ON AN EXTREME VALUE VERSION OF THE BIRNBAUM-SAUNDERS DISTRIBUTION

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Abstract:

- The Birnbaum-Saunders model is a life distribution originated from a problem of material fatigue that has been largely studied and applied in recent decades. A random variable following the Birnbaum-Saunders distribution can be stochastically represented by another random variable used as basis. Then, the Birnbaum-Saunders model can be generalized by switching the distribution of the basis variable using diverse arguments allowing to construct more general classes of models. Extreme value distributions are useful to determinate the probability of events that are more extreme than any that have already been observed. In this paper, we propose, characterize, implement and apply an extreme value version of the Birnbaum-Saunders distribution.

Key-Words:

- *Domain of attraction; Extreme data; Likelihood method; R computer language.*

AMS Subject Classification:

- 60E05, 62G32, 62N02, 62N99.

1. INTRODUCTION

Extreme value (EV) models are appropriate to establish the probability of events that are more extreme than any other that have been observed. As an example where these models can be used, suppose that a sea-wall projection requires a coastal defense from all sea levels for the next 100 years. Thus, extremal models are a precious tool that enables extrapolations of this type. Actually, the EV theory is widely used by many researchers in applied sciences when faced with modeling high values of certain phenomena. For instance, ocean wave, thermodynamics of earthquakes, wind energy, risk assessment on financial markets, and medical phenomena can be mentioned. Some books on EV theory are Leadbetter *et al.* (1983), Galambos (1987), Embrechts *et al.* (1997), Beirlant *et al.* (2004), and de Haan and Ferreira (2006). For a more practical view on this topic, see Coles (2001), and for more recent references, see Ferreira and Canto e Castro (2008), and Gomes *et al.* (2008a,b), among others.

Life distributions are usually positively skewed, unimodal, two-parameter models and with non-negative support; see Marshall and Olkin (2007) and Saunders (2007). A life distribution that has received a considerable attention in recent decades is the Birnbaum-Saunders (BS) model. This model was originated from a problem of material fatigue and has been largely applied to reliability and fatigue studies; see Birnbaum and Saunders (1969). The BS distribution relates the total time until the failure to some type of cumulative damage normally distributed. This attention for the BS distribution is due to its many attractive properties and its relationship with the normal distribution.

Extensive work has been done on the BS model with regard to its properties, inference and applications. A comprehensive treatment on this model until mid 90's can be found in Johnson *et al.* (1995, pp. 651-662). For more detail about new applications of the BS model, see Leiva *et al.* (2009a). For applications in fields beyond engineering allowing business, environmental and medical data to be analyzed by using this model, see Leiva *et al.* (2007, 2008b, 2009b, 2010a,b, 2011), Podlaski (2008), Barros *et al.* (2008), Bhatti (2010), Ahmed *et al.* (2010), and Vilca *et al.* (2010). Thus, at the present, the BS model can be widely used as a statistical distribution rather than restricted to a life distribution.

Because a random variable (r.v.) following the BS distribution can be represented by another basis r.v., generalizations of this distribution can be obtained switching the distribution of the basis variable by diverse arguments allowing to construct more general classes of models. Several generalizations of the BS distribution have been recently proposed by a number of authors, including Díaz-García and Leiva (2005), Sanhueza *et al.* (2008) and Gómez *et al.* (2009), which allow to obtain a major degree of flexibility for this distribution. Usual and generalized versions of the BS distribution are implemented in the `R` software (`http://www.R-project.org`) by packages called `bs` and `gbs`, which can be downloaded from `http://CRAN.R-project.org`; see Leiva *et al.* (2006)

and Barros *et al.* (2009). These packages contain functions for computing probabilities, estimating parameters, generating random numbers and carrying out goodness-of-fit and hazard analysis. Leiva *et al.* (2008a) studied three generators of random numbers from the BS and generalized BS (GBS) distributions.

The main aim of this work is to obtain an EV version of the BS distribution relevant not only by itself as a model, but also for a parametric statistical analysis of extreme or rare events. The paper is organized as follows. In Section 2, we provide a preliminary notion of different aspects related to BS and EV distributions. In Section 3, we characterize extreme value Birnbaum-Saunders (EVBS) distributions. In Section 4, we focus on extremal domains of attraction of a general class of BS models that we call BS type (BST) distributions. In Section 5, we carry out a hazard analysis of EVBS distributions mainly based on the hazard rate (h.r.). In Section 6, we discuss about the estimation procedure based on the maximum likelihood (ML) method and model checking. In Section 7, we conduct out the numerical application of this work, which includes an exploratory data analysis (EDA) and a parametric statistical analysis based on the EVBS distribution. Finally, In Section 8, we sketch some concluding remarks.

2. A PRELIMINARY NOTION

In this section, we provide preliminary aspects about BS, BST and EV distributions.

2.1. BS and BST distributions

An r.v. T with usual BS distribution is characterized by its shape and scale parameters $\alpha > 0$ and $\beta > 0$, respectively. This is denoted by $T \sim \mathrm{BS}(\alpha, \beta)$, where β is also the median of the distribution. BS and standard normal r.v.'s, denoted respectively by T and Z for the moment, are related by

$$T = \beta\big(\alpha\, Z/2 + \sqrt{\{\alpha\, Z/2\}^2 + 1}\big)^2 \quad \text{and} \quad Z = \big(\sqrt{T/\beta} - \sqrt{\beta/T}\big)/\alpha. \tag{2.1}$$

Let $T \sim \mathrm{BS}(\alpha, \beta)$. Then, the probability density function (p.d.f.) and cumulative distribution function (c.d.f.) of T are respectively given by

$$f_T(t) = \phi\big(a(t)\big)\, a'(t) \quad \text{and} \quad F_T(t) = \Phi\big((a(t)\big), \quad t > 0, \tag{2.2}$$

where ϕ and Φ are the standard normal p.d.f. and c.d.f., respectively,

$$a(t) \equiv a_t = (\sqrt{t/\beta} - \sqrt{\beta/t})/\alpha \quad \text{and} \quad a'(t) \equiv A_t = t^{-3/2}(t + \beta)/(2\alpha\sqrt{\beta}), \tag{2.3}$$

where $a'(t) = d\, a(t)/dt$ is the derivative of $a(t)$ with respect to t. The quantile function (q.f.) of T is expressed as

$$t(q) \equiv t_q = F_T^{-1}(q) = \beta\Big(\alpha\, \xi_q/2 + \sqrt{\{\alpha\ \xi_q/2\}^2 + 1}\Big)^2, \quad 0 < q < 1, \tag{2.4}$$

where $F_T^{-1}(t) := \inf\{x : F(x) \geq t\}$ is the generalized inverse function of the c.d.f. of T and ξ_q is the qth quantile of the r.v. $Z \sim \mathrm{N}(0,1)$. Note from (2.4) that, as mentioned, the median of T is $t_{0.5} = \beta$.

Important properties of $T \sim \mathrm{BS}(\alpha, \beta)$ are: (i) $c\,T \sim \mathrm{BS}(\alpha, c\,\beta)$, $c > 0$; (ii) $1/T \sim \mathrm{BS}(\alpha, 1/\beta)$; and (iii) $V = (T/\beta + \beta/T - 2)/\alpha^2 \sim \chi^2(1)$, i.e., V follows the χ^2 distribution with one degree of freedom (d.f.).

The assumption given in (2.1) can be relaxed supposing that Z follows any other distribution with p.d.f. f_Z. Thus, we obtain the general class of BST distributions earlier mentioned, which is denoted by $T \sim \mathrm{BST}(\alpha, \beta; f_Z)$ for an associated r.v. T and whose p.d.f. is given by

$$f_T(t) = f_Z\big(a(t)\big)\, a'(t), \quad t > 0. \tag{2.5}$$

In particular, if Z follows a standard symmetric distribution in the real number set, denoted by $Z \sim \mathrm{S}(f_Z)$, we then find the GBS distribution, i.e., $T \sim \mathrm{GBS}(\alpha, \beta; g)$, where g is the kernel of the p.d.f. of Z given by $f_Z(z) = c\, g(z^2)$, with $z \in \mathbb{R}$ and c being the normalization constant, i.e., the positive value such that $\int_{-\infty}^{+\infty} g(z^2)\, dz = 1/c$; see Díaz-García and Leiva (2005). Then, if $f_Z(z) = \phi(z) = \exp(-z^2/2)/\sqrt{2\,\pi}$, for $z \in \mathbb{R}$, the standard normal p.d.f., we obviously recover the usual BS distribution, i.e., an r.v. $T \sim \mathrm{BST}(\alpha, \beta; \phi) \equiv \mathrm{BS}(\alpha, \beta)$; see Birnbaum and Saunders (1969). For the GBS case, $V = (T/\beta + \beta/T - 2)/\alpha^2 \sim \mathrm{G}\chi^2(1; f_Z)$, i.e., V follows the generalized χ^2 class of distributions with one d.f., which has the $\chi^2(1)$ distribution as a special case if f_Z is the standard normal density; see Sanhueza *et al.* (2008).

2.2. EV distributions and extremal domains of attraction

The central limiting result in EV theory states the following. Consider an independent identically distributed sequence of r.v.'s $\{X_n,\, n \geq 1\}$, with marginal c.d.f. F. Hence, if there are constants $a_n > 0$ and $b_n \in \mathbb{R}$, and a non-degenerate c.d.f. G such that, as $n \to \infty$,

$$\mathbb{P}(\max\{X_1, \ldots, X_n\} \leq a_n x + b_n) \to G(x), \tag{2.6}$$

then G must be the c.d.f. of a generalized extreme value (GEV) r.v., depending on a parameter $\gamma \in \mathbb{R}$. The notation $X \sim \mathrm{GEV}(\gamma)$ is used in this case and the corresponding c.d.f. is given by

$$G(x) \equiv G_\gamma(x) = \begin{cases} \exp(-\{1+\gamma x\}^{-1/\gamma}); 1+\gamma\, x > 0, \gamma \in \mathbb{R}\setminus\{0\}, \\ \exp(-\exp(-x));\ x \in \mathbb{R},\ \gamma = 0, \end{cases} \tag{2.7}$$

with $G_0(x)$ obtained from $G_\gamma(x)$, for $\gamma \in \mathbb{R}\setminus\{0\}$, as $\gamma \to 0$. As a consequence, we say that F belongs to the max-domain of attraction of G_γ, in short $F \in \mathcal{D}_{\mathrm{M}}(G_\gamma)$. The parameter γ, known as the EV index, is a shape parameter that determines the right-tail behavior of F, being so a crucial parameter in EV theory. Specifically, if $\gamma < 0$, we have the Weibull max-domain of attraction, i.e., light right-tails, with a finite right endpoint. In addition, $\gamma = 0$ corresponds to the Gumbel max-domain of attraction (exponential right-tails). And if $\gamma > 0$, we have the Fréchet max-domain of attraction corresponding to heavy right-tails (polynomial tail decay), with an infinite right endpoint.

The GEV distribution with c.d.f. given in (2.7) is also known as the von Mises-Jenkinson representation. This is a general form from which we derive the three above mentioned distribution types, i.e.,

$$G_\gamma(x) = \begin{cases} \Psi_{-1/\gamma}(-1-\gamma x);\ \gamma < 0, \\ \Lambda(x);\ \gamma = 0, \\ \Phi_{1/\gamma}(1+\gamma x);\ \gamma > 0, \end{cases}$$

where, for $\varrho > 0$, $\Psi_\varrho(x) = \exp\big(-\{-x\}^\varrho\big)$ with $x < 0$ (Weibull distribution for maxima), $\Lambda(x) = \exp(-\exp(-x))$ with $x \in \mathbb{R}$ (Gumbel distribution for maxima), and $\Phi_\varrho(x) = \exp\big(-x^{-\varrho}\big)$ with $x > 0$ (Fréchet distribution for maxima). The Gumbel for maxima and the Fréchet for maxima are the commonly known Gumbel and Fréchet distributions, respectively. Location ($\mu \in \mathbb{R}$) and scale ($\sigma > 0$) parameters can be introduced in the GEV distribution by considering $G_\gamma(\{x-\mu\}/\sigma)$, denoted by $X \sim \mathrm{GEV}(\mu,\sigma,\gamma)$.

All results developed for maxima can easily be reformulated for minima because $\min\{X_1,\ldots,X_n\} = -\max\{-X_1,\ldots,-X_n\}$. Actually, if we are interested in the lower tail, we can rewrite a result similar to the one in (2.6) for minima, with a limiting c.d.f. $G(x) \equiv G^*_\gamma(x)$, which is now denoted as $X \sim \mathrm{GEV}^*(\gamma)$, such that $G^*_\gamma(x) = 1 - G_\gamma(-x)$, i.e.,

$$G^*_\gamma(x) = \begin{cases} 1-\exp\big(-\{1-\gamma x\}^{-1/\gamma}\big); 1-\gamma x > 0,\ \gamma \in \mathbb{R}\setminus\{0\}, \\ 1-\exp(-\exp(x)); x \in \mathbb{R},\ \gamma = 0. \end{cases} \tag{2.8}$$

As a consequence, we say that F belongs to the min-domain of attraction of G^*_γ, in short $F \in \mathcal{D}_\mathrm{m}(G^*_\gamma)$. Analogously to the GEV distribution, the GEV* case (minima) is a general form from which we derive the following three possible EV limiting cases:

$$G^*_\gamma(x) = \begin{cases} \Psi^*_{-1/\gamma}(1-\gamma x);\ \gamma < 0, \\ \Lambda^*(x);\ \gamma = 0, \\ \Phi^*_{1/\gamma}(-1+\gamma x);\ \gamma > 0, \end{cases}$$

where, for $\varrho > 0$, $\Phi^*_\varrho(x) = 1 - \exp\big(-\{-x\}^{-\varrho}\big)$ with $x < 0$ (Fréchet distribution for minima), $\Lambda^*(x) = 1 - \exp(-\exp(x))$ with $x \in \mathbb{R}$ (Gumbel distribution for minima), and $\Psi^*_\varrho(x) = 1 - \exp\big(-x^\varrho\big)$ with $x > 0$ (Weibull distribution for minima, commonly known as the Weibull distribution).

3. EXTREME VALUE BS DISTRIBUTIONS

In this section, we propose and characterize the EVBS model based on limiting EV models for maxima, as well as for minima, denoted as EVBS* distributions. In addition, a shape analysis for the EVBS and EVBS* distributions is provided. Specifically, consider that

$$Z \sim \text{ GEV}(\gamma) \equiv \text{GEV}(0,1,\gamma), \tag{3.1}$$

i.e., Z has c.d.f. as given in (2.7). Then,

$$T = \beta\big(\alpha Z/2 + \sqrt{\alpha^2 Z^2/4 + 1}\big)^2 \sim \text{EVBS}(\alpha,\beta,\gamma).$$

Directly from the GEV p.d.f., $g_\gamma(t) = dG_\gamma(t)/dt$, associated with the GEV c.d.f. $G_\gamma(t)$ given in (2.7), and considering $F_T(t) = G_\gamma(a_t)$, $t_q = F_T^{-1}(q)$ and $f_T(t) = A_t\, g_\gamma(a_t)$, with a_t and A_t as given in (2.3), the EVBS r.v. T can be defined in the following ways:

I. The p.d.f. of T is given by

$$f_T(t) = \begin{cases} A_t\,(1+\gamma\, a_t)^{-1-1/\gamma} \exp\big(-\{1+\gamma\, a_t\}^{-1/\gamma}\big), & \gamma \neq 0,\\ A_t \exp\left(-\exp\left(-a_t\right) - a_t\right), & \gamma = 0, \end{cases} \tag{3.2}$$

where $t > (\alpha^2\beta + 2\beta\gamma^2)/(2\gamma^2) - \sqrt{(\alpha^4\beta^2 + 4\alpha^2\beta^2\gamma^2)/\gamma^4}/2$ if $\gamma > 0$; $t > 0$ if $\gamma = 0$; and $0 < t < (\alpha^2\beta + 2\beta\gamma^2)/(2\gamma^2) + \sqrt{(\alpha^4\beta^2 + 4\alpha^2\beta^2\gamma^2)/\gamma^4}/2$ if $\gamma < 0$.

II. The c.d.f. of T is expressed as

$$F_T(t) = \begin{cases} \exp\big(-\big\{1+\gamma\, a_t\big\}^{-1/\gamma}\big), & \gamma \neq 0,\\ \exp\left(-\exp\left(-a_t\right)\right), & \gamma = 0; \end{cases} \tag{3.3}$$

III The q.f. of T is as given in (2.4) by replacing ξ_q with z_q, the qth quantile of the c.d.f. $G_\gamma(x)$, as expressed in (2.7), i.e., $z_q = (\{-\log(q)\}^{-\gamma} - 1)/\gamma$ if $\gamma \neq 0$, and $z_q = -\log\big(-\log(q)\big)$ if $\gamma = 0$.

Analogously, if we consider in (3.1) the GEV distribution for minima given in (2.8), we use the notation $T^* \sim \text{EVBS}^*(\alpha,\beta,\gamma)$ for an associated r.v. T^*, and, as before, noting that $F_{T^*}(t) = G^*_\gamma(a_t) = 1 - G_\gamma(-a_t)$ and that $f_{T^*}(t) = A_t\, g^*_\gamma(a_t) = A_t\, g_\gamma(-a_t)$, the EVBS* r.v. T^* can be defined in the following ways:

I' The p.d.f. of T^* is given by

$$f_{T^*}(t) = \begin{cases} A_t\,(1-\gamma\, a_t)^{-1-1/\gamma} \exp\big(-\{1-\gamma\, a_t\}^{-1/\gamma}\big), & \gamma \neq 0,\\ A_t \exp\left(-\exp\left(a_t\right) + a_t\right), & \gamma = 0, \end{cases} \tag{3.4}$$

where $t > (\alpha^2\beta + 2\beta\gamma^2)/(2\gamma^2) - \sqrt{(\alpha^4\beta^2 + 4\alpha^2\beta^2\gamma^2)/\gamma^4}/2$ if $\gamma < 0$; $t > 0$ if $\gamma = 0$; and $0 < t < (\alpha^2\beta + 2\beta\gamma^2)/(2\gamma^2) + \sqrt{(\alpha^4\beta^2 + 4\alpha^2\beta^2\gamma^2)/\gamma^4}/2$ if $\gamma > 0$.

II' The c.d.f. of T^* is defined as

$$F_{T^*}(t) = \begin{cases} 1 - \exp\big(-\{1 - \gamma\, a_t\}^{-1/\gamma}\big), & \gamma \neq 0, \\ 1 - \exp\left(-\exp\left(a_t\right)\right), & \gamma = 0. \end{cases} \tag{3.5}$$

III' The q.f. of T^* is also as given in (2.4), but by replacing ξ_q with $z_q^* = z_q^*(\gamma)$, the qth quantile of the c.d.f. $G_\gamma^*(x)$, as expressed in (2.8), i.e., with $z_q(\gamma)$ being the qth quantile of the c.d.f. $G_\gamma(x)$, as given in (2.7), $z_q^* = -z_{1-q}(\gamma) = (1 - \{-\log(1-q)\}^{-\gamma})/\gamma$ if $\gamma \neq 0$, and $z_q^* = \log\big(-\log(1-q)\big)$ if $\gamma = 0$.

Next, as a direct application of the change of variable method, some properties of the EVBS and EVBS* distributions are provided.

Proposition 3.1. *Let* $T \sim \mathrm{EVBS}(\alpha, \beta, \gamma)$ *and* $T^* \sim \mathrm{EVBS}^*(\alpha, \beta, \gamma)$. *Then,*

(i) $c\,T \sim \mathrm{EVBS}(\alpha, c\,\beta, \gamma)$ *and* $c\,T^* \sim \mathrm{EVBS}^*(\alpha, c\,\beta, \gamma)$, *with* $c > 0$*;*

(ii) $1/T \sim \mathrm{EVBS}^*(\alpha, 1/\beta, \gamma)$ *and* $1/T^* \sim \mathrm{EVBS}(\alpha, 1/\beta, \gamma)$.

Figure 1 (first and second panels) displays shapes for the EVBS and EVBS* densities for different values of their parameters. In all of these graphs, we consider $\beta = 1$, without loss of generality, because β is a scale parameter, such as proved in Proposition 3.1(i). In these plots, we further use the notation $\mathrm{EVBS}(\alpha, \gamma) \equiv \mathrm{EVBS}(\alpha, 1, \gamma)$. For the EVBS densities presented in Figure 1 (first panel), we see how the shape parameter α modifies the shape of this distribution. In the case of the parameter γ, we detect changes in the kurtosis, as expected. Similar aspects are observed when we consider the EVBS* densities presented in Figure 1 (second panel).

4. BST EXTREMAL DOMAINS OF ATTRACTION

In this section, we obtain the extremal domains of attraction for BST distributions.

We analyze the extreme domain of attraction of the c.d.f. of an r.v.

$$T = \beta\big(\alpha Z/2 + \sqrt{\alpha^2 Z^2/4 + 1}\big)^2, \quad \alpha > 0,\ \beta > 0, \tag{4.1}$$

not necessarily following an EVBS distribution, whenever the c.d.f. of the r.v. Z, compulsory given by

$$Z = \big(\sqrt{T/\beta} - \sqrt{\beta/T}\big)/\alpha,$$

belongs to some extreme domain of attraction either for maxima or for minima.

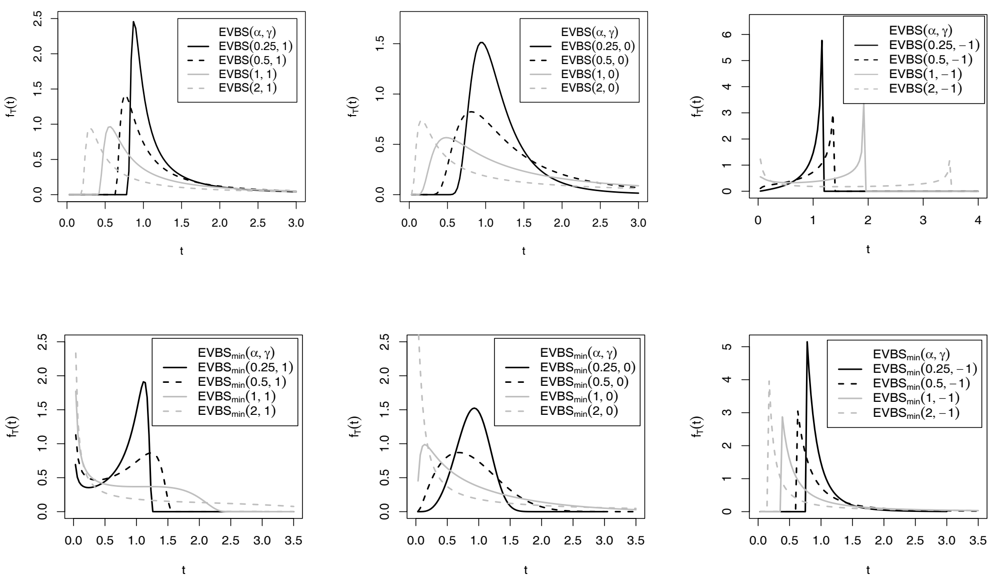

Figure 1: p.d.f. plots of the EVBS (1st panel) and EVBS* (2nd panel) distribution for $\beta = 1$ and the indicated values of (α, γ), where EVBS$^* \equiv$ EVBS$_{\text{min}}$.

4.1. Max-domains of attraction

We start with the Fréchet case and use the following necessary and sufficient condition for $F \in \mathcal{D}_{\mathrm{M}}(G_\gamma)$ with $\gamma > 0$, derived in Gnedenko (1943) (see also de Haan and Ferreira, 2006, Theorem 1.2.1-1):

$$(4.2) \qquad F \in \mathcal{D}_{\mathrm{M}}(G_\gamma),\ \gamma > 0 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-1/\gamma},$$

for all $x > 0$, and the right endpoint of F, namely $x^F := \inf\{x : F(x) \geq 1\}$, is necessarily infinity.

Theorem 4.1. *Let the c.d.f. of the r.v. Z be in the Fréchet max-domain of attraction, necessarily with a positive EV index, i.e., $\gamma_Z > 0$. Then, the c.d.f. of the r.v. T given in* (4.1) *is also in the Fréchet max-domain of attraction, i.e., $F_T \in \mathcal{D}_{\mathrm{M}}(G_{\gamma_T})$, with $\gamma_T = 2\gamma_Z$.*

Proof: By hypothesis, $F_Z \in \mathcal{D}_{\mathrm{M}}(G_{\gamma_Z})$ for $\gamma_Z > 0$. Thus, F_Z satisfies (4.2) for γ_Z. Then, we have that, as $t \to \infty$,

$$\frac{1 - F_T(tx)}{1 - F_T(t)} = \frac{1 - F_Z(a_{tx})}{1 - F_Z(a_t)} \approx \frac{1 - F_Z(\{tx/\beta\}^{1/2}/\alpha)}{1 - F_Z(\{t/\beta\}^{1/2}/\alpha)} \approx x^{-1/(2\gamma_Z)},$$

with the notation $u_t \approx v_t$ if and only if $u_t/v_t \to 1$, as $t \to \infty$. $\square$

For light right-tails, i.e., for the Weibull max-domain of attraction, we can prove a result similar to that of Theorem 4.1, if we use the following necessary and sufficient condition for $F \in \mathcal{D}_{\mathrm{M}}(G_\gamma)$ with $\gamma < 0$ (also derived in Gnedenko, 1943):

$$(4.3) \qquad F \in \mathcal{D}_{\mathrm{M}}(G_\gamma),\ \gamma < 0 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{1 - F(x^F - 1/\{tx\})}{1 - F(x^F - 1/t)} = x^{1/\gamma},$$

for all $x > 0$, and the right endpoint of F, namely x^F, is finite.

Theorem 4.2. *Let the c.d.f. of the r.v. Z be in the Weibull max-domain of attraction, necessarily with a negative EV index, i.e., $\gamma_Z < 0$. Then, the c.d.f. of the r.v. T given in* (4.1) *is also in the Weibull max-domain of attraction and $\gamma_T = \gamma_Z$.*

Proof: We have

$$\lim_{t\to\infty} \frac{1 - F_T(t^F - 1/\{tx\})}{1 - F_T(t^F - 1/t)} = \lim_{t\to\infty} \frac{1 - F_Z(a_{t^F - 1/\{tx\}})}{1 - F_Z(a_{t^F - 1/t})},$$

with t^F being the right endpoint of F_T. But we can assume, without loss of generality, that z^F, the right endpoint of F_Z, is null, i.e., $z^F = 0$. Hence, $t^F = \beta$ and, as $t \to \infty$,

$$\frac{1 - F_Z(a_{t^F - 1/\{tx\}})}{1 - F_Z(a_{t^F - 1/t})} \approx \frac{1 - F_Z(-\{\alpha\beta t x\}^{-1})}{1 - F_Z(-\{\alpha\beta t\}^{-1})} \approx x^{1/\gamma_Z}.$$

□

We next work in the slight more restrictive class of twice-differentiable c.d.f.'s $F \in \mathcal{D}_{\mathrm{M}}(G_\gamma)$, the so-called twice-differentiable domain of attraction of G_γ, denoted by $\widetilde{\mathcal{D}}_{\mathrm{M}}(G_\gamma)$. A possible characterization of the twice-differentiable domain of attraction of G_γ is due to Pickands (1986). Let us then assume that there exists F'', $f = F'$, and consider the function

$$k(x) = -f(x)/\{F(x)\log(F(x))\} = \big\{-\log(-\log F(x))\big\}'.$$

Hence, with $\gamma(x) = \{1/k(x)\}'$, we have

$$(4.4) \qquad F \in \widetilde{\mathcal{D}}_{\mathrm{M}}(G_\gamma) \quad \Longleftrightarrow \quad \lim_{x\uparrow x^F} \gamma(x) = \gamma.$$

Consequently, if $x^F = +\infty$, $\lim_{x\to\infty} x k(x) = 1/\gamma$, and if $x^F < +\infty$, $\lim_{x\uparrow x^F} (x^F - x)k(x) = -1/\gamma$, i.e., $\lim_{x\to\infty} k(x) = 0$, if $\gamma > 0$, and $\lim_{x\uparrow x^F} k(x) = +\infty$, if $\gamma < 0$. If $\gamma = 0$, we can have $k(x) \to 0$, $k(x) \to +\infty$, or $k(x) \to c$, for $0 < c < +\infty$. Observe also that, after some simple calculations, we can write

$$(4.5) \qquad \gamma(x) = F''(x)F(x)\log(F(x))/f^2(x) - \log(F(x)) - 1.$$

Theorem 4.3. *Let* $T \sim \mathrm{BST}(\alpha, \beta; f_Z)$, *with* Z *in the subset of the max-domain of attraction of* G_{γ_Z} *constituted by twice-differentiable c.d.f.'s, the so-called twice-differentiable max-domain of attraction of* G_{γ_Z}, *and assume that*

$$c = \lim_{t \uparrow t^F} F_Z(a_t) \log(F_Z(a_t)) A'_t / \{A_t^2 f_Z(a_t)\}$$

is finite, where t^F *is the right endpoint of the r.v.* T, a_t *and* A_t *are as given in* (2.3), *with* $A'_t = d\,A_t/dt$. *Then,* $F_T \in \mathcal{D}_{\mathrm{M}}(G_{\gamma_T})$, *with* $\gamma_T = \gamma_Z + c$.

Proof: By hypothesis, the necessary and sufficient condition (4.4) holds for Z, with F, γ and $\gamma(x)$ replaced with F_Z, γ_Z and $\gamma_Z(x)$, respectively. Now, just observe that, by applying (4.5), and then (2.3)-(2.5) and $A'_t = -(\sqrt{t/\beta} + 3\sqrt{\beta/t})/(4\alpha t^2)$, we have

$$\begin{aligned}\gamma_T(t) &= \frac{F''_T(t) F_T(t) \log(F_T(t))}{f_T^2(t)} - \log(F_T(t)) - 1 \\ &= \frac{F_Z(a_t) \log(F_Z(a_t)) A'_t}{A_t^2 f_Z(a_t)} + \frac{F''_Z(a_t) F_Z(a_t) \log(F_Z(a_t))}{f_Z^2(a_t)} - \log(F_Z(a_t)) - 1.\end{aligned}$$

On the basis of the limit in Theorem 4.3, the first term in the second line of (4.6) approaches c as $t \uparrow t^F$. Because the following term approaches γ_Z, the result follows. □

Corollary 4.1. *Under the conditions of* Theorem 4.3, *we have* $c = \gamma_Z$ *if* $\gamma_Z > 0$ *and* $c = 0$ *if* $\gamma_Z < 0$.

Example 4.1. We now provide a few illustrations of Corollary 4.1:

(i) If Z has Fréchet or Pareto distributions (in the Fréchet max-domain of attraction, i.e., $\gamma_Z > 0$), then the limit in Theorem 4.3 is $c = \gamma_Z$ and so $\gamma_T = 2\gamma_Z$. Indeed, as stated in Theorem 4.1, this result holds more generally in $\mathcal{D}_{\mathrm{M}}(G_{\gamma_Z})$, with $\gamma_Z > 0$.

(ii) If Z has Weibull or uniform distributions (in the Weibull max-domain of attraction, i.e., $\gamma_Z < 0$), then $c = 0$ and $\gamma_T = \gamma_Z$. In fact, as stated in Theorem 4.2, this result holds more generally in $\mathcal{D}_{\mathrm{M}}(G_{\gamma_Z})$, with $\gamma_Z < 0$.

Remark 4.1. We further conjecture that, in Corollary 4.1, we can often replace $\gamma_Z < 0$ by $\gamma_Z \leq 0$. This is supported by the examples of an r.v. Z either exponential or gamma, or Gumbel or normal, all in $\mathcal{D}_{\mathrm{M}}(G_0)$, i.e., with $\gamma_Z = 0$. Then $c = 0$ and $\gamma_T = \gamma_Z = 0$. Also, if Z has an exponential-type (ET) distribution, with a finite right endpoint, i.e., $F_Z(x) = K \exp(-c/\{z^F - x\})$, for $x < z^F$, $c > 0$, and $K > 0$ (again in the Gumbel max-domain of attraction), then also $c = 0$ and $\gamma_T = \gamma_Z = 0$.

Because in the twice-differentiable domain of attraction of G_γ the von Mises condition is a necessary and sufficient to have $\lim_{x\uparrow x^F}\gamma(x)=\gamma$, with $\gamma(x)=\{1/k(x)\}'$ (see Pickands, 1986, Theorem 5.2), we can also state that

$$F\in\widetilde{\mathcal{D}}_{\rm M}(G_\gamma) \iff \lim_{x\uparrow x^F}(1-F(x))F''(x)/(F'(x))^2=-\gamma-1. \tag{4.6}$$

Therefore, we can still write the following result.

Theorem 4.4. *Under the conditions and notations of* Theorem 4.3*, let us assume that*

$$c^*=\lim_{t\uparrow t^F}(1-F_Z(a_t))A_t'/\big(A_t^2F_Z'(a_t)\big)\;<\infty.$$

Then, $F_T\in\mathcal{D}_{\rm M}(G_{\gamma_T})$*, with* $\gamma_T=\gamma_Z-c^*$.

Proof: Just observe that

$$\begin{aligned}\frac{(1-F_T(t))F_T''(t)}{F_T'(t)^2}&=\frac{(1-F_Z(a_t))(A_t'F_Z'(a_t)+A_t^2F_Z''(a_t))}{A_t^2F_Z'(a_t)^2}\\&=\frac{(1-F_Z(a_t))A_t'}{A_t^2F_Z'(a_t)}+\frac{(1-F_Z(a_t))F_Z''(a_t)}{F_Z'(a_t)^2}.\end{aligned}\tag{4.7}$$

By hypothesis, as $t\uparrow t^F$, the last term in (4.7) converges to $-\gamma_Z-1$, and the result follows. □

Corollary 4.2. *Under the conditions and notations of* Theorem 4.4*, if we further assume that* Z *has an infinite right endpoint, then* $F_T\in\mathcal{D}_{\rm M}(G_{\gamma_T})$*, with* $\gamma_T=\gamma_Z$*, provided there exists a finite limit for* $(1-F_Z(x))/F_Z'(x)$*, as* $x\to\infty$*.*

4.2. Min-domains of attraction

We now analyze the domains of attraction for minima. To emphasize the possible difference between the right and left EV indices, we denote this last one as γ^*.

We reformulate conditions (4.2) and (4.3) for minima obtaining respectively

$$F\in\mathcal{D}_{\rm m}(G^*_{\gamma^*}),\ \gamma^*>0 \iff \lim_{t\to-\infty}\frac{F(tx)}{F(t)}=x^{-1/\gamma^*},\ \ \forall\,x>0, \tag{4.8}$$

and

$$F\in\mathcal{D}_{\rm m}(G^*_{\gamma^*}),\ \gamma^*<0 \iff \lim_{t\to-\infty}\frac{F(x_F-1/\{tx\})}{F(x_F-1/t)}=x^{1/\gamma^*},\ \ \forall\,x>0, \tag{4.9}$$

where the left endpoint $x_F := \inf\{x : F(x) > 0\}$ is finite; see, e.g., Galambos (1987, Theorem 2.1.5). Observe that a BST r.v. T cannot be in the Fréchet min-domain of attraction because its left endpoint is not $-\infty$; see, e.g., Galambos (1987, Theorem 2.1.4).

In the sequel, the notations $\text{Weibull}_{\text{min}}$, $\text{Frechet}_{\text{min}}$ and $\text{Gumbel}_{\text{min}}$ are used for denoting Weibull, Fréchet and Gumbel distributions for minima, respectively, with parameter γ^*, and z_F and t_F denoting the left endpoints of Z and T, respectively.

Theorem 4.5. *Let the c.d.f. of the r.v. Z be in the Weibull min-domain of attraction, necessarily with a negative EV index, i.e., $\gamma^*_Z < 0$. Then, the c.d.f. of the r.v. T given in* (4.1) *is in the Weibull min-domain of attraction and $\gamma^*_T = \gamma^*_Z$.*

Proof: Assume, without loss of generality, that $z_F = 0$, with z_F being the left endpoint of F_Z (i.e., $t_F = \beta$, with t_F being the left endpoint of F_T). Then,

$$\lim_{t\to-\infty} \frac{F_T(t_F - 1/\{tx\})}{F_T(t_F - 1/t)} = \lim_{t\to-\infty} \frac{F_Z(a_{t_F - 1/\{tx\}})}{F_Z(a_{t_F - 1/t})} = \lim_{t\to-\infty} \frac{F_Z(-\{\alpha\beta tx\}^{-1})}{F_Z(-\{\alpha\beta t\}^{-1})}$$

and the result follows from the fact that F_Z satisfies (4.9) for γ^*_Z. □

Theorem 4.6. *Let the c.d.f. of the r.v. Z be in the Fréchet min-domain of attraction, necessarily with a positive EV index, i.e., $\gamma^*_Z > 0$. Then, the c.d.f. of the r.v. T given in* (4.1) *is in the Weibull min-domain of attraction, and $\gamma^*_T = -2\gamma^*_Z$.*

Proof: Consider, without loss of generality, $t_F = 0$. Then, $z_F = -\infty$ and

$$\lim_{t\to-\infty} \frac{F_T(t_F - 1/\{tx\})}{F_T(t_F - 1/t)} = \lim_{t\to-\infty} \frac{F_Z(a_{-1/\{tx\}})}{F_Z(a_{-1/t})} = \lim_{t\to-\infty} \frac{F_Z(-\{-\beta tx\}^{1/2}/\alpha)}{F_Z(-\{-\beta t\}^{1/2}/\alpha)} = x^{-1/(2\gamma^*_Z)},$$

where the last step is due to the fact that F_Z satisfies (4.8) for γ^*_Z. □

Next, we again work in the slight more restrictive class of twice-differentiable c.d.f.'s, such as in Subsection 4.1. Analogously to the domain of attraction for maxima, the von Mises condition in (4.6), reformulated for minima, enables us to state that

$$F \in \widetilde{\mathcal{D}}_{\text{m}}(G^*_{\gamma^*}) \iff \lim_{x\downarrow x_F} F(x)F''(x)/(F'(x))^2 = \gamma^* + 1, \tag{4.10}$$

where x_F is the left endpoint of F and $\widetilde{\mathcal{D}}_{\text{m}}(G^*_{\gamma^*})$ denotes the twice-differentiable domain of attraction of $G^*_{\gamma^*}$.

Theorem 4.7. *Let $T \sim \mathrm{BST}(\alpha, \beta; f_Z)$, with Z in the subset of the min-domain of attraction of $G^*_{\gamma^*_Z}$ constituted by the twice-differentiable c.d.f.'s, the so-called twice-differentiable min-domain of attraction of $G^*_{\gamma^*_Z}$, and assume that*

$$d = \lim_{t \downarrow t_F} F_Z(a_t) A'_t / \big(A_t^2 F'_Z(a_t)\big)$$

is finite, where t_F is the left endpoint of T, a_t and A_t are as given in (2.3)*, with $A'_t = d\,A_t/dt$. Then, $F_T \in \mathcal{D}^*_{\mathrm{m}}(G^*_{\gamma^*_T})$, with $\gamma^*_T = \gamma^*_Z + d$.*

Proof: The result is easy to prove because

$$\frac{F_T(t) F''_T(t)}{f_T(t)^2} = \frac{F_Z(a_t)(A'_t f_Z(a_t) + A_t^2 F''_Z(a_t))}{A_t^2 f_Z(a_t)^2} = \frac{F_Z(a_t) A'_t}{A_t^2 f_Z(a_t)} + \frac{F_Z(a_t) F''_Z(a_t)}{f_Z(a_t)^2}.$$

□

Corollary 4.3. *Under the conditions of* Theorem 4.7*, we have $d = 0$ if $\gamma^*_Z < 0$ and $d = -3\gamma^*_Z$ if $\gamma^*_Z > 0$.*

Example 4.2. We next provide a few illustrations of Corollary 4.3:

(i) If Z has a Weibull distribution for minima (in the Weibull min-domain of attraction, i.e., $\gamma^*_Z < 0$), or exponential, Pareto or uniform distribution (also in the Weibull min-domain of attraction, with $\gamma^*_Z = -1$), or even a Gamma(p, q) distribution (in the Weibull min-domain of attraction, with $\gamma^*_Z = -1/p$), then $d = 0$ and $\gamma^*_T = \gamma^*_Z$. Indeed, as stated in Theorem 4.5, this result holds more generally in $\mathcal{D}_{\mathrm{m}}(G^*_{\gamma^*_Z})$, with $\gamma^*_Z < 0$.

(ii) If Z has a Fréchet distribution for minima (in the Fréchet min-domain of attraction, i.e., $\gamma^*_Z > 0$), then $d = -3\gamma^*_Z$ and $\gamma^*_T = -2\gamma^*_Z$. In fact, as stated in Theorem 4.6, this result holds more generally in $\mathcal{D}_{\mathrm{m}}(G^*_{\gamma^*_Z})$, with $\gamma^*_Z > 0$.

(iii) If Z has an ET distribution as in Remark 4.1 (in the Fréchet min-domain of attraction, with $\gamma^*_Z = 1$), then $d = -3\gamma^*_Z = -3$ and $\gamma^*_T = -2$, i.e., T belongs to the Weibull min-domain of attraction.

Remark 4.2. Similarly to what we did in Remark 4.1, we further conjecture that, in Corollary 4.3, we can often replace $\gamma^*_Z < 0$ by $\gamma^*_Z \le 0$. This is supported by the fact that if Z has a Gumbel distribution for minima, or any of the limiting distributions for maxima (Fréchet, Gumbel, Weibull), or a normal distribution (all in the Gumbel min-domain of attraction, i.e. $\gamma^*_Z = 0$), then the limit in Theorem 4.7 is $d = 0$ and $\gamma^*_T = \gamma^*_Z = 0$.

5. HAZARD ANALYSIS

We may define a hazard as a dangerous event that could conduct to an emergency or disaster. The origin of this event may be due to a situation that could have an adverse effect. Thus, a hazard is a potential and not an actual possibility, i.e., it can be statistically evaluated. A hazard analysis is the assessment of a risk that is present in a particular environment. Therefore, hazard assessment allows us to evaluate potential risk by the estimated frequency or intensity of the r.v. of interest. In this section, we study the EVBS h.r. and its change point, denoted by t_c. Such a change point, being defined as a point where the h.r., $h(t) = f(t)/(1 - F(t)$, attains either a maximum or a minimum value, is the solution of the equation $f(t_c) = -f'(t_c)/h(t_c)$, whenever F is twice-differentiable, and such a solution exists.

5.1. Hazard rate

Statistically, a hazard analysis can be carried out by the h.r. function. Apart from hazard rate, this function is also known as chance function, failure rate, force of mortality, intensity function, or risk rate, among other names. In actuarial science, for example, the h.r. is the annualized probability that a person at a specified age will die in the next instant, expressed as a death rate per year. For more details about the concept of h.r., see Marshall and Olkin (2007, pp. 10-13).

A nice property of the h.r. is that it allows us to characterize the behavior of statistical distributions. For example, the h.r. may have several different shapes such as increasing (IHR), constant (exponential distribution), decreasing (DHR), bathtube (BT), inverse bathtube (IBT or upside-down) approaching to a non-null constant and IBT approaching to zero. An incorrect specification of the h.r. could have serious consequences in the analysis; see, e.g., Bhatti (2010) for a study about this issue.

The h.r. of the r.v. T is given in general by $h_T(t) = f_T(t)/R_T(t)$, for $t > 0$, and $0 < F_T(t) < 1$, where $R_T(t) = 1 - F_T(t)$, for $t > 0$, is the reliability function (r.f.), and f_T and F_T are the p.d.f. and c.d.f. of the r.v. T.

5.2. TTT curve

The h.r. of an r.v. T can be characterized by its corresponding total time on test (TTT) function given by

$$H_T^{-1}(u) = \int_0^{F_T^{-1}(u)} (1 - F_T(y))\, dy$$

or by its scaled version given by $W_T(u) = H_T^{-1}(u)/H_T^{-1}(1)$, for $0 \leq u \leq 1$, where once again F_T^{-1} is the generalized inverse function of the c.d.f. of T.

Now, W_T can be empirically approximated, allowing to construct the empirical scaled TTT curve by plotting the consecutive points $\left[k/n, W_n(k/n)\right]$, where $W_n(k/n) = \{\sum_{i=1}^{k} T_{(i)} + (n-k)\, T_{(k)}\}/\sum_{i=1}^{n} T_{(i)}$, for $k = 1, \ldots, n$, with $T_{(i)}$ being the corresponding ith ascending order statistic, for $1 \le i \le n$.

From Figure 2 (left), we observe different theoretical shapes for the scaled TTT curve. Thus, a TTT plot expressed by a curve that is concave (or convex) corresponds to the IHR (or DHR) class. A concave (or convex) and then convex (or concave) shape for the TTT curve corresponds to a BT (or IBT) hazard rate. A TTT plot represented by a straight line is an indication that the exponential distribution must be used. Thus, a graphical plot of the empirical scaled TTT curve could provide to us the type of distribution that the data have. See also in Figure 2 the theoretical scaled TTT curves for EVBS (center) and EVBS* (right) models. In these plots, we again use the notation $\text{EVBS}(\alpha, \gamma) \equiv \text{EVBS}(\alpha, 1, \gamma)$

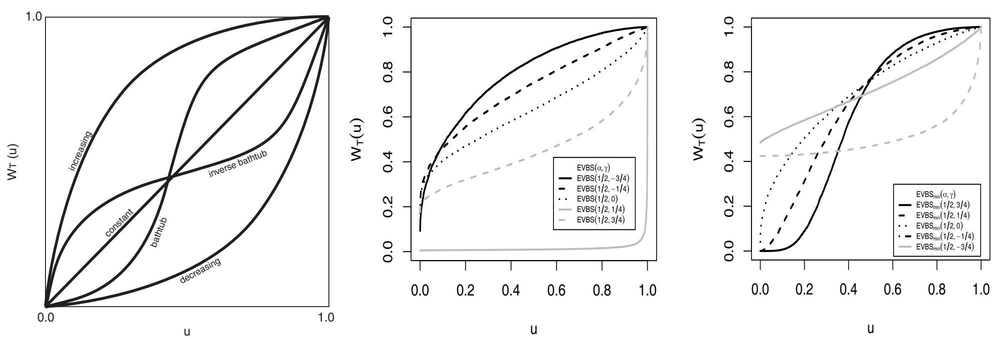

Figure 2: theoretical scaled TTT curves for a general model with the indicated h.r. shape (left), for the $\text{EVBS}(0.5, \gamma)$ model for the indicated values of γ (center), and for the $\text{EVBS}^*(0.5,\, \gamma)$ distribution for the indicated values of γ (right).

5.3. EVBS hazard rate

The normal distribution is in the IHR class. The gamma and Weibull distributions can be either in IHR or DHR classes (of course, the case of the exponential distribution with constant h.r. is considered by these two models). However, the lognormal (LN) distribution has a non-monotonic h.r., because it is initially increasing until its change point and then it decreases to zero, i.e., the LN model is in the IBT h.r. class. The BS h.r. behaves similarly to the LN h.r., i.e., it is initially increasing until its change point and then decreasing not to zero, but to a positive constant. Thus, although BS, gamma, LN and Weibull distributions have densities with similar shapes, their h.r.'s are totally different.

Let $T \sim \text{EVBS}(\alpha, \beta, \gamma)$. Then, directly from the definition of the p.d.f., $f_T(t)$, and the c.d.f., $F_T(t)$, of the r.v. $T \sim \text{EVBS}(\alpha, \beta, \gamma)$, given in (3.2) and (3.3), respectively, we have that:

A. The r.f. of T is expressed as $R_T(t) = 1 - F_T(t)$, with $F_T(t)$ given in (3.3).

B. Again with a_t and A_t as given in (2.3), the h.r. of T is defined as

$$h_T(t) = \frac{f_T(t)}{R_T(t)} = \begin{cases} A_t\,(1+\gamma\,a_t)^{-1-1/\gamma}/\big(\exp(\{1+\gamma\,a_t\}^{-1/\gamma})-1\big), & \gamma \neq 0, \\ A_t \exp(-a_t)/\exp(\exp(-a_t))-1, & \gamma = 0, \end{cases}$$

where $t > (\alpha^2\beta + 2\beta\gamma^2)/(2\gamma^2) - \sqrt{(\alpha^4\beta^2 + 4\alpha^2\beta^2\gamma^2)/\gamma^4}/2$ if $\gamma > 0$; $t > 0$ if $\gamma = 0$, and $0 < t < (\alpha^2\beta + 2\beta\gamma^2)/(2\gamma^2) + \sqrt{(\alpha^4\beta^2 + 4\alpha^2\beta^2\gamma^2)/\gamma^4}/2$ if $\gamma < 0$.

C. With the notation $b_{t_c} = 1 + \gamma a_{t_c}$, the change point t_c of the h.r. of T is obtained as the solution of the equations:

$$\begin{cases} \big\{\big[A'_{t_c} - (A_{t_c})^2(1+\gamma)b_{t_c}^{-1}\big]\big[\exp(b_{t_c}^{-1/\gamma})-1\big] \\ \qquad +(A_{t_c})^2(1+\gamma a_{t_c})^{-1-1/\gamma}\exp(b_{t_c}^{-1/\gamma})\big\}b_{t_c}^{-1-1/\gamma} = 0, & \gamma \neq 0 \\ (A_{t_c})^2\big[1 + (\exp(-a_{t_c})-1)\exp(\exp(-a_{t_c}))\big] \\ \qquad +A'_{t_c}\big[\exp(\exp(-a_{t_c}))-1\big] = 0, & \gamma = 0. \end{cases}$$

Let $T^* \sim \text{EVBS}^*(\alpha,\beta,\gamma)$. Then, now, directly from the definition of the p.d.f., $f_{T^*}(t)$, and the c.d.f., $F_{T^*}(t)$, of the r.v. $T^* \sim \text{EVBS}^*(\alpha,\beta,\gamma)$, given in (3.4) and (3.5), respectively, we have that:

D. The r.f. of T^* is expressed as $R_{T^*}(t) = 1 - F_{T^*}(t)$, with $F_{T^*}(t)$ as given in (3.5).

E. Again with a_t and A_t as given in (2.3), the h.r. of T^* is defined as

$$h_{T^*}(t) = \frac{f_{T^*}(t)}{R_{T^*}(t)} = \begin{cases} A_t\,(1-\gamma\,a_t)^{-1-1/\gamma}, & \gamma \neq 0, \\ A_t \exp(a_t), & \gamma = 0, \end{cases}$$

where $0 < t < (\alpha^2\beta + 2\beta\gamma^2)/(2\gamma^2) + \sqrt{(\alpha^4\beta^2 + 4\alpha^2\beta^2\gamma^2)/\gamma^4}/2$ if $\gamma > 0$; $t > 0$ if $\gamma = 0$, and $t > (\alpha^2\beta + 2\beta\gamma^2)/(2\gamma^2) - \sqrt{(\alpha^4\beta^2 + 4\alpha^2\beta^2\gamma^2)/\gamma^4}/2$ if $\gamma < 0$.

F. The change point t_c of the h.r. of T^* is obtained as the solution of the equations:

$$\begin{cases} (1-\gamma a_{t_c})^{-1-1/\gamma}\big[A'_{t_c} + (1+\gamma)(A_{t_c})^2(1-\gamma a_{t_c})^{-1}\big] = 0, & \gamma \neq 0 \\ (A_{t_c})^2 + A'_{t_c} = 0, & \gamma = 0. \end{cases}$$

As can be seen in Figure 3, the EVBS and EVBS* h.r.'s present several different shapes going through all the h.r. shape classes mentioned above. In these plots, once again, we use the notation $\text{EVBS}(\alpha,\gamma) \equiv \text{EVBS}(\alpha,1,\gamma)$. The h.r. of T can also approach ∞, zero or a positive constant, as $t \to \infty$. These are strong points in favor of our models, as they become interesting for modeling purposes.

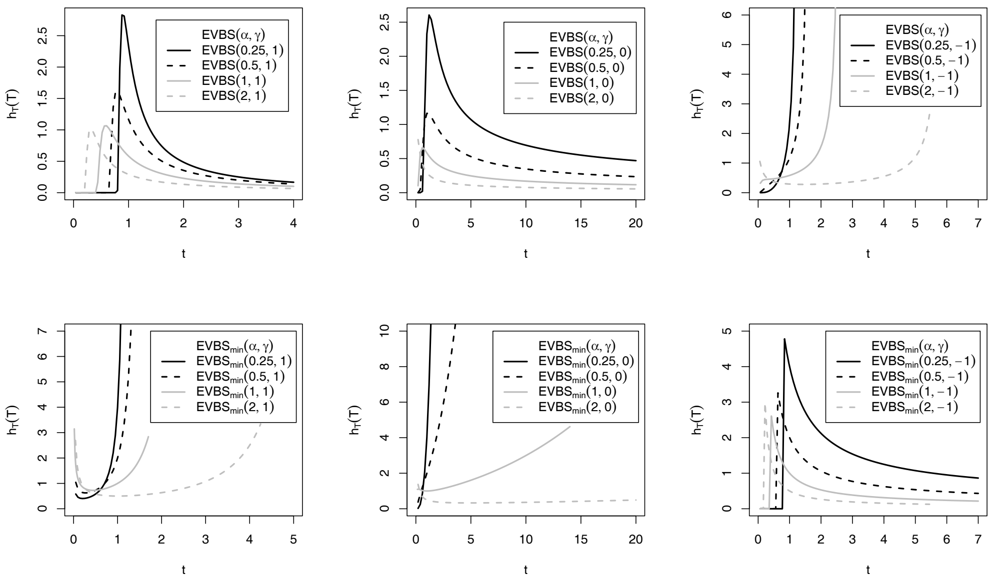

Figure 3: h.r. plots of the EVBS (1st panel) and EVBS* (2nd panel) distributions for $\beta = 1$, and the indicated values of (α, γ), where EVBS$^* \equiv$ EVBSmin.

6. ESTIMATION AND MODEL CHECKING

In this section, we present some results related to estimation aspects and model checking for EVBS distributions.

6.1. ML estimation

As is well-known, ML estimates are obtained from the solution of the system $\dot{\ell}(\boldsymbol{\theta}) \equiv \mathbf{0}$, where $\dot{\ell}(\boldsymbol{\theta})$ denotes the score vector of first derivatives of the logarithm of the likelihood function, namely $\ell(\boldsymbol{\theta})$. In our case, if we consider the EVBS model (the procedure is similar for the EVBS* model), this function is given by $\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n} \ell_i(\boldsymbol{\theta})$, where, for $i = 1, \ldots, n$,

$$\ell_i(\boldsymbol{\theta}) = \begin{cases} \log(A_{t_i}) - (1 + \frac{1}{\gamma}) \log(1 + \gamma\, a_{t_i}) - (1 + \gamma\, a_{t_i})^{-1/\gamma}, & \gamma \neq 0, \\ \log(A_{t_i}) - \exp(-a_{t_i}) - a_{t_i}, & \gamma = 0, \end{cases}$$

with $\boldsymbol{\theta} = [\alpha, \beta, \gamma]^\top$. The score vector $\dot{\ell}(\boldsymbol{\theta}) = \partial\ell(\boldsymbol{\theta})/\partial\boldsymbol{\theta} = [\dot{\ell}_{\theta_1}]$, with $\theta_1 = \alpha$, β, or γ, is given by

$$\begin{bmatrix} -\frac{n}{\alpha} + \sum_{i=1}^{n}\Big(\frac{\{1+\gamma\}a_{t_i}}{\alpha\{1+\gamma a_{t_i}\}} - \frac{1}{\alpha}a_{t_i}\{1+\gamma a_{t_i}\}^{-1-1/\gamma}\Big) \\ \sum_{i=1}^{n}\Big(-\frac{\alpha}{2\beta}\frac{a_{t_i}}{\sqrt{\frac{t_i}{\beta}}+\sqrt{\frac{\beta}{t_i}}} + \frac{\{1+\gamma\}}{2\alpha\beta\{1+\gamma a_{t_i}\}}\Big\{\sqrt{\frac{t_i}{\beta}}+\sqrt{\frac{\beta}{t_i}}\Big\} - \frac{\sqrt{\frac{t_i}{\beta}}+\sqrt{\frac{\beta}{t_i}}}{2\alpha\beta}\{1+\gamma a_{t_i}\}^{-1-1/\gamma}\Big) \\ \sum_{i=1}^{n}\big(-\frac{\{1+\gamma a_{t_i}\}^{-1/\gamma}}{\gamma^2}\big\{\log\big(1+\gamma a_{t_i}\big) - \frac{\gamma a_{t_i}}{1+\gamma a_{t_i}}\big\} - \frac{\{\gamma+1\}a_{t_i}}{\gamma\{1+\gamma a_{t_i}\}} + \frac{\log(1+\gamma a_{t_i})}{\gamma^2}\big) \end{bmatrix},$$

whenever $\gamma \neq 0$ and, if $\gamma = 0$, is given by

$$\begin{bmatrix} -\frac{n}{\alpha} - \sum_{i=1}^{n}\big(\frac{1}{\alpha}\exp(-a_{t_i})a_{t_i} - \frac{1}{\alpha}a_{t_i}\big) \\ \sum_{i=1}^{n}\big(-\frac{1}{2\alpha\beta}\exp(-a_{t_i})\{\sqrt{\frac{t_i}{\beta}}+\sqrt{\frac{\beta}{t_i}}\} + \frac{1}{2\alpha\beta}\{\sqrt{\frac{t_i}{\beta}}+\sqrt{\frac{\beta}{t_i}}\} - \frac{\alpha}{2\beta}\frac{a_{t_i}}{\sqrt{t_i/\beta}+\sqrt{\beta/t_i}}\big) \end{bmatrix}.$$

In this case, the system of likelihood equations $\dot{\ell}(\boldsymbol{\theta}) \equiv \mathbf{0}$ does not produce an explicit solution so that a numerical procedure is necessary. To this end, initial values for the parameters α, β and γ can be obtained using the methods to be described in Subsection 6.2. In addition, these likelihood equations seem to be often unstable. We propose to use the following approach for solving this problem of instability. The approach consists of obtaining the optimum value for the parameter γ assuming it to be known, for example, following a similar algorithm to that proposed by Rinne (2009, pp. 426-433) and called by him as non-failing (NF); see also Barros *et al.* (2009). In these works, they fixed values for their parameter, in our case γ, within a set of several possible values for this parameter, and they then estimate the structural parameters, in our case, α and β. Finally, we consider the fixed γ that maximizes the likelihood function. Specifically, this approach is based on a partition of the real number set into a suitable amount of sub-intervals. Fixing γ in each of these intervals, we estimate α and β by using the ML method and then we look for the value of γ that maximizes the likelihood function. In this case, the NF algorithm is given by:

NF1 For a fixed value of γ:

- **NF1.1** Estimate the parameters α and β of the EVBS model using the estimates of α and β from the procedure to be described in Subsection 6.2 as starting values.
- **NF1.2** Compute the associated likelihood function.

NF2 Choose the value of γ that maximizes the likelihood function and then consider the obtained ML estimates of α and β as result.

6.2. Starting estimation

Firstly, to find initial values for the numerical optimization procedure needed for the ML estimation of the EVBS distribution parameters described in Subsection 6.1, we introduce a graphical method analogous to the probability plots; see Leiva *et al.* (2008a). This method is useful for goodness-of-fit and can also be used as an estimation method or, at least, to find initial values for an iterative procedure. The method consists of transforming the data forming pairs of values that should follow a linear relationship if these data would come from the EVBS distribution. Then, by using a simple linear regression method, the slope and the intercept of this linear relationship are estimated. The line is used for goodness-of-fit such as a quantile versus quantile (QQ) plot. Specifically, if we consider the EVBS c.d.f. as given in (2.2), but with Φ replaced by F_Z, we have $t = \beta + \alpha\sqrt{\beta}\,\sqrt{t}\,F_Z^{-1}(F_T(t))$, where F_Z^{-1} is the generalized inverse c.d.f. of the generator EV distribution and F_T is the EVBS c.d.f. However, it is difficult to derive a linear function over t in the above expression, which is fundamental for probability plots. We consider $p = \sqrt{t}\,F_Z^{-1}(F_T(t))$ obtaining the linear function $y \approx a + b\,x$, where $x = p$, $y = t$, the intercept is $a = \beta$, and the slope is $b = \alpha\sqrt{\beta}$. Now, suppose we have n ordered observations, say $t_{(1)} \leq \ldots \leq t_{(n)}$. Because we can estimate $F_T(t_{(i)})$ by $p_i = (i-0.3)/(n+0.4)$, for $i = 1, \ldots, n$, the graphical plot of $t_{(i)}$ versus $\bar{p}_i$, where $\bar{p}_i = \sqrt{t_{(i)}}\,F_Z^{-1}(p_i)$ is approximately a straight line whenever the data come from some EVBS distribution. Goodness-of-fit can be visually and analytically studied using the coefficient of determination of the fit of regression. Therefore, the parameters α and β of this distribution can be estimated by using the least square method obtaining $\bar{\beta} = \bar{a}$ and $\bar{\alpha} = \bar{b}/\sqrt{\bar{a}}$.

Secondly, to find an initial value for the parameter γ, we can use a landmark from the EV theory. This landmark is the result about the limiting generalized Pareto (GP) behavior of the scaled excesses; see, e.g., Balkema and de Haan (1974) and Pickands (1975). This enables the development of the so-called ML EV index estimators, which we can take as an initial value for γ to be used for the numerical optimization procedure needed for the ML estimation of the EVBS distribution parameters. We refer the peaks over threshold methodology of estimation (see Smith, 1987) as well as the methodology used by Drees *et al.* (2004), named peaks over random threshold in Araújo Santos *et al.* (2006).

6.3. Model checking

Once the EVBS distribution parameters are estimated, a natural question that arises is checking how good is the fit of the model to the data. We can use the invariance property of the ML estimators for fitting the p.d.f. and c.d.f. of the EVBS model. Also, to compare the EVBS distributions to other distributions, we can use model selection procedure based on loss of information, such as Akaike (AIC), Schwarz's Bayesian (BIC) and Hannan-Quinn (HQIC) information criteria. These criteria allow us to compare models for the same data set and are given by $\text{AIC} = -2\ell(\widehat{\boldsymbol{\theta}}) + 2d$, $\text{BIC} = -2\ell(\widehat{\boldsymbol{\theta}}) + d\log(n)$, and $\text{HQIC} = -2\ell(\widehat{\boldsymbol{\theta}}) + 2d\log(\log(n))$, where, as mentioned, $\ell(\widehat{\boldsymbol{\theta}})$ is the log-likelihood function for the parameter $\boldsymbol{\theta}$ associated with the model evaluated at $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}$, n is the sample size, and d is the dimension of the parameter space.

AIC, BIC and HQIC are based on a penalization of the likelihood function as the model becomes more complex, i.e., with more parameters. Thus, a model whose information criterion has a smaller value is better. This is an important point, because the EVBS distribution has more parameters than the usual BS distribution. Because models with more parameters always provide a better fit, AIB, BIC and HQIC allow us to compare models with different numbers of parameters due to the penalization incorporated in such criteria. This methodology is very general and can be applied even to non-nested models, i.e., those models that are not particular cases of a more general model; see Vilca *et al.* (2011) and references therein.

Generally, differences between two values of the information criteria are not very noticeable. In that case, the Bayes factor (BF) can be used to highlight such differences, if they exist. To define the BF, assume the data D belong to one of two hypothetical models, namely M_1 and M_2, according to probabilities $\mathbb{P}(\mathrm{D}|\mathrm{M}_1)$ and $\mathbb{P}(\mathrm{D}|\mathrm{M}_2)$, respectively. Given probabilities $\mathbb{P}(\mathrm{M}_1)$ and $\mathbb{P}(\mathrm{M}_2) = 1 - \mathbb{P}(\mathrm{M}_1)$, the data produce conditional probabilities $\mathbb{P}(\mathrm{M}_1|\mathrm{D})$ and $\mathbb{P}(\mathrm{M}_2|\mathrm{D}) = 1-\mathbb{P}(\mathrm{M}_1|\mathrm{D})$, respectively. Then, the BF that allows to us comparing M_1 (model considered as correct) to M_2 (model to be contrasted with M_1) is given by

$$B_{12} = \frac{\mathbb{P}(\mathrm{D}|\mathrm{M}_1)}{\mathbb{P}(\mathrm{D}|\mathrm{M}_2)}. \tag{6.1}$$

Based on (6.1), we can use the approximation

$$2\log(B_{12}) \approx 2\big[\ell(\widehat{\boldsymbol{\theta}}_1) - \ell(\widehat{\boldsymbol{\theta}}_2)\big] - [d_1 - d_2]\log(n), \tag{6.2}$$

where $\ell(\widehat{\boldsymbol{\theta}}_k)$ is the log-likelihood function for the parameter $\boldsymbol{\theta}_k$ under the model M_k evaluated at $\boldsymbol{\theta}_k = \widehat{\boldsymbol{\theta}}_k$, d_k is the dimension of $\boldsymbol{\theta}_k$, for $k = 1, 2$, and n is the sample size. Notice that the approximation in (6.2) is computed subtracting the BIC value from the model M_2, given by $\mathrm{BIC}_2 = -2\ell(\boldsymbol{\theta}_2) + d_2\log(n)$, to the BIC value of the model M_1, given by $\mathrm{BIC}_1 = -2\ell(\boldsymbol{\theta}_1) + d_1\log(n)$. In addition, notice that if model M_2 is a particular case of M_1, then the procedure corresponds to applying the likelihood ratio (LR) test. In this case, $2\log(B_{12}) \approx \chi^2_{12} - \mathrm{df}_{12}\log(n)$, where χ^2_{12} is the LR test statistic for testing M_1 versus M_2 and $\mathrm{df}_{12} = d_1 - d_2$ are the d.f.'s associated with the LR test, so that one can obtain the corresponding p-value from $2\log(B_{12}) \mathrel{\dot\sim} \chi^2(d_1 - d_2)$, with $d_1 > d_2$.

In general, the BF is informative because it presents ranges of values in which the degree of superiority of one model with respect to another can be quantified. An interesting interpretation of the BF is displayed in Table 1; see Vilca *et al.* (2011) and references therein.

Table 1: interpretation of $2\log(B_{12})$ associated with the BF.

$2\log(B_{12})$	Evidence in favor of M_1
< 0	Negative (M_2 is accepted)
$[0, 2)$	Weak
$[2, 6)$	Positive
$[6, 10)$	Strong
≥ 10	Very strong

7. APPLICATION

In this section, to illustrate some of the results obtained in this study, we fit the EVBS* model (for minimum) to a real data set corresponding to air pollutant concentrations. We assume that the data are uncorrelated and independent and, therefore, a diurnal or cyclic trend analysis is not necessary. This assumption has been supported by some authors for different reasons; see, e.g., Vilca *et al.* (2010) and references therein. For example, environmental data are sometimes reported as average or total values and so spatial-time dependence is missing. In this analysis, we first discuss an implementation in `R` code of the EVBS model. Next, the data set upon analysis is introduced. Then, an EDA is produced. Finally, estimation and EVBS model checking are carried out.

7.1. Implementation in R code

Several `R` packages for analyzing data from different distributions are available from CRAN (for example, the `bs` and `gbs` packages). An `R` package named `evbs` to analyze data from EVBS models is being developed by the authors, whose "in progress" version is available upon request. This package contains diverse indicators and methodologies useful for EVBS distributions. In addition, the `evbs` package incorporates the scaled TTT curve as a descriptive tool to identify the possible shape of the hazard rate.

7.2. The data set

The data correspond to daily ozone concentrations that were collected in New York during May-September, 1973. These data were taken from Nadarajah (2008) and have been provided by the New York State Department of Conservation. This set of daily ozone level measurements (in ppb = ppm ×1000), that we call from now simply `ozone`, are: 41, 36, 12, 18, 28, 23, 19, 8, 7, 16, 11, 14, 18, 14, 34, 6, 30, 1, 11, 4, 32, 23, 45, 115, 37, 29, 71, 39, 23, 21, 37, 20, 12, 13, 49, 32, 64, 40, 77, 97, 97, 85, 10, 27, 7, 48, 35, 61, 79, 63, 16, 108, 20, 52, 82, 50, 64, 59, 39, 9, 16, 78, 35, 66, 122, 89, 110, 44, 65, 22, 59, 23, 31, 44, 21, 9, 45, 168, 73, 76, 118, 84, 85, 96, 78, 91, 47, 32, 20, 23, 21, 24, 44, 21, 28, 9, 13, 46, 18, 13, 24, 16, 23, 36, 7, 14, 30, 14, 18, 20, 11, 135, 80, 28, 73, 13.

7.3. Exploratory data analysis

Firstly, an analysis of autocorrelation indicates that there is not such autocorrelation so that the dependence over time can be discarded. Thus, the use of a methodology based on univariate random samples is adequate for ozone. Secondly, Table 2 presents a descriptive summary of these data and Figure 4 (left) displays their histogram. This table and histogram indicate a positively skewed distribution. Thirdly, the TTT plot shown in Figure 4 (right) indicates that ozone seems to have a h.r. that is coherent with that of the EVBS distributions. However, maybe the more relevant aspect of the EDA of these ozone levels is noted when we analyze the original boxplot and the adjusted boxplot for asymmetric distributions. Interestingly, the original boxplot displayed in Figure 4 (first plot on the center figure) shows some atypical observations lying on the right-tail of the distribution of ozone, but this boxplot was constructed for symmetric data. When we produce the adjusted boxplot for asymmetric distributions using ozone, there are not atypical observations on the right-tail. Nevertheless, this type of observations appear on the left-tail of the distribution of the data; see Figure 4 (second plot on the center figure). For more details about this adjusted boxplot for asymmetric data, see Hubert and Vandervieren (2008), an `R` package called `robustbase` and its function `adjbox`. Therefore, the EDA provides to us diverse evidences for supporting the use of the EVBS model to describe ozone.

Table 2: descriptive statistics for ozone (in ppb = ppm $\times$1000).

Median	Mean	SD	CV	CS	CK	Range	Min.	Max.	n
31.50	42.13	32.99	78.30%	1.21	3.11	167.00	1.00	168.00	116

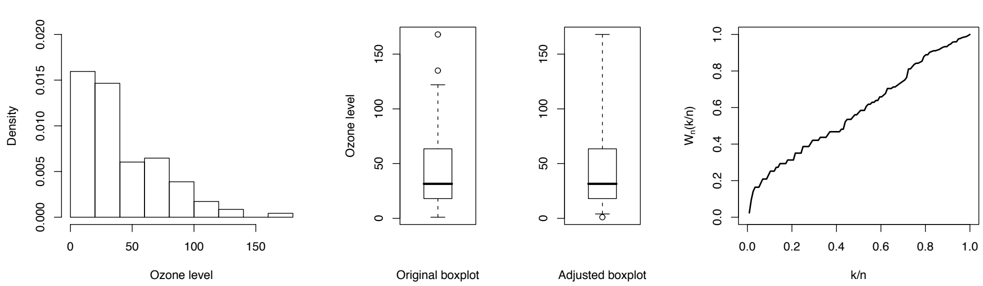

Figure 4: histogram (left), an indicated boxplot (center) and TTT plot (right) for ozone.

The EVBS* distribution should accommodate the observations concentrated on the left-tail well. Then, we think the EVBS* distribution based on the Gumbel$_{\min}$ should be an appropriate model for describing ozone. Because this model belongs to the Gumbel min-domain of attraction (see Subsection 4.2), we carry out a semi-parametric EV test to analyze whether ozone belongs to this domain or not. Specifically, we want to test H_0: $F \in \mathcal{D}_{\mathrm{m}}(G^*_\gamma)$, with $\gamma \geq 0$. For details about this test, see Dietrich *et al.* (2002). In Figure 5, we see the sample path of the test statistic as a function of the k largest order statistics and the critical value (horizontal line) above which we reject the null hypothesis. We do not reject H_0 for $1 \leq k \leq 60$, which is a credible result in EV theory to keep this hypothesis. Observe that we cannot have $\gamma > 0$ because the left endpoint $-\infty$ does not make sense for these data (the daily ozone measures must be greater or equal to 0).

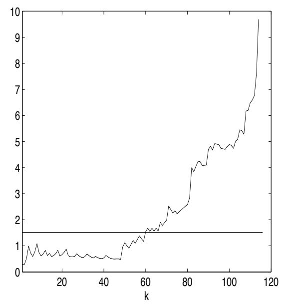

Figure 5: sample path of the extreme value condition test applied to the ozone data (horizontal line: critical value above which we reject $F \in \mathcal{D}_{\mathrm{m}}(G^*_\gamma)$, with $\gamma \geq 0$).

Next, we fit the EVBS* model based on the $\text{Gumbel}_{\min}$ distribution to ozone. The GEV and the GP models are also considered in this comparative study of alternative models. In addition, a skew-normal BS (SNBS) model is also fitted, because according to Vilca *et al.* (2011), a SNBS distribution has heavier tails than the usual BS distribution. This characteristic can also be obtained by, for instance, a BS model based on the Student-t distribution. Moreover, when the distribution of the data is concentrated on the left-tail, a SNBS distribution should be a better alternative than the usual BS distribution or a BS model based on any symmetric distribution, such as the Student-t model. In fact, if ozone comes from a SNBS model and we fit the usual BS distribution, we overestimate the lower percentiles. However, the EVBS distributions introduced here are also good alternatives for modeling data following a distribution with heavier tails than the usual BS distribution.

7.4. Estimation and checking model

To find the ML estimates of the EVBS distribution parameters, we use the procedure given in Subsections 6.1 and 6.2. Thus, based on ozone, we obtain the ML estimates along with the values of AIC, BIC, HQIC and BF used for model selection; see Table 3.

Table 3: ML estimates, information criteria and Bayes factors in the indicated models for ozone.

Distribution	$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\theta}_3$	$-\ell$	AIC	BIC	HQIC	$2\log(B_{12})$
$\text{EVBS}^*(\alpha,\beta,\gamma)$	0.80	45.68	0.00	541.31	1088.62	1088.81	1084.51	–
$\text{GEV}(\mu,\sigma,\gamma)$	24.00	18.34	0.36	543.78	1093.55	1093.75	1089.44	4.93
$\text{GP}(\sigma,\gamma)$	–	52.49	-0.25	546.19	1096.37	1096.50	1093.63	5.00
$\text{SNBS}(\alpha,\beta,\lambda)$	1.27	14.84	1.07	545.61	1097.21	1097.40	1093.10	8.59
$\text{BS}(\alpha,\beta)$	0.98	28.02	–	549.10	1102.19	1102.32	1099.45	10.82

From Table 3, we note that the EVBS* model based on the Gumbel$_{\text{min}}$ distribution has lower values of AIC, BIC and HQIC with respect to the BS, EV, GP and SNBS models for ozone data. This is a first indication of the superiority of the proposed model. Then, we use the BF to establish the magnitude of the differences between the values of the BIC of the proposed model and of its competitors. Thus, according to Table 1 and the BF's (approximated by the BIC's) also given in Table 3, we detect for the EVBS* model (i) a very strong evidence in its favor with respect to the BS model, (ii) a strong evidence with respect to the SNBS model and (iii) a positive evidence with respect to the GEV and GP models. This is a second more power indication of the superiority of the proposed model. Now, from Figure 6 (center), we see the excellent coherence between the empirical and EVBS theoretical c.d.f.'s. for ozone. Moreover, a QQ plot for the EVBS distribution shown in Figure 6 (right) confirms such a coherence between the EVBS* model and the data. In fact, the histogram and the estimated EVBS* p.d.f. based on the Gumbel$_{\text{min}}$ distribution provided in Figure 6 (left) also shows an excellent fit of the EVBS* model to these ozone data. Therefore, we conclude that the EVBS* distribution provides a much better fit than the other considered models for the ozone data analyzed in this study.

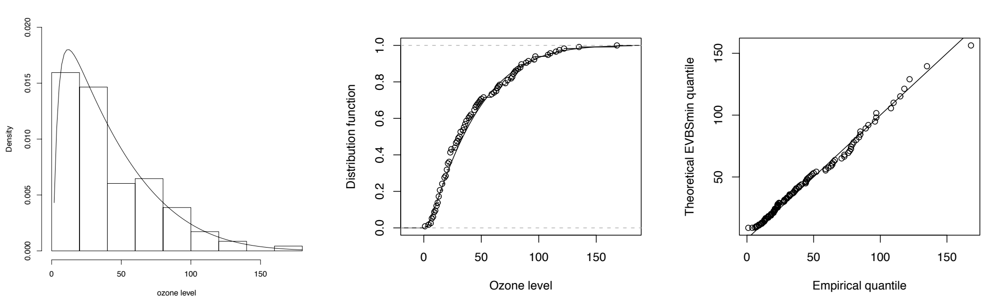

Figure 6: histogram with estimated EVBS* (Gumbel$_{\text{min}}$) density (left), empirical and theoretical c.d.f. plots (center) and QQ plot (right) for ozone.

8. CONCLUDING REMARKS

This article has dealt with an extreme value version of the Birnbaum-Saunders distribution. Specifically, we have found the density of the extreme value Birnbaum-Saunders distribution and discussed its shape. We have obtained the cumulative distribution function and quantile function of this distribution as well as highlighted some of their properties. Extremal domains of attraction for Birnbaum-Saunders type distributions have been studied. A characterization of the hazard rate of extreme value Birnbaum-Saunders distributions has been also carried out. We have developed an `R` package with the obtained results and used part of it for analyzing a real data set of ozone concentrations. This analysis has allowed us to show the adequacy of these new statistical distributions.

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