Matrix representations of a special polynomial sequence in arbitrary dimension

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Abstract

This paper provides an insight into different structures of a special polynomial sequence of binomial type in higher dimensions with values in a Clifford algebra. The elements of the special polynomial sequence are homogeneous hypercomplex differentiable (monogenic) functions of different degrees and their matrix representation allows to prove their recursive construction in analogy to the complex power functions. This property can somehow be considered as a compensation for the loss of multiplicativity caused by the non-commutativity of the underlying algebra.

1 Introduction

I cannot believe that anything so ugly as multiplication of matrices is an essential part of the scheme of nature.

Sir Arthur Eddington
(in: *Relativity Theory of Protons and Electrons*)

Considering the title of this paper, the connection between the epigraph and our work is obviously ironic. In fact, we will show that, for our concrete problem, matrix multiplication admits a type of factorization that overcomes the restrictions of non-commutativity and solves some problem connected with a special polynomial sequence. That is why matrix multiplication does not seem as ugly to us as to Eddington.

But how is it possible that the famous Eddington so vehemently rejected the elementary operation of matrix multiplication?

Probably it was a joke, but - as N. Salingaros [29] pointed out - the sentence in the epigraph has another, more serious relationship with our work. Eddington's true motive was his aversion against representation theory. Introducing E-numbers, Eddington tried to avoid representation theory and defined, already in the 1930ies, *"the complex Clifford algebra over an eight-dimensional space which today appears in formulations of supersymmetry and supergravity..." [29]*. Needless to say that a radical rejection of one or another method in Mathematics (as well as in other fields of science or daily life) goes against the spirit of working without preconceived notions, which is needed for new discoveries. Shortly after the death of Eddington, in 1944, the paper [26] appeared where the matrix form of E-numbers is described...

Today the use of quaternions or, more general, of Clifford algebras in higher dimensional analysis is not anymore an exotic mathematical subject. They have become a well established general tool, also due to their growing number of applications, not only in classical fields of mathematics, engineering and physics, but also in, for example, searching for a *"world formula"* through superstring theory, where specialists in relativistic quantum mechanics heavily rely on these non-commutative algebras. And their research is done either with or without matrix multiplication, or, in other words, with or without representation theory.

In the center of our attention are polynomial sequences in $(n+1)$ real variables with values in the real vector space of paravectors in the corresponding Clifford algebra $\mathcal{C}\ell_{0,n}$. Therefore let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ with a non-commutative product according to the multiplication rules

$$e_k e_l + e_l e_k = -2\delta_{kl},\ k,l = 1,\cdots,n,$$

where δ_{kl} is the Kronecker symbol. The set $\{e_A : A \subseteq \{1,\cdots,n\}\}$ with

$$e_A = e_{h_1} e_{h_2} \cdots e_{h_r},\ 1 \le h_1 < \cdots < h_r \le n,\ e_\emptyset = e_0 = 1,$$

forms a basis of the 2^n-dimensional Clifford algebra $\mathcal{C}\ell_{0,n}$ over $\mathbb{R}$.

Let $\mathbb{R}^{n+1}$ be embedded in $\mathcal{C}\ell_{0,n}$ by identifying $(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}$ with the algebra element $x = x_0 + \underline{x} \in \mathcal{A}_n := \mathrm{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset \mathcal{C}\ell_{0,n}$. Here

$$x_0 = \mathrm{Sc}(x)$$

and

$$\underline{x} = \mathrm{Vec}(x) = e_1 x_1 + \cdots + e_n x_n$$

are the so-called *scalar* resp. *vector* part of the *paravector* $x \in \mathcal{A}_n$. The *conjugate* of x is given by

$$\bar{x} = x_0 - \underline{x}$$

and the *norm* $|x|$ of x is defined by

$$|x|^2 = x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + \cdots + x_n^2.$$

Denoting by $\omega(\underline{x})$ the element

$$\omega(\underline{x}) = \begin{cases} \dfrac{\underline{x}}{|\underline{x}|} & , \quad \underline{x} \neq 0 \\ 0 & , \quad \underline{x} = 0 \end{cases}$$

we observe that if $\underline{x} \neq 0$, $\omega(\underline{x}) \in S^{n-1}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Moreover $\omega(\underline{x})$ behaves like an *imaginary unit*, since $\omega(\underline{x})^2 = -1$. Each paravector x can be represented either in a *complex-like form*

$$x = x_0 + \omega(\underline{x})|\underline{x}| \tag{1}$$

or in the matrix form

$$\begin{bmatrix} x_0 & -|\underline{x}| \\ |\underline{x}| & x_0 \end{bmatrix}.$$

We consider functions of the form $f(z) = \sum_A f_A(z) e_A$, where $f_A(z)$ are real valued, i.e. $\mathcal{C}\ell_{0,n}$-valued functions defined in some open subset $\Omega \subset \mathbb{R}^{n+1}$, for $n \geq 1$. The *generalized Cauchy-Riemann operator*[1] in $\mathbb{R}^{n+1}$, is defined by

$$\overline{\partial} := \tfrac{1}{2}(\partial_0 + \partial_{\underline{x}}), \quad \partial_0 := \frac{\partial}{\partial x_0}, \quad \partial_{\underline{x}} := e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}. \tag{2}$$

[1] The notation $\overline{\partial}$ has already been used in [15] and should not be misunderstood with the Dbar operator that appears in the theory of several complex variables.

C^1-functions f satisfying the equation $\overline{\partial} f = 0$ (resp. $f\overline{\partial} = 0$) are called *left monogenic* (resp. *right monogenic*). We suppose that f is hypercomplex differentiable in Ω in the sense of [16], i.e. it has a uniquely defined areolar derivative f' in the sense of Pompeiu in each point of Ω (see also [23]). If we recall the complex partial derivatives (also called Wirtinger derivatives)

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big) \text{ and } \frac{\partial f}{\partial z} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big)$$

then it is clear that we would expect that the hypercomplex (areolar) derivative f' of a monogenic function could be obtained as $f' = \partial f = \frac{1}{2}(\partial_0 - \partial_{\underline{x}})f$ where

$$\partial := \frac{1}{2}(\partial_0 - \partial_{\underline{x}})$$

is just the *conjugate generalized Cauchy-Riemann operator.* That this is really the case is far from being trivial and has been proven in [16].

The existence of the hypercomplex derivative for a monogenic function is vital for the definition of a special polynomial sequence given in the next section. Moreover, the equivalence between the hypercomplex differentiability of a given function and its monogeinity is also essential (see [16]).

The concept of a *complex areolar derivative* has its origin in the paper [27] of Dimitrie Pompeiu (1873 - 1954), a Romanian PhD-student of H. Poincaré. Adapted to our hypercomplex situation we have to introduce two differential forms; one of them is the n-form:

$$d\sigma := d\hat{x}_0 - e_1 d\hat{x}_1 + \cdots + (-1)^n e_n d\hat{x}_n = \sum_{k=0}^{n} (-1)^n e_n d\hat{x}_n$$

and the second is the $(n-1)$-form:

$$d\sigma_{(n-1)} := \sum_{k=1}^{n} (-1)^k e_k d\hat{x}_{0,k},$$

where $n \geq 2$. In the case $n = 1$, isomorphic to the complex case, we simply consider the constant form $d\sigma_{(0)} = -e_1 \cong -i$.

The notation $d\hat{x}_m,\ m = 0, \ldots, n$, means that the factor dx_m is removed in the ordered outer product of the 1-forms $dx_k,\ \ k = 0, \ldots, n,$, whereas $d\hat{x}_{0,m}\ \ (m = 1, \ldots, n)$ stands for the ordered outer product of the 1-forms $dx_k,\ \ k = 0, \ldots, n,$ where the factors dx_0 and dx_m are deleted. Then, using ∂, we have that

$$d(d\sigma_{(n-1)} f) = d\sigma_{(n)} \partial f \tag{3}$$

whence by Stokes' theorem

$$\int_{\partial \mathcal{S}} (d\sigma_{(n-1)} f) \quad = \quad \int_{\mathcal{S}} d\sigma_{(n)} \partial f. \tag{4}$$

The following theorem holds (cf. [23]).

Theorem 1. *Let $\mathcal{S} \subset \Omega$ be an oriented differentiable n-dimensional hypersurface $\mathcal{S}$ with boundary $\partial\mathcal{S}$ and z_* be a fixed point in $\mathcal{S}$. Consider a sequence of subdomains $\{\mathcal{S}_m\}$ which is shrinking to z_* if $m \to \infty$. Suppose now that f is left monogenic in Ω. Then the left hypercomplex derivative ∂f is a generalized areolar derivative in the sense of Pompeiu of the form*

$$\partial f(z_\star) \quad = \quad \lim_{m\to\infty} \Big[\int_{\mathcal{S}_m} d\sigma_{(n)}\Big]^{-1} \int_{\partial \mathcal{S}_m} (d\sigma_{(n-1)} f). \tag{5}$$

Moreover, like complex differentiable functions also a hypercomplex differentiable function f is real differentiable ([16]) and consequently f' can be expressed by the real partial derivatives as $f' = \partial f = \frac{1}{2}(\partial_0 - \partial_{\underline{x}})f$, i.e. in the form of the generalized hypercomplex Wirtinger derivative as mentioned before. Since a hypercomplex differentiable function belongs to the kernel of $\overline{\partial}$, it follows that in fact $f' = \partial_0 f = -\partial_{\underline{x}} f$ like in the complex case.

The idea to generalize the powerful methods of function theory of one complex variable in $\mathbb{R}^2$ to higher dimensions by taking advantage of algebras that include the algebra of complex numbers as a special case, is very old and goes back to the time of Weierstrass (cf. [18]). However, for almost one hundred years most of the advances in the realization of that idea were not very successfull. An exception was a first systematic breakthrough in the 1930ies by the work of R. Fueter ([12, 13, 14]). Fueter started research on the fundamentals of a hypercomplex function theory in the quaternion algebra in $\mathbb{R}^4$, but at the end of his life already worked with Clifford algebras and the well-known *Dirac operator* as a generalization of the complex Cauchy-Riemann operator. Although mainly motivated by number theoretic problems (cf. [24]) rather than by problems arising from the mainstream of physics of that time, he succeeded to build a solid foundation for further developments in higher dimension function theory.

A second breakthrough was achieved one hundred years after Weierstrass, by the publication of [4], [9] and [17] in the 1980ies and 1990ies, which widely influenced the further development of function theory in non-commutative algebras since then.

As an example we mention here the so-called *"Modified Clifford Analysis"*, a theory introduced by H. Leutwiler in [21]. The fact that $\overline{\partial}x = \frac{1-n}{2}$, i.e., that only for the complex case $(n = 1)$ the identity function (and any integer power of x) belongs to the set of monogenic functions, caught his special interest. His leading idea was to change the metric from the Euclidean one to the hyperbolic one, because the power function is the conjugate gradient of a harmonic function defined with respect to the hyperbolic metric of the upper half plane.

Some years later, G. Laville and I. Ramadanoff in [20], solved, by another method, the same problem, namely the determination of a function class that includes the power function. Thereto they introduced the class of *holomorphic Cliffordian functions* as functions

$$f : \mathbb{R}^{2m+2} \to \mathcal{C}\ell_{0,2m+1},$$

being solutions of

$$\bar{\partial}\Delta^m f = \tfrac{1}{2}\sum_{j=0}^{2m+1} e_j \frac{\partial}{\partial x_j}\Delta^m f = 0$$

with Δ as the Laplace operator in the variables $x_0, x_1, \ldots, x_{2m+1}$. Instead of changing the metric, they changed the order of the differential equation to an odd number and succeeded in this way to include the power functions in the kernel of their operator of order $2m + 1$.

A third way to solve the same problem, but now inside of the class of ordinary monogenic functions and without changing the metric or increasing the order, was realized by *constructing polynomials with the behavior of power-like functions under hypercomplex differentiation* in a series of articles by authors of this paper (cf. [11], [25], [8]). These ideas, based on Appell's concept of power-like polynomials (cf. [2]), allowed to develop a systematic study of Appell sequences as tools for other applied problems, like quasi-conformal mappings (cf. [22]) or construction of classes of generalized classical polynomials (cf. [5], [6]). Since power-like monogenic functions are of general interest in Clifford Analysis, the study of sets of Appell polynomials has developed in several directions using different methods, see e.g. [3, 19]. Of particular interest seems to us a far reaching general representation theoretic method, based on Gelfand-Tsetlin bases as used in [19].

In the next section our main concern will be a special polynomial sequence associated to ∂.

2 A sequence of homogeneous monogenic polynomials

We consider homogeneous polynomials of degree k $(k = 0, 1, \ldots)$ of the form

$$P_k^n(x) = \sum_{s=0}^{k} x_0^{k-s}\, \underline{x}^s a_s(k, n), \tag{6}$$

with coefficients $a_s(k, n) \in \mathcal{C}\ell_{0,n}$ and $a_0(k, n) \neq 0$, for all k and n.

Theorem 2. *A polynomial of the form (6) is left monogenic if and only if the coefficients $a_s(k,n)$ satisfy the relations*

$$a_s(k,n) = \begin{cases} \frac{k-s+1}{s} a_{s-1}(k,n), & \text{if } s \text{ is even} \\ \\ \frac{k-s+1}{n+s-1} a_{s-1}(k,n), & \text{if } s \text{ is odd} \end{cases}, \quad s = 1, \ldots, k. \tag{7}$$

Proof. A polynomial P_k^n of the form (6) is clearly a homogeneous polynomial of degree k. Considering the action of the operator ∂_0 on P_k^n, we obtain

$$\begin{aligned} \partial_0 P_k^n &= \sum_{s=0}^{k-1} (k-s) x_0^{k-s-1}\, \underline{x}^s a_s(k,n) \\ &= \sum_{s=0}^{\lfloor \frac{k-1}{2} \rfloor} (k-2s)\, x_0^{k-2s-1}\, \underline{x}^{2s} a_{2s}(k,n) \\ &\quad + \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor - 1} (k-2s-1)\, x_0^{k-2s-2}\, \underline{x}^{2s+1} a_{2s+1}(k,n), \end{aligned}$$

where $\lfloor\ \rfloor$ is the floor function. On the other hand, using the relations

$$\partial_{\underline{x}}\, \underline{x}^k = \begin{cases} -k\underline{x}^{k-1}, & \text{if } k \text{ is even,} \\ -(n+k-1)\underline{x}^{k-1}, & \text{if } k \text{ is odd,} \end{cases}$$

which can be verified by straightforward computations, we obtain

$$\begin{aligned} \partial_{\underline{x}} P_k^n &= \sum_{s=1}^{\lfloor \frac{k}{2} \rfloor} x_0^{k-2s} (2s)\, \underline{x}^{2s-1} a_{2s}(k,n) \\ &\quad + \sum_{s=0}^{\lfloor \frac{k-1}{2} \rfloor} x_0^{k-2s-1} (n+2s)\, \underline{x}^{2s} a_{2s+1}(k,n). \end{aligned}$$

Now, P_k^n is a left monogenic function iff $\partial_0 P_k^n = -\partial_{\underline{x}} P_k^n$, which means that

$$(k-2s) a_{2s}(k,n) = (n+2s) a_{2s+1}(k,n)$$

and

$$(k-2s-1) a_{2s+1}(k,n) = (2s+2) a_{2s+2}(k,n).$$

The result follows now easily, after simplifications. □

We underline that if $a_0(k,n)$ is chosen as a real number, then all the coefficients $a_s(k,n)$ are real and the P_k^n are $\mathcal{A}_n$-valued polynomials, since $\underline{x}^s$ $(s \geq 0)$ is real or paravector-valued.
The particular choice of $a_0(k,n) = 1$ leads, by (7), to the sequence of real numbers

$$a_s(k,n) = \binom{k}{s} c_s(n), \ s = 1, \ldots, k,$$

where

$$c_s(n) := \begin{cases} \dfrac{s!!(n-2)!!}{(n+s-1)!!}, & \text{if } s \text{ is odd,} \\ \\ c_{s-1}(n), & \text{if } s \text{ is even.} \end{cases} \tag{8}$$

We notice that for $n = 2$, the coefficients $c_s(2)$ are the generalized central binomial coefficients with weight $\frac{1}{2^s}$

$$c_s(2) = \frac{1}{2^s} \binom{s}{\lfloor \frac{s}{2} \rfloor}, \ s = 0, 1, \ldots, k.$$

As a consequence of the chosen normalization, we obtain the final form of the special homogeneous monogenic polynomials

$$\mathcal{P}_k^n(x) := \sum_{s=0}^{k} \binom{k}{s} c_s(n)\, x_0^{k-s}\, \underline{x}^s, \tag{9}$$

where the coefficients $c_k(n)$, for $k = 1, 2, \ldots$ are given by (8) and $c_0(n) = 1$, for all dimensions n.

This particular case was first considered in [10] for special $\mathcal{A}_2$-valued polynomials defined in 3-dimensional domains. Later on, this class of polynomials was generalized to higher dimensions in [11, 25].

Remark 1. We stress the fact that for $n = 1$, $c_s(1) = 1$, for all values of s and $\mathcal{P}_k^1(x) = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s} \underline{x}^s = (x_0 + e_1 x_1)^k$ are the usual powers of the holomorphic variable $z = x_0 + e_1 x_1$, if we identify, as usual, e_1 with the complex imaginary unit.

3 Some representations of a paravector-valued sequence

The polynomials $\mathcal{P}_k^n(x)$ defined by (9) admit a variety of representations. Which among them should be taken as definition, depends on the purpose and the context.

Representation in terms of x and $\bar{x}$

The aforementioned polynomials were introduced initially as functions of a paravector variable x and its conjugate $\bar{x}$ ([11, 25]).

Theorem 3. *For $n \geq 1$, the polynomials (9) can be written as*

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} T_s^k(n)\, x^{k-s}\, \bar{x}^s, \tag{10}$$

where

$$T_s^k(n) = \binom{k}{s} \frac{(\frac{n+1}{2})_{(k-s)} (\frac{n-1}{2})_{(s)}}{n_{(k)}} \tag{11}$$

and $a_{(r)} = \frac{\Gamma(a+r)}{\Gamma(a)}$, (for any integer $r > 1$) denotes the Pochhammer symbol and $a_{(0)} = 1$.

Remark 2. In connection with the representation (10)-(11) of $\mathcal{P}_k^n$ we observe the following facts.

- This form is possible, since we can express the real and vector part of x as $x_0 = \frac{x+\bar{x}}{2}$ and $\underline{x} = \frac{x-\bar{x}}{2}$.
- The case of a real variable can be formally included in the above definitions as the case $\underline{x} = 0$ or equivalently by requiring that $T_0^0(0) = 1$ and $T_s^k(0) = 0$, for $0 < s \leq k$, (see [5]).
- This representation highlights the fact that these polynomials are *special* monogenic polynomials in the sense of [1]. In that paper, a monogenic polynomial P is said to be special if there exist constants $a_{ij} \in \mathcal{A}_n$ for which
$$P(x) = \sum_{i,j}{}' \bar{x}^i x^j a_{i,j},$$
where the primed sigma stands for a finite sum.

Determinant representation in terms of x_0 and $\underline{x}$

Theorem 4. *For $n \geq 1$, the polynomials (9) can be written as*

$$\mathcal{P}_0^n(x) = c_0 = 1.$$

$$\mathcal{P}_k^n(x) = \frac{1}{\prod_{j=0}^{k-1}\binom{k}{j}c_j} \begin{vmatrix} x_0^k & x_0^{k-1}\underline{x} & x_0^{k-2}\underline{x}^2 & \dots & x_0\underline{x}^{k-1} & \underline{x}^k \\ -\binom{k}{1}c_1 & \binom{k}{0}c_0 & 0 & \dots & 0 & 0 \\ 0 & -\binom{k}{2}c_2 & \binom{k}{1}c_1 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \binom{k}{k-2}c_{k-2} & 0 \\ 0 & 0 & 0 & \dots & -\binom{k}{k}c_k & \binom{k}{k-1}c_{k-1} \end{vmatrix}, \tag{12}$$

for $k \geq 1$.

Proof. The result follows easily by expanding the determinant with respect to the first row. □

Complex-like representation

From the algebra structure, it follows that $\underline{x}^2 = -|\underline{x}|^2$ and the polynomials (9) can also be written in the *complex-like* form

$$\mathcal{P}_k^n(x) = \mathcal{P}_k^n(x_0 + \omega(\underline{x})|\underline{x}|) = u_k^n(x_0, |\underline{x}|) + \omega(\underline{x})v_k^n(x_0, |\underline{x}|)\,, \tag{13}$$

where u_k^n and v_k^n are the real valued homogeneous polynomials of degree k,

$$\begin{aligned} u_k^n(x_0, |\underline{x}|) &= \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2l}(-1)^l c_{2l}(n)\, x_0^{k-2l}\, |\underline{x}|^{2l}, \;\; k \geq 1, \\ u_0^n(x_0, |\underline{x}|) &= 1 \end{aligned} \tag{14}$$

and

$$\begin{aligned} v_k^n(x_0, |\underline{x}|) &= \sum_{l=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{2l-1}(-1)^{l-1} c_{2l-1}(n)\, x_0^{k-2l+1}\, |\underline{x}|^{2l-1}, \;\; k \geq 1, \\ v_0^n(x_0, |\underline{x}|) &= 0. \end{aligned} \tag{15}$$

In this way, $\mathcal{P}_k^n$ admits also a matrix representation similar to the matrix representation of a complex number.

4 Matrix recurrence formulas and applications

In the complex case, the iterative process of producing holomorphic powers z^k consists in multiplying by z. This elementary procedure is not possible in higher dimensions since the class of monogenic functions is not closed under multiplication. The next result provides a process for obtaining a polynomial of degree k from a polynomial of degree $k-1$ of the same class.

Theorem 5. *The real valued functions u_k^n and v_k^n in the representation (13)-(15) of $\mathcal{P}_k^n$ satisfy, for $k \geq 1$,*

$$\begin{bmatrix} u_k^n(x_0, |\underline{x}|) \\ v_k^n(x_0, |\underline{x}|) \end{bmatrix} = M_k^n(x_0, |\underline{x}|) \begin{bmatrix} u_{k-1}^n(x_0, |\underline{x}|) \\ v_{k-1}^n(x_0, |\underline{x}|) \end{bmatrix}, \tag{16}$$

where

$$M_k^n := M_k^n(x_0, |\underline{x}|) = \begin{bmatrix} x_0 & -|\underline{x}| \\ \frac{k}{k+n-1}|\underline{x}| & \frac{k}{k+n-1}x_0 \end{bmatrix}.$$

Proof. We start by first considering k even, i.e. $k = 2m,\ m \in \mathbb{N}$. From (14) and (15) we obtain

$$u_{2m}^n(x_0, |\underline{x}|) = \sum_{l=0}^{m} \binom{2m}{2l}(-1)^l c_{2l}(n) x_0^{2m-2l}|\underline{x}|^{2l}$$

and

$$v_{2m}^n(x_0, |\underline{x}|) = \sum_{l=1}^{m} \binom{2m}{2l-1}(-1)^{l-1} c_{2l-1}(n) x_0^{2m-2l+1}|\underline{x}|^{2l-1}.$$

Using the well-known relation $\binom{2m}{2l} = \binom{2m-1}{2l} + \binom{2m-1}{2l-1}$, we obtain

$$\begin{aligned} u_{2m}^n(x_0, |\underline{x}|) =& c_0(n)x_0^{2m} + \sum_{l=1}^{m-1} \binom{2m-1}{2l}(-1)^l c_{2l}(n) x_0^{2m-2l}|\underline{x}|^{2l} \\ &+ \sum_{l=1}^{m-1} \binom{2m-1}{2l-1}(-1)^l c_{2l}(n) x_0^{2m-2l}|\underline{x}|^{2l} + (-1)^m c_{2m}|\underline{x}|^{2m} \\ =& \sum_{l=0}^{m-1} \binom{2m-1}{2l}(-1)^l c_{2l}(n) x_0^{2m-2l}|\underline{x}|^{2l} + \sum_{l=1}^{m} \binom{2m-1}{2l-1}(-1)^l c_{2l}(n) x_0^{2m-2l}|\underline{x}|^{2l}. \end{aligned}$$

From (8), we have $c_{2l}(n) = c_{2l-1}(n)$ and therefore

$$\begin{aligned} u_{2m}^n(x_0, |\underline{x}|) =& x_0 \sum_{l=0}^{m-1} \binom{2m-1}{2l}(-1)^l c_{2l}(n) x_0^{2m-2l-1}|\underline{x}|^{2l} \\ &- |\underline{x}| \sum_{l=1}^{m} \binom{2m-1}{2l-1}(-1)^{l-1} c_{2l-1}(n) x_0^{2m-2l}|\underline{x}|^{2l-1}. \end{aligned}$$

Hence

$$u_{2m}^n(x_0, |\underline{x}|) = x_0 u_{2m-1}^n(x_0, |\underline{x}|) - |\underline{x}| v_{2m-1}^n(x_0, |\underline{x}|) = \begin{bmatrix} x_0 & -|\underline{x}| \end{bmatrix} \begin{bmatrix} u_{2m-1}^n(x_0, |\underline{x}|) \\ v_{2m-1}^n(x_0, |\underline{x}|) \end{bmatrix}.$$

Considering now the expression for $v_{2m}^n(x_0, |\underline{x}|)$ and the fact that

$$\binom{2m}{2l-1}\frac{2m+n-1}{2m} = \binom{2m-1}{2l-2}\frac{n+2l-2}{2l-1} + \binom{2m-1}{2l-1},$$

we obtain

$$\begin{aligned} \tfrac{2m+n-1}{2m} v_{2m}^n(x_0, |\underline{x}|) =& \sum_{l=1}^{m} \binom{2m-1}{2l-2}(-1)^{l-1} \tfrac{n+2l-2}{2l-1} c_{2l-1}(n) x_0^{2m-2l+1}|\underline{x}|^{2l-1} \\ &+ \sum_{l=1}^{m} \binom{2m-1}{2l-1}(-1)^{l-1} c_{2l-1}(n) x_0^{2m-2l+1}|\underline{x}|^{2l-1}. \end{aligned}$$

From (8) it follows that $c_{2l-1}(n) = \frac{2l-1}{n+2l-2} c_{2l-2}(n)$ and therefore

$$\begin{aligned}
\tfrac{2m+n-1}{2m} v_{2m}^n(x_0, |\underline{x}|) =& \sum_{l=1}^{m} \tbinom{2m-1}{2l-2} (-1)^{l-1} c_{2l-2}(n) x_0^{2m-2l+1} |\underline{x}|^{2l-1} \\
&+ x_0 \sum_{l=1}^{m} \tbinom{2m-1}{2l-1} (-1)^{l-1} c_{2l-1}(n) x_0^{2m-2l} |\underline{x}|^{2l-1} \\
=& |\underline{x}| \sum_{l=0}^{m-1} \tbinom{2m-1}{2l} (-1)^{l} c_{2l}(n) x_0^{2m-2l-1} |\underline{x}|^{2l} \\
&+ x_0 \sum_{l=1}^{m} \tbinom{2m-1}{2l-1} (-1)^{l-1} c_{2l-1}(n) x_0^{2m-2l} |\underline{x}|^{2l-1}.
\end{aligned}$$

Finally, using once more the relation $c_{2l-1} = c_{2l}$, we obtain

$$\begin{aligned}
v_{2m}^n(x_0, |\underline{x}|) =& \tfrac{2m}{2m+n-1} \left(|\underline{x}| u_{2m-1}^n(x_0, |\underline{x}|) + x_0 v_{2m-1}^n(x_0, |\underline{x}|) \right) \\
=& \tfrac{2m}{2m+n-1} \begin{bmatrix} |\underline{x}| & x_0 \end{bmatrix} \begin{bmatrix} u_{2m-1}^n(x_0, |\underline{x}|) \\ v_{2m-1}^n(x_0, |\underline{x}|) \end{bmatrix}.
\end{aligned}$$

The proof for k odd is similar. □

Remark 3. In the complex case, $\mathcal{P}_k^1(z) = z^k$ and the left-hand side of (16) is the vector representation of z^k. On the other hand, the right-hand side of (16) represents $z \cdot z^{k-1}$, since in this case,

$$M_k^1 = \begin{bmatrix} x_0 & -|\underline{x}| \\ |\underline{x}| & x_0 \end{bmatrix}, \ k = 1, 2, \dots,$$

is the matrix representation of the complex number $z = x_0 + \omega(\underline{x})|\underline{x}| = x_0 + e_1 x_1$. This means that, for $n = 1$, Theorem 5 describes the usual recursive procedure for the construction of the complex powers z^k.

Remark 4. Contrary to the complex case, the matrices M_k^n $(n \neq 1)$ depend on the degree k. However, these matrices can be written as the product of a nonsingular k-dependent diagonal matrix D_k^n and a matrix $M_{\mathbb{C}}(n)$ with a structure similar to the matrix M_k^1. More precisely,

$$M_k^n = \begin{bmatrix} 1 & 0 \\ 0 & \frac{k}{k+n-1} \end{bmatrix} \begin{bmatrix} x_0 & -|\underline{x}| \\ |\underline{x}| & x_0 \end{bmatrix} = D_k^n M_{\mathbb{C}}(n).$$

Theorem 5 gives a possibility of constructing the sequence $\mathcal{P}_k^n$ from its constant term. In fact, starting from the representation $[1 \ \ 0]^T$ of $\mathcal{P}_0^n$, we obtain

$$\begin{bmatrix} u_1^n(x_0, |\underline{x}|) \\ v_1^n(x_0, |\underline{x}|) \end{bmatrix} = \begin{bmatrix} x_0 & -|\underline{x}| \\ \frac{1}{n}|\underline{x}| & \frac{1}{n} x_0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_0 \\ \frac{1}{n}|\underline{x}| \end{bmatrix} \tag{17}$$

as the vector representation of $\mathcal{P}_1^n(x) = x_0 + \omega(\underline{x}) \frac{1}{n} |\underline{x}|$ and

$$\begin{bmatrix} u_2^n(x_0, |\underline{x}|) \\ v_2^n(x_0, |\underline{x}|) \end{bmatrix} = \begin{bmatrix} x_0 & -|\underline{x}| \\ \frac{2}{n+1}|\underline{x}| & \frac{2}{n+1} x_0 \end{bmatrix} \begin{bmatrix} x_0 \\ \frac{1}{n}|\underline{x}| \end{bmatrix} = \begin{bmatrix} x_0^2 - \frac{1}{n}|\underline{x}|^2 \\ \frac{2}{n} x_0 |\underline{x}|^2 \end{bmatrix}$$

as the vector representation of $\mathcal{P}_2^n(x) = x_0^2 - \frac{1}{n}|\underline{x}|^2 + \omega(\underline{x}) \frac{2}{n} x_0 |\underline{x}|^2$ and so on.

Using the matrix representation given by Theorem 5, it is possible to prove, as in the real and complex case, that each polynomial $\mathcal{P}_k^n$ has only a trivial zero.

Corollary 6. *If $\mathcal{P}_k^n$ are the polynomials (9) then, for all dimensions $n \geq 0$, $\mathcal{P}_k^n(x) = 0$, ($k = 1, 2, \dots$) if and only if $x = 0$.*

Proof. Clearly $\mathcal{P}_k^n(0) = 0$, for $k \neq 0$. Consider now a paravector $x = x_0 + \omega(\underline{x})|\underline{x}|$ such that $\mathcal{P}_k(x) = 0$, i.e.

$$u_k^n(x_0, |\underline{x}|) + \omega(\underline{x}) v_k^n(x_0, |\underline{x}|) = 0,$$

where u_k^n and v_k^n are the real valued functions (14) and (15). Then

$$u_k^n(x_0, |\underline{x}|) = v_k^n(x_0, |\underline{x}|) = 0.$$

Recalling that $\mathcal{P}_0^n(x) = 1$ and using Theorem 5 recursively we obtain

$$\begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} = M_k^n \begin{bmatrix} u_{k-1}^n(x_0, |\underline{x}|) \\ \\ v_{k-1}^n(x_0, |\underline{x}|) \end{bmatrix} = M_k^n M_{k-1}^n \begin{bmatrix} u_{k-2}^n(x_0, |\underline{x}|) \\ \\ v_{k-2}^n(x_0, |\underline{x}|) \end{bmatrix} = \cdots$$

$$= M_k^n M_{k-1}^n \cdots M_1^n \begin{bmatrix} u_0^n(x_0, |\underline{x}|) \\ \\ v_0^n(x_0, |\underline{x}|) \end{bmatrix} = \mathcal{M}_k \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix},$$

where $\mathcal{M}_k = \prod_{j=1}^k M_j^n$. The homogeneous system $\mathcal{M}_k y = 0$ has non trivial solutions if and only if $\det \mathcal{M}_k = 0$. Since

$$\det M_j^n = |x|^2 \frac{j}{j+n-1},$$

we have

$$\det \mathcal{M}_k = |x|^{2k} \frac{k!}{n_{(k)}}$$

and this in turn implies that $\det \mathcal{M}_k$ vanishes if and only if $x = 0$. □

Corollary 7. *If $\mathcal{P}_k^n$ are the polynomials (9) then, for all dimensions $n \geq 0$,*

$$|\mathcal{P}_k^n(x)| \leq |x|^k, \;\; k = 0, 1, \dots. \tag{18}$$

Proof. For $k = 0$, the result is trivial. Writing, for $k \neq 0$,

$$\begin{bmatrix} u_k^n(x_0, |\underline{x}|) \\ \\ v_k^n(x_0, |\underline{x}|) \end{bmatrix} = \mathcal{M}_k \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix},$$

where $\mathcal{M}_k = \prod_{j=1}^k M_j^n$, we obtain, using the well-known properties of the Euclidean norm in $\mathbb{R}^2$ and the corresponding induced matrix norm

$$\|(u_k^n(x_0, |\underline{x}|), v_k^n(x_0, |\underline{x}|)\|_2 \leq \|\mathcal{M}_k\|_2 \|(1,0)\|_2 \leq \prod_{j=1}^k \|M_j^n\|_2.$$

We notice that $M_j^n (M_j^n)^T$ is the diagonal matrix

$$M_j^n (M_j^n)^T = \begin{bmatrix} |x|^2 & 0 \\ \\ 0 & \left(\frac{k}{k+n-1}\right)^2 |x|^2 \end{bmatrix},$$

whose largest eigenvalue is $|x|^2$. Then $\|M_j^n\|_2 = |x|$ and the result is proved, since $\|(u_k^n(x_0, |\underline{x}|), v_k^n(x_0, |\underline{x}|)\|_2 = |\mathcal{P}_k^n(x)|$. □

Remark 5.
- This result was already proved in [5] by writing the polynomials in the form (10)-(11) and using the property $\sum_{s=0}^{k} T_s^k(n) = 1$.
- Inequality (18) ensures that if we can express a monogenic function in a power series of $\mathcal{P}_k^n$, the resulting series converges absolutely in as much as the corresponding complex or real series is absolutely convergent. Representations of that type were considered in [5, 6] to construct generalized exponential and other related functions.

Theorem 8. *The homogeneous polynomials (9) can be obtained from the recurrence relations*

$$(n+k+1)\mathcal{P}_{k+2}^n(x) - ((2k+n+2)x_0 + \omega(\underline{x})|\underline{x}|)\mathcal{P}_{k+1}^n(x) + (k+1)|x|^2\mathcal{P}_k^n(x) = 0, \tag{19}$$

$$\mathcal{P}_0^n(x) = 1, \qquad \mathcal{P}_1^n(x) = x_0 + \omega(\underline{x})\frac{|\underline{x}|}{n}. \tag{20}$$

Proof. Representing the paravector $(2k+n+2)x_0 + \omega(\underline{x})|\underline{x}|$ in the matrix form

$$X_k^n = \begin{bmatrix} (2k+n+2)x_0 & -|\underline{x}| \\ |\underline{x}| & (2k+n+2)x_0 \end{bmatrix}$$

and the polynomials $\mathcal{P}_{k+2}^n$, $\mathcal{P}_{k+1}^n$ and $\mathcal{P}_k^n$ $(k = 2, 3, \dots)$ in the form (16), the left hand side of (19) can be written as

$$\left((n+k+1)M_{k+2} - X_k^n M_{k+1} + (k+1)|x|^2 I\right) \begin{bmatrix} u_k^n(x_0, |\underline{x}|) \\ \\ v_k^n(x_0, |\underline{x}|) \end{bmatrix}.$$

The final result follows by noting that

$$(n+k+1)M_{k+2} - X_k^n M_{k+1} = -(k+1)|x|^2 I.$$

□

Remark 6. Substituting in (19) and (20), x_0 for $\frac{x+\bar{x}}{2}$ and $\underline{x}$ for $\frac{x-\bar{x}}{2}$, we obtain the relations

$$(n+k+1)\mathcal{P}_{k+2}^n(x) - \left(\tfrac{2k+n+1}{2}\bar{x} + \tfrac{2k+n+3}{2}x\right)\mathcal{P}_{k+1}^n(x) + (k+1)x\bar{x}\mathcal{P}_k^n(x) = 0,$$

$$\mathcal{P}_0^n(x) = 1, \qquad \mathcal{P}_1^n(x) = \frac{x+\bar{x}}{2} + \frac{x-\bar{x}}{2n},$$

which is a generalization of the result in [3] for the particular case $n = 2$.

5 Applications to the construction of Appell sequences

In [2], Appell considered a sequence of polynomials $P_0(x),\ P_1(x), \dots,\ P_{n-1}(x), P_n(x), \dots$ such that $P_k(x)$ is of exact degree k, for each $k = 0, 1, \dots$ and moreover two consecutive terms are linked by the relation

$$P_k'(x) = kP_{k-1}(x),\ k = 1, 2, \dots. \tag{21}$$

In addition he observed that sequences of polynomials satisfying (21) have the general form

$$P_k(x) = \sum_{s=0}^{k} \alpha_s \binom{k}{s} x^{k-s},\ k = 0, 1, \dots, \tag{22}$$

with $\alpha_s, s = 0, \dots, k$ real arbitrary coefficients $(\alpha_0 \neq 0)$. Nowadays such sequences are usually known as Appell sequences. The basic idea is that the polynomials of an Appell sequence behave like power-law functions under the differentiation operation. The classical Hermite, Bernoulli and Euler polynomials are well-known examples of Appell sequences.

In the hypercomplex context, Appell sequences or Appell sets were first considered in order to construct the monogenic polynomials $\mathcal{P}_k^n$ introduced in the previous sections (see [10, 11, 25] for details). In these cases, the considered derivative is the hypercomplex derivative, and a sequence of $\mathcal{A}_n$-valued monogenic polynomials $(\mathcal{F}_k)_{k\geq 0}$ is called an Appell sequence if $\mathcal{F}_k$ is of exact degree k and

$$\mathcal{F}' := \partial \mathcal{F}_k = k\,\mathcal{F}_{k-1},\;\; k = 1, 2, \ldots, \tag{23}$$

In previous work of the authors (see e.g. [10, 11, 25]), the aforementioned polynomials were constructed in the form (10) and the coefficients T_s^k in (11) were computed in order to obtain polynomials for which condition (23) is fulfilled. The proof that a sequence of polynomials of the form (9) is an Appell sequence, is trivial. We will give now an alternative proof of the Appell property, based on the equivalent representations (12) and (16) of $\mathcal{P}_k^n$.

Theorem 9. *If $\mathcal{P}_k^n$ are monogenic polynomials of the form (12), then $(\mathcal{P}_k^n)'(x) = k\mathcal{P}_{k-1}^n(x)$, for $k = 1, 2, \ldots$.*

Proof. For $k \geq 1$, we obtain by differentiating (12) and expanding the correspondent determinant along the last column,

$$(\mathcal{P}_k^n)'(x) = \frac{\det D}{\prod_{j=0}^{k-2}\binom{k}{j}c_j},$$

where D is the $k \times k$ matrix

$$D = \begin{bmatrix} kx_0^{k-1} & (k-1)x_0^{k-2}\underline{x} & (k-2)x_0^{k-3}\underline{x}^2 & \ldots & 2x_0\underline{x}^{k-2} & \underline{x}^{k-1} \\ -\binom{k}{1}c_1 & \binom{k}{0}c_0 & 0 & \ldots & 0 & 0 \\ 0 & -\binom{k}{2}c_2 & \binom{k}{1}c_1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \binom{k}{k-3}c_{k-3} & 0 \\ 0 & 0 & 0 & \ldots & -\binom{k}{k-1}c_{k-1} & \binom{k}{k-2}c_{k-2} \end{bmatrix}.$$

The result follows after multiplying the i-th row of D by $(k-i+2)(k-i+1)$, $(i = 2, \ldots, k)$ and the j-th column by $\frac{1}{k-j+1}$, $(j = 1, \ldots, k)$. □

Theorem 10. *If $\mathcal{P}_k^n$ are monogenic polynomials of the form (13), such that relation (16) holds, then $(\mathcal{P}_k^n)'(x) = k\mathcal{P}_{k-1}^n(x)$, for $k = 1, 2, \ldots$.*

Proof. Writing

$$\begin{bmatrix} u_k^n(x_0, |\underline{x}|) \\ v_k^n(x_0, |\underline{x}|) \end{bmatrix} = D_k^n M_{\mathbb{C}}(n) \begin{bmatrix} u_{k-1}^n(x_0, |\underline{x}|) \\ v_{k-1}^n(x_0, |\underline{x}|) \end{bmatrix}, \tag{24}$$

where

$$D_k^n = \begin{bmatrix} 1 & 0 \\ 0 & \frac{k}{k+n-1} \end{bmatrix} \quad \text{and} \quad M_{\mathbb{C}}(n) = \begin{bmatrix} x_0 & -|\underline{x}| \\ |\underline{x}| & x_0 \end{bmatrix}$$

and recalling that $(\mathcal{P}_k^n)' = \partial_0 \mathcal{P}_k^n$, we obtain

$$\begin{bmatrix} \partial_0 u_k^n(x_0, |\underline{x}|) \\ \partial_0 v_k^n(x_0, |\underline{x}|) \end{bmatrix} = D_k^n \begin{bmatrix} u_{k-1}^n(x_0, |\underline{x}|) \\ v_{k-1}^n(x_0, |\underline{x}|) \end{bmatrix} + D_k^n M_{\mathbb{C}}(n) \begin{bmatrix} \partial_0 u_{k-1}^n(x_0, |\underline{x}|) \\ \partial_0 v_{k-1}^n(x_0, |\underline{x}|) \end{bmatrix}.$$

We want to prove that $\partial_0 \mathcal{P}_k^n = k\mathcal{P}_{k-1}^n$, for $k = 1, 2, \ldots$, i.e.

$$\begin{bmatrix} \partial_0 u_k^n(x_0, |\underline{x}|) \\ \\ \partial_0 v_k^n(x_0, |\underline{x}|) \end{bmatrix} = k \begin{bmatrix} u_{k-1}^n(x_0, |\underline{x}|) \\ \\ v_{k-1}^n(x_0, |\underline{x}|) \end{bmatrix}. \tag{25}$$

We prove (25) by induction on the degree k of $\mathcal{P}_k^n$. The result is obviously true for $k = 1$ (see (17)). We assume now that the statement (25) is true for k, and note that

$$\begin{bmatrix} \partial_0 u_{k+1}^n(x_0, |\underline{x}|) \\ \\ \partial_0 v_{k+1}^n(x_0, |\underline{x}|) \end{bmatrix} = D_{k+1}^n \left(\begin{bmatrix} u_k^n(x_0, |\underline{x}|) \\ \\ v_k^n(x_0, |\underline{x}|) \end{bmatrix} + M_{\mathbb{C}}(n) \begin{bmatrix} \partial_0 u_k^n(x_0, |\underline{x}|) \\ \\ \partial_0 v_k^n(x_0, |\underline{x}|) \end{bmatrix} \right).$$

On the other hand, the induction hypothesis and relation (24) imply that

$$M_{\mathbb{C}}(n) \begin{bmatrix} \partial_0 u_k^n(x_0, |\underline{x}|) \\ \\ \partial_0 v_k^n(x_0, |\underline{x}|) \end{bmatrix} = kM_{\mathbb{C}}(n) \begin{bmatrix} u_{k-1}^n(x_0, |\underline{x}|) \\ \\ v_{k-1}^n(x_0, |\underline{x}|) \end{bmatrix} = k(D_k^n)^{-1} \begin{bmatrix} u_k^n(x_0, |\underline{x}|) \\ \\ v_k^n(x_0, |\underline{x}|) \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} \partial_0 u_{k+1}^n(x_0, |\underline{x}|) \\ \\ \partial_0 v_{k+1}^n(x_0, |\underline{x}|) \end{bmatrix} = D_{k+1}^n(I + k(D_k^n)^{-1}) \begin{bmatrix} u_k^n(x_0, |\underline{x}|) \\ \\ v_k^n(x_0, |\underline{x}|) \end{bmatrix}.$$

Since $D_{k+1}^n(I + k(D_k^n)^{-1}) = (k+1)I$, we proved statement (25) for $k + 1$. □

Appell sequences in $\mathbb{P}_k^n := \mathrm{span}_{\mathbb{R}}\{\mathcal{P}_0^n, \mathcal{P}_1^n, \ldots, \mathcal{P}_k^n\}$ are linked with the polynomials $\mathcal{P}_k^n$ in a way similar to the classical case (cf. (22)).

Theorem 11. *A sequence of polynomials $\mathcal{F}_k$ in $\mathbb{P}_k^n$ is an Appell sequence if and only if*

$$\mathcal{F}_k(x) = \sum_{s=0}^{k} \alpha_s \binom{k}{s} \mathcal{P}_{k-s}^n, \; k = 0, 1, \ldots, \tag{26}$$

where $\alpha_k = \mathcal{F}_k(0)$ and $\alpha_0 \neq 0$.

Proof. Due to the Appell property of $\mathcal{P}_k^n$, it is clear that any sequence of polynomials of the form (26) is an Appell sequence.

Now, assume that $\mathcal{F}_k$ is any polynomial in $\mathbb{P}_k^n$ that is Appell, i.e.

$$\mathcal{F}_k(x) = \sum_{s=0}^{k} \beta_s^k \mathcal{P}_{k-s}^n(x) \quad \text{and} \quad \mathcal{F}_k^{'}(x) = k\mathcal{F}_{k-1}(x).$$

We are going to use induction on the degree k of $\mathcal{F}_k$ to prove that

$$\beta_s^k = \alpha_s \binom{k}{s}, \; s = 0, 1, \ldots, k. \tag{27}$$

Since $\mathcal{F}_0(x) = \alpha_0$, result (27) is true for $k = 0$. Assume that the same result is true for k, i.e.

$$\mathcal{F}_k(x) = \sum_{s=0}^{k} \alpha_s \binom{k}{s} \mathcal{P}_{k-s}^n.$$

From the Appell property, we obtain

$$\mathcal{F}_{k+1}^{'}(x) = \sum_{s=0}^{k} \beta_s^{k+1}(k - s + 1)\mathcal{P}_{k-s}^n(x) = (k+1)\mathcal{F}_k(x).$$

The polynomials $\mathcal{P}_k^n$ are linearly independent and therefore we get from the last two expressions

$$(k+1)\alpha_s\binom{k}{s} = \beta_s^{k+1}(k-s+1),\ s=0,\dots,k,$$

which means that

$$\beta_s^{k+1} = \alpha_s\binom{k+1}{s},\ s=0,\dots,k.$$

For $s=k+1$, we have $\beta_{k+1}^{k+1} = \mathcal{F}_{k+1}(0) = \alpha_{k+1}$ and the result is proved. □

Using this result, we can extend the determinant representation of the Appell sequence $(\mathcal{P}_k^n)_{k\geq 0}$ given by (12) to any Appell sequence $(\mathcal{F}_k^n)_{k\geq 0}$ in $\mathbb{P}_k^n$ and the proof follows *mutatis mutandis*.

Theorem 12. *For each $k=1,2,\dots$, $\mathcal{F}_k \in \mathbb{P}_k^n$ can be represented by*

$$\mathcal{F}_k^n(x) = \frac{1}{\prod_{j=0}^{k-1}\binom{k}{j}\alpha_j}\begin{vmatrix} \mathcal{P}_k^n & \mathcal{P}_{k-1}^n & \mathcal{P}_{k-2}^n & \dots & \mathcal{P}_1^n & \mathcal{P}_0^n \\ -\binom{k}{1}\alpha_1 & \binom{k}{0}\alpha_0 & 0 & \dots & 0 & 0 \\ 0 & -\binom{k}{2}\alpha_2 & \binom{k}{1}\alpha_1 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \binom{k}{k-2}\alpha_{k-2} & 0 \\ 0 & 0 & 0 & \dots & -\binom{k}{k}\alpha_k & \binom{k}{k-1}\alpha_{k-1} \end{vmatrix}, \tag{28}$$

with $\alpha_0 = \mathcal{F}_0(0) = 1$.

As an application of the aforementioned form of Appell sequences we can refer to the recently constructed monogenic Hermite and Laguerre polynomials ([5, 7]). The expression of the Hermite polynomials is given by

$$H_k^n(x) = \sum_{r=0}^{\lfloor \frac{k}{2}\rfloor} \frac{1}{r!}\frac{k!}{(k-2r)!}\frac{(-1)^r}{4^r}\gamma_r(n)\mathcal{P}_{k-2r}^n(x),\ k=0,1,\dots$$

where $\gamma_r(n) = \sum_{s=0}^{r}\binom{r}{s}c_s(n)$ denotes the binomial transform of the sequence $(c_s)_{s\geq 0}$ given by (8). The sequence $(H_k^n)_{k\geq 0}$ is clearly an Appell sequence and therefore it can be written as

$$H_k^n(x) = \sum_{s=0}^{k}\alpha_s\binom{k}{s}\mathcal{P}_{k-s}^n,\ k=0,1,\dots,$$

where, in this case,

$$\alpha_{2s} = \frac{(-1)^s(2s)!}{4^s s!}\gamma_s \quad \text{and} \quad \alpha_{2s+1}=0,\ s=0,\dots,\lfloor \tfrac{k}{2}\rfloor.$$

The Laguerre polynomials presented in [5] are not an Appell sequence. However, using the associated Laguerre polynomials, also constructed in [5], and given by

$$L_k^{n,(\alpha)}(x) = \sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}\frac{1}{2^r\,k!}\,\gamma_r(n)\,\tfrac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha-r+1)}\mathcal{P}_{k-r}^n(x).$$

it is clear that the sequence of polynomials $\mathcal{L}_k^{(j)} = (-1)^k k! L_k^{n,(j-k)}$, with $j\in\mathbb{N}$, is an Appell sequence which can be written as

$$\mathcal{L}_k^{(j)}(x) = \sum_{s=0}^{k}\alpha_s\binom{k}{s}\mathcal{P}_{k-s}^n(x),\ k=0,1,\dots,$$

with

$$\alpha_s = \frac{(-1)^s j!}{2^s(j-s)!}\gamma_s,\ s=0,1,\dots,k.$$

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