac1p,p}(\Gamma)$ and $W^{\frac1p,p'}(\Gamma)$ and $\gamma_n(\boldsymbol{v})$ is the trace of $\boldsymbol{v}$, which will be, from now on, denoted by $\boldsymbol{v}\cdot\boldsymbol{n}_{|\Gamma}$. We represent the kernel of γ_n by $\boldsymbol{W}^p_0(\nabla\cdot,\Omega)$.

Defining

$$\boldsymbol{W}^p(\nabla\times,\Omega)=\big\{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega):\ \nabla\times\boldsymbol{v}\in\boldsymbol{L}^p(\Omega)\big\},$$

with the norm $\|\boldsymbol{v}\|_{\boldsymbol{W}^p(\nabla\times,\Omega)}=\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$, we have, for $\boldsymbol{v}\in\mathscr{D}(\bar\Omega)$ and $\boldsymbol{\varphi}\in\boldsymbol{W}^{1,p'}(\Omega)$,

$$\int_\Omega \boldsymbol{v}\cdot\nabla\times\boldsymbol{\varphi}-\int_\Omega\nabla\times\boldsymbol{v}\cdot\boldsymbol{\varphi}=\int_\Gamma \boldsymbol{v}\times\boldsymbol{n}\cdot\boldsymbol{\varphi},$$

which we extend, by density, to

$$\int_\Omega\nabla\times\boldsymbol{v}\cdot\boldsymbol{\varphi}-\int_\Omega\boldsymbol{v}\cdot\nabla\times\boldsymbol{\varphi}=\langle\gamma_\tau(\boldsymbol{v}),\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-\frac1p,p}(\Gamma)\times\boldsymbol{W}^{\frac1p,p'}(\Gamma)},\ \forall\boldsymbol{v}\in\boldsymbol{W}^p(\nabla\times,\Omega),\ \forall\boldsymbol{\varphi}\in\boldsymbol{W}^{1,p'}(\Omega),$$

where $\boldsymbol{\gamma}_\tau(\boldsymbol{v})$ is the trace of $\boldsymbol{v}$, denoted, from now on, by $\boldsymbol{v}\times\boldsymbol{n}_{|\Gamma}$. We represent the kernel of $\boldsymbol{\gamma}_\tau$ by $\boldsymbol{W}^p_0(\nabla\times,\Omega)$.

We denote

$$\boldsymbol{W}^p_T(\Omega)=\big\{\boldsymbol{v}\in\boldsymbol{W}^p(\nabla\times,\Omega):\ \nabla\cdot\boldsymbol{v}=0\text{ in }\Omega,\boldsymbol{v}\cdot\boldsymbol{n}_{|\Gamma}=0,\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Sigma_j}=0,\ j=1,\dots,J\big\},$$

where the brackets $\langle\cdot,\cdot\rangle_{\Sigma_j}$ represent the duality pairing between $W^{-\frac1p,p}(\Sigma_j)$ and $W^{\frac1p,p'}(\Sigma_j)$, and

$$\boldsymbol{W}^p_N(\Omega)=\big\{\boldsymbol{v}\in\boldsymbol{W}^p(\nabla\times,\Omega):\ \nabla\cdot\boldsymbol{v}=0\text{ in }\Omega,\ \boldsymbol{v}\times\boldsymbol{n}_{|\Gamma}=0,\ \langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0,\ i=1,\dots,I\big\},$$

where, $\langle\cdot,\cdot\rangle_{\Gamma_i}$ represents the duality pairing between $W^{-\frac1p,p}(\Gamma_i)$ and $W^{\frac1p,p'}(\Gamma_i)$. These spaces are subspaces of $\boldsymbol{W}^{1,p}(\Omega)$ and the semi-norm $\|\nabla\times\cdot\|_{\boldsymbol{L}^p(\Omega)}$ induces on them a norm equivalent to their natural norms and to the one induced from the $\boldsymbol{W}^{1,p}$-norm (see [2] and [25]). The spaces $\boldsymbol{W}^p_T(\Omega)$ and $\boldsymbol{W}^p_N(\Omega)$ are closed in $\boldsymbol{W}^{1,p}(\Omega)$ and so they are reflexive and separable. In addition, for $\boldsymbol{v}\in\boldsymbol{W}^p_T(\Omega)\cup\boldsymbol{W}^p_N(\Omega)$, the following Sobolev type inequality is verified

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^q(\Omega)}\le C_q\,\|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)},\tag{6}$$

where C_q is a positive constant and

$$q=\frac{3p}{3-p}\ \text{ if } 1<p<3,\quad q<\infty\ \text{ if } p=3,\quad q=\infty\ \text{ if } p>3\tag{7}$$

and also the trace result

$$\big\|\boldsymbol{v}_{|\Gamma}\big\|_{\boldsymbol{L}^r(\Gamma)}\le C_r\,\|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)},\tag{8}$$

holds with

$$r=\frac{2p}{3-p}\ \text{ if } 1<p<3,\quad r<\infty\ \text{ if } p=3,\quad r=\infty\ \text{ if } p>3.\tag{9}$$

From now on we denote by $\boldsymbol{W}^p(\Omega)$ either the space $\boldsymbol{W}^p_T(\Omega)$ or $\boldsymbol{W}^p_N(\Omega)$.

Let $\boldsymbol{a}:\Omega\times\mathbb{R}^3\longrightarrow\mathbb{R}^3$ be a Carathéodory function satisfying the structural conditions (10a), (10b) and (10c) or (10c')

$$\boldsymbol{a}(x,\boldsymbol{u})\cdot\boldsymbol{u}\ge a_*|\boldsymbol{u}|^p,\tag{10a}$$

$$|\boldsymbol{a}(x,\boldsymbol{u})|\le a^*|\boldsymbol{u}|^{p-1},\tag{10b}$$

$$\big(\boldsymbol{a}(x,\boldsymbol{u})-\boldsymbol{a}(x,\boldsymbol{v})\big)\cdot(\boldsymbol{u}-\boldsymbol{v})>0,\ \text{ if } \boldsymbol{u}\neq\boldsymbol{v},\tag{10c}$$

$$\big(\boldsymbol{a}(x,\boldsymbol{u})-\boldsymbol{a}(x,\boldsymbol{v})\big)\cdot(\boldsymbol{u}-\boldsymbol{v})\ge\begin{cases}a_*|\boldsymbol{u}-\boldsymbol{v}|^p & \text{if } p\ge2,\\ a_*\big(|\boldsymbol{u}|+|\boldsymbol{v}|\big)^{p-2}|\boldsymbol{u}-\boldsymbol{v}|^2 & \text{if } p<2,\end{cases}\tag{10c'}$$

for given constants $0 < a_* < a^*$, for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$ and a.e. $x \in \Omega$.

Given $\varphi \in L^\infty(\Omega)$, $\varphi \geq 0$, let

$$\mathbb{K}_\varphi = \{\boldsymbol{v} \in \boldsymbol{W}^p(\Omega) : \left|\nabla \times \boldsymbol{v}\right| \leq \varphi \text{ a.e. in } \Omega\}.$$

For q and r defined by (7) and (9), respectively, let

$$\boldsymbol{f} \in \boldsymbol{L}^{q'}(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \boldsymbol{L}^{r'}(\Gamma) \tag{11}$$

and consider the following problem: to find $\boldsymbol{h} \in \mathbb{K}_\varphi$ such that

$$\int_\Omega \boldsymbol{a}(x, \nabla \times \boldsymbol{h}) \cdot \nabla \times (\boldsymbol{v} - \boldsymbol{h}) \geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{h}) + \int_\Gamma \boldsymbol{g} \cdot (\boldsymbol{v} - \boldsymbol{h}), \qquad \forall \boldsymbol{v} \in \mathbb{K}_\varphi. \tag{12}$$

Note that according to whether $\boldsymbol{W}^p(\Omega)$ is $\boldsymbol{W}^p_T(\Omega)$ or $\boldsymbol{W}^p_N(\Omega)$, the boundary condition is (4) or (5), respectively.

Proposition 2.1 *Let $\varphi \in L^\infty(\Omega)$, $\varphi \geq 0$, $\boldsymbol{f}$ and $\boldsymbol{g}$ verifying (11). If $\boldsymbol{a}$ satisfies assumptions (10a,10b,10c), problem (12) has a unique solution.*

Proof The operator $A : \boldsymbol{W}^p(\Omega) \longrightarrow \boldsymbol{W}^p(\Omega)'$ defined by

$$\langle A\boldsymbol{h}, \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{a}(x, \nabla \times \boldsymbol{h}) \cdot \nabla \times \boldsymbol{v},$$

is bounded, hemicontinuous, monotone and coercive, since $\|\nabla \times \cdot\|_{\boldsymbol{L}^p(\Omega)}$ is a norm, equivalent to the norm of $\boldsymbol{W}^p(\Omega)$.

The linear form $L : \boldsymbol{W}^p(\Omega) \longrightarrow \mathbb{R}$ defined by

$$L(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} + \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{v}$$

is continuous. So the variational inequality (12) has a unique solution (see Theorem 8.2, p247 of [14]). □

The proofs presented from now on follow the steps of [17], where these questions were considered only in the framework $\boldsymbol{W}^p_T(\Omega)$, for simply connected domains and $p > \frac{6}{5}$.

Proposition 2.2 *For $i = 1, 2$, given data $\boldsymbol{f}_i$, $\boldsymbol{g}_i$ verifying (11), $\varphi_i \in L^\infty(\Omega)$ with a positive lower bound and $\boldsymbol{a}$ verifying assumptions (10a,10b,10c'), the solutions $\boldsymbol{h}_i$ of problem (12) satisfy*

$$\|\boldsymbol{h}_1 - \boldsymbol{h}_2\|^{p\vee 2}_{\boldsymbol{W}^p(\Omega)} \leq C\big(\|\boldsymbol{f}_1 - \boldsymbol{f}_2\|^{p'\wedge 2}_{\boldsymbol{L}^{q'}(\Omega)} + \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|^{p'\wedge 2}_{\boldsymbol{L}^{r'}(\Gamma)} + \|\varphi_1 - \varphi_2\|_{L^\infty(\Omega)}\big),$$

where C is a positive constant, $\alpha \vee \beta$ denotes $\max\{\alpha, \beta\}$ and $\alpha \wedge \beta$ denotes $\min\{\alpha, \beta\}$.

Proof Let φ_* be a positive lower bound of φ_1 and φ_2 and denote $\mu = \|\varphi_1 - \varphi_2\|_{L^\infty(\Omega)}$. For $i, j = 1, 2$ and $j \neq i$, given $\boldsymbol{v}_i \in \mathbb{K}_{\varphi_i}$ the function $\widehat{\boldsymbol{v}}_j = \frac{\varphi_*}{\varphi_* + \mu}\boldsymbol{v}_i$ belongs to $\mathbb{K}_{\varphi_j}$ and

$$\|\boldsymbol{v}_i - \widehat{\boldsymbol{v}}_j\|^p_{\boldsymbol{W}^p(\Omega)} = \frac{\mu^p}{(\varphi_* + \mu)^p}\int_\Omega \left|\nabla \times \boldsymbol{v}_i\right|^p \leq C_1\mu^p, \tag{13}$$

where $C_1 = \left(\frac{\|\boldsymbol{v}_i\|_{\boldsymbol{W}^p(\Omega)}}{\varphi_*}\right)^p$.

Also note that, choosing $\boldsymbol{v} = \boldsymbol{0}$ as a test function in (12) and using (6) and (8), we get

$$\|\boldsymbol{h}_i\|_{\boldsymbol{L}^q(\Omega)} \leq C_q\,\|\nabla \times \boldsymbol{h}_i\|_{\boldsymbol{L}^p(\Omega)} \leq a_*^{-\frac{p'}{p}}\left(C_q\|\boldsymbol{f}_i\|_{\boldsymbol{L}^{q'}(\Omega)} + C_r\|\boldsymbol{g}_i\|_{\boldsymbol{L}^{r'}(\Gamma)}\right)^{\frac{p'}{p}}. \tag{14}$$

Using $\widehat{\boldsymbol{h}}_i$ as a test function in problem (12) with data $\boldsymbol{f}_i$, $\boldsymbol{g}_i$ and φ_i, we have

$$\int_\Omega \boldsymbol{a}(x, \nabla \times \boldsymbol{h}_i) \cdot \nabla \times (\widehat{\boldsymbol{h}}_i - \boldsymbol{h}_i) \geq \int_\Omega \boldsymbol{f}_i \cdot (\widehat{\boldsymbol{h}}_i - \boldsymbol{h}_i) + \int_\Gamma \boldsymbol{g}_i \cdot (\widehat{\boldsymbol{h}}_i - \boldsymbol{h}_i)$$

and so

$$\int_\Omega \boldsymbol{a}(x,\nabla\times\boldsymbol{h}_i)\cdot\nabla\times(\boldsymbol{h}_j-\boldsymbol{h}_i)\geq\int_\Omega \boldsymbol{f}_i\cdot(\boldsymbol{h}_j-\boldsymbol{h}_i)+\int_\Gamma \boldsymbol{g}_i\cdot(\boldsymbol{h}_j-\boldsymbol{h}_i)$$
$$+\int_\Omega \boldsymbol{a}(x,\nabla\times\boldsymbol{h}_i)\cdot\nabla\times(\boldsymbol{h}_j-\widehat{\boldsymbol{h}}_i)+\int_\Omega \boldsymbol{f}_i\cdot(\widehat{\boldsymbol{h}}_i-\boldsymbol{h}_j)+\int_\Gamma \boldsymbol{g}_i\cdot(\widehat{\boldsymbol{h}}_i-\boldsymbol{h}_j).$$

Then

$$\int_\Omega \big(\boldsymbol{a}(x,\nabla\times\boldsymbol{h}_1)-\boldsymbol{a}(x,\nabla\times\boldsymbol{h}_2)\big)\cdot\nabla\times(\boldsymbol{h}_1-\boldsymbol{h}_2)\leq\int_\Omega(\boldsymbol{f}_1-\boldsymbol{f}_2)\cdot(\boldsymbol{h}_1-\boldsymbol{h}_2)$$
$$+\int_\Gamma(\boldsymbol{g}_1-\boldsymbol{g}_2)\cdot(\boldsymbol{h}_1-\boldsymbol{h}_2)+\Theta, \quad (15)$$

where

$$\Theta=a^*\int_\Omega|\nabla\times\boldsymbol{h}_1|^{p-1}|\nabla\times(\widehat{\boldsymbol{h}}_1-\boldsymbol{h}_2)|+\int_\Omega \boldsymbol{f}_1\cdot(\boldsymbol{h}_2-\widehat{\boldsymbol{h}}_1)+\int_\Gamma \boldsymbol{g}_1\cdot(\boldsymbol{h}_2-\widehat{\boldsymbol{h}}_1)$$
$$+a^*\int_\Omega|\nabla\times\boldsymbol{h}_2|^{p-1}|\nabla\times(\widehat{\boldsymbol{h}}_2-\boldsymbol{h}_1)|+\int_\Omega \boldsymbol{f}_2\cdot(\boldsymbol{h}_1-\widehat{\boldsymbol{h}}_2)+\int_\Gamma \boldsymbol{g}_2\cdot(\boldsymbol{h}_1-\widehat{\boldsymbol{h}}_2).$$

Notice that, using the Hölder inequality, (6) and (8), as well as (14) and (13),

$$\Theta\leq\Big(a^*\|\nabla\times\boldsymbol{h}_1\|^{p-1}_{\boldsymbol{L}^p(\Omega)}+C_q\|\boldsymbol{f}_1\|_{L^{q'}(\Omega)}+C_r\|\boldsymbol{g}_1\|_{L^{r'}(\Gamma)}\Big)\,\|\nabla\times(\widehat{\boldsymbol{h}}_1-\boldsymbol{h}_2)\|_{\boldsymbol{L}^p(\Omega)}$$
$$+\Big(a^*\|\nabla\times\boldsymbol{h}_2\|^{p-1}_{\boldsymbol{L}^p(\Omega)}+C_q\|\boldsymbol{f}_2\|_{L^{q'}(\Omega)}+C_r\|\boldsymbol{g}_2\|_{L^{r'}(\Gamma)}\Big)\,\|\nabla\times(\widehat{\boldsymbol{h}}_2-\boldsymbol{h}_1)\|_{\boldsymbol{L}^p(\Omega)}$$
$$\leq D\,\|\varphi_1-\varphi_2\|_{L^\infty(\Omega)}$$

and

$$\int_\Omega(\boldsymbol{f}_1-\boldsymbol{f}_2)\cdot(\boldsymbol{h}_1-\boldsymbol{h}_2)+\int_\Gamma(\boldsymbol{g}_1-\boldsymbol{g}_2)\cdot(\boldsymbol{h}_1-\boldsymbol{h}_2)$$
$$\leq\Big(C_q\|\boldsymbol{f}_1-\boldsymbol{f}_2\|_{\boldsymbol{L}^{q'}(\Omega)}+C_r\|\boldsymbol{g}_1-\boldsymbol{g}_2\|_{\boldsymbol{L}^{r'}(\Gamma)}\Big)\,\|\nabla\times(\boldsymbol{h}_1-\boldsymbol{h}_2)\|_{\boldsymbol{L}^p(\Omega)}.$$

Going back to (15), applying (10c') and the previous inequalities, we can find, in the case $p\geq 2$, a positive constant D_1 such that

$$\int_\Omega|\nabla\times(\boldsymbol{h}_1-\boldsymbol{h}_2)|^p\leq D_1\Big(\|\boldsymbol{f}_1-\boldsymbol{f}_2\|^{p'}_{\boldsymbol{L}^{q'}(\Omega)}+\|\boldsymbol{g}_1-\boldsymbol{g}_2\|^{p'}_{\boldsymbol{L}^{r'}(\Gamma)}+\|\varphi_1-\varphi_2\|_{L^\infty(\Omega)}\Big)$$

and, in the case $1<p<2$,

$$a_*\int_\Omega\big(|\nabla\times\boldsymbol{h}_1|^{p-2}+|\nabla\times\boldsymbol{h}_2|^{p-2}\big)|\nabla\times(\boldsymbol{h}_1-\boldsymbol{h}_2)|^2$$
$$\leq\Big(C_q\|\boldsymbol{f}_1-\boldsymbol{f}_2\|_{\boldsymbol{L}^{q'}(\Omega)}+C_r\|\boldsymbol{g}_1-\boldsymbol{g}_2\|_{\boldsymbol{L}^{r'}(\Gamma)}\Big)\,\|\nabla\times(\boldsymbol{h}_1-\boldsymbol{h}_2)\|_{\boldsymbol{L}^p(\Omega)}+D\,\|\varphi_1-\varphi_2\|_{L^\infty(\Omega)}.$$

Applying, in the last case, the reverse Hölder inequality with $s=\frac{p}{2}$ and $s'=\frac{p}{p-2}$, we obtain

$$\int_\Omega\big(|\nabla\times\boldsymbol{h}_1|+|\nabla\times\boldsymbol{h}_2|\big)^{p-2}\big|\nabla\times(\boldsymbol{h}_1-\boldsymbol{h}_2)\big|^2\geq\Big(\int_\Omega\big(|\nabla\times\boldsymbol{h}_1|+|\nabla\times\boldsymbol{h}_2|\big)^p\Big)^{\frac{p-2}{p}}\,\|\nabla\times(\boldsymbol{h}_1-\boldsymbol{h}_2)\|^2_{\boldsymbol{L}^p(\Omega)}.$$

By inequality (14)

$$\Big(\int_\Omega\big(|\nabla\times\boldsymbol{h}_1|+|\nabla\times\boldsymbol{h}_2|\big)^p\Big)^{\frac{2-p}{p}}\leq D_2\big(\|\nabla\times\boldsymbol{h}_1\|^p_{\boldsymbol{L}^p(\Omega)}+\|\nabla\times\boldsymbol{h}_2\|^p_{\boldsymbol{L}^p(\Omega)}\big)^{\frac{2-p}{p}}\leq D_3,$$

where D_2 and D_3 are positive constants. Finally we get

$$\|\nabla\times(\boldsymbol{h}_1-\boldsymbol{h}_2)\|^2_{\boldsymbol{L}^p(\Omega)}\leq D_4\big(\|\boldsymbol{f}_1-\boldsymbol{f}_2\|^2_{\boldsymbol{L}^{q'}(\Omega)}+\|\boldsymbol{g}_1-\boldsymbol{g}_2\|^2_{\boldsymbol{L}^{r'}(\Gamma)}+\|\varphi_1-\varphi_2\|_{L^\infty(\Omega)}\big),$$

for a positive constant D_4. □

Consider a function $F:\mathbb{R}\longrightarrow\mathbb{R}^+$ and define the quasivariational inequality: to find $\boldsymbol{h}\in\mathbb{K}_{F(|\boldsymbol{h}|)}$ such that

$$\int_\Omega \boldsymbol{a}(x,\nabla\times\boldsymbol{h})\cdot\nabla\times(\boldsymbol{v}-\boldsymbol{h})\geq\int_\Omega\boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{h})+\int_\Gamma\boldsymbol{g}\cdot(\boldsymbol{v}-\boldsymbol{h}),\qquad\forall\boldsymbol{v}\in\mathbb{K}_{F(|\boldsymbol{h}|)}.\tag{16}$$

Theorem 2.3 *Let $\boldsymbol{f}$ and $\boldsymbol{g}$ verify (11) and assume that F is continuous and $\boldsymbol{a}$ satisfies (10a,10b,10c'). Suppose, in addition, that if $1<p\leq 3$, there exist positive constants c_0 and c_1 such that*

$$F(s)\leq c_0+c_1|s|^\alpha,\quad\forall\, s\in\mathbb{R},\tag{17}$$

where $\alpha\geq 0$ if $p=3$ and $0\leq\alpha<\frac{p}{3-p}$ if $1<p<3$.

Then the quasivariational inequality (16) has a solution.

Proof The proof of this theorem follows ideas of [13]. Consider first the case $p>3$. Given $\varphi\in\mathscr{C}(\bar\Omega)$ we denote by $\boldsymbol{h}_\varphi$ the solution of the variational inequality (12) with $\mathbb{K}_{F(\varphi)}$ replacing $\mathbb{K}_\varphi$. As the space $\boldsymbol{W}^p(\Omega)$ is a closed subspace of $\boldsymbol{W}^{1,p}(\Omega)$, by the Sobolev embedding theorem, the inclusion $i:\boldsymbol{W}^p(\Omega)\longrightarrow\boldsymbol{\mathscr{C}}(\bar\Omega)$ is continuous and compact. The continuity of the operator $T:\mathscr{C}(\bar\Omega)\longrightarrow\boldsymbol{W}^p(\Omega)$, such that $T(\varphi)=\boldsymbol{h}_\varphi$, is a consequence of the previous proposition. So, the operator $S:\mathscr{C}(\bar\Omega)\longrightarrow\mathscr{C}(\bar\Omega)$ defined by $S(\varphi)=|i(T(\varphi))|$ is continuous and compact.

From (14) we have, for $1<p<\infty$,

$$\|\boldsymbol{h}_\varphi\|_{\boldsymbol{W}^p(\Omega)}\leq {a_*}^{-\frac{2}{p}}\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{q'}(\Omega)}+\|\boldsymbol{g}\|_{\boldsymbol{L}^{r'}(\Omega)}\big)^{\frac{p'}{p}}.\tag{18}$$

On the other hand, there exists $C_1>0$ such that, for any $\boldsymbol{v}\in\boldsymbol{W}^p(\Omega)$, $\|\,|\boldsymbol{v}|\,\|_{\mathscr{C}(\bar\Omega)}\leq C_1\|\boldsymbol{v}\|_{\boldsymbol{W}^p(\Omega)}$, and then

$$\|\,|\boldsymbol{h}_\varphi|\,\|_{\mathscr{C}(\bar\Omega)}\leq C_1\,{a_*}^{-\frac{2}{p}}\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{q'}(\Omega)}+\|\boldsymbol{g}\|_{\boldsymbol{L}^{r'}(\Omega)}\big)^{\frac{p'}{p}}=R.$$

Denoting the disc with center in the origin and radius R, in $\mathscr{C}(\bar\Omega)$, by $D_R(0)$, we have $S(D_R(0))\subseteq D_R(0)$ and we may apply the Schauder fixed point theorem concluding the existence of a fixed point for S. The image by T of this fixed point solves the quasivariational inequality (16).

Consider now the case $1<p\leq 3$. To prove that T is continuous let $\varphi\in\mathscr{C}(\bar\Omega)$ and $M>0$ be such that $\|F\circ\psi\|_{\mathscr{C}(\bar\Omega)}\leq\|F\circ\varphi\|_{\mathscr{C}(\bar\Omega)}+1$ if $\|\varphi-\psi\|_{\mathscr{C}(\bar\Omega)}\leq M$. For those ψ and $s>3$ we have,

$$\begin{aligned}\|\boldsymbol{h}_\varphi-\boldsymbol{h}_\psi\|^s_{\boldsymbol{W}^s(\Omega)}&=\int_\Omega|\nabla\times(\boldsymbol{h}_\varphi-\boldsymbol{h}_\psi)|^{s-p}|\nabla\times(\boldsymbol{h}_\varphi-\boldsymbol{h}_\psi)|^p\\&\leq\int_\Omega\big(F(\varphi)+F(\psi)\big)^{s-p}|\nabla\times(\boldsymbol{h}_\varphi-\boldsymbol{h}_\psi)|^p\leq\big(2\,\|F\circ\varphi\|_{\mathscr{C}(\bar\Omega)}+1\big)^{s-p}\|\boldsymbol{h}_\varphi-\boldsymbol{h}_\psi\|^p_{\boldsymbol{W}^p(\Omega)}\end{aligned}$$

which, by Proposition 2.2, proves the continuity of T.

The function $S=i\circ T$ with the codomain of T replaced by $\boldsymbol{W}^s(\Omega)$ is continuous, by Proposition 2.2, and compact.

After showing that $\mathcal{A}=\big\{\varphi\in\mathscr{C}(\bar\Omega):\ \varphi=\lambda T(\varphi)\text{ for some }\lambda\in[0,1]\big\}$ is bounded, the Leray-Schauder fixed point theorem gives us the desired result.

For $\varphi\in\mathcal{A}$, there exists $\lambda\in[0,1]$ such that $\varphi=\lambda T(\varphi)=\lambda|\boldsymbol{h}_\varphi|$. Using assumption (17),

$$\begin{aligned}\|\varphi\|^s_{\mathscr{C}(\bar\Omega)}=\lambda^s\|\,|\boldsymbol{h}_\varphi|\,\|^s_{\mathscr{C}(\bar\Omega)}\leq c\,\|\boldsymbol{h}_\varphi\|^s_{\boldsymbol{W}^s(\Omega)}=c\int_\Omega|\nabla\times\boldsymbol{h}_\varphi|^s\\\leq c\int_\Omega|F(\varphi)|^s\leq c\int_\Omega\big(c_0+c_1\varphi^\alpha\big)^s=\tilde c_0+\tilde c_1\lambda^{s\alpha}\int_\Omega|\boldsymbol{h}_\varphi|^{s\alpha}\leq\tilde c_0+c_2\|\boldsymbol{h}_\varphi\|^{s\alpha}_{\boldsymbol{W}^p(\Omega)},\end{aligned}$$

by the Sobolev inclusion $\boldsymbol{W}^{1,p}(\Omega)\subset\boldsymbol{L}^{s\alpha}(\Omega)$, choosing any $s>3$ if $p=3$ and $s=\frac{3p}{3-p}\frac{1}{\alpha}$ if $p<3$. The boundedness of the set $\mathcal{A}$ follows then directly from the inequality (18). □

3 The problem with gradient constraint

Let Ω be a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary Γ. In this section we study variational and quasivariational inequalities, defined by an operator $\boldsymbol{a} = \boldsymbol{a}(x, \nabla u) : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$, satisfying structural assumptions of p-laplacian type, defined in (10), with 3 replaced by N, in a convex set of functions with a variable gradient constraint. Non-homogeneous Neumann or Dirichlet boundary condition will be considered.

Given $v \in W^{1,p}(\Omega)$, we consider the following well-known Sobolev inequality

$$\|v\|_{L^q(\Omega)} \leq C_q \|v\|_{W^{1,p}(\Omega)},$$

where C_q is a positive constant and

$$q = \frac{Np}{N-p} \text{ if } 1 < p < N, \qquad q < \infty \text{ if } p = N, \qquad q = \infty \text{ if } p > N \tag{19}$$

and also the trace result

$$\|v_{|\Gamma}\|_{L^r(\Gamma)} \leq C_r \, \|v\|_{W^{1,p}(\Omega)} \,,$$

$$r = \frac{(N-1)p}{N-p} \text{ if } 1 < p < N, \qquad r < \infty \text{ if } p = N, \qquad r = \infty \text{ for } p > N. \tag{20}$$

3.1 The Neumann boundary condition case

For q and r defined in (19) and (20) respectively, let

$$f \in L^{q'}(\Omega), \ g \in L^{r'}(\Gamma) \text{ and } c \in L^\infty(\Omega), \ c \geq c_* > 0. \tag{21}$$

Given $\varphi \in L^\infty(\Omega)$, $\varphi \geq 0$, we define the closed convex subset of $W^{1,p}(\Omega)$,

$$K_\varphi = \big\{ v \in W^{1,p}(\Omega) : |\nabla v| \leq \varphi \text{ a.e. in } \Omega \big\}$$

and we consider the variational inequality: to find $u \in K_\varphi$ such that

$$\int_\Omega \boldsymbol{a}(x, \nabla u) \cdot \nabla(v-u) + \int_\Omega c\,|u|^{p-2} u(v-u) \geq \int_\Omega f(v-u) + \int_\Gamma g(v-u), \qquad \forall v \in K_\varphi. \tag{22}$$

Proposition 3.1 *Let $\varphi \in L^\infty(\Omega)$, $\varphi \geq 0$ and assume that f, g and c verify (21). If $\boldsymbol{a}$ satisfies assumptions (10a,10b,10c) then problem (22) has a unique solution.*

Proof We remark that the operator $A : W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)'$ defined by

$$\langle Au, v \rangle = \int_\Omega \boldsymbol{a}(x, \nabla u) \cdot \nabla v + \int_\Omega c\,|u|^{p-2} uv,$$

is bounded, monotone, hemicontinuous and coercive. Thus the result is a direct consequence of Theorem 8.2, p247 of [14]. □

We present now a continuous dependence result.

Proposition 3.2 *For $i = 1, 2$, given data f_i, g_i, c_i and φ_i satisfying the assumptions of Proposition 3.1, φ_i with positive lower bound and $\boldsymbol{a}$ verifying (10a,10b,10c'), the solutions u_i of problem (22) satisfy*

$$\|u_1 - u_2\|^{p \vee 2}_{W^{1,p}(\Omega)} \leq C\big(\|f_1 - f_2\|^{p' \wedge 2}_{L^{q'}(\Omega)} + \|g_1 - g_2\|^{p' \wedge 2}_{L^{r'}(\Gamma)} + \|c_1 - c_2\|_{L^\infty(\Omega)} + \|\varphi_1 - \varphi_2\|_{L^\infty(\Omega)} \big),$$

where C is a positive constant.

Proof Defining $\widehat{u}_i$ as in the proof of Proposition 2.2 and using it as a test function in problem (22) with data f_i, g_i, c_i and φ_i, by simple calculations we have

$$\int_\Omega \big(\boldsymbol{a}(x,\nabla u_1)-\boldsymbol{a}(x,\nabla u_2)\big)\cdot\nabla(u_1-u_2)+\int_\Omega c_1\left(|u_1|^{p-2}u_1-|u_2|^{p-2}u_2\right)(u_1-u_2)$$
$$\leq \int_\Omega (f_1-f_2)(u_1-u_2)+\int_\Gamma (g_1-g_2)(u_1-u_2)+\int_\Omega (c_2-c_1)|u_2|^{p-2}u_2(u_1-u_2)+\Theta,$$

where

$$\Theta\leq a^*\int_\Omega|\nabla u_1|^{p-1}|\nabla(\widehat{u}_1-u_2)|+\int_\Omega f_1(u_2-\widehat{u}_1)+\int_\Gamma g_1(u_2-\widehat{u}_1)+\|c_1\|_{L^\infty(\Omega)}\int_\Omega|u_1|^{p-1}\,|\widehat{u}_1-u_2|$$
$$+\,a^*\int_\Omega|\nabla u_2|^{p-1}|\nabla(\widehat{u}_2-u_1)|+\int_\Omega f_2(u_1-\widehat{u}_2)+\int_\Gamma g_2(u_1-\widehat{u}_2)+\|c_2\|_{L^\infty(\Omega)}\int_\Omega|u_2|^{p-1}\,|\widehat{u}_2-u_1|\,.$$

Using the Hölder inequality, we have

$$\int_\Omega (c_1-c_2)|u_2|^{p-2}u_2(u_1-u_2)\leq\|c_1-c_2\|_{L^\infty(\Omega)}\|u_2\|^{p-1}_{L^p(\Omega)}\|u_1-u_2\|_{L^p(\Omega)}.$$

For a positive lower bound c_* of c_i, using $v=0$ as a test function in (22) we obtain

$$(a_*\wedge c_*)\|u_i\|^{p-1}_{W^{1,p}(\Omega)}\leq C_q\|f_i\|_{L^{q'}(\Omega)}+C_r\|g_i\|_{L^{r'}(\Gamma)}. \tag{23}$$

The operator $b(u)=|u|^{p-2}u$ satisfies the structural condition (10c') with a_* replaced by $b_*>0$.
For $p\geq 2$,

$$\int_\Omega \big(\boldsymbol{a}(x,\nabla u_1)-\boldsymbol{a}(x,\nabla u_2)\big)\cdot\nabla(u_1-u_2)+\int_\Omega c_1\big(|u_1|^{p-2}u_1-|u_2|^{p-2}u_2\big)(u_1-u_2)$$
$$\geq (a_*\wedge b_*)\|u_1-u_2\|^p_{W^{1,p}(\Omega)}$$

and, if $1<p<2$,

$$\int_\Omega \big(\boldsymbol{a}(x,\nabla u_1)-\boldsymbol{a}(x,\nabla u_2)\big)\cdot\nabla(u_1-u_2)+\int_\Omega c_1\big(|u_1|^{p-2}u_1-|u_2|^{p-2}u_2\big)(u_1-u_2)$$
$$\geq a_*\int_\Omega(|\nabla u_1|+|\nabla u_2|)^{p-2}|\nabla(u_1-u_2)|^2+b_*\int_\Omega(|u_1|+|u_2|)^{p-2}|u_1-u_2|^2$$

Applying the reverse Hölder inequality to both terms of the right-hand side we obtain

$$\int_\Omega\big(|\nabla u_1|+|\nabla u_2|\big)^{p-2}|\nabla(u_1-u_2)|^2\geq\left(\int_\Omega\big(|\nabla u_1|+|\nabla u_2|\big)^p\right)^{\frac{p-2}{p}}\left(\int_\Omega|\nabla(u_1-u_2)|^p\right)^{\frac{2}{p}}$$

and

$$\int_\Omega\big(|u_1|+|u_2|\big)^{p-2}|u_1-u_2|^2\geq\left(\int_\Omega\big(|u_1|+|u_2|\big)^p\right)^{\frac{p-2}{p}}\left(\int_\Omega|u_1-u_2|^p\right)^{\frac{2}{p}}.$$

From the inequality (23), there exists a positive constant D_1 such that

$$\left(\int_\Omega\big(|\nabla u_1|+|\nabla u_2|\big)^p\right)^{\frac{2-p}{p}}\leq D_1\quad\text{and}\quad\left(\int_\Omega\big(|u_1|+|u_2|\big)^p\right)^{\frac{2-p}{p}}\leq D_1.$$

So

$$\int_\Omega \big(\boldsymbol{a}(x,\nabla u_1)-\boldsymbol{a}(x,\nabla u_2)\big)\cdot\nabla(u_1-u_2)+\int_\Omega c_1\big(|u_1|^{p-2}u_1-|u_2|^{p-2}u_2\big)(u_1-u_2)$$
$$\geq\frac{a_*\wedge b_*}{D_1}\left(\left(\int_\Omega|\nabla(u_1-u_2)|^p\right)^{\frac{2}{p}}+\left(\int_\Omega|u_1-u_2|^p\right)^{\frac{2}{p}}\right)\geq D_2\|u_1-u_2\|^2_{W^{1,p}(\Omega)}$$

and the conclusion follows as in Proposition 2.2. □

Consider a function $F:\mathbb{R}\to\mathbb{R}^+$ and the quasivariational inequality: to find $u\in K_{F(u)}$ such that

$$\int_\Omega \boldsymbol{a}(x,\nabla u)\cdot\nabla(v-u)+\int_\Omega c\,|u|^{p-2}u(v-u)\geq\int_\Omega f(v-u)+\int_\Gamma g(v-u),\qquad \forall v\in K_{F(u)}. \tag{24}$$

Theorem 3.3 *Assume that f, g, c verify (21), F is continuous and $\boldsymbol{a}$ satisfies assumptions (10a,10b,10c'). If $p\leq N$ suppose, in addition, that there exist positive constants c_0 and c_1 such that*

$$F(s)\leq c_0+c_1|s|^\alpha,\qquad \forall s\in\mathbb{R}, \tag{25}$$

being $\alpha\geq 0$ if $p=N$ and $0\leq\alpha<\frac{p}{N-p}$ if $p<N$.

Then the quasivariational inequality (24) has a solution.

Proof The case $p>N$ is treated as in the proof of Theorem 2.3, with 3 replaced by N.

If $p\leq N$, let $k\in\mathbb{N}$ be such that $\frac{N}{k+1}<p\leq\frac{N}{k}$ and consider $(p_m)_{0\leq m\leq k}$, iterations of the critical Sobolev exponent, as follows

$$p_0=p,\;\; N<p_k<\infty \text{ if } p=\frac{N}{k},\;\; p_m=\frac{Np_{m-1}}{N-p_{m-1}}\;\; \text{otherwise.}$$

For convenience, if $p=N$ and $\alpha>1$ we choose $p_1=\alpha N$.

Note that $p_m=\frac{Np}{N-mp}$ if $m<k$ or if $m=k$ and $p<\frac{N}{k}$. In particular $p_m>N$ if and only if $m=k$. Applying repeatedly the Sobolev inequality we have,

$$\exists C>0\;\forall m\leq k\;\forall u\in W^{1,p_m}(\Omega)\quad \|u\|_{W^{1,p_m}(\Omega)}\leq C\left(\|u\|_{L^p(\Omega)}+\|\nabla u\|_{L^{p_m}(\Omega)}\right). \tag{26}$$

Let $s=p_k$ if $\alpha\leq 1$ and $s=\frac{p_k}{\alpha}$ if $\alpha>1$ and note that $N<s<p_k$ and $\alpha s\leq p_k$. Observe that, if $\varphi\in\mathscr{C}(\bar\Omega)$ and $u\in K_{F(\varphi)}$ then $u\in W^{1,s}(\bar\Omega)$, as $\nabla u\in L^\infty(\Omega)$. In particular, the operator $T:\mathscr{C}(\bar\Omega)\longrightarrow W^{1,s}(\Omega)$ such that $T(\varphi)=u_\varphi$, where u_φ is the solution of problem (22) with $K_{F(\varphi)}$ replacing K_φ, is well-defined.

To prove that T is continuous consider, as in the proof of Theorem 2.3, $\varphi\in\mathscr{C}(\bar\Omega)$ and $M>0$ such that $\|F\circ\psi\|_{\mathscr{C}(\bar\Omega)}\leq\|F\circ\varphi\|_{\mathscr{C}(\bar\Omega)}+1$ if $\|\varphi-\psi\|_{\mathscr{C}(\bar\Omega)}\leq M$. For those ψ we have,

$$\begin{aligned}\|u_\varphi-u_\psi\|_{W^{1,s}(\Omega)}&\leq C\|u_\varphi-u_\psi\|_{W^{1,p_k}(\Omega)}\leq C_1\left(\|u_\varphi-u_\psi\|_{L^p(\Omega)}+\|\nabla u_\varphi-\nabla u_\psi\|_{L^{p_k}(\Omega)}\right)\\&\leq C_2\left(\|u_\varphi-u_\psi\|_{L^p(\Omega)}+\left(2\|F\circ\varphi\|_{\mathscr{C}(\bar\Omega)}+1\right)^{\frac{p_k-p}{p_k}}\|\nabla u_\varphi-\nabla u_\psi\|_{L^p(\Omega)}^{\frac{p}{p_k}}\right)\\&\leq C_2\left(\|u_\varphi-u_\psi\|_{W^{1,p}(\Omega)}+\left(2\|F\circ\varphi\|_{\mathscr{C}(\bar\Omega)}+1\right)^{\frac{p_k-p}{p_k}}\|u_\varphi-u_\psi\|_{W^{1,p}(\Omega)}^{\frac{p}{p_k}}\right)\end{aligned}$$

and then, using the previous proposition, T is continuous.

In order to apply the Leray-Schauder fixed point theorem we consider $S=i\circ T:\mathscr{C}(\bar\Omega)\longrightarrow\mathscr{C}(\bar\Omega)$, where i is the compact inclusion of $W^{1,s}(\Omega)$ in $\mathscr{C}(\bar\Omega)$, and we are going to prove the boundedness of the set $\mathcal{A}=\left\{\varphi\in\mathscr{C}(\bar\Omega):\,\varphi=\lambda S(\varphi)\text{ for some }\lambda\in[0,1]\right\}$. As i is compact it is enough to prove that $\mathcal{A}$ is bounded in $W^{1,s}(\Omega)$. Note that, as in Theorem 2.3, we can obtain an inequality similar to (18), proving that $\mathcal{A}$ is bounded in $W^{1,p}(\Omega)$.

As, for $\varphi\in\mathscr{C}(\bar\Omega)$, $|\nabla u_\varphi|\leq F(\varphi)\leq c_0+c_1|\varphi|^\alpha=c_0+c_1\lambda^\alpha|u_\varphi|^\alpha$ then, for $r>p$, there exist $A_r,B_r>0$ such that

$$\|\nabla u_\varphi\|_{L^r(\Omega)}\leq A_r+B_r\|u_\varphi\|^\alpha_{L^{\alpha r}(\Omega)}. \tag{27}$$

If $\alpha>1$, then $\frac{p}{N-p}>1$ and therefore $k=1$. We apply (27) with $r=s$, the inclusions $L^{p_1}(\Omega)\subseteq L^s(\Omega)$ and $W^{1,p}(\Omega)\subseteq L^{p_1}(\Omega)$ to prove that there exists $A,B>0$ such that

$$\|u_\varphi\|_{W^{1,s}(\Omega)}\leq A+B\|u_\varphi\|_{W^{1,p}(\Omega)},$$

showing the boundedness of $\mathcal{A}$ in $W^{1,s}(\Omega)$.

If $\alpha\leq 1$, using (26), (27) for $r=p_m$, the inclusions $L^{p_m}(\Omega)\subseteq L^{\alpha p_m}(\Omega)$ and $W^{1,p_{m-1}}(\Omega)\subseteq L^{p_m}(\Omega)$ for $1\leq m\leq k$, there exists $\tilde A,\tilde B>0$ such that

$$\|u_\varphi\|_{W^{1,p_m}(\Omega)}\leq C\big(\|u\|_{W^{1,p}(\Omega)}+\tilde A+\tilde B\|u_\varphi\|^\alpha_{W^{1,p_{m-1}}(\Omega)}\big),$$

which shows that $\mathcal{A}$ is bounded in $W^{1,p_m}(\Omega)$ if it is bounded in $W^{1,p_{m-1}}(\Omega)$. So, by an iterative process, the conclusion follows. □

3.2 The Dirichlet boundary condition case

We define, for $\varphi \in L^\infty(\Omega)$, $\varphi \geq 0$, and $g \in W^{\frac{1}{p'},p}(\Gamma)$, the closed convex subset of $W^{1,p}(\Omega)$,

$$\mathcal{K}_\varphi = \big\{ v \in W^{1,p}(\Omega) : |\nabla v| \leq \varphi \text{ a.e. in } \Omega,\ v_{|_\Gamma} = g \big\}.$$

To define a variational inequality in the convex set $\mathcal{K}_\varphi$, we need to guarantee that this set is not empty, imposing a compatibility condition between φ and g (see [15], p116). In fact, if for $x,\, y \in \bar{\Omega}$, we denote

$$L_\varphi(x,y) = \inf \Big\{ \int_0^T \varphi\big(\xi(s)\big)\, ds : T > 0,\ \xi : [0,T] \longrightarrow \Omega \text{ smooth},\ \xi(0) = x,\ \xi(T) = y,\ |\xi'| \leq 1 \Big\},$$

a function g defined on Γ is admissible as trace of a function belonging to $\mathcal{K}_\varphi$ as long as

$$|g(x) - g(y)| \leq L_\varphi(x,y) \qquad \text{for } x, y \in \Gamma. \tag{28}$$

This implies, in particular, that g admits an extension to $\bar{\Omega}$, belonging to $W^{1,\infty}(\Omega)$, which is a solution of the Hamilton-Jacobi equation

$$|\nabla v| = \varphi \text{ in } \Omega, \tag{29a}$$
$$v = g \text{ on } \Gamma. \tag{29b}$$

Given $f \in L^{q'}(\Omega)$, q as in (19), we define the variational inequality that consists on finding $u \in \mathcal{K}_\varphi$ such that

$$\int_\Omega \boldsymbol{a}(x, \nabla u) \cdot \nabla (v - u) \geq \int_\Omega f(v-u), \qquad \forall\, v \in \mathcal{K}_\varphi. \tag{30}$$

Proposition 3.4 *Let $\varphi \in L^\infty(\Omega)$, $\varphi \geq 0$, q as in (19), $f \in L^{q'}(\Omega)$, g defined on Γ verifying (28). If $\boldsymbol{a}$ satisfies assumptions (10a,10b,10c) then problem (30) has a unique solution.*

Proof This result is an immediate consequence of Theorem 8.2, p247 of [14]. □

Given a function $F : \mathbb{R} \to \mathbb{R}^+$ we define the quasivariational inequality: to find $u \in \mathcal{K}_{F(u)}$ such that

$$\int_\Omega \boldsymbol{a}(x, \nabla u) \cdot \nabla (v - u) \geq \int_\Omega f(v-u), \qquad \forall\, v \in \mathcal{K}_{F(u)}. \tag{31}$$

In order to guarantee that the convex set $\mathcal{K}_{F(u)}$ is nonempty, the inequality (28) needs to be satisfied for $\varphi = F(u)$. With this goal we assume that F has a positive lower bound F_* and

$$\exists\, 0 < k < 1 : \ |g(x) - g(y)| \leq k\, L_{F_*}(x,y) \qquad \text{for } x, y \in \Gamma. \tag{32}$$

Theorem 3.5 *Let $f \in L^{q'}(\Omega)$, q as in (19), g defined on Γ verifying (32). Suppose that F is a continuous function such that $F_* = \inf F > 0$ and $\boldsymbol{a}$ satisfies assumptions (10a,10b,10c').*

If $p \leq N$ assume, in addition, that there exist positive constants c_0 and c_1 such that

$$F(s) \leq c_0 + c_1 |s|^\alpha, \qquad \forall s \in \mathbb{R},$$

being $\alpha \geq 0$ if $p = N$ and $0 \leq \alpha \leq \frac{p}{N-p}$ if $p < N$.

Then the quasivariational inequality (31) has a solution.

Proof The proof follows the steps of the proof of Theorem 2.3, using N instead of 3. The main difference consists in the proof of the continuity of the operator $T : \mathscr{C}(\bar{\Omega}) \longrightarrow W^{1,p}(\Omega)$, where $T(\varphi)$ is the solution of problem (30), with $F(\varphi)$ in the place of φ. Let $\varphi \in \mathscr{C}(\bar{\Omega})$ and $(\varphi_n)_n$ a sequence converging in $\mathscr{C}(\bar{\Omega})$ to φ. The convergence of $(T(\varphi_n))_n$ to $T(\varphi)$ is an immediate consequence of a result of [19], if we prove the Mosco convergence of the family of convex sets $\mathcal{K}_{F(\varphi_n)}$ to $\mathcal{K}_{F(\varphi)}$. So, we only need to prove the following two conditions:

$$\forall v \in \mathcal{K}_{F(\varphi)}\ \forall n \in \mathbb{N}\ \ \exists\, v_n \in \mathcal{K}_{F(\varphi_n)} : \quad v_n \underset{n}{\longrightarrow} v \ \text{ in } W^{1,p}(\Omega), \tag{33a}$$

$$\text{if, for all } n \in \mathbb{N}, v_n \in \mathcal{K}_{F(\varphi_n)} \text{ and } \ v_n \underset{n}{\longrightarrow} v \ \text{ in } W^{1,p}(\Omega), \text{ then } v \in \mathcal{K}_{F(\varphi)}. \tag{33b}$$

To prove (33a) consider, for given $v \in \mathcal{K}_{F(\varphi)}$ and $n \in \mathbb{N}$, $G_n = F(\varphi_n) \wedge F(\varphi)$ and $v_n = b_n(v-g)+g$, where

$$b_n = \min_{x\in\bar\Omega} \frac{G_n(x) - kF_*}{F(\varphi(x)) - kF_*}.$$

We observe that, for all $n \in \mathbb{N}$, $0 < b_n \leq 1$ and also $(G_n - kF_*)_n$ converges to $F(\varphi) - kF_*$ in $\mathscr{C}(\bar\Omega)$. Then, as $F(\varphi) - kF_* \geq (1-k)F_* > 0$, we conclude that $\left(\frac{G_n - kF_*}{F(\varphi)-kF_*}\right)_n$ converges to 1 in $\mathscr{C}(\bar\Omega)$ which implies that $b_n \underset{n}{\longrightarrow} 1$.

Using (32) we can define an extension of g, still denoted by g, such that $|\nabla g| = k\,F_*$ (see (29a)). Note that $v_n \in \mathcal{K}_{F(\varphi_n)}$ as $v_{n|_\Gamma} = g$ and

$$\begin{aligned} |\nabla v_n(x)| &= |b_n \nabla v(x) + (1-b_n)\nabla g(x)| \\ &\leq b_n F(\varphi(x)) + (1-b_n)kF_* \\ &\leq G_n(x), \end{aligned}$$

because $b_n \leq \frac{G_n(x)-kF_*}{F(\varphi(x))-kF_*}$.

On the other hand

$$\int_\Omega |\nabla(v_n - v)|^p = (1-b_n)^p \int_\Omega |\nabla(g-v)|^p \underset{n}{\longrightarrow} 0.$$

To prove (33b), let $(v_n)_n$ be a sequence in $\mathcal{K}_{F(\varphi_n)}$, converging weakly in $W^{1,p}(\Omega)$ to v. As $v_{n|_\Gamma} = g$ then $v_{|_\Gamma} = g$. Given any measurable set $\omega \subset \Omega$,

$$\int_\omega |\nabla v| \leq \liminf_n \int_\omega |\nabla v_n| \leq \liminf_n \int_\omega F(\varphi_n) = \int_\omega F(\varphi),$$

so $|\nabla v| \leq F(\varphi)$ a.e. in Ω, which means $v \in \mathcal{K}_{F(\varphi)}$. This concludes the proof of the continuity of T.

In order to follow the steps of the proof of Theorem 2.3, we are going to obtain an *a priori* estimate for $u_\varphi = T(\varphi)$, similar to the estimates (18), obtained for $\boldsymbol{h}_\varphi$.

We choose g as a test function in (30). Then

$$\int_\Omega \boldsymbol{a}(x,\nabla u_\varphi)\cdot\nabla u_\varphi \leq \int_\Omega \boldsymbol{a}(x,\nabla u_\varphi)\cdot\nabla g + \int_\Omega f u_\varphi - \int_\Omega fg$$

and, for $\varepsilon > 0$,

$$\begin{aligned} a_*\|\nabla u_\varphi\|^p_{L^p(\Omega)} \leq a^*\Big(&\frac{\varepsilon^{p'}}{p'}\|\nabla u_\varphi\|^p_{L^p(\Omega)} + \frac{1}{\varepsilon^p p}\|\nabla g\|^p_{L^p(\Omega)}\Big) \\ &+ \frac{1}{\varepsilon^{p'}p'}\|f\|^{p'}_{L^{p'}(\Omega)} + \frac{\varepsilon^p}{p}C\big(\|\nabla u_\varphi\|^p_{L^p(\Omega)} + \|g\|^p_{L^p(\Gamma)}\big) \\ &+ \frac{1}{p'}\|f\|^{p'}_{L^{p'}(\Omega)} + \frac{1}{p}\|g\|^p_{L^p(\Omega)}, \end{aligned}$$

where C is the Poincaré constant.

Choosing ε conveniently and using the continuity of the trace operator in $W^{1,p}(\Omega)$ there exists a positive constant C_1 such that

$$\|\nabla u_\varphi\|^p_{L^p(\Omega)} \leq C_1\big(\|f\|^{p'}_{L^{p'}(\Omega)} + \|g\|^p_{W^{1,p}(\Omega)}\big).$$

Applying again the Poincaré inequality,

$$\|u_\varphi\|^p_{W^{1,p}(\Omega)} \leq C_2\big(\|f\|^{p'}_{L^{p'}(\Omega)} + \|g\|^p_{W^{1,p}(\Omega)}\big),$$

with $C_2 > 0$. □

4 Existence of a Lagrange multiplier in the case of gradient constraint and $p = 2$

Let Ω be a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary Γ. In this section we consider the variational inequality with gradient constraint and homogeneous Dirichlet boundary condition in the special case $p = 2$ and $\boldsymbol{a}(x, \nabla u) = \nabla u$. We prove the equivalence of this problem with a Lagrange multiplier problem, for general source term and for any smooth strictly positive gradient constraint φ.

Given f and φ in appropriate spaces, we consider the problem of finding λ and u such that

$$-\nabla \cdot \big(\lambda \nabla u\big) = f \quad \text{in } \mathscr{D}'(\Omega), \tag{34a}$$
$$u = 0 \quad \text{on } \Gamma, \tag{34b}$$
$$|\nabla u| \leq \varphi \quad \text{in } \Omega, \tag{34c}$$
$$\lambda \geq 1 \quad \text{in } L^\infty(\Omega)', \tag{34d}$$
$$(\lambda - 1)(|\nabla u| - \varphi) = 0 \quad \text{in } L^\infty(\Omega)'. \tag{34e}$$

Concerning equality (34a) we will prove the following slightly stronger weak formulation

$$\langle \lambda, \nabla u \cdot \nabla v \rangle_{L^\infty(\Omega)' \times L^\infty(\Omega)} = \int_\Omega f v, \ \forall v \in W_0^{1,\infty}(\Omega).$$

We intend to show that problem (34) is equivalent to the following variational inequality: to find $u \in \mathcal{K}_\varphi$ such that

$$\int_\Omega \nabla u \cdot \nabla (v - u) \geq \int_\Omega f(v - u), \quad \forall v \in \mathcal{K}_\varphi, \tag{35}$$

where $\mathcal{K}_\varphi = \{v \in H_0^1(\Omega) : |\nabla v| \leq \varphi \text{ a.e. in } \Omega\}$.

The main difficulty of the proof of this result consists on the lack of regularity of the Lagrange multiplier λ. We will prove that $\lambda \in L^\infty(\Omega)'$ and $\nabla u \in L^\infty(\Omega)$, but the approach used, which consists on the approximation of problem (34) by a family of problems using the penalization proposed in [11], already mentioned in the Introduction, does not allow the direct identification of the limit. The identification of the limit $\nabla \cdot (\lambda \nabla u)$ in $\mathscr{D}'(\Omega)$ is the main step to prove (34).

Theorem 4.1 *Given $f \in L^2(\Omega)$ and $\varphi \in W^{2,\infty}(\Omega)$ with a positive lower bound, problem (34) has a solution $(u, \lambda) \in W^{1,\infty}(\Omega) \times L^\infty(\Omega)'$. In addition, if (u, λ) solves (34), then u solves the variational inequality (35).*

To prove this theorem we start by considering a family of approximated problems. Given the data f and φ as above and $0 < \varepsilon < 1$, let us consider the problem of finding u^ε such that

$$-\nabla \cdot \big(k_\varepsilon(|\nabla u^\varepsilon|^2 - \varphi^2)\nabla u^\varepsilon\big) = f_\varepsilon \qquad \text{in } \Omega, \tag{36a}$$
$$u^\varepsilon = 0 \qquad \text{on } \Gamma, \tag{36b}$$

where $k_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$ is a $\mathscr{C}^2$ nondecreasing convex function such that

$$k_\varepsilon(s) = \begin{cases} 1 & \text{if } s \leq 0 \\ e^{\frac{s}{\varepsilon}} & \text{if } s \geq \varepsilon \end{cases}$$

and $f_\varepsilon = f * \rho_\varepsilon$, being ρ_ε a mollifier.

Proposition 4.2 *For $f \in L^2(\Omega)$ and $\varphi \in L^\infty(\Omega)$, with $\varphi > 0$, problem (36) has a unique solution, $u^\varepsilon \in \mathscr{C}^{2,\alpha}(\Omega) \cap \mathscr{C}(\bar{\Omega})$.*

Proof Let $K_\varepsilon(s) = \displaystyle\int_0^s k_\varepsilon(\tau)\, d\tau$. By the assumptions on k_ε, the functional $F_\varepsilon(v) = \displaystyle\int_\Omega \big(\tfrac{1}{2} K_\varepsilon(|\nabla v|^2 - \varphi^2) - fv\big)$ has a minimizer in $H_0^1(\Omega)$, so problem (36) has a solution belonging to $H_0^1(\Omega)$.

The regularity of u_ε is a consequence of a result of Marcellini [16].

To prove the uniqueness, let u_1 and u_2 be two solutions of problem (36). Then

$$\int_\Omega k_\varepsilon(|\nabla u_1|^2-\varphi^2)\nabla u_1\cdot\nabla(u_2-u_1)=\int_\Omega k_\varepsilon(|\nabla u_2|^2-\varphi^2)\nabla u_2\cdot\nabla(u_2-u_1) \tag{37}$$

and so

$$\int_\Omega \big(k_\varepsilon(|\nabla u_1|^2-\varphi^2)|\nabla u_1|-k_\varepsilon(|\nabla u_2|^2-\varphi^2)|\nabla u_2|\big)(|\nabla u_1|-|\nabla u_2|)\le 0.$$

As the function $\Phi(x,t)=k_\varepsilon(t^2-\varphi^2(x))t$ is strictly increasing in the variable t, we conclude that $|\nabla u_1|=|\nabla u_2|$ a.e. in Ω. Finally, going back to (37), we get

$$\int_\Omega \big(k_\varepsilon(|\nabla u_1|^2-\varphi^2)|\nabla(u_1-u_2)|^2=0$$

and so $u_1=u_2$ a.e. in Ω. □

The following lemma gives us some useful estimates.

Lemma 4.3 *For $f\in L^2(\Omega)$ and $\varphi\in L^\infty(\Omega)$ with a positive lower bound and $1\le q<\infty$, there exist positive constants C and C_q such that, for $0<\varepsilon<1$, the solution u^ε of the approximated problem (36) verifies*

$$\|k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla u^\varepsilon|^2\|_{L^1(\Omega)}\le C, \tag{38}$$

$$\|k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\|_{L^1(\Omega)}\le C, \tag{39}$$

$$\|k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\nabla u^\varepsilon\|_{\boldsymbol{L}^\infty(\Omega)'}\le C, \tag{40}$$

$$\|\nabla u^\varepsilon\|_{\boldsymbol{L}^q(\Omega)}\le C_q. \tag{41}$$

Proof Multiplying equation (36a) by u^ε, integrating in Ω and applying Young and Poincaré inequalities, we obtain

$$\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla u^\varepsilon|^2=\int_\Omega f_\varepsilon u^\varepsilon\le C_1\int_\Omega|f_\varepsilon|^2+\tfrac12\int_\Omega|\nabla u^\varepsilon|^2$$

and so

$$\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla u^\varepsilon|^2\le 2C_1\int_\Omega f_\varepsilon^2,$$

proving (38). Observing that, if φ_* is a positive lower bound of φ,

$$\begin{aligned}\varphi_*^2\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)&=\varphi_*^2\int_{\{|\nabla u^\varepsilon|<\varphi\}}k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)+\varphi_*^2\int_{\{|\nabla u^\varepsilon|\ge\varphi\}}k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\\&\le\varphi_*^2\int_{\{|\nabla u^\varepsilon|<\varphi\}}1+\int_{\{|\nabla u^\varepsilon|\ge\varphi\}}k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla u^\varepsilon|^2\\&\le\varphi_*^2|\Omega|+\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla u^\varepsilon|^2\\&\le\varphi_*^2|\Omega|+2C_1\|f_\varepsilon\|^2_{L^2(\Omega)},\end{aligned}$$

obtaining (39) and

$$\begin{aligned}\|k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\nabla u^\varepsilon\|_{\boldsymbol{L}^\infty(\Omega)'}&=\sup_{\|\boldsymbol{v}\|_{\boldsymbol{L}^\infty(\Omega)}\le1}\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\nabla u^\varepsilon\cdot\boldsymbol{v}\\&\le\sup_{\|\boldsymbol{v}\|_{\boldsymbol{L}^\infty(\Omega)}\le1}\|k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla u^\varepsilon|^2\|^{\frac12}_{L^1(\Omega)}\|k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\|^{\frac12}_{L^1(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{L}^\infty(\Omega)}\le C,\end{aligned}$$

which proves (40).

Let us now consider the set $A_\varepsilon=\{x\in\Omega:|\nabla u^\varepsilon(x)|^2>\varphi^2(x)+\varepsilon\}$ and let q be an even integer. Splitting the integral

$$\int_\Omega|\nabla u^\varepsilon|^q=\int_{\Omega\setminus A_\varepsilon}|\nabla u^\varepsilon|^q+\int_{A_\varepsilon}|\nabla u^\varepsilon|^q,$$

we have

$$\int_{\Omega\setminus A_\varepsilon}|\nabla u^\varepsilon|^q \le \int_\Omega(\varphi^2+1)^{\frac{q}{2}} \le |\Omega|(\|\varphi\|^2_{L^\infty(\Omega)}+1)^{\frac{q}{2}}$$

and

$$\int_{A_\varepsilon}|\nabla u^\varepsilon|^q = \int_{A_\varepsilon}(|\nabla u^\varepsilon|^2-\varphi^2+\varphi^2)^{\frac{q}{2}} \le 2^{\frac{q-2}{2}}\left(\int_{A_\varepsilon}(|\nabla u^\varepsilon|^2-\varphi^2)^{\frac{q}{2}}+\int_{A_\varepsilon}\varphi^q\right).$$

When $s\ge\varepsilon$ we have

$$k_\varepsilon(s)=e^{\frac{s}{\varepsilon}} \ge \frac{(\frac{s}{\varepsilon})^{\frac{q}{2}}}{(\frac{q}{2})!}$$

and so

$$s^{\frac{q}{2}} \le \varepsilon^{\frac{q}{2}}(\tfrac{q}{2})!\,k_\varepsilon(s).$$

Taking into account the previous inequality and the definition of A_ε we obtain

$$\int_{A_\varepsilon}|\nabla u^\varepsilon|^q \le 2^{\frac{q-2}{2}}\left(\varepsilon^{\frac{q}{2}}(\tfrac{q}{2})!\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)+\int_\Omega\varphi^q\right)$$

and, using (39), we obtain (41). □

Proposition 4.4 *For $f\in L^2(\Omega)$ and $\varphi\in L^\infty(\Omega)$ with a positive lower bound, the family $(u^\varepsilon)_\varepsilon$ of solutions of the approximated problems (36) converges weakly in $H^1_0(\Omega)$ to the solution of the variational inequality (35).*

Proof As $(u^\varepsilon)_\varepsilon$ is bounded in $H^1_0(\Omega)$ by (41), there exists $u\in H^1_0(\Omega)$ such that, at least for a subsequence, $u^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} u$ in $H^1_0(\Omega)$. We start by proving that u belongs to the convex set $\mathcal{K}_\varphi$.

For $0<\varepsilon<1$ let us consider the set

$$A_\varepsilon=\{x\in\Omega : |\nabla u^\varepsilon(x)|^2>\varphi^2(x)+\sqrt{\varepsilon}\}.$$

The measure of A_ε tends to zero with ε. Indeed, recalling that k_ε is a non decreasing function and taking into account the estimate (39) we have

$$|A_\varepsilon|=\int_{A_\varepsilon}1 \le \int_{A_\varepsilon}\frac{k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)}{e^{\frac{1}{\sqrt{\varepsilon}}}} \le \int_\Omega\frac{k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)}{e^{\frac{1}{\sqrt{\varepsilon}}}} \le Ce^{-\frac{1}{\sqrt{\varepsilon}}}$$

and so $|A_\varepsilon| \underset{\varepsilon\to 0}{\longrightarrow} 0$.

Observing that

$$\begin{aligned}\int_\Omega(|\nabla u|^2-\varphi^2)^+ &\le \liminf_{\varepsilon\to 0}\int_\Omega(|\nabla u^\varepsilon|^2-\varphi^2-\sqrt{\varepsilon})^+\\ &=\liminf_{\varepsilon\to 0}\int_{A_\varepsilon}(|\nabla u^\varepsilon|^2-\varphi^2-\sqrt{\varepsilon})\\ &\le \lim_{\varepsilon\to 0}\Big(\int_\Omega(|\nabla u^\varepsilon|^2-\varphi^2-\sqrt{\varepsilon})^2\Big)^{\frac12}|A_\varepsilon|^{\frac12}\\ &=0 \quad \text{by (41)},\end{aligned}$$

the conclusion follows.

Let us now prove that u solves the variational inequality (35). Multiplying (36a) by $v-u^\varepsilon$, with $v\in\mathcal{K}_\varphi$ and integrating in Ω, we obtain

$$\int_\Omega(k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\nabla u^\varepsilon\cdot\nabla(v-u^\varepsilon)=\int_\Omega f_\varepsilon(v-u^\varepsilon).$$

Observing that $v\in\mathcal{K}_\varphi$ and taking into account the definition and the monotonicity of k_ε we have

$$\begin{aligned}&k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\nabla u^\varepsilon\cdot\nabla(v-u^\varepsilon)\\ &\quad=\big(k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\nabla u^\varepsilon-k_\varepsilon(|\nabla v|^2-\varphi^2)\nabla v\big)\cdot\nabla(v-u^\varepsilon)+k_\varepsilon(|\nabla v|^2-\varphi^2)\nabla v)\cdot\nabla(v-u^\varepsilon)\\ &\qquad\le\nabla v\cdot\nabla(v-u^\varepsilon),\end{aligned}$$

so

$$\int_\Omega \nabla v\cdot\nabla(v-u^\varepsilon)\geq\int_\Omega f_\varepsilon(v-u^\varepsilon)$$

and, letting $\varepsilon\to 0$,

$$\int_\Omega \nabla v\cdot\nabla(v-u)\geq\int_\Omega f(v-u).$$

For $w\in\mathcal{K}_\varphi$, let $v=u+\xi(w-u)$, $0\leq\xi\leq 1$. Then

$$\int_\Omega \nabla\big(u+\xi(w-u)\big)\cdot\nabla(w-u)\geq\int_\Omega f(w-u)$$

and, letting $\xi\to 0$,

$$\int_\Omega \nabla u\cdot\nabla(w-u)\geq\int_\Omega f(w-u).$$

We observe that, as u is the unique solution of the variational inequality (35), the family $(u^\varepsilon)_\varepsilon$ converges weakly to u in $H^1_0(\Omega)$. □

Theorem 4.5 *If $f\in L^2(\Omega)$ and $\varphi\in W^{2,\infty}(\Omega)$ with a positive lower bound, then the solution of the variational inequality (35) belongs to $H^2_{loc}(\Omega)$.*

Proof Let u^ε be the solution of the approximated problem (36). We will prove the uniform boundedness of $(u^\varepsilon)_\varepsilon$ in $H^2_{\mathrm{loc}}(\Omega)$.

Given $\Omega'\subset\subset\Omega$, let $\eta\in\mathscr{D}(\Omega)$ be nonnegative and $\eta_{|\Omega'}=1$.

In this proof we omit, for simplicity, the subscripts and superscripts ε, we denote by u_{x_i} the partial derivative of u with respect to x_i and we adopt the summation convention for repeated indices.

Multiplying equation (36a) by $u_{x_kx_k}\eta^2$, for a fixed $k\in\{1,\dots,N\}$ and integrating in Ω, we obtain

$$\int_\Omega \big(k(|\nabla u|^2-\varphi^2)u_{x_i}\big)_{x_i}u_{x_kx_k}\eta^2=-\int_\Omega f u_{x_kx_k}\eta^2. \tag{42}$$

Integrating by parts we obtain

$$\begin{aligned}\int_\Omega \big(k(|\nabla u|^2-\varphi^2)u_{x_i}\big)_{x_i}u_{x_kx_k}\eta^2 &= -\int_\Omega k(|\nabla u|^2-\varphi^2)u_{x_i}u_{x_kx_kx_i}\eta^2-\int_\Omega k(|\nabla u|^2-\varphi^2)u_{x_i}u_{x_kx_k}(\eta^2)_{x_i}\\ &=\int_\Omega \big(k(|\nabla u|^2-\varphi^2)u_{x_i}\eta^2\big)_{x_k}u_{x_kx_i}-\int_\Omega k(|\nabla u|^2-\varphi^2)u_{x_i}u_{x_kx_k}(\eta^2)_{x_i}\\ &=\int_\Omega \big(k(|\nabla u|^2-\varphi^2)\big)_{x_k}u_{x_i}u_{x_kx_i}\eta^2+\int_\Omega k(|\nabla u|^2-\varphi^2)u^2_{x_ix_k}\eta^2\\ &\quad+\int_\Omega k(|\nabla u|^2-\varphi^2)u_{x_i}u_{x_kx_i}(\eta^2)_{x_k}-\int_\Omega k(|\nabla u|^2-\varphi^2)u_{x_i}u_{x_kx_k}(\eta^2)_{x_i}.\end{aligned}$$

Returning to (42) we get

$$\begin{aligned}\int_\Omega k(|\nabla u|^2-\varphi^2)u^2_{x_ix_k}\eta^2 &= -\int_\Omega f u_{x_kx_k}\eta^2-\frac{1}{2}\int_\Omega \big(k(|\nabla u|^2-\varphi^2)\big)_{x_k}(|\nabla u|^2)_{x_k}\eta^2\\ &\quad-2\int_\Omega (k(|\nabla u|^2-\varphi^2)u_{x_i}u_{x_kx_i}\eta\eta_{x_k}+2\int_\Omega k(|\nabla u|^2-\varphi^2)u_{x_i}u_{x_kx_k}\eta\eta_{x_i}.\end{aligned}$$

Applying the Young inequality we obtain, for $\delta>0$,

$$\begin{aligned}\int_\Omega k(|\nabla u|^2-\varphi^2)u^2_{x_ix_k}\eta^2 &\leq \frac{1}{2\delta}\int_\Omega f^2\eta^2+\frac{\delta}{2}\int_\Omega u^2_{x_kx_k}\eta^2\\ &\quad-\frac{1}{2}\int_\Omega \big(k(|\nabla u|^2-\varphi^2)\big)_{x_k}(|\nabla u|^2-\varphi^2)_{x_k}\eta^2-\frac{1}{2}\int_\Omega \big(k(|\nabla u|^2-\varphi^2)\big)_{x_k}(\varphi^2)_{x_k}\eta^2\\ &\quad+\frac{1}{\delta}\int_\Omega k(|\nabla u|^2-\varphi^2)u^2_{x_i}\eta^2_{x_k}+\delta\int_\Omega k(|\nabla u|^2-\varphi^2)u^2_{x_kx_i}\eta^2\\ &\quad+\frac{1}{\delta}\int_\Omega k(|\nabla u|^2-\varphi^2)u^2_{x_i}\eta^2_{x_i}+\delta\int_\Omega k(|\nabla u|^2-\varphi^2)u^2_{x_kx_k}\eta^2.\end{aligned} \tag{43}$$

Observing that

$$\big(k(|\nabla u|^2-\varphi^2)\big)_{x_k}(|\nabla u|^2-\varphi^2)_{x_k}=k'(|\nabla u|^2-\varphi^2)(|\nabla u|^2-\varphi^2)^2_{x_k}\geq 0$$

and choosing $\delta=\frac{1}{3}$, from the inequality (43) we have

$$\frac{1}{3}\int_\Omega k(|\nabla u|^2-\varphi^2)u^2_{x_ix_k}\eta^2\leq\frac{3}{2}\int_\Omega f^2\eta^2+\frac{1}{6}\int_\Omega u^2_{x_kx_k}\eta^2 \\ +\frac{1}{2}\int_\Omega k(|\nabla u|^2-\varphi^2)\big((\varphi^2)_{x_k}\eta^2\big)_{x_k}+6\int_\Omega k(|\nabla u|^2-\varphi^2)|\nabla u|^2|\nabla\eta|^2.$$

As $k(s)\geq 1$, we obtain

$$\frac{1}{6}\int_\Omega u^2_{x_ix_k}\eta^2\leq\frac{3}{2}\int_\Omega f^2\eta^2+\frac{1}{2}\int_\Omega k(|\nabla u|^2-\varphi^2)\big((\varphi^2)_{x_k}\eta^2\big)_{x_k}+6\int_\Omega k(|\nabla u|^2-\varphi^2)|\nabla u|^2|\nabla\eta|^2,$$

and so $(u^\varepsilon)_\varepsilon$ is bounded in $H^2_{\text{loc}}(\Omega)$ by (38) and (39). Passing to the weak limit in ε the conclusion holds for the solution of the variational inequality. □

Remark 4.6 *If $\Gamma\in\mathscr{C}^2$ and $\varphi\in\mathscr{C}^2(\bar\Omega)$, Williams proved in [27] that $u\in H^2(\Omega)$. In addition, if $f\in L^p(\Omega)$, $p>2$, then $u\in W^{2,p}(\Omega)$.*

Proposition 4.7 *If $f\in L^2(\Omega)$, $\varphi\in W^{2,\infty}(\Omega)$ with a positive lower bound, u^ε is the solution of the approximated problem (36) and u is the solution of the variational inequality (35), then*

$$u^\varepsilon\underset{\varepsilon\to0}{\longrightarrow}u\quad\text{in}\quad W^{1,p}_0(\Omega),\ 1\leq p<\infty$$

and also in $\mathscr{C}^{0,\alpha}(\bar\Omega)$, for $0\leq\alpha<1$.

Proof By Proposition 4.4, $u^\varepsilon\underset{\varepsilon\to0}{\rightharpoonup}u$ in $H^1_0(\Omega)$.

For Ω' open, $\Omega'\subset\subset\Omega$, as $\|u^\varepsilon\|_{H^2(\Omega')}\leq C$, $u^\varepsilon\underset{\varepsilon\to0}{\rightharpoonup}u$ in $H^2(\Omega')$ and so, by compactness, $u^\varepsilon\underset{\varepsilon\to0}{\longrightarrow}u$ in $H^1(\Omega')$.

In order to prove that $u^\varepsilon\underset{\varepsilon\to0}{\longrightarrow}u$ in $H^1(\Omega)$ we fix $n\in\mathbb{N}$ and choose an open subset Ω_n of Ω such that $\Omega_n\subset\subset\Omega$ and $|\Omega\setminus\Omega_n|\leq\frac{1}{n}$.

Observing that

$$\int_{\Omega\setminus\Omega_n}|\nabla(u^\varepsilon-u)|^2=\int_\Omega|\nabla(u^\varepsilon-u)|^2\chi_{\Omega\setminus\Omega_n}\leq\|\nabla(u^\varepsilon-u)\|^2_{L^4(\Omega)}|\Omega\setminus\Omega_n|^{\frac{1}{2}}\leq\tfrac{C}{\sqrt{n}},$$

where C is independent of ε and n, we obtain

$$\int_\Omega|\nabla(u^\varepsilon-u)|^2=\int_{\Omega_n}|\nabla(u^\varepsilon-u)|^2+\int_{\Omega\setminus\Omega_n}|\nabla(u^\varepsilon-u)|^2\leq\int_{\Omega_n}|\nabla(u^\varepsilon-u)|^2+\tfrac{C}{\sqrt{n}}$$

and so, since u^ε converges to u in $H^1(\Omega_n)$,

$$\lim_{\varepsilon\to0}\int_\Omega|\nabla(u^\varepsilon-u)|^2\leq\tfrac{C}{\sqrt{n}}.$$

Noting that the last inequality is valid for any $n\in\mathbb{N}$, we conclude that

$$\lim_{\varepsilon\to0}\int_\Omega|\nabla(u^\varepsilon-u)|^2=0.$$

For $p>2$ we have, applying the Hölder inequality and the inequality (41),

$$\int_\Omega|\nabla(u^\varepsilon-u)|^p\leq\|\nabla(u^\varepsilon-u)\|^{p-1}_{L^{2p-2}(\Omega)}\|\nabla(u^\varepsilon-u)\|_{L^2(\Omega)}\leq C\|\nabla(u^\varepsilon-u)\|_{L^2(\Omega)}$$

and so we have

$$u^\varepsilon \xrightarrow[\varepsilon\to 0]{} u \text{ in } W^{1,p}(\Omega),\ 1\le p<\infty.$$

To conclude it suffices to recall that for $0\le\alpha<1$ there exists p large enough such that $W^{1,p}(\Omega)\subset\mathscr{C}^{0,\alpha}(\bar\Omega)$.
□

We are now able to prove Theorem 4.1.

Proof of Theorem 4.1 From the estimates obtained in Lemma 4.3 and the Banach-Alaoglu-Bourbaki theorem we have

$$k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\nabla u^\varepsilon \xrightarrow[\varepsilon\to 0]{} \boldsymbol{\Upsilon} \text{ weak-}* \text{ in } \boldsymbol{L}^\infty(\Omega)'$$

and

$$k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2) \xrightarrow[\varepsilon\to 0]{} \lambda \text{ weak-}* \text{ in } L^\infty(\Omega)',$$

at least for a subsequence.

From now on, in order to simplify the notations,

$$\langle\alpha,\beta\rangle \text{ will represent } \langle\alpha,\beta\rangle_{L^\infty(\Omega)'\times L^\infty(\Omega)}$$

and

$$\boldsymbol{\langle\alpha,\beta\rangle} \text{ will represent } \sum_{i=1}^N\langle\alpha_i,\beta_i\rangle.$$

Recall that, from the previous propositions, $\nabla u^\varepsilon \xrightarrow[\varepsilon\to 0]{} \nabla u$ in $L^2(\Omega)$ and $|\nabla u|\le\varphi$ a.e. in Ω.

Multiplying (36a) by $v\in W_0^{1,\infty}(\Omega)$ we get

$$\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\nabla u^\varepsilon\cdot\nabla v=\int_\Omega f_\varepsilon v \tag{44}$$

and so, passing to the limit in ε,

$$\boldsymbol{\langle\Upsilon,\nabla v\rangle}=\int_\Omega fv.$$

Replacing v by u^ε in (44), we have

$$\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla u^\varepsilon|^2=\int_\Omega f_\varepsilon u^\varepsilon \xrightarrow[\varepsilon\to 0]{} \int_\Omega fu=\boldsymbol{\langle\Upsilon,\nabla u\rangle}.$$

Observing that

$$(k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)-1)(|\nabla u^\varepsilon|^2-\varphi^2)\ge 0,$$

integrating in Ω,

$$\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla u^\varepsilon|^2-\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\varphi^2\ge\int_\Omega(|\nabla u^\varepsilon|^2-\varphi^2)$$

and, letting $\varepsilon\to 0$,

$$\boldsymbol{\langle\Upsilon,\nabla u\rangle}-\langle\lambda,\varphi^2\rangle\ge\int_\Omega|\nabla u|^2-\int_\Omega\varphi^2.$$

Hence

$$\boldsymbol{\langle\Upsilon,\nabla u\rangle}\ge\langle\lambda-1,\varphi^2\rangle+\int_\Omega|\nabla u|^2=\langle\lambda-1,\varphi^2-|\nabla u|^2\rangle+\langle\lambda,|\nabla u^2|\rangle.$$

Taking into account that

$$\langle\lambda-1,\varphi^2-|\nabla u|^2\rangle=\lim_{\varepsilon\to 0}\int_\Omega\big(k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)-1\big)(\varphi^2-|\nabla u|^2)\ge 0, \tag{45}$$

we conclude

$$\boldsymbol{\langle\Upsilon,\nabla u\rangle}\ge\langle\lambda,|\nabla u|^2\rangle. \tag{46}$$

As $\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla(u^\varepsilon-u)|^2\geq 0$, then

$$\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla u^\varepsilon|^2-2\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\nabla u^\varepsilon\cdot\nabla u+\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla u|^2\geq 0$$

and so,

$$\langle\!\langle \boldsymbol{\Upsilon},\nabla u\rangle\!\rangle-2\langle\!\langle \boldsymbol{\Upsilon},\nabla u\rangle\!\rangle+\langle\lambda,|\nabla u|^2\rangle\geq 0,$$

thus

$$\langle\lambda,|\nabla u|^2\rangle\geq\langle\!\langle \boldsymbol{\Upsilon},\nabla u\rangle\!\rangle. \tag{47}$$

From (46) and (47) we obtain

$$\langle\!\langle \boldsymbol{\Upsilon},\nabla u\rangle\!\rangle=\langle\lambda,|\nabla u|^2\rangle.$$

Using (44) we obtain

$$\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\nabla(u^\varepsilon-u)\cdot\nabla v+\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\nabla u\cdot\nabla v=\int_\Omega f_\varepsilon v\quad\forall v\in W^{1,\infty}_0(\Omega). \tag{48}$$

Applying the Hölder inequality we get

$$\left|\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\nabla(u^\varepsilon-u)\cdot\nabla v\right|\leq\left(\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla(u^\varepsilon-u)|^2\right)^{\frac12}\left(\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla v|^2\right)^{\frac12}.$$

Consequently

$$\begin{aligned}\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla(u^\varepsilon-u)|^2&=\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla u^\varepsilon|^2\\&\quad-2\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla u^\varepsilon\cdot\nabla u+\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla u|^2\\&\quad\xrightarrow[\varepsilon\to0]{}\langle\!\langle \boldsymbol{\Upsilon},\nabla u\rangle\!\rangle-2\langle\!\langle \boldsymbol{\Upsilon},\nabla u\rangle\!\rangle+\langle\lambda,|\nabla u|^2\rangle=0\end{aligned}$$

and so we conclude that

$$\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\nabla(u^\varepsilon-u)\cdot\nabla v\xrightarrow[\varepsilon\to0]{}0.$$

Passing to the limit in ε in (48) we have

$$\langle\lambda,\nabla u\cdot\nabla v\rangle=\int_\Omega fv,\quad\forall v\in W^{1,\infty}_0(\Omega)$$

and so (34a) is satisfied.

As

$$\int_\Omega\big(k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)-1\big)v\geq 0,\quad\forall v\in L^\infty(\Omega),\ v\geq 0,$$

we obtain (34d) letting $\varepsilon\to0$.

By construction we have

$$\big(k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)-1\big)\big(\varphi^2-|\nabla u^\varepsilon|^2\big)^+=0,$$

and so,

$$\begin{aligned}0&=\int_\Omega\big(k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)-1\big)\big(\varphi^2-|\nabla u^\varepsilon|^2\big)^+v^+\geq\int_\Omega\big(k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)-1\big)\big(\varphi^2-|\nabla u^\varepsilon|^2\big)v^+\\&=\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\varphi^2v^+-\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla u^\varepsilon|^2v^+-\int_\Omega(\varphi^2-|\nabla u^\varepsilon|^2)v^+,\quad\forall v\in L^\infty(\Omega).\end{aligned}$$

But

$$\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla u^\varepsilon|^2 v^+ = \int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla(u^\varepsilon-u)|^2 v^+ + 2\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\nabla(u^\varepsilon-u)\cdot\nabla u\, v^+ + \int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla u|^2 v^+ \underset{\varepsilon\to 0}{\longrightarrow} \langle\lambda, |\nabla u|^2 v^+\rangle$$

since

$$\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla(u^\varepsilon-u)|^2 v^+ \le \|v^+\|_{L^\infty(\Omega)}\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla(u^\varepsilon-u)|^2 \underset{\varepsilon\to 0}{\longrightarrow} 0$$

and

$$\left|\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)\nabla(u^\varepsilon-u)\cdot\nabla u\, v^+\right| \le \|v^+\|_{L^\infty(\Omega)}\left(\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla(u^\varepsilon-u)|^2\right)^{\frac12}\left(\int_\Omega k_\varepsilon(|\nabla u^\varepsilon|^2-\varphi^2)|\nabla u|^2\right)^{\frac12} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

So

$$0\ge\langle\lambda,\varphi^2 v^+\rangle-\langle\lambda,|\nabla u|^2 v^+\rangle-\int_\Omega(\varphi^2-|\nabla u|^2)v^+=\langle\lambda-1,(\varphi^2-|\nabla u|^2)v^+\rangle\ge 0,$$

by (34d), concluding that

$$\langle\lambda-1,(\varphi^2-|\nabla u|^2)v^+\rangle=0\quad\forall v\in L^\infty(\Omega).$$

Given $w\in L^\infty(\Omega)$, if we choose $v=\dfrac{w}{\varphi+|\nabla u|}\in L^\infty(\Omega)$, because $\varphi\ge\varphi_*>0$, we conclude that

$$\langle\lambda-1,(\varphi-|\nabla u|)w^+\rangle=0\quad\forall w\in L^\infty(\Omega),$$

which implies (34e).

To conclude, it remains to prove that if (u,λ) solves (34) then u solves the variational inequality (35).

Given $v\in\mathbb{K}_\varphi$, as

$$\nabla u\cdot\nabla(v-u)\le|\nabla u||\nabla v|-|\nabla u|^2\le|\nabla u|\big(\varphi-|\nabla u|\big),$$

we have

$$\langle\lambda-1,\nabla u\cdot\nabla(v-u)\rangle\le\langle\lambda-1,|\nabla u|(\varphi-|\nabla u|)\rangle=0,$$

and so,

$$\int_\Omega f(v-u)=\langle\lambda,\nabla u\cdot\nabla(v-u)\rangle\le\int_\Omega\nabla u\cdot\nabla(v-u),$$

which concludes the proof. □

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