The Moore-Penrose inverse of a companion matrix*

Pedro Patrício[a]

March 7, 2012

[a]Departamento de Matemática e Aplicações, Universidade do Minho, 4710-057 Braga, Portugal.

e-mail: pedro@math.uminho.pt

Abstract

Necessary and sufficient conditions are given for the Moore-Penrose inverse of a companion matrix over an arbitrary ring to exist.

Keywords: Companion matrix, Moore-Penrose inverse, von Neumann inverse, rings

AMS classification: 15A09, 16E50, 16W10

1 Introduction

Let R be a ring with 1 and involution $\bar{(\cdot)}$. That is, for all $a, b \in R$, the equalities $\bar{1} = 1$, $\overline{(a+b)} = \bar{a} + \bar{b}$ and $\overline{(ab)} = \bar{b}\bar{a}$ hold. The involution $\bar{(\cdot)}$ in R endows an involution $*$ in the set $\mathcal{M}(R)$ of (finite) matrices over R, defined as $[a_{ij}]^* = [\overline{a_{ji}}]$.

A matrix A is said to be Moore-Penrose invertible with respect to $*$ provided there is $A^\dagger$ such that

$$AA^\dagger A = A, A^\dagger AA^\dagger = A^\dagger, (AA^\dagger)^* = AA^\dagger, (A^\dagger A)^* = A^\dagger A.$$

If such a matrix $A^\dagger$ exists, then it is well known it is unique (see [1]).

We say $a \in R$ is regular if $a \in aRa$, or equivalently $axa = a$ is a ring consistent equation. A particular solution is denoted by a^- and called a von Neumann inverse of a. A regular ring is a ring whose elements are regular. It is a standard fact that if R is a regular ring then the ring of $m \times m$ matrices over R is again regular (see, for instance, [2]).

We will use the following known fact:

*This research was financed by FEDER Funds through "Programa Operacional Factores de Competitividade – COMPETE" and by Portuguese Funds through FCT - "Fundação para a Ciência e a Tecnologia", within the project PEst-C/MAT/UI0013/2011.

Lemma 1.1. *Given $x, y \in R$, then $1+xy$ is a unit if and only if $1+yx$ is a unit, and in this case*

$$(1+xy)^{-1} = 1 - x(1+yx)^{-1}y.$$

Lemma 1.1 has a useful extension for rectangular matrices which we will need later on. Given $n \times k$ matrices B and C, then

$$I_n + BC^T \text{ is invertible if and only if } I_k + C^T B \text{ is invertible.} \quad (1)$$

Versions of this relation for generalized inverses can be found in [3] and [4].

Using von Neumann inverses, it was shown in [6], [8], [9] how to characterize the existence of a Moore-Penrose inverse by means of units. The equivalence between the existence of $M^\dagger$, the invertibility of $U = MM^* + I - MM^-$, and the invertibility of $V = M^*M + I - M^-M$ will play an important role throughout this paper.

Theorem 1.1. *Let $a \in R$ be a regular element, and a^- a von Neumann inverse of a. The following conditions are equivalent:*

(a) $a^\dagger$ *exists;*

(b) $s = a\bar{a}aa^- + 1 - aa^-$ *is a unit;*

(c) $h = a^-a\bar{a}a + 1 - a^-a$ *is a unit;*

(d) $v = \bar{a}a + 1 - a^-a$ *is a unit;*

(e) $u = a\bar{a} + 1 - aa^-$ *is a unit.*

In this case,

$$a^\dagger = \overline{(s^{-1}a)} = \overline{(ah^{-1})} = \overline{(u^{-1}a)} = \overline{(av^{-1})}.$$

Proof. The equivalences $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ follow from [8, Theorem 2], as well as the first two expressions for $a^\dagger$.

$(b) \Leftrightarrow (d)$. Write $s = a\bar{a}aa^- + 1 - aa^- = 1 - a(-\bar{a}aa^- + a^-) = 1 - yx$ with $y = a$ and $x = -\bar{a}aa^- + a^-$. Then $v = 1 - xy = \bar{a}a + 1 - a^-a$ and the equivalence follows using Lemma 1.1, with $v^{-1} = 1 + xs^{-1}y = 1 + (-\bar{a}aa^- + a^-)s^{-1}a = a^-s^{-1}a + 1 - \bar{a}s^{-1}a$ and $s^{-1} = (1-yx)^{-1} = 1 + y(1-xy)^{-1}x = 1 + yv^{-1}x = 1 + av^{-1}(-\bar{a}aa^- + a^-) = av^{-1}a^- + 1 - av^{-1}\bar{a}aa^-$.

$(c) \Leftrightarrow (e)$. Now, write $u = a\bar{a} + 1 - aa^- = 1 - a(-a^-a\bar{a} + a^-) = 1 - xy$ with $x = a$ and $y = -a^-a\bar{a} + a^-$. Then $h = 1 - yx = 1 - (-a^-a\bar{a} + a^-)a = a^-a\bar{a}a + 1 - a^-a$ and the equivalence follows using Lemma 1.1, with $u^{-1} = 1 + xh^{-1}y = 1 + ah^{-1}(-a^-a\bar{a} + a^-) = ah^{-1}a^- + 1 - ah^{-1}a^-a\bar{a} = ah^{-1}a^- + 1 - ah^{-1}\bar{a}$ and $h^{-1} = (1-yx)^{-1} = 1 + y(1-xy)^{-1}x = 1 + (-a^-a\bar{a} + a^-)u^{-1}a = a^-u^{-1}a + 1 - a^-a\bar{a}u^{-1}a$.

We now derive the expressions for the $a^\dagger$. From [8, Theorem 2], $a^\dagger = \overline{(ah^{-1})}$. Since

$$\begin{aligned} ah^{-1} &= aa^- u^{-1} a + a - a\bar{a}u^{-1}a \\ &= aa^- u^{-1} a + uu^{-1}a - a\bar{a}u^{-1}a \\ &= (aa^- - a\bar{a} + u)u^{-1}a \\ &= u^{-1}a \end{aligned}$$

then $a^\dagger = \overline{(u^{-1}a)}$.

Finally, since $ua = a\bar{a}a = av$ then $u^{-1}a = av^{-1}$ and $a^\dagger = \overline{(av^{-1})}$. □

Consider the $(n+1)\times(n+1)$ companion matrix

$$M = \begin{bmatrix} 0 & a \\ I_n & \boldsymbol{b} \end{bmatrix},$$

with $a \in R$ and $\boldsymbol{b} \in R^n$. In this paper, we are interested on characterizing the existence of $M^\dagger$ by means of units in R. For the group inverse of M the reader is referred to [5] and [10].

We will reduce the Moore-Penrose inverse of the companion matrix M to the lower triangular case, by using the factorization $M = AP$ where

$$A = \begin{bmatrix} a & 0 \\ \boldsymbol{b} & I_n \end{bmatrix} \text{ and } P = \begin{bmatrix} 0 & 1 \\ I_n & 0 \end{bmatrix}.$$

Since M is unitarily equivalent to A, then M has a Moore-Penrose inverse exactly when A is Moore-Penrose invertible. Futhermore,

$$M^\dagger = P^* A^\dagger.$$

In this paper, we will assume a to be regular in R, that is, there exists $a^- \in R$ for which $aa^-a = a$. Given solutions (possibly distinct) $a^-, a^=$ to $axa = a$ in R, then one can construct a reflexive inverse of a, that is, a common solution to $axa = a$ and $xax = x$, by taking $a^+ = a^= a a^-$.

Note that the Moore-Penrose invertibility of A does not imply a is Moore-Penrose invertible. Indeed, consider R the ring of 2×2 complex matrices with transposition as the involution, and set

$$a = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} a & 0 \\ \boldsymbol{b} & I_2 \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right].$$

Using [11], $A^\dagger$ exists since $rk(A) = rk(AA^T) = rk(A^TA) = 3$, but $a^\dagger$ does not since $aa^T = 0$. Consequently, the Moore-Penrose invertibility of the companion matrix M does not imply the Moore-Penrose invertibility of a.

On the other hand, a may be Moore-Penrose invertible and $M^\dagger$ may not exist. As an example, consider R the ring of 2×2 complex matrices with transposition as the involution,

$$a=\begin{bmatrix}1&0\\0&0\end{bmatrix}=a^\dagger,\boldsymbol{b}=\begin{bmatrix}1&i\\0&0\end{bmatrix},M=\begin{bmatrix}0&a\\I_2&\boldsymbol{b}\end{bmatrix}=\begin{bmatrix}0&0&1&0\\0&0&0&0\\1&0&1&i\\0&1&0&0\end{bmatrix}.$$

Then $rk(M)=3\neq 2=rk(M^TM)$ for M as a 4×4 complex matrix, and $M^\dagger$ does not exist.

We will divide this paper in two parts. In the first, we will assume $a^\dagger$ exists, and in the second we just assume regularity of a.

2 The case $a^\dagger$ exists

Suppose $a^\dagger$ exists and consider the unit

$$u=a\bar{a}+1-aa^\dagger,\text{ with } u^{-1}=\bar{a}^\dagger a^\dagger+1-aa^\dagger.$$

Note that $u^{-1}a=\bar{a}^\dagger$ and $\bar{a}u^{-1}=a^\dagger$.

The matrix

$$A=\begin{bmatrix}a&0\\\boldsymbol{b}&I_n\end{bmatrix}$$

is Moore-Penrose invertible if and only if $U=AA^*+I_{n+1}-AA^-$ is invertible for one, and hence, all choices of von Neumann inverses A^- of A, by Theorem 1.1. Applying [7, Theorem 1], we may take

$$A^-=\begin{bmatrix}a^\dagger&0\\-\boldsymbol{b}a^\dagger&I_n\end{bmatrix},$$

for which choice we obtain

$$AA^-=\begin{bmatrix}aa^\dagger&0\\0&I_n\end{bmatrix}$$

and

$$U=\begin{bmatrix}a\bar{a}+1-aa^\dagger&a\boldsymbol{b}^*\\\boldsymbol{b}\bar{a}&\boldsymbol{b}\boldsymbol{b}^*+I_n\end{bmatrix}=\begin{bmatrix}1&0\\\boldsymbol{b}\bar{a}u^{-1}&I_n\end{bmatrix}\begin{bmatrix}u&0\\0&Z\end{bmatrix}\begin{bmatrix}1&u^{-1}a\boldsymbol{b}^*\\0&I_n\end{bmatrix},$$

where

$$\begin{aligned}Z&=\boldsymbol{b}\boldsymbol{b}^*+I_n-\boldsymbol{b}\bar{a}u^{-1}a\boldsymbol{b}^*\\&=I_n+\boldsymbol{b}(1-\bar{a}u^{-1}a)\boldsymbol{b}^*\\&=I_n+\boldsymbol{b}(1-a^\dagger a)\boldsymbol{b}^*\end{aligned}$$

Now, the invertibility of Z is equivalent to $z = 1 + \boldsymbol{b}^*\boldsymbol{b}(1 - a^\dagger a)$ being a unit of R, by the equivalence (1). Writing $\boldsymbol{b} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$, this is the same as

$$z = 1 + \sum_{i=1}^{n} \bar{b_i} b_i (1 - a^\dagger a) \tag{2}$$

being a unit of R.

Theorem 2.1. *Given $a \in R$ such that $a^\dagger$ exists and $\boldsymbol{b} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$, then the following are equivalent:*

(a) The companion matrix $M = \begin{bmatrix} 0 & a \\ I_n & \boldsymbol{b} \end{bmatrix}$ is Moore-Penrose invertible.

(b) $1 + (1 - a^\dagger a)\boldsymbol{b}^*\boldsymbol{b}(1 - a^\dagger a)$ is a unit of R.

(c) $1 + \boldsymbol{b}^*\boldsymbol{b}(1 - a^\dagger a)$ is a unit of R.

(d) $1 + (1 - a^\dagger a)\boldsymbol{b}^*\boldsymbol{b}$ is a unit of R.

We now carry out the construction of the Moore-Penrose inverse of the companion matrix, in the case $a^\dagger$ exists.

Using Theorem 1.1,

$$A^\dagger = \left(U^{-1}A\right)^*$$

which leads to

$$\begin{bmatrix} 0 & a \\ I_n & \boldsymbol{b} \end{bmatrix}^\dagger = \begin{bmatrix} 0 & I_n \\ 1 & 0 \end{bmatrix} A^\dagger = \begin{bmatrix} 0 & I_n \\ 1 & 0 \end{bmatrix} \left(U^{-1}A\right)^*.$$

Note that u, U and Z are symmetric, and hence also are their inverses. Therefore,

$$\begin{aligned} U^{-1} &= \begin{bmatrix} 1 & -\bar{a}^\dagger \boldsymbol{b}^* \\ 0 & I_n \end{bmatrix} \begin{bmatrix} u^{-1} & 0 \\ 0 & Z^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \boldsymbol{b}a^\dagger & I_n \end{bmatrix} \\ &= \begin{bmatrix} u^{-1} + \bar{a}^\dagger \boldsymbol{b}^* Z^{-1} \boldsymbol{b} a^\dagger & -\bar{a}^\dagger \boldsymbol{b}^* Z^{-1} \\ -Z^{-1}\boldsymbol{b}a^\dagger & Z^{-1} \end{bmatrix}, \end{aligned}$$

with $Z^{-1} = \left(I_n + \boldsymbol{b}(1 - a^\dagger a)\boldsymbol{b}^*\right)^{-1} = I_n - \boldsymbol{b}(1 - a^\dagger a) z^{-1} (1 - a^\dagger a)\boldsymbol{b}^*$ and $z = 1 + (1 - a^\dagger a)\boldsymbol{b}^*\boldsymbol{b}(1 - a^\dagger a)$. Then

$$\begin{aligned} A^\dagger &= A^*(U^*)^{-1} \\ &= \begin{bmatrix} \bar{a}u^{-1} + a^\dagger a \boldsymbol{b}^* Z^{-1} \boldsymbol{b} a^\dagger - \boldsymbol{b}^* Z^{-1} \boldsymbol{b} a^\dagger & -a^\dagger a \boldsymbol{b}^* Z^{-1} + \boldsymbol{b}^* Z^{-1} \\ -Z^{-1}\boldsymbol{b}a^\dagger & Z^{-1} \end{bmatrix} \\ &= \begin{bmatrix} a^\dagger - (1 - a^\dagger a)\boldsymbol{b}^* Z^{-1} \boldsymbol{b} a^\dagger & (1 - a^\dagger a)\boldsymbol{b}^* Z^{-1} \\ -Z^{-1}\boldsymbol{b}a^\dagger & Z^{-1} \end{bmatrix}. \end{aligned}$$

Finally,

$$\begin{aligned}\begin{bmatrix} 0 & a \\ I_n & \boldsymbol{b} \end{bmatrix}^{\dagger} &= \begin{bmatrix} 0 & I_n \\ 1 & 0 \end{bmatrix} A^{\dagger} \\ &= \begin{bmatrix} -Z^{-1}\boldsymbol{b}a^{\dagger} & Z^{-1} \\ a^{\dagger} - (1-a^{\dagger}a)\boldsymbol{b}^*Z^{-1}\boldsymbol{b}a^{\dagger} & (1-a^{\dagger}a)\boldsymbol{b}^*Z^{-1} \end{bmatrix}.\end{aligned}$$

3 The case a is regular

We note that the companion matrix

$$\begin{bmatrix} 0 & a \\ I_n & \boldsymbol{b} \end{bmatrix}$$

is regular if and only if a is regular. This follows from the factorization

$$\begin{bmatrix} 0 & a \\ I_n & \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ I_n & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} I_n & \boldsymbol{b} \\ 0 & 1 \end{bmatrix}.$$

Suppose a is regular and let a^+ be any reflexive inverse of a.

The matrix

$$A = \begin{bmatrix} a & 0 \\ \boldsymbol{b} & I_n \end{bmatrix}$$

is Moore-Penrose invertible if and only if $V = A^*A + I_{n+1} - A^-A$ is invertible for one, and hence, all choices of von Neumann inverses A^- of A, by Theorem 1.1. Applying [7, Theorem 1], we may take

$$A^- = \begin{bmatrix} a^+ & 0 \\ -\boldsymbol{b}a^+ & I_n \end{bmatrix},$$

for which choice we obtain

$$A^-A = \begin{bmatrix} a^+a & 0 \\ -\boldsymbol{b}a^+a + \boldsymbol{b} & I_n \end{bmatrix}$$

and

$$V = \begin{bmatrix} \bar{a}a + 1 - a^+a + \boldsymbol{b}^*\boldsymbol{b} & \boldsymbol{b}^* \\ \boldsymbol{b}a^+a & I_n \end{bmatrix} = \begin{bmatrix} 1 & \boldsymbol{b}^* \\ 0 & I_n \end{bmatrix} \begin{bmatrix} \zeta & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \boldsymbol{b}a^+a & I_n \end{bmatrix},$$

where

$$\begin{aligned}\zeta &= \bar{a}a + 1 - a^+a + \boldsymbol{b}^*\boldsymbol{b}(1-a^+a) \\ &= \bar{a}a + 1 - a^+a + \sum_{i=1}^{n} \bar{b}_i b_i (1-a^+a),\end{aligned}$$

with $\boldsymbol{b} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$.

Theorem 3.1. *Given $a \in R$ and $\boldsymbol{b} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$, then the companion matrix $M = \begin{bmatrix} 0 & a \\ I_n & \boldsymbol{b} \end{bmatrix}$ is Moore-Penrose invertible if and only if a is regular and, for some reflexive inverse a^+ of a, the element*

$$\zeta = \bar{a}a + 1 - a^+a + \sum_{i=1}^{n} \bar{b}_i b_i (1 - a^+a) \tag{3}$$

is a unit of R.

We now construct the Moore-Penrose inverse of the companion matrix, in the case a is regular.

Using Theorem 1.1, the Moore-Penrose inverse of A is given by

$$A^\dagger = \left(AV^{-1}\right)^*$$

where

$$\begin{aligned} V^{-1} &= \left(A^*A + I_{n+1} - A^-A\right)^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ -\boldsymbol{b}a^+a & I_n \end{bmatrix} \begin{bmatrix} \zeta^{-1} & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 1 & -\boldsymbol{b}^* \\ 0 & I_n \end{bmatrix} \end{aligned}$$

and $\zeta = \bar{a}a + 1 - a^+a + \boldsymbol{b}^*\boldsymbol{b}(1 - a^+a)$. Then

$$\left(V^{-1}\right)^* = \begin{bmatrix} \bar{\zeta}^{-1} & -\bar{\zeta}^{-1}\overline{(a^+a)}\boldsymbol{b}^* \\ -\boldsymbol{b}\bar{\zeta}^{-1} & I_n + \boldsymbol{b}\bar{\zeta}^{-1}\overline{(a^+a)}\boldsymbol{b}^* \end{bmatrix}.$$

Substituting in the expression of $A^\dagger$,

$$\begin{aligned} A^\dagger &= \left(V^{-1}\right)^* A^* \\ &= \begin{bmatrix} \bar{\zeta}^{-1} & -\bar{\zeta}^{-1}\overline{(a^+a)}\boldsymbol{b}^* \\ -\boldsymbol{b}\bar{\zeta}^{-1} & I_n + \boldsymbol{b}\bar{\zeta}^{-1}\overline{(a^+a)}\boldsymbol{b}^* \end{bmatrix} \begin{bmatrix} \bar{a} & \boldsymbol{b}^* \\ 0 & I_n \end{bmatrix} \\ &= \begin{bmatrix} \bar{\zeta}^{-1}\bar{a} & \bar{\zeta}^{-1}(1 - \overline{(a^+a)})\boldsymbol{b}^* \\ -\boldsymbol{b}\bar{\zeta}^{-1}\bar{a} & I_n - \boldsymbol{b}\bar{\zeta}^{-1}(1 - \overline{(a^+a)})\boldsymbol{b}^* \end{bmatrix} \end{aligned}$$

from which we deduce

$$\begin{aligned} \begin{bmatrix} 0 & a \\ I_n & \boldsymbol{b} \end{bmatrix}^\dagger &= \begin{bmatrix} 0 & I_n \\ 1 & 0 \end{bmatrix} A^\dagger \\ &= \begin{bmatrix} -\boldsymbol{b}\bar{\zeta}^{-1}\bar{a} & I_n - \boldsymbol{b}\bar{\zeta}^{-1}(1 - \overline{(a^+a)})\boldsymbol{b}^* \\ \bar{\zeta}^{-1}\bar{a} & \bar{\zeta}^{-1}(1 - \overline{(a^+a)})\boldsymbol{b}^* \end{bmatrix} \end{aligned}$$

4 Questions and remarks

1. If $a^\dagger$ exists and $\boldsymbol{b}^*\boldsymbol{b} \in a^\dagger \boldsymbol{b}^*\boldsymbol{b}R$ then $\begin{bmatrix} 0 & a \\ I_n & \boldsymbol{b} \end{bmatrix}^\dagger$ exists. Indeed, if $\boldsymbol{b}^*\boldsymbol{b} = a^\dagger \boldsymbol{b}^*\boldsymbol{b}x$ for some x in R then $a^\dagger a \boldsymbol{b}^*\boldsymbol{b} = a^\dagger \boldsymbol{b}^*\boldsymbol{b}x = \boldsymbol{b}^*\boldsymbol{b}$.

2. If a is regular and $b_i \in Rb_i a$ then $\begin{bmatrix} 0 & a \\ I_n & \boldsymbol{b} \end{bmatrix}^\dagger$ exists, with $\boldsymbol{b} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$, if and only if $a^\dagger$ exists. Indeed, if $b_i = xb_i a$ then $b_i a^+ a = xb_i a = b_i$, from which the element ζ in equation (3) collapses to $\zeta = \bar{a}a + 1 - a^+ a$, which is a unit exactly when $a^\dagger$ exists.

3. How can the invertible elements defined in equations (2) and (3) be directly related, in the case $a^\dagger$ exists?

Ackowledgement

This problem was proposed by Professor Roland Puystjens more than 10 years ago. The question was to give a similar characterization of the one provided for group invertible companion matrices as in [10].

The author wishes to thank one of the referees for his/her valuable comments and corrections, and namely for providing the simplification for the expression of the Moore-Penrose inverse stated in Theorem 1.1.

References

[1] Ben Israel, A.; Greville, T. N. E.; *Generalized Inverses, Theory and Applications*, 2nd Edition, Springer, New York 2003.

[2] Brown, B.; McCoy, N. H.; The maximal regular ideal of a ring. *Proc. Amer. Math. Soc.* 1, (1950), 165–171.

[3] Castro-González, N.; Mendes-Araújo, C.; Patrício, P.; Generalized inverses of a sum in rings. *Bull. Aust. Math. Soc.* 82 (2010), no. 1, 156–164.

[4] Cvetkovic-Ilic, D.; Harte, R.; On Jacobson's lemma and Drazin invertibility. *Appl. Math. Lett.* 23 (2010), no. 4, 417–420,

[5] Hartwig, R. E.; Shoaf, J.; Group inverses and Drazin inverses of bidiagonal and triangular Toeplitz matrices. *J. Austral. Math. Soc. Ser. A*, 24 (1977), no. 1, 10–34.

[6] Patrício, Pedro.; The Moore-Penrose inverse of von Neumann regular matrices over a ring. *Linear Algebra Appl.* 332/334 (2001), 469–483.

[7] Patrício, Pedro; Puystjens, Roland; About the von Neumann regularity of triangular block matrices. *Linear Algebra Appl.* 332/334 (2001), 485–502.

[8] Patrício, Pedro; The Moore-Penrose inverse of a factorization. *Linear Algebra Appl.* 370 (2003), 227–235.

[9] Patrício, Pedro; Mendes-Araújo, C.; Moore-Penrose invertibility in involutory rings: the case $aa^\dagger = bb^\dagger$. *Linear Multilinear Algebra* 58 (2010), no. 3-4, 445–452.

[10] Puystjens, R.; Hartwig, R. E.; The group inverse of a companion matrix. *Linear and Multilinear Algebra*, 43 (1997), no. 1-3, 137–150,

[11] Pearl, M. H.; Generalized inverses of matrices with entries taken from an arbitrary field. *Linear Algebra and Appl.* 1 (1968), 571–587.