Should Mathematics remain invisible?

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Mathematical literacy, broadly understood as the ability to reason in terms of abstract models and the effective use of logical arguments and mathematical calculation, became a condition for democratic citizenship. This paper discusses argumentation *and* proof *as two main ingredients in strategies for achieving a higher degree of mathematical fluency in both social and professional life.*

Introduction

In the *brave, new world* of Information Society, mathematical literacy became a condition for democratic citizenship. Actually, skills as basic as the ability to think and reason in terms of abstract models and the effective use of logical arguments and mathematical calculation in normal, daily business practice are on demand. Actually this concerns not only highly skilled IT professionals, who are expected to successfully design complex systems at ever-increasing levels of reliability and security, but also specialised workers monitoring, for example, CNC machines.

Even more it concerns, in general, everyone, who, surrounded by ubiquitous and interacting computing devices, has an unprecedent computational power at her fingers' tips to turn on effective power and self-control of her own life and work. Neologism *info-excluded* is often used to denote fundamental difficulties in the use of IT technologies. More fundamentally, from our perspective, it should encompass mathematical illiteracy and lack of precise reasoning skills rooted in formal logic.

Irrespective of its foundational role in all the technology on which modern life depends, Mathematics seems absent, or invisible, from the dominant cultural practices. Regarded as *difficult* or *boring*, its clear and ordered mental discipline seems to conflict with the superposition of images and multiple *rationales* of post-modern way of living. Maybe just a minor symptom of this state of affairs, but *mathphobia*, which seems to be spreading everywhere, has become a hot spot for the media. Our societies, as noticed by E. W. Dijkstra a decade ago, are through an *ongoing process of becoming more and more "amathematical"* [11]. On the surface, at least.

Under it, however, Mathematics is playing the dominant role, and failing to recognize that and training oneself in its discipline, will most probably result in people impoverished in their interaction with the global *polis* and diminished citizenship.

In such a context, this paper aims at contributing to the debate on strategies for achieving a higher degree of *mathematical fluency*. By this we do not have in mind the exclusive development of numerical, operative competences, but the ability to resort to the mathematical language and method to build models of problems, and reason effectively within them. Our claim is that such strategies should be directed towards *unveiling* mathematics contents by rediscovering the relevance of both

- *argumentation* skills, broadly understood as the ability to formulate and structure relationships, justifications and explanations to support an argument;
- and *proof*, as the formal certification of an argument, which encompasses the effective development of proof design and manipulation skills.

Although both aspects are often emphasized separately, the development of educational strategies to bind them together in learning contexts may have an impact in empowering people reasoning skills and, therefore, their ability to survive in a complex world.

The study of *mathematical arguments* is still an issue in Mathematics Education (see, e.g., [1, 16]). On the other hand, the rediscovery of the essential role played by *proofs* (and the associated relevance given to formal logic), has been raised, for the last 3 decades, in a very particular context: that of Computing Science. Actually, the quest for programs whose *correctness* (with respect to a specified intended behaviour) could be established by mathematical reasoning, which has been around for a long time as a research agenda, has recently emerged as a key concern for the Information Society. More and more, our way of living depends on software whose reliability is crucial for our work, security, privacy, and even life (cf, for example wide spread computer-controlled medical instrumentation). Industry is recognising this fact and becoming aware that, at present, *proofs pay V.A.T.*: they are no more an academic activity or an exotic detail, but simply part of the business [9].

The remaining of this paper addresses *argumentation* and *proof*, stressing the need for making them *explicit* in mathematical training at all levels (from middle and high school curricula to profissional education in industrial contexts). Section 2 addresses mathematical argumentation and the development of adequate skills. Part of the discussion is illustrated through the analysis of a class episode registered in the context of a collaborative research project on mathematical communication coordinated by the first author and partially documented in [14]. Section 3, on the other hand, goes from argumentation to proof, building on developments in Computing Science with potential impact for reinvigorating mathematics education. In particular, we focus on the centrality of formal logic and the proposal of a calculational, goal-directed reasoning style which has proven to scale up from the school desk to the engineer's desk tackling complex, real-life problems.

Argumentation

Mathematical learning requires a stepwise construction of a reference framework through which students construct their own personal account of mathematics in a dynamic tension between old and newly acquired knowledge. This is achieved along the countless interaction processes taking place in the classroom. In particular, the nature of the questions posed by the teacher may lead to, or inhibit, the development of argumentation and reasoning skills [3]. A student who is given the opportunity to share what she already knows, her conjectures, and explain the way she though about a problem, will develop higher levels of mathematical literacy in the broad sense proposed in section 1. Team work, which entails the need for each participant to expose his views, argue and try to convince the others, is an excellent strategy to achieve this goal.

Strategies which call students to analyze their arguments and identify its strengths and weaknesses are also instrumental to this aim [13]. Reference [15] singles out a number of basic issues in the development of what is called a *reflexive* mathematical discourse: the ability to go back (either to recover previous arguments in a discussion or to introduce new view points) and the ability to share different sorts of images supporting argumentation (eg, sketches, tables, etc.).

Training argumentation skills is not easy, but certainly an essential task if one cares about mathematical literacy in modern societies. Teacher's role can not be neglected. She/he is responsible for stimulating a friendly, open discussion environment [1], avoiding rejection and helping students to recognize implications and eventual contradictions in their arguments to go ahead [10, 18]. Her role is also to make explicit what is implicit in the students formulation [5], helping them to build up intuitions, asking for generalizations or confronting them with specific particular cases.

Often in school practice conceptual disagreements are avoided (let alone encouraged!), with negative effects in the development of suitable argumentation skills. On the contrary, such skills benefit from exposition to diverse arguments, their attentive consideration and elicitation, as empirically documented in, e.g., [20].

The following episode was recorded in the context mentioned above, in a class of 10 years old students. Although very short, this excerpt illustrates both argumentation in a class and the way a teacher can promote vivid discussions without ruling out any participant. The context was a general discussion in the class on the result of some team work tackling the following problem: *a gardner wants to sow new plants in the flower bed depicted in Figure 1. How much seed should he buy if* 10 *g are required for each square meter of the flower bed?*.

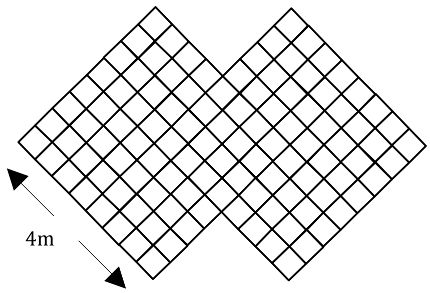

Figure 1: The garden problem.

The excerpt illustrates what [16] calls emphposition-oriented discussions in which the teacher tries to identify and promote different explicit viewpoints on the problem. She is supposed to analyze and make explicit the logic structure of the arguments in presence and act as a source of both criticism and confidence for all students involved.

T:	Let's see this group's solution. Why did you compute the area, instead of, let's say, the perimeter or the volume?
Magda:	We need to kwon the area because we were told that $10\,g$ of seed would do for $1\,m^2$.
T:	Ok. And then?
Magda:	Then we divided the area of the central square by two. We had to get rid of one half.
T:	Why that?
Magda:	We already had the area of one of the big squares ...
T:	Sure. This one for example [*pointing to one of them*].
Magda:	The problem was they were not complete...
T:	Who was not complete?
Magda:	The squares were not complete.
T:	That's true: this one is not complete, neither is the other.
Magda:	Actually the central square is imaginary. We needed to know the area and divide it by two to get it removed from the area of each of the big squares.

Notice the teacher's effort to make students clarify their reasoning and consider all aspects of the problem. Magda's group strategy was to consider two big squares partially overlapping: the overlap area is refered to as the imaginary central square. Then they took half of latter to subtract to the sum of the areas of the two big squares taken independently. The strategy is correct and shows a clear spatial perception. It is not easy for her, however, to put it into words when other students express doubts about the result. Sue tries to help: she understood why Magda calls the central square imaginary, but also why other students claim it can't exist:

T: Nice observation. Can you repeat it louder?

Ann: If it were imaginary nobody would have sketched it on the figure.

T: Then we have two theories! One group claims it is an imaginary square, and, being imaginary it does not exist. The other claims the opposite. Any help?

Sue: Ok, it is imaginary because it is there just to signal that one of the big squares is above the other.

T: You mean this square is above the other?

Sue: Yes, it is an incomplete square.

T: What do you think Magda? Would you like to defend your position?

Magda: We claim it is imaginary because otherwise, if it was a real flower bed in a garden, it had to be perceived as such ... and it isn't there, is it?

T: You mean, if you were parallel to the garden you couldn't see the central square, right?

Magda: Exactly.

The debate went on. Just notice, in the following small excerpt, how making an effort to be concrete and come back to the original gardening problem, can help to build up the correct intuitions.

Rachel: Sue claims one of the squares is incomplete, but actually they can be both complete and just one of them be over the other.

T: Ok, you can imagine one square overlapping the other ... your colleagues say this is not possible ...

Rachel: Oh, yes, sure! I forgot the earth in the flower bed. They are right: if the two squares really overlap, the bed will not be flat, but a little higher in the overlapping area.

As a final remark note how classroom interactions can shape the mathematical universe of students. Actually, school mathematics is an iceberg, of which students often only sees what emerges at surface (typically, definitions and procedures). Rendering explicit what is hidden under water is the role of effective mathematical training in argumentation.

Proof

If the development of suitable *argumentation skills* is a first step to a Mathematics-aware citizenship, mastering *proof technology* is essential in a context where, as explained above, *proofs pay V.A.T.* Such is the context of software industry and the increasing demand for quality certified software, namely in safety-critical applications. But what contributions may Computing Science bring to such a discipline? And how could they improve current standards in mathematical education?

As a contribution to a wider debate, we would like to single out in this paper the emphasis on the central role of *formal logic* and the development of a *calculational* style of reasoning. The former is perhaps the main consequence to Mathematics of Computing Science development. An indicator of this move is the almost universal presence of a course on formal logic in every computing undergraduate curriculum.

Proficiency in mathematics, however, would benefit from an earlier introduction and explicit use of logic in middle and high school. Note this is usually not the case in most European countries; the justification for such an omission is that *logic is implicit in Mathematics and therefore does not need to be taught as an independent issue*. Such an argument was used in Portugal to eliminate logic from the high-school curriculum in the nineties. The damage it caused is still to be assessed, but it is certainly not alien to the appalling indicators in what concerns the country overall ranking in mathematics education [17].

High-valued programmers are heavy users of logic. At another scale, this is also true of whoever tries to use and master information in modern IT societies: the explicit use of logic enables critical and secure reasoning and decision making. On the other hand, a heavy use of logic entails the need for more concise ways of expression and notations amenable to formal, systematic manipulation.

The so-called *calculational style* [2, 19] for structuring mathematical reasoning and proof emerged from two decades of research on *correct-by-construction* program design, starting with the pioneering work of Dijkstra and Gries [6, 12], and in particular, through the development of the so-called *algebra of programming* [4]. This style emphasizes the use of systematic mathematical calculation in the design of algorithms. This was not new, but routinely done in algebra and analysis, albeit subconsciously and not always in a systematic fashion. The realization that such a style is equally applicable to logical arguments [6, 12] and that it can greatly improve on traditional verbose proofs in natural language has led to a systematization that can, in return, also improve exposition in the more classical branches of mathematics. In particular, lengthy and verbose proofs (full of *dot-dot* notation, case analyses, and natural language explanations for "obvious" steps) are replaced by easy-to-follow calculations presented in a standard layout which replaces classical implication-first logic by variable-free algebraic reasoning [19, 11].

Let us illustrate with a very simple example what we mean by a *calculational* proof. Suppose we are given the task to find out *whether* $ln(2) + ln(7)$ *is greater than, or lesser than* $ln(3) + ln(5)$. The 'classical' response consists of first formulating the hypothesis $ln(2) + ln(7) \leq ln(3) + ln(5)$ and then verifying it as follows:

(1)	function ln is strictly increasing	
(2)	$ln(x \times y) = ln(x) + ln(y)$	
(3)	$14 < 15$	
(4)	$14 = 2 \times 7$ and $15 = 3 \times 5$	
(5)	$ln(14) < ln(15)$	by (1) e (3)
(6)	$ln(2) + ln(7) < ln(3) + ln(5)$	by (2), (4) e (5)

The proof is easy to follow, but, in the end, the intuition it provides on the problem is quite poor. Moreover, it is hard memorize or reproduce. Most probably it was not made, originally , by the order in which it is presented. This may explain why, in general, this sort of proofs, although dominant in the current mathematical discourse, fail to attract students enthusiasm.

Consider, now, a *calculational* approach to the same problem. The main, initial difference is easy to spot and has an enormous impact: its starting point is not an hypothesis to verify, formulated in a more or less diligent way, but the original problem itself. The proof starts by identifying an unknown $\Box$ which stands, not for a number as students are used in school mathematics, but for an order relation. Then it proceeds by the identification and application of whatever known properties are useful in its determination. The whole proof, being essentially syntax driven, builds intuition and meaning.

$$
\begin{array}{cl}
 & ln(2) + ln(7) \;\; \Box \;\; ln(3) + ln(5) \\
= & \quad \{ \text{ function } ln \text{ distributes over multiplication } \} \\
 & ln(2 \times 7) \;\; \Box \;\; ln(3 \times 5) \\
= & \quad \{ \text{ routine arithmetic } \} \\
 & ln(14) \;\; \Box \;\; ln(15) \\
= & \quad \{ \; 14 < 15 \text{ and function } ln \text{ is strictly increasing } \} \\
 & \Box \quad \text{is} \quad <
\end{array}
$$

Empirical evidence gathered within MATHIS[1] suggests the systematization of such a calculational style of reasoning can greatly improve on the way proofs are presented. In particular it may help to overcome the typical justification for omitting proofs in school mathematics: that they are difficult to follow for all but exceptional students.

[1]A research Portuguese project which started in 2009, aiming at exploiting the interplay between Mathematics and Computer Science.

The MATHIS project is ’refactoring’ several pieces of school mathematics, systematically introducing this sort of *proofs by calculation*. Although it is too early to draw general conclusions (preliminary results, however, appeared in [7] and [8]), this effort shows how the formalization of topics arising in different contexts results in formulae with the same *flavour*, which can be manipulated thereafter by the same rules of the predicate calculus, without reference to a ‘domain specific’ interpretation of such formulae in their original area of discourse. This is the essence of formal manipulation, and yields proofs that are shorter, explicit, independent of hidden assumptions, easy to re-construct, check and generalize.

Conclusions and Future Work

Understood, more and more, as a condition for democratic citizenship in modern Information Societies, mathematical literacy has to be taken as a serious concern for the years to come. From our perspective this entails the need for a systematic (and, given *l’esprit du temps*, courageous) *unveiling* of Mathematics. That is, to make mathematical reasoning explicit at all levels of human argumentation and develop, through adequate teaching strategies, the skills suitable to empower correct reasoning in all sorts of social, cultural or professional contexts.

This paper focused on two main issues in this process: empowering *mathematical argumentation*, by developing adequate teaching strategies, and *proof*, made simpler, easier to produce and more systematic through a new calculation style which has proved successful in reasoning about complex software. The latter may be, so we believe, a contribution of Computing Science to reinvigorating mathematical education.

A final word is in order on the above mentioned relationship of Mathematics and Computing. Actually, the latter is probably the paradigm of an area of knowledge from which a popular and effective technology emerged long before a solid, specific, scientific methodology, let alone formal foundations, have been put forward. Often, as our readers may notice, in software industry the whole software production seems to be totally biased to specific technologies, encircling, as a long term effect, the company’s culture in quite strict limits. For example, mastering of particular, often ephemeral, technologies appears as a decisive requirement for recruitment policies.

This state of affairs is, however, only the surface of the iceberg. Companies involved in the development of safety-critical or mission-critical software have already recognized that mathematical rigorous reasoning is, not only the key to success in market, but also the warrantee of their own survival. With a long experience in training software engineers and collaborating with software industry, the authors can only claim the need for a double change:

- in the Mathematics *middle school curriculum*, in which the notion of *proof* and the development of argumentation skills are virtually absent;

- in a popular, but pernicious, technology-driven computing education which fails to provide effective training in tackling rigorously the overwhelming complex problems software is supposed to solve.

Future research, specially in the context of the MATHIS project, goes exactly in this direction. In particular, we are currently working on strategies for developing argumentation and calculational proof skills in probabilistic reasoning. As researchers in Education and Computing Science, respectively, the authors see their job as E. W. Dijkstra once put it, *We must give industry not what it wants, but what it needs.*. Mathematics should not, definitively, remain hidden.

Acknowledgements. On-going collaboration with J. N. Oliveira and R. Backhouse on calculational approaches to mathematics is deeply acknowledged. This work is funded by ERDF - European Regional Development Fund through the COMPETE Programme (operational programme for competitiveness) and by National Funds through the FCT - Fundao para a Cincia e a Tecnologia (Portuguese Foundation for Science and Technology) within project `FCOMP-01-0124-FEDER-007254`.

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