9^{th} International Conference on Clifford Algebras
and their Applications in Mathematical Physics
K. Gürlebeck (ed.)
Weimar, Germany, 15–20 July 2011

MONOGENIC GENERALIZED LAGUERRE AND HERMITE POLYNOMIALS AND RELATED FUNCTIONS

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Keywords: Clifford analysis, generalized Laguerre polynomials, generalized Hermite polynomials, exponential operators.

Abstract. *In recent years classical polynomials of a real or complex variable and their generalizations to the case of several real or complex variables have been in a focus of increasing attention leading to new and interesting problems. In this paper we construct higher dimensional analogues to generalized Laguerre and Hermite polynomials as well as some based functions in the framework of Clifford analysis. Our process of construction makes use of the Appell sequence of monogenic polynomials constructed by Falcão/Malonek and stresses the usefulness of the concept of the hypercomplex derivative in connection with the adaptation of the operational approach, developed by Gould et al. in the 60's of the last century and by Dattoli et al. in recent years for the case of the Laguerre polynomials. The constructed polynomials are used to define related functions whose properties show the application of Special Functions in Clifford analysis.*

1 INTRODUCTION

On the real line, special classes of polynomials such as Hermite and Laguerre are a well studied subject that arises in numerous problems of theoretical and applied mathematics. Usually they are defined by their generating functions or by their differential equations or by their Rodrigues formulas and so on. Our goal is to generalize those polynomials to higher dimensions in the framework of Clifford analysis, following a probably less known operational definition. On the real line, the well-known Hermite polynomials H_k, of degree k ($k = 0, 1, \ldots$), can be explicitly defined by

$$H_k(x) = \sum_{r=0}^{[\frac{k}{2}]} (-1)^r \frac{1}{r!} \frac{k!}{2^r(k-2r)!} x^{k-2r}, \; x \in \mathbb{R}, \tag{1}$$

and belong to a more general class of polynomials $H_{k,m}^{\lambda}$, of integer order m and real parameter λ. This family of generalized Hermite polynomials was defined by Gould et al. [11] by the operational identity

$$H_{k,m}^{\lambda}(x) := e^{\lambda(\frac{d}{dx})^m} x^k, \quad x \in \mathbb{R}. \tag{2}$$

In fact, the Hermite polynomials (1) correspond to the choices $m = 2$ and $\lambda = -\frac{1}{2}$ in (2).

On the other hand, the well-known real-valued generalized Laguerre polynomials $L_k^{(\alpha)}$ of degree k ($k = 0, 1, \ldots$) and real parameter α (sometimes called associated Laguerre) are explicitly given by

$$L_k^{(\alpha)}(x) = \sum_{r=0}^{k} (-1)^{k-r} \frac{\Gamma(k+\alpha+1)}{k!\,\Gamma(k+\alpha-r+1)} x^{k-r}, \; x \in \mathbb{R}. \tag{3}$$

The ordinary Laguerre polynomials L_k are obtained from (3) considering $\alpha = 0$.

More recently also an operational approach to the generalized Laguerre polynomials $L_k^{(\alpha)}(x, y)$ of two real variables x and y was studied by Dattoli [8], using the so-called Laguerre derivative operator $D_L := \frac{d}{dx} x \frac{d}{dx}$ [9]. By considering $L_k^{(\alpha)}(x) = L_k^{(\alpha)}(x, 1)$, Dattoli's procedure yields to the generalized Laguerre derivative operator $D_L^{(\alpha)} := D_L + \alpha \frac{d}{dx}, \; \alpha \in \mathbb{R}$, and to the definition of the generalized Laguerre polynomials $L_k^{(\alpha)}$ as

$$L_k^{(\alpha)}(x) := e^{-D_L^{(\alpha)}} \left(-\frac{x^k}{\Gamma(k+\alpha+1)} \right), \quad x \in \mathbb{R}. \tag{4}$$

Multidimensional analogues to (2) and (4) can be defined in the hypercomplex context of generalized holomorphic function theory by considering an appropriate hypercomplex exponential operator and a basic set of polynomials that can replace x^k. Generalized holomorphic function theory (more frequently called *monogenic* function theory) generalizes to higher dimensions the theory of holomorphic functions of one complex variable by using Clifford Algebras. Since in this context the set of monogenic functions is not closed with respect to the usual multiplication, a basic and crucial question is the replacement of x^k. Noting that $(x^k)_{k \in \mathbb{N}}$ is an Appell sequence with respect to the derivative operator $\frac{d}{dx}$, an idea for this replacement is to consider a monogenic Appell sequence with respect to the hypercomplex derivative operator (see [13, 14]). In this work we consider the monogenic Appell sequence defined in [10, 16] that contains the usual real and complex powers as particular cases.

The hypercomplex counterpart of the exponential operator used in (2) and (4) is defined through the monogenic exponential function considered in [10, 16] that is based naturally on the constructed Appell sequence.

2 BASIC NOTIONS

Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ with a non-commutative product according to the multiplication rules

$$e_k e_l + e_l e_k = -2\delta_{kl},\ k, l = 1, \cdots, n,$$

where δ_{kl} is the Kronecker symbol. The set $\{e_A : A \subseteq \{1, \cdots, n\}\}$ with

$$e_A = e_{h_1} e_{h_2} \cdots e_{h_r},\ 1 \leq h_1 < \cdots < h_r \leq n,\ e_\emptyset = e_0 = 1,$$

forms a basis of the 2^n-dimensional Clifford algebra $\mathcal{C}\ell_{0,n}$ over $\mathbb{R}$.

The space $\mathbb{R}^{n+1}$ is embedded in $\mathcal{C}\ell_{0,n}$ by identifying $\vec{x} = (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}$ with $x = x_0 + \underline{x} \in \mathcal{A}_n$, where $\mathcal{A}_n := \mathrm{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset \mathcal{C}\ell_{0,n}$. The elements of $\mathcal{A}_n$ are called paravectors and $x_0 = \mathrm{Sc}(x)$, $\underline{x} = \mathrm{Vec}(x) = e_1 x_1 + \cdots + e_n x_n$ are the so-called scalar resp. vector part of the paravector $x \in \mathcal{A}_n$. The conjugate of x is given by $\bar{x} = x_0 - \underline{x}$ and the norm $|x|$ of x is defined by $|x|^2 = x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + \cdots + x_n^2$.

We consider functions of the form $f(z) = \sum_A f_A(z) e_A$, where $f_A(z)$ are real valued, i.e. $\mathcal{C}\ell_{0,n}$-valued functions defined in some open subset $\Omega \subset \mathbb{R}^{n+1}$. Continuity and real-differentiability are defined componentwise.

The generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1},\ n \geq 1$, is defined by

$$\overline{\partial} := \partial_0 + \partial_{\underline{x}}, \quad \partial_0 := \frac{\partial}{\partial x_0}, \quad \partial_{\underline{x}} := e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}$$

and $C^1(\Omega)$-functions f satisfying the equation

$$\overline{\partial} f = 0 \ \ (\text{resp.} f\overline{\partial} = 0)$$

are called *left monogenic* (resp. *right monogenic*). We suppose that f is hypercomplex differentiable in Ω in the sense of [13, 14, 15] i.e. it has a uniquely defined areolar derivative f' in each point of Ω. Then f is real differentiable and f' can be expressed by the real partial derivatives as $f' = \frac{1}{2}\partial$, where $\partial := \partial_0 - \partial_{\underline{x}}$ is the conjugate Cauchy-Riemann operator. Since a hypercomplex differentiable function belongs to the kernel of $\overline{\partial}$, it follows that in fact $f' = \partial_0 f$ like in the complex case.

3 A BASIC SEQUENCE OF MONOGENIC POLYNOMIALS

In this section, we consider the Appell sequence $(\mathcal{P}_k^n)_{k \in \mathbb{N}}$ of monogenic homogeneous polynomials defined and studied in [10, 16], that can be represented by the binomial-type formula (see [4], Theorem 1),

$$\mathcal{P}_k^n(x) \ = \ \sum_{s=0}^{k} \binom{k}{s} c_s(n)\, x_0^{k-s}\, \underline{x}^s, \tag{5}$$

where

$$c_k(n) := \begin{cases} \dfrac{k!!(n-2)!!}{(n+k-1)!!}, & \text{if } k \text{ is odd} \\ & \\ c_{k-1}(n), & \text{if } k \text{ is even} \end{cases} \tag{6}$$

and $c_0(n) = 1$. As usual, we define $(-1)!! = 0!! = 1$.

We remark that $\mathcal{P}_0^n(x) = 1$ and $\mathcal{P}_k^n(0) = 0,\ k > 0$, in consequence of the homogeneity of these functions.

Particular cases:

1. For $\underline{x} = 0$ (real case), the polynomials (5) are the usual powers in the real variable x_0, for each $k = 0, 1, 2, \ldots$.

2. For $x_0 = 0$, we obtain
$$\mathcal{P}_k^n(\underline{x}) = c_k(n)\, \underline{x}^k.$$

Finally, we remark that in the complex case ($n = 1$), the polynomials $\mathcal{P}_k^1$ coincide, as expected, with the usual powers z^k. In fact, from (6), we get $c_k(1) = 1$, for all k. Then, the binomial-type formula (5) permits to state that

$$\mathcal{P}_k^1(x) = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s}\, \underline{x}^s = (x_0 + e_1 x_1)^k \simeq z^k.$$

4 A MONOGENIC EXPONENTIAL FUNCTION

Monogenic exponential functions and other special functions are mainly relying on the Cauchy-Kovalevskaya extension principle (see [1, 2, 12]) or the Fueter-Sce mapping-with some restriction on the space dimension-(see [1, 12]). The approach followed in [6, 10, 12, 16] defines a monogenic exponential function f as a solution of the simple first order differential equation $f' = f$, with $f(0) = 1$, where f' stands for the hypercomplex derivative of f. The combination of this approach with the constructed Appell sequence $(\mathcal{P}_k^n)_{k\in\mathbb{N}}$ leads to the monogenic exponential function in $\mathbb{R}^{n+1}$ defined by

$$\mathrm{Exp}_n(x) = \sum_{k=0}^{\infty} \frac{\mathcal{P}_k^n(x)}{k!}. \tag{7}$$

Considering $\boldsymbol{\omega} := \omega(x) = \dfrac{\underline{x}}{|\underline{x}|}$, a closed formula for the monogenic exponential (7) in terms of Bessel functions of integer or half-integer orders (depending on the dimension n) can be obtained:

Theorem 1. [10] The Exp_n-function can be written in terms of Bessel functions of the first kind, $J_a(x)$, for orders $a = \frac{n}{2} - 1, \frac{n}{2}$ as

$$\mathrm{Exp}_n(x_0 + \underline{x}) = e^{x_0}\Gamma(\tfrac{n}{2}) \left(\frac{2}{|\underline{x}|}\right)^{\frac{n}{2}-1} \left(J_{\frac{n}{2}-1}(|\underline{x}|) + \omega(x) J_{\frac{n}{2}}(|\underline{x}|)\right).$$

5 AN OPERATIONAL APPROACH TO MONOGENIC HERMITE AND LAGUERRE POLYNOMIALS

Let U_1 and U_2 be (right) linear modules over $\mathcal{C}\ell_{0,n}$ and $\hat{T} : U_1 \to U_2$ be a hypercomplex (right) linear operator. The exponential function in $\mathbb{R}^{n+1}$ defined above permits to consider the exponential operator

$$\mathrm{Exp}_n(\lambda\hat{T}) = \sum_{k=0}^{\infty} \frac{\mathcal{P}_k^n(\hat{T})}{k!}\lambda^k, \ \lambda \in \mathbb{R} \tag{8}$$

as a multidimensional counterpart in hypercomplex function theory of the usual exponential operator $e^{\lambda Q} = \sum_{k=0}^{\infty} \frac{Q^k}{k!}\lambda^k$.

The hypercomplex exponential operator (8) in connection with the basic Appell sequence $(\mathcal{P}_k^n)_{k\in\mathbb{N}}$ can be used now to generalize the Gould-Hopper's (2) and the Dattoli's (4) operational approaches to higher dimensions in the framework of Clifford analysis.

5.1 The generalization of Gould-Hopper's operational approach

The counterpart of Gould-Hopper's operational approach (2) to define the hypercomplex generalized Hermite polynomials $H_{k,m}^{(\lambda)}$ of integer order m and real parameter λ is given by

$$\begin{aligned} H_{k,m}^{(\lambda)}(x) &:= \mathrm{Exp}_n\left(\lambda\left(\tfrac{1}{2}\partial\right)^m\right)(\mathcal{P}_k^n(x)) \\ &= \sum_{r=0}^{[\frac{k}{m}]} \frac{1}{r!}\frac{k!}{(k-mr)!}\frac{\lambda^r}{2^r}\gamma_r(n)\mathcal{P}_{k-mr}^n(x), \ k = 0,1,\ldots \end{aligned} \tag{9}$$

where $\gamma_r(n) = \sum_{s=0}^{r}\binom{r}{s}c_s(n)$ is the binomial transform of the sequence $(c_s)_{s\geq 0}$. The constant polynomial $H_{0,m}^{(\lambda)}(x) \equiv 1$ (for any m, λ) is included in a natural way in (9), since $\gamma_0(n) = 1$ and $\mathcal{P}_0^n(x) \equiv 1$, independently of the dimension n.

For details on the construction of these polynomials see [7].

Special cases:

1. The real case can be formally included in the definition (9) as the case $n = 0$ (see [4] for details). Then it follows that $c_s(0) = 1$ $(s = 0,1,2,\ldots)$ and $\gamma_r(0) = \sum_{s=0}^{r}\binom{r}{s} = 2^r$. Taking into account also that $\mathcal{P}_k^0(x) = x_0^k$, from (9) we obtain the known generalized Hermite polynomials or Gould-Hopper polynomials in the real variable x_0,

$$H_{k,m}^{(\lambda)}(x) = \sum_{r=0}^{[\frac{k}{m}]} \frac{1}{r!}\frac{k!}{(k-mr)!}\lambda^r x_0^{k-mr}.$$

2. In the complex case $(n = 1)$, the polynomials $\mathcal{P}_k^1$ are isomorphic to the complex powers z^k $(k = 0,1,2,\ldots)$ and $c_s(1) = 1$, for arbitrary s. Therefore $\gamma_r(1) = 2^r$ and (9) gives the holomorphic Gould-Hopper polynomials in the complex variable $x = x_0 + e_1 x_1$.

3. The special choices of $m = 2$ and $\lambda = -\frac{1}{2}$ in (9) lead to

$$H_k^n(x) := H_{k,2}^{(-1/2)}(x) = \sum_{r=0}^{[\frac{k}{2}]}(-1)^r\frac{1}{r!}\frac{k!}{(k-2r)!}\frac{1}{4^r}\gamma_r(n)\mathcal{P}_{k-2r}^n(x) \tag{10}$$

which corresponds to the generalization for the hypercomplex case of the well-known Hermite polynomials (1) defined on the real line. In fact, if $\underline{x} \equiv 0$ (or $n = 0$) then $\mathcal{P}_{k-2r}^0(x) = x_0^{k-2r}$, $\gamma_r(0) = 1$ $(r = 0,1,2,\ldots)$ and (9) coincides with (1).

4. The choices $m = 2$ and $\lambda = -1$ in (9) as well as the consideration of the variable $2x$ instead of x give the polynomials

$$H_{k,2}^{(-1)}(2x) \quad = \quad \sum_{r=0}^{[\frac{k}{2}]}(-1)^r \frac{1}{r!}\frac{k!}{(k-2r)!}\frac{1}{2^r}\gamma_r(n)\mathcal{P}_{k-2r}^n(2x), \tag{11}$$

that for the particular case of $n = 0$ (real case) coincides with the ordinary Hermite polynomials used frequently in physics and related to the quantum harmonic oscillator.

5.2 The generalization of Dattoli's operational approach

The hypercomplex counterpart of Dattoli's operational approach (4) is not trivial because the set of monogenic functions is not closed under multiplication and composition and an alternative way to define the generalized hypercomplex Laguerre derivative should be considered. Notice that $(x^k)_{k\in\mathbb{N}}$ is a sequence of eigenfunctions of the operator $D_x := x\,\frac{d}{dx}$ involved in the Laguerre derivative operator $D_L := \frac{d}{dx}x\frac{d}{dx}$. Taking into account that in higher dimensions any homogeneous polynomial is an eigenfunction of the Euler operator $\mathbb{E} := \sum_{k=0}^n x_k\frac{\partial}{\partial x_k}$, it seems natural to replace the operator D_x by the Euler operator $\mathbb{E}$ in the hypercomplex version of the generalized Laguerre derivative operator $D_L^{(\alpha)} := D_L + \alpha\frac{d}{dx}$. In this context, we consider the hypercomplex generalized Laguerre derivative operator ${}_LD^{(\alpha)} :={}_L D + \alpha\frac{1}{2}\partial$, $\alpha \in \mathbb{R}$, and we define the monogenic Laguerre polynomials by

$$\begin{aligned}\mathcal{L}_k^{n,(\alpha)} \quad &:= \quad \mathrm{Exp}_n({}_LD^{(\alpha)})\left(\frac{\mathcal{P}_k^n}{\Gamma(k+\alpha+1)}\right)\\ &= \quad \sum_{r=0}^{k}\binom{k}{r}\frac{1}{2^r}\,\gamma_r(n)\,\frac{\mathcal{P}_{k-r}^n}{\Gamma(k+\alpha-r+1)},\ k=0,1,\ldots.\end{aligned}$$

For details on the construction, we refer to [4].

The consideration of the normalizing factor $\dfrac{\Gamma(k+\alpha+1)}{k!}$ and the substitution of x by $-x$ in the expression of $\mathcal{L}_k^{n,(\alpha)}$, i.e., the consideration of the polynomials

$$L_k^{n,(\alpha)}(x) := \frac{\Gamma(k+\alpha+1)}{k!}\mathcal{L}_k^{n,\alpha}(-x), \tag{12}$$

gives for the particular case $n = 0$ (or $\underline{x} = 0$) the generalized Laguerre polynomials defined by (3). For that reason, we call *monogenic generalized Laguerre polynomials* of degree k ($k = 0, 1, 2, \ldots$) to the polynomials (12).

6 MONOGENIC RELATED FUNCTIONS

In this section we consider some monogenic functions related to the constructed monogenic generalized Hermite and Laguerre polynomials. Considering the monogenic Gould-Hopper polynomials (9) with parameters $m = 2$ and $\lambda = -\frac{1}{t}$ ($t > 0$) and the scaled variable $2x$, we can define the monogenic Chebyshev polynomials of first and second kinds respectively by

$$\begin{aligned}T_k^n(x) \quad &:= \quad \frac{1}{2(k-1)!}\int_0^\infty e^{-t}\,t^{k-1}\,H_{k,2}^{(-1/t)}(2x)\,dt\\ &= \quad \frac{k}{2}\sum_{r=0}^{[\frac{k}{2}]}(-1)^r\frac{(k-1-r)!}{r!(k-2r)!}\frac{1}{2^r}\gamma_r(n)\mathcal{P}_{k-2r}^n(2x),\quad k=1,2,\ldots\end{aligned}$$

and

$$\begin{aligned} U_k^n(x) &:= \frac{1}{k!}\int_0^\infty e^{-t}\, t^k\, H_{k,2}^{(-1/t)}(2x)\, dt \\ &= \sum_{r=0}^{[\frac{k}{2}]}(-1)^r \frac{(k-r)!}{r!(k-2r)!}\frac{1}{2^r}\gamma_r(n)\mathcal{P}_{k-2r}^n(2x), \quad k=1,2,\ldots. \end{aligned}$$

For $n=0$ and $n=1$ we obtain the Chebyshev polynomials of first and second kinds in the real and complex variables, respectively, as particular cases of T_k^n and U_k^n $(k=1,2,\ldots)$.

As a consequence of the homogeneity of the polynomials $\mathcal{P}_k^n$ $(k=0,1,2,\ldots)$ and the fact that they form an Appell sequence, the monogenic Chebyshev polynomials of first and second kinds are related by the equality

$$\frac{1}{2}\partial\, T_k^n(x) = k\, U_{k-1}^n(x), \quad k=1,2,\ldots,$$

analogously to the real case.

Related to the monogenic generalized Laguerre polynomials and to the hypercomplex generalized Laguerre derivative operator, we can define monogenic Laguerre-type exponentials as well as Laguerre-type circular and hyperbolic functions. Those functions can be expressed in terms of different Special Functions according to the dimension of the considered Euclidean space (see [4, 5]). Adapting to the hypercomplex setting the approach developed in [9], we define the monogenic Laguerre-type exponential (or L-exponential) by

$$_{LD^{(\alpha)}}\mathrm{Exp}_n(x) := \sum_{k=0}^{\infty}\frac{1}{k!}\frac{\mathcal{P}_n^k(x)}{\Gamma(k+\alpha+1)}, \; \alpha\in\mathbb{R}, \tag{13}$$

which is an eigenfunction of the operator $_LD^{(\alpha)}$. In the case $\alpha=0$ the monogenic L-exponential (13) is given by

$$_{LD}\mathrm{Exp}_n(x) := \sum_{k=0}^{\infty}\frac{\mathcal{P}_k^n(x)}{(k!)^2} \tag{14}$$

and can be generalized to an arbitrary m-th Laguerre-type exponential (or mL-exponential) as

$$_{mL}\mathrm{Exp}_n(x) := \sum_{k=0}^{\infty}\frac{\mathcal{P}_k^n(x)}{(k!)^{m+1}}, \tag{15}$$

which is an eigenfunction of the operator $_{mL}D := \frac{1}{2}\partial\mathbb{E}^m$. Notice that for the case $m=0$, the $0L$-exponential coincides with the exponential function (7) and if x is a real number, then the function $_{mL}\mathrm{Exp}_0$ gives the generalized real-valued Laguerre-type exponential presented in [9]. For that case and considering $m=1$, $_L\mathrm{Exp}_0(x_0)$ can be written in terms of modified Bessel functions of the first kind I_0,

$$_L\mathrm{Exp}_0(x_0) = I_0(2\sqrt{x_0}).$$

When $x=\underline{x}$, an explicit representation of (14) can also be obtained.

Theorem 2. [4] The $_{LD}\mathrm{Exp}_n$-function (14) can be written in terms of hypergeometric functions as

$$_{LD}\mathrm{Exp}_n(\underline{x}) = {}_0\mathrm{F}_3\left(\,;\frac{1}{2},1,\frac{n}{2};-\frac{|\underline{x}|^2}{16}\right) + \boldsymbol{\omega}\,\frac{|\underline{x}|}{n}\,{}_0\mathrm{F}_3\left(\,;1,\frac{3}{2},\frac{n}{2}+1;-\frac{|\underline{x}|^2}{16}\right).$$

Based on the ${}_{mL}\mathrm{Exp}_n$-function (15) and following [9] we can go further and define, in a natural way, the corresponding monogenic m-th Laguerre circular (or mL-circular) and m-th Laguerre hyperbolic (or mL-hyperbolic) functions. Hence

$$
{}_{mL}\mathrm{Cos}_n(x) := \sum_{k=0}^{\infty} \frac{(-1)^k \mathcal{P}_{2k}^n(x)}{((2k)!)^{m+1}} \quad \text{and} \quad {}_{mL}\mathrm{Sin}_n(x) := \sum_{k=0}^{\infty} \frac{(-1)^k \mathcal{P}_{2k+1}^n(x)}{((2k+1)!)^{m+1}}
$$

$$
{}_{mL}\mathrm{Cosh}_n(x) := \sum_{k=0}^{\infty} \frac{\mathcal{P}_{2k}^n(x)}{((2k)!)^{m+1}} \quad \text{and} \quad {}_{mL}\mathrm{Sinh}_n(x) := \sum_{k=0}^{\infty} \frac{\mathcal{P}_{2k+1}^n(x)}{((2k+1)!)^{m+1}}
$$

Some examples of these functions are shown in the Table 1 for $x_0 = 0$ and particular values of m and n. It is clear that the mL-hyperbolic functions are related to the mL-exponential function by

$$
{}_{mL}\mathrm{Exp}_n(\underline{x}) = {}_{mL}\mathrm{Cosh}_n(\underline{x}) + {}_{mL}\mathrm{Sinh}_n(\underline{x}) \ \text{ and } \ {}_{mL}\mathrm{Exp}_n(-\underline{x}) = {}_{mL}\mathrm{Cosh}_n(\underline{x}) - {}_{mL}\mathrm{Sinh}_n(\underline{x}).
$$

Therefore we obtain the Euler-type formulae

$$
{}_{mL}\mathrm{Cosh}_n(\underline{x}) = \frac{{}_{mL}\mathrm{Exp}_n(\underline{x}) + {}_{mL}\mathrm{Exp}_n(-\underline{x})}{2}
$$

and

$$
{}_{mL}\mathrm{Sinh}_n(\underline{x}) = \frac{{}_{mL}\mathrm{Exp}_n(\underline{x}) - {}_{mL}\mathrm{Exp}_n(-\underline{x})}{2}.
$$

7 CONCLUDING REMARKS

The chosen operational approach for the construction of generalized Hermite and Laguerre polynomials in the framework of Clifford Algebras was mainly motivated by two aspects:

- on one hand, the possibility of obtaining the corresponding real and holomorphic polynomials as particular cases, in contrast to the approaches followed earlier by some authors (see [17, 3]) where this compatibility is not directly visible;

-on the other hand, the chosen procedure permits in a very natural way the definition of some related monogenic functions, such as the Chebyshev polynomials and also generalized exponential functions, stressing once more the usefulness of the hypercomplex derivative operator in monogenic function theory.

ACKNOWLEDGMENTS

Financial support from “Center for research and development in Mathematics and Applications” of the University of Aveiro, through the Portuguese Foundation for Science and Technology (FCT), is gratefully acknowledged.

Table 1: Some examples of L-exponential, L-circular and L-hyperbolic functions

		$m=0$	$m=1$
${}_{mL}\mathrm{Exp}_n(\underline{x})$	$n=1$	$\cos(\lvert\underline{x}\rvert)+\boldsymbol{\omega}\sin(\lvert\underline{x}\rvert)$	$\mathrm{Ber}_0(2\sqrt{\lvert x_1\rvert})+\boldsymbol{\omega}\,\mathrm{Bei}_0(2\sqrt{\lvert x_1\rvert})$
	$n=2$	$J_0(\lvert\underline{x}\rvert)+\boldsymbol{\omega}J_1(\lvert\underline{x}\rvert)$	$I_0(\sqrt{2\lvert\underline{x}\rvert})J_0(\sqrt{2\lvert\underline{x}\rvert})+\boldsymbol{\omega}I_1(\sqrt{2\lvert\underline{x}\rvert})J_1(\sqrt{2\lvert\underline{x}\rvert})$
	$n>2$	$(\frac{2}{\lvert\underline{x}\rvert})^{\frac{1-n}{2}-1}\Gamma(\frac{n}{2})\left(J_{\frac{n}{2}-1}(\lvert\underline{x}\rvert)+\boldsymbol{\omega}J_{\frac{n}{2}}(\lvert\underline{x}\rvert)\right)$	${}_0\mathrm{F}_3\left(\,;\frac{1}{2},1,\frac{n}{2};-\frac{\lvert\underline{x}\rvert^2}{16}\right)+\boldsymbol{\omega}\frac{\lvert\underline{x}\rvert}{n}\,{}_0\mathrm{F}_3\left(\,;1,\frac{3}{2},\frac{n}{2}+1;-\frac{\lvert\underline{x}\rvert^2}{16}\right)$
${}_{mL}\mathrm{Cos}_n(\underline{x})$	$n=1$	$\cosh(\lvert\underline{x}\rvert)$	$\frac{1}{2}\left(J_0(\sqrt{2\lvert\underline{x}\rvert})+I_0(\sqrt{2\lvert\underline{x}\rvert})\right)$
	$n=2$	$I_0(\lvert\underline{x}\rvert)$	$\frac{1}{2\sqrt{\lvert\underline{x}\rvert}}\left(J_1(\sqrt{2\lvert\underline{x}\rvert})+I_1(\sqrt{2\lvert\underline{x}\rvert})\right)$
	$n>2$	$(\frac{2}{\lvert\underline{x}\rvert})^{\frac{n}{2}-1}\Gamma(\frac{n}{2})I_{\frac{n-2}{2}(\lvert\underline{x}\rvert)}$	$\frac{1}{2}\lvert\underline{x}\rvert^{\frac{1-n}{2}}\Gamma(n)\left(J_{n-1}(2\sqrt{\lvert\underline{x}\rvert})+I_{n-1}(2\sqrt{\lvert\underline{x}\rvert})\right)$
${}_{mL}\mathrm{Sin}_n(\underline{x})$	$n=1$	$\boldsymbol{\omega}\sinh(\lvert\underline{x}\rvert$	$\boldsymbol{\omega}\frac{1}{2}\left(I_0(\sqrt{2\lvert\underline{x}\rvert})-J_0(\sqrt{2\lvert\underline{x}\rvert})\right)$
	$n=2$	$\boldsymbol{\omega}I_1(\lvert\underline{x}\rvert)$	$\boldsymbol{\omega}\left(\mathrm{Ber}_1(2\lvert\underline{x}\rvert)+\mathrm{Bei}_1(2\lvert\underline{x}\rvert)\right)$
	$n>2$	$\boldsymbol{\omega}(\frac{2}{\lvert\underline{x}\rvert})^{\frac{n}{2}-1}\Gamma(\frac{n}{2})I_{\frac{n}{2}(\lvert\underline{x}\rvert)}$	$\boldsymbol{\omega}\frac{\lvert\underline{x}\rvert}{n}{}_0\mathrm{F}_3\left(\,;1,\frac{3}{2},\frac{n}{2}+1;-\frac{\lvert\underline{x}\rvert^2}{16}\right)$
${}_{mL}\mathrm{Cosh}_n(\underline{x})$	$n=1$	$\cos(\lvert\underline{x}\rvert)$	$\mathrm{Ber}_0(2\sqrt{\lvert x_1\rvert})$
	$n=2$	$J_0(\lvert\underline{x}\rvert)$	$I_0(\sqrt{2\lvert\underline{x}\rvert})J_0(\sqrt{2\lvert\underline{x}\rvert})$
	$n>2$	$(\frac{2}{\lvert\underline{x}\rvert})^{\frac{1-n}{2}-1}\Gamma(\frac{n}{2})J_{\frac{n}{2}-1}(\lvert\underline{x}\rvert)$	${}_0\mathrm{F}_3\left(\,;\frac{1}{2},1,\frac{n}{2};-\frac{\lvert\underline{x}\rvert^2}{16}\right)$
${}_{mL}\mathrm{Sinh}_n(\underline{x})$	$n=1$	$\boldsymbol{\omega}\sin(\lvert\underline{x}\rvert)$	$\boldsymbol{\omega}\,\mathrm{Bei}_0(2\sqrt{\lvert x_1\rvert})$
	$n=2$	$\boldsymbol{\omega}J_1(\lvert\underline{x}\rvert)$	$\boldsymbol{\omega}I_1(\sqrt{2\lvert\underline{x}\rvert})J_1(\sqrt{2\lvert\underline{x}\rvert})$
	$n>2$	$\boldsymbol{\omega}(\frac{2}{\lvert\underline{x}\rvert})^{\frac{1-n}{2}-1}\Gamma(\frac{n}{2})J_{\frac{n}{2}}(\lvert\underline{x}\rvert)$	$\boldsymbol{\omega}\frac{\lvert\underline{x}\rvert}{n}\,{}_0\mathrm{F}_3\left(\,;1,\frac{3}{2},\frac{n}{2}+1;-\frac{\lvert\underline{x}\rvert^2}{16}\right)$

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