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LINKING CLIFFORD ANALYSIS AND COMBINATORICS THROUGH BIJECTIVE METHODS

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Abstract. *The application of Clifford Analysis methods in Combinatorics has some peculiarities due to the use of noncommutative algebras. But it seems natural to expect from here some new results different from those obtained by using approaches based on several complex variable. For instances, the fact that in Clifford Analysis the point-wise multiplication of monogenic functions as well as their composition are not algebraically closed in this class of generalized holomorphic functions causes serious problems. Indeed, this is one of the reasons why in polynomial approximation almost every problem needs the construction of specially adapted polynomial bases. Our aim is to show that the analysis and comparison of different representations of the same polynomial or entire function allow to link Clifford Analysis and Combinatorics by means of bijective methods. In this context we also stress the central role of the hypercomplex derivative for power series representations in connection with the concept of Appell sequences as analytic tools for establishing this link.*

1 INTRODUCTION

"Remember, it's supposed to be fun"
Richard Feynman, quoted in
Some time with Feynman
by L. Mlodinow, 2003

Combinatorics as a branch of mathematics concerning the study of finite or countable discrete structures has enjoyed in the last decades a rapid growth, partially influenced by new connections to other branches like algebra, topology, geometry, probability theory, and others. It has many applications, for example, in graph theory, optimization, computer science, and statistical physics.

Several combinatorial subjects like sums, recurrences, binomial identities, elementary number theory, generating functions, or asymptotic methods can also be successfully treated by methods of complex analysis (of one or several complex variables). A most comprehensive, though less known, work by Egorychev on these methods is [10] (originally published in Russian in 1977) where an intensive use of residue theory is made for bridging the continuous and the discrete. For this purpose he systematically worked with integral transformations of combinatorial sums (involving binomial coefficients but also non-hypergeometric expressions). Directly or after simplifications or substitutions the residue-calculus is applied. Often one can compute combinatorial sums to which classical algorithms are not applicable. Those manipulations of integral representations also allow a more systematic treatment of combinatorial sums as those of [24], for instance, and implies a direct approach to asymptotical methods.

But in this connection reference should be made to the feeling that between *combinatorial analysts* and *true combinatorialists* exists a deep trench. It is worth knowing that almost ten years before Egorychev published his book, one of the most famous specialists of that time in enumerative combinatorics, *John Riordan*, wrote in the Preface to his book *Combinatorial Identities* [24] (1968) that

> *The identity is verified, apart from its putative combinatorial origin, by operations in which properties of the binomial coefficients are employed. Combinatorialists use recurrence, generating functions, and such transformations as the Vandermonde convolutions; others,* to my horror*, use contour integrals, differential equations, and other resources of mathematical analysis.*

There's no question that after 50 years that opinion has mostly changed and on the occasion of the Waterloo Workshop in Computer Algebra 2008 in honor of Georgy P. Egorychev and his 70th birthday D. Zeilenberger called Egorychev a *Bridge-Builder between the Discrete and the Continuous* [25]. He mentioned

> *Eight years after I finished my doctorate, I came across Egorychev's fascinating modern classic [10] about using the methods of complex analysis to evaluate (discrete) combinatorial sums. That was a pioneering ecumenical work, that influenced me greatly. Its content, of course, but especially its spirit and philosophy.*

In light of this praise the question whether Clifford Analysis methods could be useful for combinatorial problems or not seems merely rhetoric, since Clifford Analysis is a far reaching generalization of complex analysis of one variable to higher dimensions, different from several complex variable approaches. The paper [27] is an example of more detailed studies on the corresponding hypercomplex residuum theory. Other details are explained in [17], but the authors call attention to the fact that the corresponding general theory is not sufficiently developed, so far. We are convinced that the application to combinatorial analysis of a well developed residuum calculus in Clifford Analysis could be very promising. Our aim here is more modest and tries only to call attention to the subject via some examples relying on a generalized formal power series approach.

But even such a seemingly elementary approach suffers from the drawback that in Clifford Analysis the point-wise multiplication of monogenic functions as well as their composition are not algebraically closed in this class. This causes serious problems for the use of corresponding formal power series, for the development of a suitable generating function approach to special monogenic polynomials, or for establishing relations to corresponding hypergeometric functions etc. It is also the reason why in polynomial approximation of Clifford Analysis almost every problem needs the development of different adapted polynomial bases (e.g. [3], [5], [6], [7], [19], [22]).

However, the analysis of all those possible different representations led in the past to a deeper understanding and the construction of a monogenic hypercomplex exponential function which plays the same central role in applications as the ordinary exponential function of a real or complex variable. This concerns particularly its application as exponential generating function [12]. Previous constructions of hypercomplex exponential functions and other special functions like, for instance, a monogenic Gaussian distribution function, are mainly relying on the Cauchy-Kovalevskaya extension principle or - with some restriction on the space dimension - the so-called Fueter-Sce mapping (see [4, 16, 23]). The latter connects holomorphic functions with solutions of bi-harmonic or higher order equations. The former is based on the analytic continuation of complex or, in general, Clifford algebra valued functions of purely imaginary, respectively purely vectorial, arguments and therefore lacks the direct compatibility with the real or complex case as explained in [8].

The crucial idea for constructing a hypercomplex exponential function as shown in [12] (which stresses at the same time the central role of the hypercomplex derivative) was the construction of a monogenic hypercomplex exponential function as solution of an ordinary hypercomplex differential equation. The adequate multiple power series representations in connection with the classical concept of Appell sequences [2] (c.f. for applications [3, 7, 8, 18]) allowed, for instance, to develop new hypercomplex analytic tools for linking Clifford Analysis with combinatorics.

After introducing in Section 2 the necessary notations from Clifford analysis, we describe briefly in Section 3 a basic set of homogeneous polynomials which gained in the last years special interest in Clifford Analysis as generalization of the complex power function $w = z^k; z \in \mathbb{C};\ k = 1, 2, \ldots$. In the way as we use it for illustrating some interesting combinatorial relations, it was introduced in 2006 in [11]. In connection with the theory of basic sets of polynomials, introduced by J. M. Whittaker and B. Cannon for one complex variable, the paper [1] refers to the same set, but only in terms of a hypercomplex non-monogenic variable and its conjugate. Neither its representation in terms of monogenic variables nor its intrinsic properties had been studied therein. At the time of publication of [1] the concept of hypercomplex differentiability or the corresponding use of the hypercomplex derivative, first published in [19] resp. [15], have not been at disposal for the investigation of Appell sequences of monogenic polynomials. Since the construction of power-like monogenic functions was of general interest in Clifford Analysis, the study of sets of Appell polynomials developed in several directions and has been realized with different methods and by different authors as can be seen, for instance, in [3], [7] or [18]. The closing Section 4 contains references to some interesting combinatorial properties of the considered basic set of homogeneous polynomials. We combine, in some sense, the knowledge of different hypercomplex polynomial bases systems with the problem of establishing and proving new combinatorial identities by bijective methods. As far as we know, enumerative combinatorics did not deal until now with the corresponding function classes arising in Clifford Analysis.Here we can only have a glimpse of the basic ideas. In the talk we will explain them in more detail and with more and different approaches to the basic set and its relatives.

2 PRELIMINARIES

Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ with a non-commutative product according to the multiplication rules

$$e_k e_l + e_l e_k = -2\delta_{kl},\ k, l = 1, \cdots, n,$$

where δ_{kl} is the Kronecker symbol. The set $\{e_A : A \subseteq \{1, \cdots, n\}\}$ with

$$e_A = e_{h_1} e_{h_2} \cdots e_{h_r},\ 1 \leq h_1 < \cdots < h_r \leq n,\ e_\emptyset = e_0 = 1,$$

forms a basis of the 2^n-dimensional Clifford algebra $\mathcal{C}\ell_{0,m}$ over $\mathbb{R}$. Let $\mathbb{R}^{n+1}$ be embedded in $\mathcal{C}\ell_{0,m}$ by identifying $(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}$ with the elements (sometimes also denoted by z for underlining the relation to the complex case) $x = x_0 + \underline{x} \in \mathcal{A}_n := \text{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset \mathcal{C}\ell_{0,m}$. Here $x_0 = \text{Sc}(x) = \text{Sc}(z)$ and $\underline{x} = \text{Vec}(x) = \text{Vec}(z) = e_1 x_1 + \cdots + e_n x_n$ are the so-called scalar resp. vector part of the paravector $x \in \mathcal{A}_n$. The conjugate of x is given by $\bar{x} = \bar{z} = x_0 - \underline{x}$ and the norm $|x|$ of x is defined by $|x|^2 = x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + \cdots + x_n^2$. Denoting by $\omega(x) = \frac{\underline{x}}{|\underline{x}|} \in S^n$, where S^n is the unit sphere in $\mathbb{R}^n$, each paravector x can be written as $x = x_0 + \omega(x)|\underline{x}|$.

We consider functions of the form $f(z) = \sum_A f_A(z) e_A$, where $f_A(z)$ are real valued, i.e. $\mathcal{C}\ell_{0,m}$-valued functions defined in some open subset $\Omega \subset \mathbb{R}^{n+1}$. The generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1},\ n \geq 1$, is defined by

$$\overline{\partial} := \partial_0 + \partial_{\underline{x}}, \quad \partial_0 := \frac{\partial}{\partial x_0}, \quad \partial_{\underline{x}} := e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}.$$

C^1-functions f satisfying the equation $\overline{\partial} f = 0$ (resp. $f\overline{\partial} = 0$) are called *left monogenic* (resp. *right monogenic*). We suppose that f is hypercomplex differentiable in Ω in the sense of [15, 19], i.e. has a uniquely defined areolar derivative f' in each point of Ω (see also [21]). Then f is real differentiable and f' can be expressed by the real partial derivatives as $f' = \frac{1}{2}\partial$, where $\partial := \partial_0 - \partial_{\underline{x}}$ is the conjugate Cauchy-Riemann operator. Since a hypercomplex differentiable function belongs to the kernel of $\overline{\partial}$, it follows that in fact $f' = \partial_0 f$ like in the complex case.

We call attention to the fact, that powers of z, i.e, $f(z) = z^k, k = 2, \ldots$, are not monogenic, consequently they cannot be considered as appropriate for hypercomplex generalizations of the complex power $z^k, z \in \mathbb{C}$. Even the function $f(z) = z$ is monogenic only if $n = 1$, that is, if $\mathcal{A} = \mathbb{C}$. These facts justify the use of generalized power series of a special structure.

We consider also a hypercomplex structure for $\mathbb{R}^{n+1}$ based on an isomorphism between $\mathbb{R}^{n+1}$ and

$$\mathcal{H}^n = \{\vec{z} : \vec{z} = (z_1, \ldots, z_n), z_k = x_k - x_0 e_k, \quad x_0, x_k \in \mathbb{R}, k = 1, \ldots, n\},$$

(cf. [19]).

The hypercomplex variables z_k themselves are monogenic, but the same is not true for their ordinary products $z_i z_k, i \neq k$. However, this problem can be overcome by the introduction of their permutational (symmetric) product [19]:

Definition 1 *Let $V_{+,\cdot}$ be a commutative or non-commutative ring, $a_k \in V$ $(k = 1, \ldots, n)$, then the symmetric "$\times$"-product is defined by*

$$a_1 \times a_2 \times \cdots \times a_n = \frac{1}{n!} \sum_{\pi(i_1, \ldots, i_n)} a_{i_1} a_{i_2} \cdots a_{i_n} \tag{1}$$

where the sum runs over ***all*** *permutations of all $(i_1, \ldots, i_n)$.*

Convention: If the factor a_j occurs σ_j-times in (1), we briefly write

$$\underbrace{a_1 \times \cdots \times a_1}_{\sigma_1} \times \cdots \times \underbrace{a_n \times \cdots \times a_n}_{\sigma_n} = a_1{}^{\sigma_1} \times \cdots \times a_n{}^{\sigma_n} = \vec{a}^{\,\sigma} \tag{2}$$

where $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{N}_0^n$ and set parentheses if the powers are understood in the ordinary way (cf.[19], [20]).

The symmetric product along with the established convention permit to deal with a polynomial formula exactly in the same way as in the case of several commutative variables. It holds (see [20], [21])

$$(z_1 + \cdots + z_n)^k = \sum_{|\sigma|=k} \binom{k}{\sigma} z_1^{\sigma_1} \times \cdots \times z_n^{\sigma_n} = \sum_{|\sigma|=k} \binom{k}{\sigma} \vec{z}^{\sigma}, \quad k \in \mathbb{N} \tag{3}$$

with polynomial coefficients defined as usual by $\binom{k}{\sigma} = \frac{k!}{\sigma!}$ where $\sigma! = \sigma_1! \cdots \sigma_n!$.

The generalized powers $f(z) = \vec{z}^{\sigma}$, are left and right monogenic and $Cl_{0,n}$ - linear independent. Therefore they can be used as basis for generalized power series. Following [19] and [20] it has been shown, that the generalized power series of the form

$$P(\vec{z}) = \sum_{k=0}^{\infty} \left(\sum_{|\sigma|=k} \vec{z}^{\sigma} c_{\sigma} \right), c_{\sigma} \in Cl_{0,n}$$

generates in the neighborhood of the origin a monogenic function $f(\vec{z})$ and coincides in the interior of its domain of convergence with the Taylor series of $f(\vec{z})$, i.e, in a neighborhood of the origin we have

$$f(\vec{z}) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{|\sigma|=k} \vec{z}^{\sigma} \binom{k}{\sigma} \frac{\partial^{|\sigma|} f(\vec{0})}{\partial \vec{x}^{\sigma}} \right),$$

where $\vec{x} = (x_1, \ldots, x_n)$.

3 A BASIC SET OF POLYNOMIALS

Following the paper [8], where we introduced Appell sequences in Clifford Analysis in their general operational setting, we refer to

Definition 2 *Let U_1 and U_2 be (right) modules over $\mathcal{C}\ell_{0,n}$ and let $\hat{T} : U_1 \longrightarrow U_2$ be a hypercomplex (right) linear operator. A sequence of monogenic polynomials $(\mathcal{F}_k)_{k\geq 0}$ is called a $\hat{T}$-Appell sequence if $\hat{T}$ is a lowering operator with respect to the sequence, i.e., if*

$$\hat{T}\mathcal{F}_k = k\,\mathcal{F}_{k-1},\ k = 1, 2, \ldots,$$

and $\hat{T}(1) = 0$.

Since the operator $\frac{1}{2}\partial$ defines the hypercomplex derivative of monogenic functions, a sequence of monogenic polynomials that is $\frac{1}{2}\partial$-Appell is a hypercomplex counterpart of a classical Appell sequence and it is simply called Appell sequence or Appell set. The well known strong relation of Appell sequences to combinatorial problems, particularly to combinatorial sums or identities, is also manifest in the Clifford Analysis setting, where we have the following theorem

Theorem 1 ([8]) *A monogenic polynomial sequence $(\mathcal{F}_k)_{k\geq 0}$ is an Appell set if and only if it satisfies the binomial-type identity*

$$\mathcal{F}_k(x) = \mathcal{F}_k(x_0 + \underline{x}) = \sum_{s=0}^{k} \binom{k}{s} \mathcal{F}_{k-s}(\underline{x}) x_0^s,\ x \in \mathcal{A}_n. \tag{4}$$

We refer now in some detail to the basic set of Appell polynomials, mentioned in the Introduction. It was first considered in [11] for $\mathcal{A}_2$-valued polynomials defined in 3-dimensional domains and later on generalized to higher dimensions in [12, 22]. The polynomials under consideration are functions of the form

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} T_s^k(n)\, x^{k-s}\, \bar{x}^s,\ n \geq 1 \tag{5}$$

where

$$T_s^k(n) = \frac{k!}{n_{(k)}} \frac{(\frac{n+1}{2})_{(k-s)}(\frac{n-1}{2})_{(s)}}{(k-s)!s!}, \tag{6}$$

and $a_{(r)}$ denotes the Pochhammer symbol, i.e. $a_{(r)} = \frac{\Gamma(a+r)}{\Gamma(a)}$, for any integer $r > 1$, and $a_{(0)} = 1$.

The case of the real variable $x = x_0$ (i.e. $\underline{x} = 0$) is formally included in the above definitions as the case $n = 0$ with

$$T_0^k(0) = 1 \quad \text{and} \quad T_s^k(0) = 0, \quad \text{for } 0 < s \leq k, \tag{7}$$

so that the polynomials (5) are defined in $\mathbb{R}^{n+1}$, for all $n \geq 0$.

The infinite array of numbers (6) resembles in a lot of aspects a set of non-symmetric generalized binomial coefficients. Several intrinsic properties of this set can be obtained and are indicators of more advanced combinatorial relations in the next section. We highlight the following essential properties (see [11, 12, 22] and the references therein for details):

Proposition 1 *The triangle numbers $T_s^k(n)$ defined by (6) satisfy*

$$\sum_{s=0}^{k} T_s^k(n) = 1,\ \textit{for } n, k \geq 0. \tag{8}$$

Moreover, denoting by $c_k(n)$ the alternating sum $\sum_{s=0}^{k}(-1)^s T_s^k(n)$, then for $n \geq 1$ and $k = 1, 2, \ldots,$

$$c_k(n) = \begin{cases} \frac{k!!(n-2)!!}{(n+k-1)!!}, & \textit{if } k \textit{ is odd} \\ \\ c_{k-1}(n), & \textit{if } k \textit{ is even} \end{cases} \tag{9}$$

and $c_0(n) = 1$, for $n \geq 0$. As usual, we define $(-1)!! = 0!! = 1$.

Remark 1 *It is clear, from (8), that the polynomials $\mathcal{P}_k^n$ satisfy the normalization condition $\mathcal{P}_k^n(1) = 1$, for $k = 0, 1, \cdots$ and $n \geq 0$.*

We can now state the following fundamental result.

Theorem 2 ([9]) *The sequence of polynomials $\mathcal{P} := (\mathcal{P}_k^n)_{k\geq 0}$ is an Appell sequence.*

Theorems 1 and 2 lead to the binomial-type formula

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s} \mathcal{P}_s^n(\underline{x}) = \sum_{s=0}^{k} \binom{k}{s} c_s(n)\, x_0^{k-s}\, \underline{x}^s. \tag{10}$$

Remark 2 *The same result is obtained directly by the possibility to generalize the complex power through a suitable generalization of its binomial expansion (which is an essential property of Appell sets)*

$$z^k = (x+iy)^k = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s}(iy)^s.$$

Therefore one should consider a polynomial in x_0 and $\underline{x}$ of the form

$$\mathcal{P}_k^n(x_0,\underline{x}) = \sum_{s=0}^{k} d_s(n)\binom{k}{s} x_0^{k-s}(\underline{x})^s = \sum_{s=0}^{k} d_s(n)\binom{k}{s} x_0^{k-s}|\underline{x}|^s(\omega)^s$$

with some unknown coefficients $d_s(n)$. Demanding monogenicity and the natural initial values

$$\mathcal{P}_k^n(1) = 1, \;\; \textit{for} \;\; k = 0,1,\cdots \;\; \textit{and} \;\; n \geq 0, \;\; \textit{as well} \;\; \mathcal{P}_k^n(x_0) = x_0^k, k \geq 1$$

it can be shown that $d_s(n) \equiv c_s(n)$ due to the uniqueness of the hypercomplex Taylor series.

Formula (10) can be used to derive immediately the following properties:

Proposition 2

1. $\mathcal{P}_k^n(x_0) = x_0^k$ *, for all* $x_0 \in \mathbb{R}, \; n \geq 0.$

2. $\mathcal{P}_k^n(\underline{x}) = c_k(n)\underline{x}^k, \; n \geq 1.$

3. $\mathcal{P}_k^n(x) = \mathcal{P}_k^n(x_0 + \omega(x)|\underline{x}|) = u(x_0,|\underline{x}|) + \omega(x)v(x_0,|\underline{x}|)$*, where* $n \geq 1$ *and* u *and* v *are the real valued functions*

$$u(x_0,|\underline{x}|) = \sum_{s=0}^{[\frac{k}{2}]} \binom{k}{2s} (-1)^s x_0^{k-2s} c_{2s}(n)\, x_0^{k-2s}\, |\underline{x}|^{2s}$$

and

$$v(x_0,|\underline{x}|) = \sum_{s=0}^{[\frac{k-1}{2}]} \binom{k}{2s+1} (-1)^s x_0^{k-2s-1} c_{2s+1}(n)\, x_0^{k-2s-1}\, |\underline{x}|^{2s+1}$$

The second result of Proposition 2 can be seen as the essential property which characterizes the difference to the complex case. Nevertheless, the polynomials $\mathcal{P}_k^1$ coincide, as expected, with the usual powers z^k, since we get from (9), $c_k(1) = 1$, for all k. Furthermore, observing that $\omega^2(x) = -1$, we can consider that $\boldsymbol{\omega} := \omega(x)$ behaves like the imaginary unit, which means that the last property gives a representation of $\mathcal{P}_k^n$ in terms of a scalar part and an "imaginary" part.

4 A SECOND BIJECTIVE REPRESENTATION AND BINOMIAL IDENTITIES

The fact that we can write the polynomials

$$\mathcal{P}_k^n(x) = \mathcal{P}_k^n(x_0 + \underline{x})$$

in a bijective way also in terms of the hypercomplex variables $(z_1,\cdots,z_n)$

$$\mathcal{P}_k^n(x) = \mathbf{P}_k(z_1,\cdots,z_n)$$

allow now to prove two propositions of combinatorial character by comparing these different representations of one and the same polynomial. For this purpose we notice that the coefficients $c_k(n)$ defined as the alternating sum $\sum_0^k(-1)^s\mathcal{T}_s^k(n)$ of $T_s^k(n)$ appear in the second form of the polynomial representation, namely we have

Theorem 3 ([11]) *In terms of the generalized powers $z_1^{\nu_1} \times \cdots \times z_n^{\nu_n}$ the Appell set $\{\mathcal{P}_k^n\}$ is given by*

$$\mathcal{P}_k^n(x) = \mathbf{P}_k(z_1, \cdots, z_n) = c_k(n) \sum_{|\nu|=k} z_1^{\nu_1} \times \cdots \times z_n^{\nu_n} \binom{k}{\nu} e_1^{\nu_1} \times \cdots \times e_n^{\nu_n},$$

where $\binom{k}{\nu} = \frac{k!}{\nu_1! \ldots \nu_n!}$.

Together with the properties explained in the previous section, the next proposition gives a purely algebraic characterization of the coefficients $c_k(n)$.

Proposition 3 *The $c_k(n)$ in the representation of the generalized power function $\mathcal{P}_k^n(x) = \mathbf{P}_k(z_1, \cdots, z_n)$ are related to the algebraic generators $e_1, \ldots, e_n$ of the Clifford Algebra by*

$$\sum_{|\nu|=k} (-1)^k \binom{k}{\nu} (e_1^{\nu_1} \times e_2^{\nu_2} \times \cdots \times e_n^{\nu_n})^2 = \frac{1}{c_k(n)}.$$

and fulfill the binomial identity

$$c_k(n) = \frac{k!}{(n)_k} \sum_{s=0}^{k} \binom{-\frac{n+1}{2}}{k-s} \binom{\frac{1-n}{2}}{s}.$$

Proposition 4 *In the case $n = 2$, the $c_k = c_k(2)$ are weighted central binomial coefficients and we have*

$$c_k = \frac{1}{2^k} \binom{k}{\lfloor \frac{k}{2} \rfloor} = \left[\sum_{|\nu|=k} (-1)^k \binom{k}{\nu} (e_1^{\nu_1} \times e_2^{\nu_2})^2 \right]^{-1} = \frac{1}{k+1} \sum_{s=0}^{k} \binom{-\frac{3}{2}}{k-s} \binom{\frac{-1}{2}}{s}.$$

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