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ON A QUASI-CONFORMAL JOUKOWSKI TYPE TRANSFORMATION OF SECOND ORDER IN $\mathbb{R}^{M+1}$

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Abstract. *The classical Joukowski transformation plays an important role in different applications of conformal mappings, in particular in the study of flows around airfoils. Generalizations of this transformation where used in the 1920s by J. L. Walsh in order to approximate a continuous function on the boundary of its domain. Later, in the 1980s, H. Haruki and M. Barran studied generalized Joukowski transformations of higher order in the complex plane but from the perspective of functional equations. The aim of our contribution is to present a second order Joukowski type transformation in* $\mathbb{R}^{m+1}$*, but the construction also shows how to proceed in the case of higher orders. Like in the complex plane it still preserves some of the main properties of the ordinary Joukowski transformation (thereby justifying to be called a Joukowski type transformation), but also reveals some new and less expected properties. We deal in some detail only with the 3D-case corresponding to* $m = 2$ *and discuss its properties and visualizations for different geometric configurations.*

1 GENERALIZED JOUKOWSKI TRANSFORMATIONS IN THE COMPLEX PLANE

In the decade of 20 J. L. Walsh (c.f. [16]) studied the problem of approximating an arbitrary function of a complex variable by rational functions. Concretely, if the function is known to be continuous (analyticity is not reacquired) on a Jordan curve enclosing the origin then, on this curve the function can be approximated as closely as desired by a polynomial $R_n(z) = \sum_{k=1}^{n} a_k \left(z^k + \frac{1}{z^k}\right)$ (see also [15]). In fact, restricting to the unit circle we have

$$\sin k\theta = \frac{z^k - z^{-k}}{2i}, \qquad \cos k\theta = \frac{z^k + z^{-k}}{2},$$

revealing that every trigonometric polynomial is a polynomial in z and $\frac{1}{z}$. A classical example in this category of trigonometric polynomials are the Chebychev polynomials. Its complex parametrization over the unit circle in the form $T_k(x) = \cos k\theta = \frac{1}{2}(z^k + z^{-k})$ with $x = \frac{1}{2}(z + z^{-1})$, already revels the link between the classical Joukowski transformation (where $k = 1$) and its generalization of higher order in approximation theory.

In the decade of 80, the same functions referred above played again an important role in another subject. H. Haruki and M. Barran, with [11] and [1], studied specific functional equations whose unique solution is given by

$$\widetilde{w} = \widetilde{w}(z) = \frac{1}{2}(z^k + z^{-k}), \tag{1}$$

where k is a positive integer. The function $\widetilde{w} = w_0 + iw_1$ is said to be a *generalized Joukowski transformation of order k*.

Among other properties it maps the unit circle into the interval $[-1, 1]$ of the real axis in the $\widetilde{w}$-plane traced $2k$ times. Moreover, for $k = 1$, i.e. the classical case, symmetric and unsymmetric airfoils are obtained as images of circles with centers sufficiently near to the origin. Following [5], [2] and [3], the generalization to higher dimensions[1] indicates some modification in the use of the standard polar coordinates as well as the function representation itself. If we use modified polar coordinates in the form $z = \rho e^{i\left(\frac{\pi}{2} - \varphi\right)} = \rho(\sin\varphi + i\cos\varphi)$, for $\varphi \in [0, 2\pi]$, then we obtain the interval $[-i, i]$ as the image of the unit circle S^1 under the mapping

$$w = w(z) = \frac{1}{2}(z^k - z^{-k}). \tag{2}$$

Moreover the real and imaginary parts of w are obtained in the following form

$$w_0 = \frac{1}{2}\left(\rho^k - \frac{1}{\rho^k}\right)\cos(\frac{k}{2}\pi - k\varphi), \qquad w_1 = \frac{1}{2}\left(\rho^k + \frac{1}{\rho^k}\right)\sin(\frac{k}{2}\pi - k\varphi).$$

Circles of radius $\rho \neq 1$ are transformed onto confocal ellipses with semi-axis

$$a = \frac{1}{2}\left|\rho^k - \frac{1}{\rho^k}\right|, \qquad b = \frac{1}{2}\left(\rho^k + \frac{1}{\rho^k}\right)$$

and foci $w = i$ and $w = -i$.

[1] For $k = 1$ see [2] or [3], where this modified treatment of the Joukowski transformation was used for the first time. It allows to connect the 2D case more directly with the corresponding hypercomplex 3D case, where the unit sphere S^2 has a purely vector-valued image in analogy to the purely imaginary image of S^1.

Complex Case - Order $k = 2$

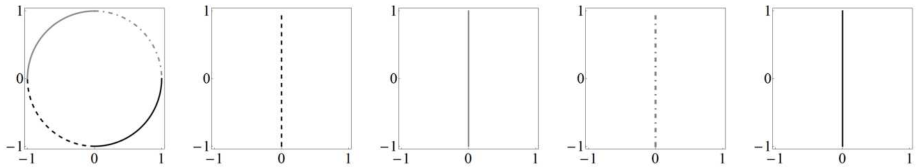

Figure 1: Images of the quarter-disks

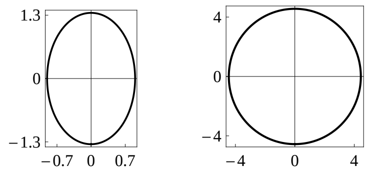

Figure 2: Images of semi-disks of radius $\rho = 1.5$ and $\rho = 3$

Figures 1 and 2 show the images of disks with radii equal or greater than one under the mapping w, for $k = 2$. In this case, the mapping is 4-fold when $\rho = 1$ and 2-fold for $\rho > 1$. For this reason, to stress the 4-fold covering of the segment $[-i, i]$, in the case $\rho = 1$, we present the images of the four quarter-disks separately.

A suitably higher dimensional analogue of the generalized Joukowski transformation w can be obtained using Clifford Analysis. In order to obtain monogenic analogues to z^k and z^{-k} in $\mathbb{R}^{m+1}$ we present in the next section some preliminaries.

2 CLIFFORD ANALYSIS - SOME BASIC NOTATIONS AND DEFINITIONS

Let $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^m$ with the non-commutative product according to the multiplication rules $e_k e_l + e_l e_k = -2\delta_{kl},\ k, l = 1, \ldots, m$, where δ_{kl} is the Kronecker symbol. The set $\{e_A : A \subseteq \{1, \ldots, m\}\}$ with $e_A = e_{h_1} e_{h_2} \ldots e_{h_r},\ 1 \leq h_1 < \cdots < h_r \leq m,\ e_\emptyset = e_0 = 1$, forms a basis of the 2^m-dimensional Clifford algebra $\mathcal{C}\ell_{0,m}$ over $\mathbb{R}$. Let $\mathbb{R}^{m+1}$ be embedded in $\mathcal{C}\ell_{0,m}$ by identifying $(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$ with the algebra's element $x = x_0 + \underline{x} \in \mathcal{A} := \text{span}_{\mathbb{R}}\{1, e_1, \ldots, e_m\} \subset \mathcal{C}\ell_{0,m}$. The elements of $\mathcal{A}$ are called paravectors and $x_0 = \text{Sc}(x)$ and $\underline{x} = \text{Vec}(x) = e_1 x_1 + \cdots + e_m x_m$ are the scalar resp. vector part of the paravector x. The conjugate of x is given by $\bar{x} = x_0 - \underline{x}$ and the norm $|x|$ of x is defined by $|x|^2 = x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + \cdots + x_m^2$. Consequently, any non-zero x has an inverse defined by $x^{-1} = \frac{\bar{x}}{|x|^2}$.

We consider functions of the form $f(z) = \sum_A f_A(z) e_A$, where $f_A(z)$ are real valued, i.e. $\mathcal{C}\ell_{0,m}$-valued functions defined in some open subset $\Omega \subset \mathbb{R}^{m+1}$. Continuity and real differentiability of f in Ω are defined componentwise. The generalized Cauchy-Riemann operator in $\mathbb{R}^{m+1},\ m \geq 1$, is defined by

$$\overline{\partial} := \partial_0 + \partial_{\underline{x}},$$

where

$$\partial_0 := \frac{\partial}{\partial x_0}, \quad \partial_{\underline{x}} := e_1 \frac{\partial}{\partial x_1} + \cdots + e_m \frac{\partial}{\partial x_m}.$$

The higher dimensional analogue of an holomorphic function is usually defined as $\mathscr{C}^1(\Omega)$-

function f satisfying the equation $\overline{\partial} f = 0$ (resp. $f\overline{\partial} = 0$) which is the hypercomplex form of a generalized Cauchy-Riemann system. By historical reasons it is called *left monogenic* (resp. *right monogenic*) [4]. An equivalent definition of monogenic functions is that f is hypercomplex differentiable in Ω in the sense of [8], i.e. that for f exists a uniquely defined areolar derivative f' in each point of Ω (see also [14]). Then f is automatically real differentiable and f' can be expressed by the real partial derivatives as $f' = 1/2\partial f$, where $\partial := (\partial_0 - \partial_{\underline{x}})$ is the conjugate Cauchy-Riemann operator. Since a hypercomplex differentiable function belongs to the kernel of $\overline{\partial}$, it follows that in fact $f' = \partial_0 f = -\partial_{\underline{x}} f$ like in the complex case. Conformal mappings in real Euclidean spaces of dimension higher than 2 are restricted to Möbius transformations (Liouville's theorem) which are not monogenic functions. But obviously, this does not mean that monogenic functions cannot play an important role in applications to the more general class of quasi-conformal mappings. The advantage of hypercomplex methods applicable to Euclidean spaces of arbitrary real dimensions (not only of even dimensions like in the case of $\mathbb{C}^n$-methods) is already evident for one of the most important case in practical applications, i.e. the lowest odd dimensional case of $\mathbb{R}^{2+1}$. It allows directly visualization of all geometric mapping properties. We compare the complex 2D case with 3D-plots produced with *Mathematica* for their visualization. A deeper function theoretic analysis, for instance, the relationship of the hypercomplex derivative and the Jacobian matrix, was not the aim of this work. However, in [2], the reader can find the corresponding results for the hypercomplex case of generalized Joukowski transformations of order $k = 1$.

A detailed discussion of the relationship between the hypercomplex derivative and the Jacobian matrix for $k = 2$ would exceed the purpose of this article. It can be done in the same way as in [3] for the generalized Joukowski transformation of order $k = 1$, which is 4-quasiconformal for $\rho > \sqrt[3]{4}$. Proposition 2 is linked to the behavior of the Jacobian matrix in different ranges of ρ.

3 MONOGENIC POLYNOMIALS AND THEIR KELVIN TRANSFORM

In [2] the higher dimensional analogue of the classical Joukowski transform for $m \geq 1$ and $k = 1$ has been studied in detail for the first time. For its generalization to the case of arbitrary order $k \geq 1$, we use the fundamental solution of the generalized Cauchy Riemann operator $E_m(x) = \frac{\bar{x}}{|x|^{m+1}}$, defined for $x \neq 0$, and the monogenic paravector-valued function defined for $m \geq 1$ by

$$\mathcal{P}_k^m(x) = \sum_{s=0}^{k} c_s(m) \binom{k}{s} x_0^{k-s} \underline{x}^s = \sum_{s=0}^{k} T_s^k(m) x^{k-s} \bar{x}^s \tag{3}$$

where $c_k(m) = \sum_{s=0}^{k} (-1)^s T_s^k(m)$ and

$$T_s^k(m) = \frac{k!}{m_{(k)}} \frac{\left(\frac{m+1}{2}\right)_{(k-s)} \left(\frac{m-1}{2}\right)_{(s)}}{(k-s)!s!},$$

with $m_{(k)}$ denoting the Pochhammer symbol (see [6] and [7]). The polynomials $\mathcal{P}_k^m$ generalize the complex powers z^k and, restricted to the real case, coincide with x_0^k. We recall also the definition of the Kelvin transform.

Definition 1 *Given a monogenic, paravector-valued, and homogeneous function f of degree k, then its Kelvin transform is defined, for $x \neq 0$, by*

$$I[f](x) := E_m(x) f(x^{-1}). \tag{4}$$

The following proposition shows the connection between the Kelvin transform applied to the polynomials $\mathcal{P}_k^m$ and the hypercomplex derivative of E_m.

Proposition 1 *Let $\mathcal{P}_k^m$ be the polynomials defined by (3). Then, for $x \neq 0$*

$$E_m^{(k)}(x) = (-1)^k m_{(k)} I[\mathcal{P}_k^m](x). \tag{5}$$

Proof: The factorization of the fundamental solution in the form

$$\frac{\bar{x}}{|x|^{m+1}} = \left(\frac{\bar{x}}{|x|^2}\right)^{\frac{m+1}{2}} \left(\frac{x}{|x|^2}\right)^{\frac{m-1}{2}}$$

allows the use of Leibniz' differentiation rule in order to obtain

$$\begin{aligned}
E_m^{(k)}(x) &= \frac{\partial^k}{\partial x_0^k} \frac{\bar{x}}{|x|^{m+1}} \\
&= \sum_{s=0}^{k} \binom{k}{s} (-1)^k \left(\frac{m+1}{2}\right)_{(k-s)} \left(\frac{m-1}{2}\right)_{(s)} \left(\frac{\bar{x}}{|x|^2}\right)^{\frac{m+1}{2}+k-s} \left(\frac{x}{|x|^2}\right)^{\frac{m-1}{2}+s} \\
&= (-1)^k k! \frac{\bar{x}}{|x|^{m+2k+1}} \sum_{s=0}^{k} \frac{\left(\frac{m+1}{2}\right)_{(k-s)}}{(k-s)!} \frac{\left(\frac{m-1}{2}\right)_{(s)}}{s!} \bar{x}^{k-s}\, x^s \\
&= (-1)^k m_{(k)} \frac{\bar{x}}{|x|^{m+2k+1}} \mathcal{P}_k^m(\bar{x}).
\end{aligned} \tag{6}$$

On the other hand, applying the Kelvin transform (4), we obtain

$$I[\mathcal{P}_k^m](x) = \frac{\bar{x}}{|x|^{m+1}} \mathcal{P}_k^m\left(\frac{\bar{x}}{|x|^2}\right) = \frac{\bar{x}}{|x|^{m+2k+1}} \mathcal{P}_k^m(\bar{x}) \tag{7}$$

and the final result follows now at once. □

4 GENERALIZED JOUKOWSKI TRANSFORMATIONS IN HIGHER DIMENSIONS

Analogously to [2], the proposed higher dimensional analogue of the Joukowski transformation is given by

Definition 2 *Let $x = x_0 + \underline{x} \in \mathcal{A} \cong \mathbb{R}^{m+1} \subset \mathcal{C}\ell_{0,m}$, with $x \neq 0$. The generalized hypercomplex Joukowski transformation of order k is defined as*

$$J_k^m(x) = \alpha_k \left(\mathcal{P}_k^m(x) + \frac{(-1)^k}{m_{(k-1)}} E_m^{(k-1)}(x) \right), \tag{8}$$

where α_k is a real normalization constant and $E_m^{(k-1)}(x)$ denotes the hypercomplex derivative of order $(k-1)$, for $k \geq 1$.

Formula (8) with $\alpha_1 = \frac{2}{3}$ is the generalized hypercomplex Joukowski transformation considered in [2].

In what follows we focus on the case $m = 2$ and $k = 2$, i.e. $\mathbb{R}^3$, and write briefly $\mathcal{P}_2^2(x) = \mathcal{P}_2(x)$ and $J_2^2(x) = J_2(x)$. This means, that we consider now the generalized hypercomplex Joukowski transformation of second order in the form

$$J_2(x) = \alpha_2 \left(\mathcal{P}_2(x) + \frac{1}{2} E'(x) \right) = \alpha_2 \left(\mathcal{P}_2(x) - I[\mathcal{P}_1](x) \right).$$

Then the image of the unit sphere $S^2 = \{x = x_0 + \underline{x} : \, |x|^2 = 1\}$ under J_2 is given by:

$$J_2(S^2) = \alpha_2 \frac{5}{2} x_0 \underline{x} \tag{9}$$

For $m, k \geq 3$, $J_k^m(S^m)$ has a paravector-valued expression, i.e., the image of the unit ball S^m in $\mathbb{R}^{m+1}$ is mapped into the hyperplane $w_0 = 0$, however, the corresponding proof relies on more difficult expressions of the $c_k(m)$ (see [6]) and for this reason has been omitted here. In fact, for $m = 2$, the normalization constant α_k in (8) is determined in such a way that S^1 in the hyperplane $w_0 = 0$ is the image of the unit sphere S^2. For that, and analogously to Section 1, we recall the geographic spherical coordinates in the 3-dimensional space. They allow to describe easily the mapping properties of J_1 as explained in [2] and [3]. Therefore, let (ρ, φ, θ) be radius, latitude, and longitude respectively, so that we work with

$$x_1 = \rho \cos\varphi \cos\theta, \quad x_2 = \rho \cos\varphi \sin\theta, \quad x_0 = \rho \sin\varphi$$

where $\rho > 0$, $-\pi < \theta \leq \pi$ and $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$.

For $k = 2$ we have in terms of spherical coordinates

$$\left| J_2(S^2) \right|^2 = \alpha_2^2 \left(\frac{5}{2} \right)^2 \cos^2\varphi \sin^2\varphi = \alpha_2^2 \left(\frac{5}{4} \right)^2 \sin^2(2\varphi). \tag{10}$$

Finally one easily observes that $\alpha_2 = \frac{4}{5}$ is the desired normalization value. In the same way it is, in principle, possible to determine for every k the corresponding value of α_k in such a way that S^1 in the hyperplane $w_0 = 0$ is the image of the unit sphere S^2. However, since the solution of algebraic equations of higher order becomes involved, it will obviously be more complicated than in those lower dimensional cases.

5 3D MAPPINGS BY GENERALIZED JOUKOWSKI TRANSFORMATIONS OF SECOND ORDER

We now focus on some basic geometric mapping aspects of the transformation J_2 in $\mathbb{R}^3$. Using the normalization factor $\alpha_2 = \frac{4}{5}$:

$$J_2(x) = \frac{4}{5} \left(x_0^2 + x_0 \underline{x} + \frac{1}{2} \underline{x}^2 \right) + \frac{2}{5} \left(\frac{x_0 - \underline{x}}{|x|^3} \right)'$$

we obtain for the components of $J_2 = w_0 + w_1 e_1 + w_2 e_2$ the following expressions:

$$w_0 = \frac{2}{5} \left(1 - \frac{1}{\rho^5} \right) \rho^2 (-1 + 3 \sin^2\varphi), \tag{11}$$

$$w_1 = \frac{4}{5} \left(1 + \frac{3}{2\rho^5} \right) \rho^2 \sin\varphi \cos\varphi \cos\theta, \tag{12}$$

$$w_2 = \frac{4}{5} \left(1 + \frac{3}{2\rho^5} \right) \rho^2 \sin\varphi \cos\varphi \sin\theta, \tag{13}$$

Case $\mathbb{R}^3$ - Order 2

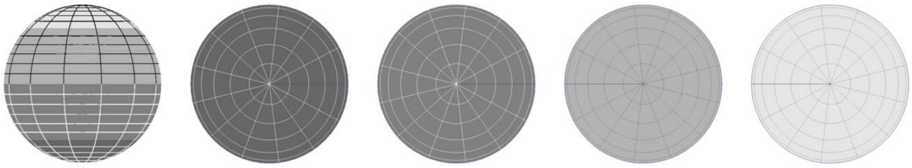

Figure 3: Images of the unit sphere

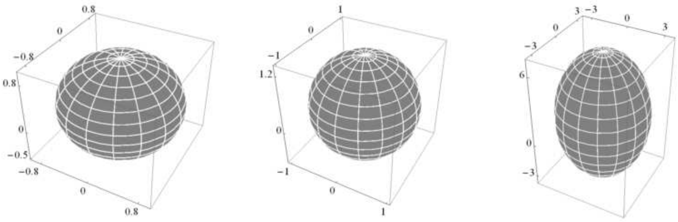

Figure 4: Images of hemispheres of radius $\rho = 1.3,\ \rho = \sqrt[5]{6}$ and $\rho = 3$

As we expected, spheres in $\mathbb{R}^3$ with radius $\rho \neq 1$ are transformed into spheroids, but this time, we obtain a 2-fold mapping. It is also possible to detect another new property, different from the case $k = 1$, namely the effect that the center of the spheroids does not anymore remain on the origin. The shift of the center from the origin occurs in direction of the real w_0-axis and is equal to $\frac{1}{5}\rho^2\left(1 - \frac{1}{\rho^5}\right)$. Therefore the polar radius is given by $b = \frac{3}{5}\rho^2\left|1 - \frac{1}{\rho^5}\right|$ and the equatorial radius by $a = \frac{1}{5}\rho^2\left(2 + \frac{3}{\rho^5}\right)$, so that we have:

$$\frac{\left[w_0 - \frac{1}{5}\rho^2\left(1 - \frac{1}{\rho^5}\right)\right]^2}{\left[\frac{3}{5}\rho^2\left(1 - \frac{1}{\rho^5}\right)\right]^2} + \frac{w_1^2}{\left[\frac{1}{5}\rho^2\left(2 + \frac{3}{\rho^5}\right)\right]^2} + \frac{w_2^2}{\left[\frac{1}{5}\rho^2\left(2 + \frac{3}{\rho^5}\right)\right]^2} = 1.$$

The following proposition summarizes some properties of the mapping J_2 (see [5]):

Proposition 2

1. *Spheres with radius* $1 < \rho < \sqrt[5]{6}$ *are 2-folded transformed into oblate spheroids.*
2. *The sphere with radius* $\rho = \sqrt[5]{6}$ *is 2-folded transformed into the sphere with center* $\left(0, 0, \frac{1}{\sqrt[5]{6^3}}\right)$.
3. *Spheres with radius* $\sqrt[5]{6} < \rho$ *are 2-folded transformed into prolate spheroids.*
4. *The unit sphere* S^2 *is 4-folded mapped onto the unit circle (including its interior) in the hyperplane* $w_0 = 0$.

Figure 3 shows the images of four zones of the unit sphere under the mapping J_2 as consequence of the 4-fold mapping of the unit sphere S^2 to S^1. Analogously to $k = 1$, Fig. 4 is the result of mapping one of the hemispheres with several radii greater than one.

Hypercomplex Case - Order $k = 1$

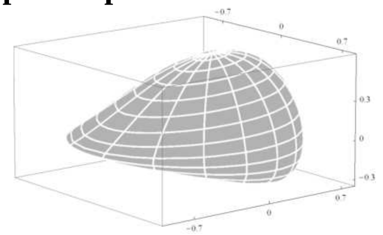

Figure 5: Image of a sphere of radius $\rho = 1 + |d|$ and center $d = 0.15e_0 + 0.1e_1 + 0.2e_2$

Figure 6: Cuts parallel to the hyperplane $w_1 = w_2$

6 FINAL REMARKS ON JOUKOWSKI TYPE 3D AIRFOILS OF SECOND ORDER

The classical Joukowski transformation $w(z) = \frac{1}{2}(z + z^{-1})$ plays in Aerodynamics an important role in the study of flows around so-called Joukowski airfoils, since it maps circles with centers sufficiently near to the origin into airfoils. In the classical *Dictionary of Conformal Representations* [13], for example, or the more recent book *Computational Conformal Mapping* [12], specially dedicated to computational aspects, one can find a lot of details about those symmetric or unsymmetric airfoils.

In the hypercomplex case, the paper [3] includes images produced with *Maple* of spheres in $\mathbb{R}^3$ centered at points of one of the axes x_1 or x_2 with a small displacement and passing through the endpoints of the unit vectors e_1 and e_2, respectively. If the displacement of the center of the sphere is done in all three directions unsymmetrically with three different values of the center coordinates, then we get a mapping like the one presented in the Fig. 5. We interpret this figure as some kind of unsymmetric Joukowski airfoils generalized in 3D. Figure 6 which shows some cuts of the domain illustrated in Fig. 5 parallel to the hyperplane $w_1 = w_2$ illustrates this situation even more.

Finally we compare some mappings for the case $k = 2$ in 2D and 3D. Due to the higher order of singularities in the origin we should also be aware of more complicated images of circles and spheres, respectively, with radii different from $\rho = 1$ (Figures 7-8). Nevertheless, we would not exclude the possibility, that they could be useful for mathematical models working with more complicated geometric configurations with some singularities, particularly in $\mathbb{R}^3$.

Resuming this steps towards a more systematic study of 3D mappings realized by generalized hypercomplex Joukowski transformations we would like to mention that hypercomplex methods seem to us in general a promising tool for quasi-conformal mappings in $\mathbb{R}^3$ ([17]).

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Complex Case - Order $k = 2$

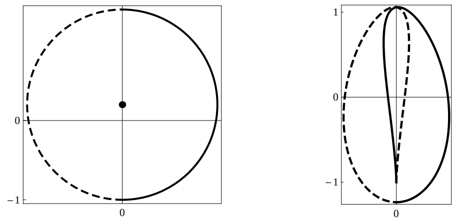

Figure 7: The image of a disk with radius $\rho = 1.2$ and center $d = (\rho - 1)i$

Hypercomplex Case - Order $k = 2$

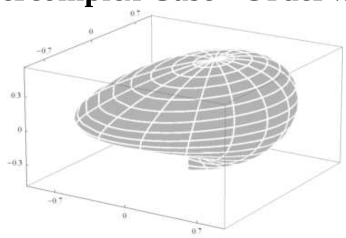

Figure 8: The image of a sphere of radius $\rho = 1 + |d|$ and center $d = 0.1e_1 + 0.1e_2$

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