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# SPECIAL FUNCTIONS VERSUS ELEMENTARY FUNCTIONS IN HYPERCOMPLEX FUNCTION THEORY

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**Abstract.** *In recent years special hypercomplex Appell polynomials have been introduced by several authors and their main properties have been studied by different methods and with different objectives. Like in the classical theory of Appell polynomials, the generating function of hypercomplex Appell polynomials is a hypercomplex exponential function. The observation that this generalized exponential function has, for example, a close relationship with Bessel functions confirmed the practical significance of investigation on special classes of hypercomplex differentiable functions. Its usefulness for combinatorial studies has also been investigated. Moreover, an extension of those ideas led to the construction of complete sets of hypercomplex Appell polynomial sequences. Here we show how this opens the way for a more systematic study of the relation between some classes of Special Functions and Elementary Functions in Hypercomplex Function Theory.*

## 1 INTRODUCTION

Looking back to a time without computers, let's say to the middle of the last century, one can recognize that the role of the so called Special Functions in the mathematical practice must have been enormous. Indeed, in a lot of mathematical and engineering courses Special Functions played an important part of instruction and the ability to work with their tables were mandatory. Superseded by other emerging disciplines of that time like, for instance, topology, measure theory or functional analysis, the Special Functions have lost their impact. But mainly the development of big computers with their capacity of dealing instantaneously with almost all differential equations contributed for some time to the fact that they fall into oblivion. Nevertheless, their relative revival in the time of Scientific Computing is not surprising. If we have in mind that Special Functions are omnipresent in the theoretical background of any differential, integral or functional equation and that the use of symbolic algebraic calculations simplifies the detection of interesting unknown properties, than we can understand that the amount of papers written about Special Functions and related subjects is again increasing. From this general point of view it also seems natural to ask for the role and the application of Special Functions in Clifford Analysis and, more concretely, in hypercomplex function theory (HFT). Since the theory of *monogenic* or *hypercomplex differentiable* or *Clifford holomorphic* functions (see [6],[16],[13]) has its origin in complex function theory one can expect that almost all properties of Special Function theory should have its counterpart in HFR and that therefore a deeper study would not be interesting. Particularly this could be the case for the well known relationship between Special Functions and Elementary Functions of one variable. A first superficial look may confirm those positions, but two arguments exist that are in our opinion essential for a different judgement about the usefulness of studies on Special Functions in Clifford Analysis. First, it seems that the possible contribution to a different way of dealing with functions in several real variables could enrich the multidimensional theory of Special Functions. Second, the use of the non-commutative Clifford Algebra promises results which cannot be obtained in the usual multidimensional commutative setting. Otherwise, defending the opinion that everything can be done with real analytic methods, one would also question the value of complex analytic methods in the plane case. In the following we would like to invite the reader to a first glimpse about related questions by surveying some older (see [5], [22], [23]) and more recent (see [15], [13], [19], [9], [20], [7], [4]) developments in direction to a better understanding of the role of Special Functions in Clifford Analysis.

After introducing very briefly the necessary notations we motivate the further results by an elementary observation, which to our knowledge has not been used so far. It is based on the construction of a special exponential function in HFR analogously to the complex one and on the deduction of its relationship with Bessel functions by deducing the corresponding differential equations for the scalar and vector part of the exponential function. Since the main tool for our approaches are series expansions of holomorphic functions, in Section 1, we refer to a special set of Appell type polynomials which also stresses the central role of the hypercomplex derivative in problems of this type. The next step is the connection of the special set of Appell type polynomials to its exponential generating function and the corresponding Special Functions. Section 2 develops these methods further and reveals us connections of other types of Special Functions with other Elementary Functions in the sense of HFR. The last section is dedicated to application to combinatorial identities.

## 1.1 PRELIMINARIES

Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal base of the Euclidean vector space $\mathbb{R}^n$ with a non-commutative product according to the multiplication rules

$$e_k e_l + e_l e_k = -2\delta_{kl},\ k, l = 1, \cdots, n,$$

where $\delta_{kl}$ is the Kronecker symbol. The set $\{e_A : A \subseteq \{1, \cdots, n\}\}$ with

$$e_A = e_{h_1} e_{h_2} \cdots e_{h_r}, \quad 1 \leq h_1 \leq \cdots \leq h_r \leq n,\ e_\emptyset = e_0 = 1,$$

forms a basis of the $2^n$-dimensional Clifford algebra $Cl_{0,n}$ over $\mathbb{R}$. Let $\mathbb{R}^{n+1}$ be embedded in $Cl_{0,n}$ by identifying $(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}$ with the element $x = x_0 + \underline{x}$ of the algebra, where $\underline{x} = e_1 x_1 + \cdots + e_n x_n$. The conjugate of $x$ is $\bar{x} = x_0 - \underline{x}$ and the norm $|x|$ of $x$ is defined by $|x|^2 = x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + \cdots + x_n^2$.

We consider functions of the form $f(z) = \sum_A f_A(z) e_A$, where $f_A(z)$ are real valued, i.e. $Cl_{0,n}$-valued functions defined in some open subset $\Omega \subset \mathbb{R}^{n+1}$. We suppose that $f$ is hypercomplex differentiable in $\Omega$ in the sense of [12], [17], i.e. has a uniquely defined areolar derivative $f'$ in each point of $\Omega$ (see [18]). Then $f$ is real differentiable and $f'$ can be expressed by the real partial derivatives as $f' = 1/2(\partial_0 - \partial_{\underline{x}})f$, where

$$\partial_0 := \frac{\partial}{\partial x_0}, \quad \partial_{\underline{x}} := e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}.$$

With $D := \partial_0 + \partial_{\underline{x}}$ as the generalized Cauchy-Riemann operator, obviously holds $f' = 1/2\bar{D}f$. Since a hypercomplex differentiable function belongs to the kernel of $D$, i.e. satisfies $Df = 0$ or $0 = fD$ ($f$ is a *left resp. right monogenic function* in the sense of Clifford Analysis), then it follows that in fact $f' = \partial_0 f$ like in the complex case. We also need to consider the monogenic polynomials as functions of the hypercomplex monogenic variables

$$z_k = x_k - x_0 e_k = -\frac{x e_k + e_k x}{2},\ k = 1, 2, \cdots, n.$$

This implies the use of so called *generalized powers* of degree $m$ that are by convention symbolically written as

$$z_1^{\mu_1} \times \cdots z_n^{\mu_n}$$

and defined as an *m-nary* symmetric product by

$$z_1^{\mu_1} \times \cdots z_n^{\mu_n} = \frac{1}{m!} \sum_{\pi(i_1, \ldots, i_n)} z_{i_1} \cdots z_{i_n},$$

where the sum is taken over *all* $m! = |\mu|!$ permutations of $(i_1, \ldots, i_n)$, (see [17] and [18]).

As a starting point for the discussion on Elementary Functions in HFT, one could take the simple problem of a generalized holomorphic exponential function. Due to the special role of the (real or complex) exponential function in all areas of pure and applied analysis, the idea to ask for the existence of a generalized holomorphic exponential function was from the beginning on a principal question in HFT and reflects the main differences with the complex case. Why? Of course, it is not possible to distinguish among all functions in the kernel of $D$ a generalized holomorphic exponential function $f = f(x), \quad x \in \mathbb{R}^{n+1}$ , if one relies only on the fact that

$Df = 0$ or $fD = 0$. The characterization should be based on some analogy with the ordinary complex exponential function $f(z) = e^z, \quad z \in \mathbb{C}$.

Depending from the main property that one would like to preserve for the higher dimensional case, one could demand a series representation similar to $f(z) := \sum_{k=0}^{\infty} \frac{z^n}{k!}$.

If the conservation of the addition theorem would be in the center of attention then one could, for example, demand that the functional equation $f(z+w) = f(z)f(w)$ should be fulfilled.

A third possibility would be through an analytic continuation approach (Cauchy-Kowalevskaya extension) starting from the exponential function with real argument and asking for a complex holomorphic function that equals in the case of its restriction to the real axis exactly the exponential function with real argument.

But it would also be possible to define a generalized holomorphic exponential function $f$ as a solution of the simple first order differential equation $f' = f, \quad f(0) = 1$. Here, of course, the derivative means the hypercomplex derivative $f' = 1/2\bar{D}f$.

In the beginning of HFT only the Riemann approach to generalized holomorphic (monogenic) functions as solutions of $Df = 0$ or $fD = 0$ was at hand, being even considered as the only one possible approach (see [6], [13], [18]). That is why one can not find any remark on a regular quaternionic exponential function in the work of R. Fueter (see e.g. [10]).

The fact, that $Dx^n = 1 - n$, i.e., that only for the complex case $n = 1$ an integer power of $x$ belongs to the set of monogenic functions, caused problems by taking the adapted series expansion

$$f(z) := \sum_{k=0}^{\infty} \frac{x^n}{k!}, \quad x \in \mathbb{R}^{n+1}$$

as defining relation. Nevertheless, Sprößig discussed this situation in [23] (see also [13]) and showed how far one could come with discussing such a series as defining property for a non holomorphic analogue to the complex exponential. Of course, some algebraic manipulations allow to define a non holomorphic analogue to the complex cosine or sine function etc., too.

The next mentioned hypothetic possibility, i.e., the objective to preserve the functional equation $f(z+w) = f(z)f(w)$ in the higher dimensional case in this form is illusorily, because the set of monogenic functions is not closed with respect to multiplication.

The first attempts towards a meaningful definition of an exponential function in the context of HFT have been [5], [22]. Both papers rely on the Cauchy-Kowalevskaya extension approach (see also [6]).

In the previous list of possibilities we mentioned at the end the way of defining a generalized exponential functions as a solution of a first order differential equation. This approach is very natural and well motivated by the fact, that hypercomplex differentiability is granted for our type of solutions of generalized Cauchy-Riemann systems. In [19], [9], [20], [7] we have chosen exactly this approach, but still combined with the idea that through the construction of a set of special polynomials with differential properties like $x^n$ also an easy to handle series representation should be obtained. Therefore we combined the differential equation approach with the construction of Appell sequences of holomorphic polynomials. The advantage of such method is to have an easy way to construct other Elementary Functions, too.

To be complete we mention also the paper [15] where an interesting integral operator method for defining a generalized exponential function is presented.

### 1.2 THE APPEARANCE OF SPECIAL FUNCTIONS

Before coming to the main part, we will show the appearance of the connection with certain Special Function by a simple observation. Therefore we ask for a Clifford algebra-valued function $E = E(x)$, defined and hypercomplex differentiable in $\mathbb{R}^{n+1}$, such that

$$\begin{aligned} E' &= E \\ E(0) &= 1. \end{aligned}$$

Instead of $x = x_0 + \underline{x} = x_0 + e_1x_1 + \cdots + e_nx_n$ we can also write $x = x_0 + \omega|\underline{x}|$ with $\omega = \frac{\underline{x}}{|\underline{x}|}$ if $\underline{x} \neq 0$. Applying to $\omega$ and $|\underline{x}|$ the vectorial part

$$\partial_{\underline{x}} := e_1\frac{\partial}{\partial x_1} + \cdots + e_n\frac{\partial}{\partial x_n}$$

of the differential operator $D$ we observe that

$$\begin{aligned} \partial_{\underline{x}}|\underline{x}| &= \omega \\ \text{and} \qquad \partial_{\underline{x}}\omega &= \frac{1-n}{|\underline{x}|}. \end{aligned}$$

In the special case of $n = 2$ the second equation leads to

$$\partial_{\underline{x}}\omega = \frac{-1}{|\underline{x}|}. \tag{1}$$

The simplicity of those relations together with the fact that $\omega^2 = -1$ allows now to determine $E = E(x)$ for $n = 2$ in the most suggestive generalized complex form, namely as

$$E(x) = E(x_0 + \underline{x}) = E(x_0 + \omega|\underline{x}|) = e^{x_0}(F(|\underline{x}|) + \omega G(|\underline{x}|))$$

with $F(|\underline{x}|)$ and $G(|\underline{x}|))$ as real valued functions of their arguments and $F(\underline{0}) = 1; G(\underline{0}) = 0$. In this case the initial value $E(0) = 1$ is obviously automatically fulfilled.

To determine $F$ and $G$ we have to consider the equation

$$DE(x) = (\partial_0 + \partial_{\underline{x}})[e^{x_0}(F(|\underline{x}|) + \omega G(|\underline{x}|))] = 0,$$

which is equivalent to the system

$$F - \frac{1}{|\underline{x}|} - G' = 0 \tag{2}$$

$$G + F' = 0. \tag{3}$$

Differentiating equation (3) we see that with equation (2) follows that

$$F'' + \frac{1}{|\underline{x}|}F' + F = 0. \tag{4}$$

But (4) is nothing else than the differential equation for the Bessel function of the first kind $F(|\underline{x}|) = J_0(|\underline{x}|)$.

Analogously, differentiating equation (2) together with equation (3) we end up with a differential equation for the unknown function $G$, i.e. we obtain

$$G'' + \frac{1}{|\underline{x}|}G' + (1 - \frac{1}{|\underline{x}|^2})G = 0. \tag{5}$$

and (5) is nothing else than the differential equation for the Bessel function of the first kind $G(|\underline{x}|) = J_1(|\underline{x}|)$. Obviously, we also have that the hypercomplex derivative of a hypercomplex holomorphic function $f$ given by $f' = \partial_0 f$ applied to $f = E(x)$ leads really to the differential equation of an exponential function, i.e. we have $E'(x) = E(x)$, and all together means that the generalized exponential function we have asked for in the case $n = 2$ can be obtained as

$$Exp(x) := E(x) = e^{x_0}(J_0(|\underline{x}|) + \omega J_1(|\underline{x}|)).$$

This result coincides with the results of the previously mentioned papers [5], [22], and [15], but only relies on some elementary formal generalization of the complex exponential function $e^z = e^{x_0}[\cos y + i \sin y]$.

The following sections are now giving a survey on the subject and compile mainly the results obtained in [19], [9], and [20].

## 2 SPECIAL POLYNOMIALS AND GENERATING FUNCTIONS

The essential ideas of our approach can be shown for the case $n = 2$, since the general structure of the considered polynomial remains one and the same for different values of $n \geq 2$.

We recall that a sequence of polynomials $P_0(x),\ P_1(x), \cdots$ is said to form a Appell sequence if

i. $P_k(x)$ is of exact degree $k$, for each $k = 0, 1, \cdots$;

ii. $P_k'(x) = kP_{k-1}(x)$, for each $k = 1, 2, \cdots$.

The basic idea is that the polynomials of an Appell sequence behave like power-law functions under the differentiation operation (see e.g. [2], [8], [3]). In our case the polynomials will be holomorphic. As usual, the sequence will be normalized by demanding that $P_0(x) \equiv 1$. It is evident, that only the use of a hypercomplex derivative enables us to speak about an Appell sequence in the setting of Clifford Analysis. We stress the fact that treating such polynomials exclusively as solutions of a generalized Cauchy-Riemann system would not allow to obtain an analogue to the concept of an Appell sequence in the real or complex case.

Independent of the dimension $n$, we are looking for an Appell sequence of monogenic polynomials $\mathcal{P}_k(x)$ of the form

$$\mathcal{P}_k(x) = \sum_{s=0}^{k} T_s^k \, x^{k-s} \, \bar{x}^s,$$

where $T_s^k$ are suitable defined real numbers.[1]

[1]Notice that special monogenic polynomials in terms of $x$ and $\bar{x}$ similar to (16) have been considered before in [1]. But the paper [1] is concerned with the extension of the theory of basic sets of polynomials in one complex variable, as introduced by J. M. Whittaker and B. Cannon, to the setting of Clifford analysis. At the time of publication of [1] the concept of hypercomplex differentiability or the corresponding use of the hypercomplex derivative, first published in [16] resp. [12], have not been at disposal for the investigation of Appell sequences of monogenic polynomials.

In the complex case, corresponding to $n = 1$ with $e_1 := i$, for polynomials $\mathcal{P}_k(x)$ normalized by $\mathcal{P}_k(1) = 1,\ k = 0, 1, \ldots$ follows immediately that $T_0^k \equiv 1$ and $T_s^k \equiv 0$, for $s > 0$, since holomorphic functions in $\mathbb{C}$ have a series expansion which involves only the powers of $z = x_0 + ix_1$ and not the conjugate variable $\bar{z} = x_0 - ix_1$. In the hypercomplex case, and particularly in the case $n = 2$ which is in the center of our attention, the $\mathcal{P}_k$ a priori may depend on the values of $T_s^k$ not only for the trivial case $s = 0$. This can already be seen by the following

**Theorem 1** *Consider in the case $n = 2$ the variable $x = x_0 + x_1 e_1 + x_2 e_2$ and its conjugate $\bar{x} = x_0 - x_1 e_1 - x_2 e_2$ . The homogeneous polynomial of degree $k;\ k = 0, 1, \cdots$ ,*

$$\mathcal{P}_k(x) = \sum_{s=0}^{k} T_s^k x^{k-s} \bar{x}^s, \tag{6}$$

*normalized by*

$$\mathcal{P}_k(1) = 1, \tag{7}$$

*is monogenic if and only if the alternating sum*

$$c_k := \sum_{s=0}^{k} T_s^k (-1)^s \tag{8}$$

*satisfies*

$$c_k = \left[ \sum_{|\nu|=k} (-1)^k \binom{k}{\nu} (e_1^{\nu_1} \times e_2^{\nu_2})^2 \right]^{-1}. \tag{9}$$

The explicit expression of the uniquely defined $c_k$ relies on the fact that the polynomials $\mathcal{P}_k(x)$ in terms of the corresponding hypercomplex monogenic variables $z_k = x_k - x_0 e_k,\ \ k = 1, 2$ are obtained as

$$\mathcal{P}_k(x) = \mathbf{P}_k(z_1, z_2) = c_k \sum_{k=0}^{n} z_1^{n-k} \times z_2^k \binom{n}{k} e_1^{n-k} \times e_2^k. \tag{10}$$

The normalization condition (7), i.e. $\mathcal{P}_k(1) = \mathbf{P}_k(-e_1, -e_2) = 1$ then implies that

$$c_k = \left[ \sum_{|\nu|=k} (-1)^k \binom{k}{\nu} (e_1^{\nu_1} \times e_2^{\nu_2})^2 \right]^{-1}.$$

Suppose now that $\mathcal{P}_k'(x) = k\mathcal{P}_{k-1}(x);\ k = 1, 2, \cdots$ . Then it is possible to prove that the values of $T_s^k,\ s = 0, \cdots, k$, can be determined recursively from the values of $T_s^{k-1},\ j = 0, \cdots, k-1$ and $c_k$. In other words, we have a recursion formula for the $\mathcal{P}_k(x)$.

**Theorem 2** *The coefficients* $T_s^k,\ s=0,\cdots,k$ *and* $T_s^{k-1},\ j=0,\cdots,k-1$ *satisfy the* $(k+1)\times(k+1)$ *system of algebraic equations*

$$M_k \begin{pmatrix} T_0^k \\ T_1^k \\ T_2^k \\ \vdots \\ T_{k-2}^k \\ T_{k-1}^k \\ T_k^k \end{pmatrix} = k \begin{pmatrix} T_0^{k-1} \\ T_1^{k-1} \\ T_2^{k-1} \\ \vdots \\ T_{k-2}^{k-1} \\ T_{k-1}^{k-1} \\ c_k \end{pmatrix}. \tag{11}$$

*where*

$$M_k := \begin{pmatrix} k & 1 & 0 & 0 & & 0 & 0 \\ 0 & k-1 & 2 & 0 & & 0 & 0 \\ 0 & 0 & k-2 & 3 & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & & k-1 & 0 \\ 0 & 0 & 0 & 0 & & 1 & k \\ 1 & -1 & 1 & -1 & \cdots & (-1)^{k-1} & (-1)^k \end{pmatrix}.$$

The system is uniquely solvable since

$$\det(M_k) = (-1)^k k! 2^k \neq 0,\ k=0,1,\ldots.$$

As a corollary it is possible to relate for every fixed value of $k \geq 0$ the vector $\{T_s^k\}$ to the vector $\{c_s\},\ s=0,\cdots,k.$

**Corollary 1** *For every* $k \geq 0$ *the values of* $T_s^k$ *and* $c_s$*;* $s=0,1,\cdots,k$ *are related by*

$$\begin{pmatrix} T_0^k \\ \vdots \\ T_k^k \end{pmatrix} = N_k \begin{pmatrix} c_0 \\ \vdots \\ c_k \end{pmatrix}, \tag{12}$$

*where*

$$N_k = M_k^{-1} \left( \begin{array}{c|c} kN_{k-1} & 0 \\ \hline 0 & 1 \end{array} \right);\ k=1,2,\cdots \qquad \textit{and} \qquad N_0 = 1.$$

It is also possible to prove other intrinsic properties of the set $\{T_s^k,\ s=0,\cdots,k\}$, which are of own interest in combinatorial questions, since they resemble in a lot of aspects a set of non-symmetric generalized binomial coefficients.

This all together leads to the following

**Theorem 3** *Monogenic polynomials of the form*

$$\mathcal{P}_k(x) = \sum_{s=0}^{k} T_s^k\, x^{k-s}\, \bar{x}^{\ s}, \quad \textit{with} \quad T_s^k = \frac{1}{k+1} \frac{(\frac{3}{2})_{(k-s)}(\frac{1}{2})_{(s)}}{(k-s)!s!}, \tag{13}$$

*where $a_{(r)}$ denotes the Pochhammer symbol (raising factorial) form an Appell sequence of monogenic polynomials.*

*In terms of generalized powers these polynomials are of the form*

$$\mathcal{P}_k(x) = \mathbf{P}_k(z_1, z_2) = c_k \sum_{k=0}^{n} z_1^{n-k} \times z_2^k \binom{n}{k} e_1^{n-k} \times e_2^k, \tag{14}$$

*where*

$$c_k := \sum_{s=0}^{k} T_s^k (-1)^s = \begin{cases} \frac{k!!}{(k+1)!!}, & \text{if } k \text{ is odd} \\ \\ c_{k-1}, & \text{if } k \text{ is even} \end{cases} \tag{15}$$

Let us now consider the case of arbitrary $n \geq 1$.

In the case $x = x_0 + \underline{x} = x_0 + x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$, we denote by $\mathcal{P}_k^n(x)$ the homogeneous monogenic polynomial of degree $k$, generalizing the particular case $n = 2$ . It can be proved that for arbitrary $n \geq 1$, the polynomials

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} T_s^k(n)\, x^{k-s}\, \bar{x}^s, \quad \text{with} \quad T_s^k(n) = \frac{k!}{(n)_k} \frac{\left(\frac{n+1}{2}\right)_{(k-s)} \left(\frac{n-1}{2}\right)_{(s)}}{(k-s)!s!}, \tag{16}$$

form an Appell sequence. In terms of generalized powers these polynomials are of the form

$$\mathcal{P}_k^n(x) = \mathbf{P}_k(z_1, \cdots, z_n) = c_k(n) \sum_{|\nu|=n} z_1^{\nu_1} \times \cdots \times z_n^{\nu_n} \binom{n}{\nu} e_1^{\nu_1} \times \cdots \times e_n^{\nu_n}, \tag{17}$$

where $\nu = (\nu_1, \cdots, \nu_n)$ is a multiindex and

$$c_k(n) := \sum_{s=0}^{k} (-1)^s T_s^k(n) = \begin{cases} \dfrac{k!}{n_{(k)}} \left(\dfrac{n+1}{2}\right)_{\left(\frac{k-1}{2}\right)} \dfrac{1}{\left(\frac{k-1}{2}\right)!}, & \text{if } k \text{ is odd} \\ \\ \dfrac{k!}{n_{(k)}} \left(\dfrac{n+1}{2}\right)_{\left(\frac{k}{2}\right)} \dfrac{1}{\left(\frac{k}{2}\right)!}, & \text{if } k \text{ is even} \end{cases} \tag{18}$$

We would like to remark, that the number of hypercomplex linear independent polynomials of homogeneous degree $k$ is $\binom{n+k-1}{k}$, which is easy to check. Our $\mathcal{P}_k^n(x)$ written in terms of $x$ and $\bar{x}$ is only one special representative for arbitrary degree $k$ of such homogeneous polynomials. But our aim was not to discuss the completeness of a system of Appell sequences, for example in the space of square integrable hypercomplex functions. This question has been principally solved in [7] with respect to a representation in terms of the hypercomplex variables $z_k, \quad k = 1, \ldots n$. It is interesting to notice that in that obtained system our special polynomials $\mathcal{P}_k^n(x)$ not occur, i.e., its structure seems really to be very special.

A detailed and profound discussion of completeness of a system of Appell sequences in the space of square integrable hypercomplex functions is given in the very recent paper [4]. It contains a very interesting construction of those systems including orthogonality and their applications and relations to Fourier series.

Here our intention were more the intrinsic structure of our special polynomials. Notice that several important properties of the $\mathcal{P}_k^n$ are independent from $n$. If we look, for instance, to their representation in terms of $x_0$ and $\underline{x}$, then the following binomial type theorem holds:

$$\mathcal{P}_k(x_0+\underline{x}) = \sum_{s=0}^{k} \binom{k}{s} \mathcal{P}_{k-s}(x_0)\mathcal{P}_s(\underline{x}). \tag{19}$$

From (19) and $c_0(n) \equiv 1$ for any dimension $n$ follows in the case of $\underline{x} \equiv 0$ that

$$\mathcal{P}_k(x_0) = x_0^k$$

as consequence of the construction of $\{\mathcal{P}_k^n\}$ as Appell set with $\mathcal{P}_k^n(0) = c_0(n) \equiv 1$ for $\forall n$.

In the case of $x_0 \equiv 0$ we obtain the essential property which characterizes the difference to the complex case:

$$\mathcal{P}_k(\underline{x}) = c_k(n)\underline{x}^k.$$

With these relations and the binomial type theorem (19) we can obtain the remarkable formula

$$\mathcal{P}_k(x) = \sum_{s=0}^{k} c_s \binom{k}{s} \left(\frac{x+\bar{x}}{2}\right)^{k-s} \left(\frac{x-\bar{x}}{2}\right)^{s}.$$

Observe that for Legendre polynomials $L_k(x)$ holds a similar formula with $\binom{k}{s}^2$ instead of $c_s\binom{k}{s}$

$$L_k(x) = \sum_{s=0}^{k} \binom{k}{s}^2 \left(\frac{x-1}{2}\right)^{k-s} \left(\frac{x+1}{2}\right)^{s}$$

as a consequence of Rodrigues' formula

$$L_k(x) = \frac{1}{2^k k!}\frac{d}{dx^k}(x^2-1)^k.$$

Now if we define

$$\mathrm{Exp}_n(x) := \sum_{k=0}^{\infty} \frac{\mathcal{P}_k^n(x)}{k!} \tag{20}$$

then we can prove the following result.

**Theorem 4** *The* $\mathrm{Exp}_n$*-function can be written in terms of Bessel functions of the first kind,* $J_a(x)$*, for orders* $a = \frac{n}{2}-1, \frac{n}{2}$ *as*

$$\mathrm{Exp}_n(x_0+\underline{x}) = e^{x_0}\Gamma(\frac{n}{2}) \left(\frac{2}{|\underline{x}|}\right)^{\frac{n}{2}-1} \left(J_{\frac{n}{2}-1}(|\underline{x}|) + \omega(x) J_{\frac{n}{2}}(|\underline{x}|)\right). \tag{21}$$

For $n=1$ and $n=2$, (21) gives the ordinary complex exponential and the reduced quaternion valued $\mathrm{Exp}$-function, respectively. Moreover, it is easy to conclude that this function is different from zero everywhere.

For the particular case $n=3$, leads to

$$\mathrm{Exp}_3(x_0+\underline{x}) = e^{x_0}\left(\frac{\sin(|\underline{x}|)}{|\underline{x}|} + \omega(x)\frac{\sin(|\underline{x}|) - |\underline{x}|\cos(|\underline{x}|)}{|\underline{x}|}\right). \tag{22}$$

This $\mathrm{Exp}_3$-function coincides with the one referred by W. Sprößig in [23].

Due to the fact that Appell sequences imply automatically a direct link to a corresponding exponential function ([3]) we have the following result.

**Theorem 5** *Let $J_a(x)$ be Bessel functions of the first kind for orders $a = \frac{n}{2} - 1, \frac{n}{2}$. Then*

$$\mathrm{Exp}_n(x\,t) = \widetilde{\mathrm{Sc}}\ [\mathrm{Exp}_n] + \widetilde{\mathrm{Vec}}\ [\mathrm{Exp}_n]\ \underline{x}\ = \sum_{k=0}^{\infty} \frac{\mathcal{P}_k^n(x)t^k}{k!}$$

*is the exponential generating function of the special monogenic polynomials $\mathcal{P}_k^n(x)$ with*

$$\begin{aligned} \widetilde{\mathrm{Sc}}\ [\mathrm{Exp}_n] &= e^{x_0\,t}\Gamma(\frac{n}{2})\left(\frac{2}{|\underline{x}|t}\right)^{\frac{n}{2}-1} \left(J_{\frac{n}{2}-1}(|\underline{x}|t)\right), \\ \widetilde{\mathrm{Vec}}\ [\mathrm{Exp}_n] &= \frac{1}{|\underline{x}|}e^{x_0\,t}\Gamma(\frac{n}{2})\left(\frac{2}{|\underline{x}|t}\right)^{\frac{n}{2}-1} \left(J_{\frac{n}{2}}(|\underline{x}|t)\right). \end{aligned}$$

## 3 SPECIAL FUNCTIONS VERSUS ELEMENTARY FUNCTIONS

The results of the previous section suggest now to introduce the following series as a hypercomplex cosine function

$$COS_n(\underline{x}) := \sum_{k=0}^{\infty}(-1)^k \frac{\mathcal{P}_{2k}^n(\underline{x})}{(2k)!} = \Gamma(\frac{n}{2})\left(\frac{2}{|\underline{x}|}\right)^{\frac{n}{2}-1} J_{\frac{n}{2}-1}(|\underline{x}|)$$

and the hypercomplex sine function

$$SIN_n(\underline{x}) := \sum_{k=0}^{\infty}(-1)^k \frac{\mathcal{P}_{2k+1}^n(\underline{x})}{(2k+1)!} = \Gamma(\frac{n}{2})\left(\frac{2}{|\underline{x}|}\right)^{\frac{n}{2}-1} J_{\frac{n}{2}}(|\underline{x}|).$$

Analogously, by substituting the power $z^k$ by $\mathcal{P}_k(x)$ in the complex series of $\cosh$ and $\sinh$ we can introduce the hypercomplex hyperbolic cosine and hypercomplex hyperbolic sine as

$$COSH_n(\underline{x}) := \sum_{k=0}^{\infty} \frac{\mathcal{P}_{2k}^n(\underline{x})}{(2k)!} = \Gamma(\frac{n}{2})\left(\frac{2}{|\underline{x}|}\right)^{\frac{n}{2}-1} I_{\frac{n}{2}-1}(|\underline{x}|)$$

and

$$SINH_n(\underline{x}) := \sum_{k=0}^{\infty} \frac{\mathcal{P}_{2k+1}^n(\underline{x})}{(2k+1)!} = \Gamma(\frac{n}{2})\left(\frac{2}{|\underline{x}|}\right)^{\frac{n}{2}-1} I_{\frac{n}{2}}(|\underline{x}|)\omega(\underline{x}),$$

respectively. The change to modified Bessel function of the first kind shows the role of this type of Special Functions for expressions as Elementary Functions in the HFT setting. The last result coincides with results in [15] obtained by a so called Bessel integral transform and splitting the generalized exponential function in its even and odd part.

We notice that this type of combination of Special Functions in expressions of hypercomplex functions differs from the corresponding combinations of Bessel functions in the form of Hankel functions (Bessel functions of the third kind), obtained as complex combination of Bessel functions of the first and second kind.

Finally we mention, that more complicated relations are obtained by the same procedure for the hypercomplex analogue of, for example, $\arccos$ or $\arcsin$, which lead to expressions involving elliptic integrals of type **K** and **E**.

The corresponding series for $\arctan$ leads to expressions involving the LerchPhi-function etc.

## 4 APPLICATIONS: COMBINATORIAL IDENTITIES

At the end and without a more detailed explanation we would like to remark, how the developed relationships between Elementary and Special Functions could be used for deriving new combinatorial identities. Notice, for example, that for $n = 2$ in the generalized exponential function (21) the involved Bessel functions are just $J_0$ and $J_1$ with derivatives very easy to handle. It implies, that also the determination of the generated polynomials in terms of the derivatives of $J_0$ and $J_1$ is a relatively easy task. And this is just the point for their use in the deduction of combinatorial identities arising from the comparison with expressions derived from (13) for special values of $x_0$ and $|\underline{x}|$. Take, for instance $x_0 = 1$ and $|\underline{x}| = 1$. This leads on one side to the binomial sum

$$\begin{aligned}
\binom{2l}{0} - \binom{2l}{2}\frac{1}{2} + \binom{2l}{4}\frac{1\cdot 3}{2\cdot 4} - \cdots + (-1)^l \binom{2l}{2l}\frac{(2l-1)!!}{(2l)!!} &= A \\
\binom{2l}{1}\frac{1}{2} - \binom{2l}{3}\frac{1\cdot 3}{2\cdot 4} + \cdots + (-1)^{(l-1)} \binom{2l}{2l-1}\frac{(2l-1)!!}{(2l)!!} &= B,
\end{aligned}$$

but on the other side to the determination of the corresponding values of the $k$-th coefficients in the well known series development of $J_0$ and $J_1$. The corresponding expressions of A and B in terms of binomial coefficients form together with the corresponding left sides examples of two combinatorial identities. It is evident that this method works for any $n \geq 2$ and is not less and not more than a generalization to $\mathbb{R}^{n+1}$ of the use of $z^k$ for the determination of binomial sums (see [11], [14], [24]).

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