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A Nonstandard Approach to the Stochastic Nonhomogeneous Navier-Stokes Equations

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Abstract

This thesis is concerned with finding weak solutions for the stochastic Nonhomogeneous Navier-Stokes Equations. The method employed is that of Nonstandard analysis, in particular Loeb space theory. In Chapter 5, a new existence result is proven, this generalises the existing results since feedback is incorporated into the equation. The method found in this work is the approach developed by Capiński and Cutland (see [CaCu 95]) to solve the homogenous case.

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Finally I thank my family for there unwavering support and all they have done for me.

Dedicated to the memory of my Dad, Paul Enright 1939-1998 whose love and support was unquestionable.
Also to the life to come of my nice Olivia Spencer.

Contents

Chapter 1

Introduction

This work is concerned with the application of nonstandard methods to obtain new results for a stochastic equation of hydromechanics.
The main object of study is the classical Nonhomogeneous Navier-Stokes equations:

$$\rho\frac{\partial u}{\partial t} + < \rho u, \nabla > u = \nu \Delta u - \nabla p + \rho f$$

$$\frac{\partial \rho}{\partial t} + < u, \nabla > \rho = 0$$

$$\text{div } u = 0$$

$$u|_{t=0} = u_0 \text{ and } \rho|_{t=0} = \rho_0$$

These equations describe the motion of a viscous, incompressible, nonhomogeneous fluid confined to a bounded domain $D \subset \mathbb{R}^3$. The unknowns in the equations are the velocity u, the density ρ and the pressure p; the fluid is subjected to a known external force denoted by f. The equation governing the time derivative of ρ will be referred to as the density equation, and is derived by a consideration of the law of conservation of matter.

In many physical situations there is a degree of uncertainty, or randomness at play, to mirror this we introduce a random feature into the classical equations. It is the force term f that we allow to exhibit some random behaviour, this results in the formation of a stochastic partial differential equation:

$$\rho\frac{\partial u}{\partial t} + < \rho u, \nabla > u = \nu \Delta u - \nabla p + \rho f(t,u) + \rho g(t\ ,u)\frac{dw}{dt}$$

$$\frac{\partial \rho}{\partial t} + < u, \nabla > \rho = 0$$

$$\text{div } u = 0$$

$$u|_{t=0} = u_0 \text{ and } \rho|_{t=0} = \rho_0$$

where $\frac{dw}{dt}$ denotes white noise.

1.1 Background

The equations are studied in the well known Hilbert space framework, first developed by Leray, Temam and others in the 1930's; this pioneering work has led to a vast literature on the subject spawning much mathematical interest in equations of this type.
The nonstandard methodology used in this work was developed by Capiński and Cutland and was applied to the homogeneous Navier-Stokes equations. Their work can be found in a series of papers starting with [CaCu 91]; this worked culminated in a self contained presentation of their methods and new results in [CaCu 95].
The aim of this work is to twofold. Firstly to apply the methods found in [CaCu 95] to the nonhomogeneous Navier-Stokes equations, and demonstrate an existence theorem; this is not a new result (see [AKM 90] for example), however the new proof is more natural and requires less labour than those found in the literature. Secondly we apply the method of [CaCu 95] to the general stochastic Nonhomogeneous deterministic Navier-Stokes equations, and prove an existence theorem; this is a new result, the equation considered being more general that those found, for example in [Ya 92]; in particular, 'feedback' is incorporated into the equations.

Most of the literature concerning the Navier-Stokes equations is concerned with the homogeneous case. Physically this signifies that all particles within the fluid exhibit the same material properties. The equations describing such a system can be obtained from the nonhomogeneous case by requiring that the density ρ is constant in both time and space.
Thus, the resulting well known homogeneous classical equations are, with say $\rho \equiv 1$:

$$\frac{\partial u}{\partial t} + < u, \nabla > u = \nu \Delta u - \nabla p + f$$

$$\text{div } u = 0$$

$$u|_{t=0} = u_0$$

Therefore the most obvious difficulty associated with the nonhomogeneous case, as opposed to the homogeneous case, is the introduction of another equation; the density equation, this clearly being redundant when ρ is taken to be a constant. Thus the coupling of the two equations signifies a complication.
Secondly care has to taken with the often occurring product ρu, this is not an element of the usual solenoidal Hilbert spaces, but is however an element of the fundamental space, $L^2(D, \mathbb{R}^3)$.
One may expect the solution to the nonhomogeneous equations to be less regular than that of the homogeneous case, and in fact this is the case. Whereas it is well known that any solution to the homogeneous case must be weakly continuous in H, the solution found to the nonhomogeneous equations displays no such continuity. However there is a counterpart; it will be seen that the projection of ρu onto the space H is weakly continuous.

1.2 An Overview of the Nonstandard Approach

As indicated, the tools used in this work to solve the equations are drawn from nonstandard analysis. This theory provides us with a proper field extension ${}^*\mathbb{R}$ of $\mathbb{R}$ that contains infinitesimals, and infinite numbers. Basic knowledge of the theory is assumed, for more details see [AFHL 86], [Cu 88], [AFHL 86] and in particular [Li 88].

The theory allows the formation of a nonstandard space H_N constructed from the standard space H, in which the equations are set. Here N is an infinite natural number; the powerful transfer principle implies that this space behaves 'exactly' like the standard finite dimensional projections H_n where $n \in \mathrm{N}$. Moreover, there is a natural mapping from well behaved elements of H_N, to elements which are 'close' in H; this is analogous to the *standard part map* from ${}^*\mathbb{R}$ to $\mathbb{R}$, and shares with it, its name.

The standard approach to solving the equations (for example in [AKM 90]) is to form finite dimensional approximations to the equations, the so called Galerkin approximations on H_n; solving these yields a sequence $\{u_n\}_{n\in\mathrm{N}}$ of approximate solutions. In this work, the same initial approach is adopted, however due to the presence of infinite natural numbers, it is possible to form the Galerkin approximation of hyperfinite dimension N; then the transfer of the standard theory of ordinary differential equations provides a solution U on H_N.
The difficulty in the standard approach is then to ensure that a limit of such a sequence of approximations exists, and that this limit is indeed a solution; the non-linearity of the equations makes this a difficult step to take, consequently specialised compactness theorem are proved to overcome this.
However the nonstandard approach provides a very general limit U, and if this is proven to be sufficiently regular it can be 'projected' onto H to give a standard solution u to the equations, circumventing the need for compactness theorems. It is worth noting that compactness of the standard sequence $\{u_n\}$ is a consequence of the proof presented here, and is not needed within it. This illustrates the richness of the approach.

When solving the stochastic equations the same approach is taken, resulting in a stochastic internally adapted process U; it is then shown that this process is close to a standard adapted process u. This work relies heavily on the tools of Loeb measure theory. In particular the fact that we can move canonically from an internal process carried on a internal measure space to a process u carried on a standard (Loeb) measure space, this notion being at the very heart of the theory initiated Loeb in [Lo 75]. The Loeb measure spaces constructed from internal measure spaces are extremely rich, and are standard measure space, all be it constructed on a exotic probability space. The richness of the Loeb spaces along with the elegant theory which allows movement between the nonstandard world, and the standard world, is what makes such an approach effective.

Chapter 2

Standard Preliminaries

This chapter contains the necessary standard preliminaries and is based on the more detailed exposition given in [CaCu 95] .
The first section is devoted to functional analysis, in which the usual Hilbert space framework is described. This framework is well-known and can be found in detail in, for instance [Te 79] or [CoFo 88]. The second section is devoted to stochastic analysis, setting the scene in order that we may study the stochastic nonhomogeneous Navier-Stokes equations.

2.1 Functional Analysis

Throughout this work D will represent a fixed bounded open domain of $\mathbb{R}^3$, it will be assumed that the boundary is of class C^2.
The notation $L^2(D,\mathbb{R})$ is used for the vector space of real valued square integrable functions; whilst the notation $L^2(D)$ is reserved for the space $\{(L^2(D)\}^3 = L^2(D,\mathbb{R}^3)$. Denote by $|\cdot|$ the norm on $L^2(D)$ i.e.

$$| u |= (u,u)^{\frac{1}{2}}$$

where

$$(u,v) = \sum_{i=1}^{3} \int_D u^i(x)v^i(x)dx.$$

The space $\mathcal{V}$ is defined as follows

$$\mathcal{V} = \left\{u \in C_0^\infty(D,\mathbb{R}^3) : \text{div } u = 0\right\}$$

Let H denote the closure of the space $\mathcal{V}$ in the norm $|\cdot|$, then H is a Hilbert space with the inner product $(\cdot,\cdot)$. The topology induced on H by the norm $|\cdot|$ will be referred to as the strong topology (or norm topology), we will also be considering the space H equipped with the weak topology.
Let V denote the closure of the space $\mathcal{V}$ in the norm $|\cdot| + ||\cdot||$, where $|| u ||= ((u,u))^{\frac{1}{2}}$ and

$$((u,v)) = \sum_{i=1}^{3} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right).$$

Then V is a Hilbert space with the inner product $((\cdot,\cdot))$.
Let V' denote the dual space of V, then by identifying H with its dual we have the following well-known series of inclusions

$$V \subset H \subset V'$$

with continuous injections.
Thus, in particular we have

$$|\, v \,| \leq c \,||\, v \,|| \qquad \text{for all } v \in V. \tag{2.1.1}$$

Let A denote the Friedrichs self adjoint extension of the operator $-\Delta$ densely defined in H. Since D is bounded, A^{-1} is compact so there is an orthonormal basis $\{e_k\}$ of its eigenfunctions with corresponding eigenvalues λ_k i.e.

$$Ae_k = \lambda_k e_k \qquad \text{for all } k \in \mathrm{N}. \tag{2.1.2}$$

Further we may assume that $\lambda_k > 0$ for all $k \in \mathrm{N}$ and that $\{\lambda_k\}_{k\in\mathrm{N}}$ is a nondecreasing sequence diverging to $+\infty$.
Given $u \in H$ then u_k is used to denote the kth coordinate i.e.

$$u_k = (u, e_k).$$

Let H_m denote the finite dimensional subspace of H spanned by the first m basis vectors i.e.

$$H_m = \mathrm{span}\{e_1, \ldots, e_m\}.$$

The notation Pr_m is used to denote the projection from H onto H_m, so that

$$\mathrm{Pr}_m : H \to H_m \text{ and } \mathrm{Pr}_m(u) = \sum_{k=1}^{m} u_k e_k.$$

Note that a consequence of the regularity of the Eigen functions e_k is that for any $m \in \mathrm{N}$

$$H_m \subset C^1_B(D) \tag{2.1.3}$$

where C^1_B is the space of bounded continuous functions on D with bounded continuous derivatives on D. A scale of subspaces of H can be defined as follows, for $r \geq 0$

$$H^r = \left\{ u \in H : \sum_{k=1}^{\infty} \lambda_k^r u_k^2 < \infty \right\}$$

and so $H^r = \mathrm{dom}A^{\frac{r}{2}}$. For $u \in H^r$ the norm $|\, u \,|_r$ is given by

$$|\, u \,|_r = \left(\sum_{k=1}^{\infty} \lambda_k^r u_k^2 \right)^{\frac{1}{2}}.$$

And so $H^0 = H$, with $|\, u \,|_0 = |\, u \,|$; $H^1 = V$, with $|\, u \,|_1 = ||\, u \,||$.
Note also that for all $u \in \mathrm{dom}A$, $v \in V$

$$(Au, v) = ((u, v)).$$

The nonlinear term in the Navier-Stokes equations is represented by the following trilinear form:

$$b(u,v,z)=\sum_{i,j=1}^{3}\int_D u^j(x)\frac{\partial v^i}{\partial x^j}(x)z^i(x)dx=(<u,\nabla>v,z) \tag{2.1.4}$$

whenever the integral makes sense.
Using the Hölderinequality, it follows from (2.1.4) that

$$| b(u,v,z) |\leq| u |_{L^4(D)} \| v \| \; | z |_{L^4(D)} \tag{2.1.5}$$

There are many well-known inequalities for b that arise from various Sobolev embedding theorems, for instance

$$V \subset L^4(D) \tag{2.1.6}$$

and the embedding is continuous i.e.

$$| v |_{L^4(D)} \leq c \| v \| . \tag{2.1.7}$$

Thus, from (2.1.5) and (2.1.7) we can deduce the following basic inequality

$$| b(u,v,z) |\leq c \| u \| \, \| v \| \, \| z \| .$$

In this work we will often encounter b in the following form

$$b(\theta u,v,z)$$

where $\theta \in L^\infty(D,\mathbb{R})$. From the derivation of (2.1.5) it is clear that we have

$$| b(\theta u,v,z) |\leq| \theta |_{L^\infty(D,\mathbb{R})} \, | u |_{L^4(D)} \| v \| \; | z |_{L^4(D)}$$

and hence, using (2.1.7) we arrive at

$$| b(\theta u,v,z) |\leq c_1 \, | \theta |_{L^\infty(D)} \, \| u \| \, \| v \| \, \| z \| . \tag{2.1.8}$$

Remark 2.1.1 Note the trivial, but often useful identity

$$b(\theta u,v,z)=b(u,v,\theta z).$$

More general inequalities can be derived from the following result.

Proposition 2.1.2 *Let $m_i \geq 0$, $i=1,2,3$, and suppose that $m_1+m_2+m_3 \geq \frac{3}{2}$ if $m_i \neq \frac{3}{2}$ for all i, and $m_1+m_2+m_3 > \frac{3}{2}$ if some $m_i = \frac{3}{2}$. Then the form b is well defined on $Y^{m_1} \times H^{1+m_2} \times Y^{m_3}$ and*

$$| b(\theta_1 u,v,\theta_2 z) |\leq c \, | \theta_1 |_{L^\infty(D)} \, | \theta_2 |_{L^\infty} \, | u |_{m_1} | v |_{1+m_2} | z |_{m_3}$$

where $Y^{m_i}=\{\theta u : u \in H^{m_i}, \theta \in L^\infty(D)\}$.

This is a slight generalization of Proposition 1.1.1 [CaCu 95] . The proof presented there can be applied to prove Proposition 2.1.2 by first noting that

$$| \theta u |_{L^p(D)} \leq | \theta |_{L^\infty(D)} | u |_{L^p(D)}$$

and then employing the relevant embedding. For more details see [CaCu 95] and [Te 83].
Particular consequences of Proposition 2.1.2 are that

$$| b(\theta u, v, z) | \leq c | \theta |_{L^\infty(D)} | u | \| v \| | Az | . \tag{2.1.9}$$

$$| b(\theta u, v, z) | \leq c | \theta |_{L^\infty(D)} | u | | Av | \| z \| . \tag{2.1.10}$$

The following Lemma by Gronwall is needed; it is presented here in the form in which it is used.

Lemma 2.1.3 (Gronwall's Lemma) *If x and h are locally integrable functions on $[0,\infty)$, with dh/dt also locally integrable on $(0,\infty)$ and for all $t \geq 0$*

$$h(t) \leq c + \int_0^t h(t)x(t)dt$$

then for all $t \geq 0$

$$h(t) \leq c \exp\left(\int_0^t x(s)ds\right)$$

Finally for a very useful inequality:

Proposition 2.1.4 (Young's inequality) *If a, b, p, $\epsilon > 0$, then*

$$ab \leq \frac{\epsilon}{p}a^p + \frac{1}{q\epsilon^{q/p}}b^q$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

2.2 Stochastic Analysis

In this section it is assumed that the reader is familiar with the finite dimensional Itô integral (i.e with respect to a Wiener process). This theory shall be needed when we consider the hyperfinite approximation to the stochastic equation, some of the highlights are presented below, for further details see [KaSh 88] or the more readable [Ma 97].
We shall describe an extension of the Itô integral to our particular Hilbert space setting; this is based on [Ic 82], again more details can be found in [CaCu 95] . But firstly, let us take a very brief look at finite dimensional stochastic analysis.
Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space, that is $(\mathcal{F}_t)_{t\geq 0}$ is an increasing sequence of sub $\sigma-$algebras of $\mathcal{F}$. Suppose that on this space is an $\mathbb{R}^n$ valued Wiener process $w(t)$ adapted to $(\mathcal{F}_t)_{t\geq 0}$ i.e.

$$w(t) = (w_1(t), \ldots, w_n(t))$$

where $\{w_i(t)\}_{1\leq i\leq n}$ are real valued mutually independent Wiener process. Denote the covariance matrix of $w(t)$ by Q_n. Then for an adapted $\mathbb{R}^{n\times n}$ (i.e. $n\times n$ matrix) valued process $\zeta = \zeta(t,\omega)$ with

$$E\left(\int_0^t \operatorname{tr}(\zeta^T(s,\omega)\zeta(s,\omega))ds\right) < \infty \tag{2.2.11}$$

(here tr is the trace and ζ^T is the transpose of ζ) we may define the Itô integral

$$\eta(t) = \int_0^t \zeta(s)dw(s)$$

and we write

$$d\eta(t) = \zeta(t)dw(t)$$

which is called a stochastic differential. The process η is a $\mathbb{R}^n$ valued, zero mean process that satisfies

$$E(\eta(t)|\mathcal{F}_s) = \eta(s) \qquad \text{for } s\leq t$$

and thus is a martingale.
Also, if $\eta(t)$ is of this form then $[\eta](t)$, the quadratic variation of $\eta(t)$, is given by

$$[\eta](t) = \int_0^t \operatorname{tr}\left[\zeta^T(s)Q_n\zeta(s)\right] ds$$

Now let η be of a more general form i.e.

$$\eta(t) = \int_0^t f(s)ds + \int_0^t g(s)dw(s)$$

with f an adapted $\mathbb{R}^n$ valued process with integrable trajectories and g is a $n\times n$ matrix valued adapted process satisfying (2.2.11). Now let $F:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}$, be continuously differentiable in t and twice Fréchet differentiable in x; then we have the following stochastic chain rule, usually referred to as Itô's formula.

$$\begin{aligned} F(t,\eta(t)) = \quad & F(0,\eta(0)) + \int_0^t F_t(s,\eta(s)) + F_x(s,\eta(s))f(s)ds \\ & + \tfrac{1}{2}\int_0^t \operatorname{tr}\left[Q_n g^T(s)F_{xx}(s,\eta(s))g(s)\right] ds + \int_0^t F_x(s,\eta(s))g(s)dw(s). \end{aligned}$$

where F_t, F_x, F_{xx} denote the partial derivatives of F; note that F_x is a $n-$tuple and that F_{xx} is an $n\times n$ matrix.
Consider the following stochastic differential equation (really a stochastic integral equation):

$$\eta(t) = \eta_0 + \int_0^t f(s,\eta(s))ds + \int_0^t g(s,\eta(s))dw(s)$$

usually written symbolically as the following stochastic differential:

$$d\eta(t) = f(t,\eta(t))dt + g(t,\eta(t))dw(t)$$

$$\eta(0) = \eta_0.$$

Now if the functions f, g are continuous and Lipschitz with respect to the second variable then there is a unique adapted process η that solves this equation (see for example [Ma 97]).
We shall need the following well known inequality concerning the quadratic variation of a $\mathbb{R}^n$ valued martingale.

Proposition 2.2.1 (Burkholder-Davis-Gundy inequality)

$$E\left(\sup_{t\leq T} \mid \eta(t) \mid\right) \leq \kappa E\left([\eta](T)^{\frac{1}{2}}\right)$$

The crucial thing here for us is that the constant κ is not only independent of T, but is also independent of the dimension n.

Now to the extension of the Itô integral to our particular Hilbert space setting. The goal is to define the integral

$$\int_0^t g(s)dw(s)$$

for certain adapted $g(s) : H \to H$ and a Wiener process w in H.
Let $Q : H \to H$ be a linear, nonnegative, trace class operator, this analogous to the matrix Q_n. Firstly, we define what is meant by a H valued Wiener process.

Definition 2.2.2 *An H valued square integrable stochastic process $w(t)$, $t \geq 0$, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is a Wiener process with covariance operator Q if:*

1) $w(0) = 0$,

2) $Ew(t) = 0$, $Cov[w(t) - w(s)] = \mid t - s \mid Q$, *for all* s, $t \geq 0$,

3) $w(t) - w(s)$ *is independent of* $\mathcal{F}_s$, $s \leq t$,

4) w *has continuous paths,*

5) w *is adapted to* $(\mathcal{F}_t)$.

Such a process is Gaussian and has the following structure: Let $\{d_i\}$ be an orthonormal set of eigenvectors of Q with corresponding eigenvalues γ_i, so that

$$\mathrm{tr}Q = \sum_{i=1}^{\infty} \gamma_i.$$

Then

$$w(t) = \sum_{i=1}^{\infty} \beta_i(t) d_i,$$

where β_i are mutually independent real Wiener processes with $E(\beta_i^2(t)) = \gamma_i t$. For any Hilbert space Y denote by $\mathcal{I}(Y)$ the space of all stochastic processes

$$g : [0,T] \times \Omega \to L(H,Y)$$

such that

$$E\left(\int_0^T \mid g(t) \mid_{H,Y}^2 dt < \infty\right)$$

and such that for all $h \in H$, $g(t)h$ is a Y valued stochastic process adapted to the filtration $\mathcal{F}_t$.
The stochastic integral is then defined for all $g \in \mathcal{I}(Y)$ by

$$Y \ni \int_0^t g(s)dw(s) := L^2 - \lim_{m\to\infty} \int_0^t g(s)d_i d\beta_i(s) \tag{2.2.12}$$

where the integral of an H valued function with respect to a real Wiener process is defined in the same way as in the 1 dimensional case.
If $g \in L(H,H)$ and $u \in H$ then we shall write (u,g) for $g^T u$ (g^T is the adjoint to g) and so

$$((u,g),v) = (u,gv) \qquad \text{for } v \in H$$

and so $(u,g) \in L(H,\mathbb{R})$. Thus the following integral is well defined ($Y = \mathbb{R}$):

$$\int_0^t (u, g(s))dw(s)$$

and is real valued.
The general integral defined in (2.2.12) has properties analogous to the finite dimensional case, for instance the integral has mean zero. For other properties see [CaCu 95] or for more details see [Ic 82] and [DPZa 92].

Chapter 3

Nonstandard Preliminaries

This chapter is devoted to the nonstandard preliminaries needed for the subsequent chapters and is based on [CaCu 95] . The aim here is to highlight the most important ideas, definitions and notation that we will need later. It is assumed that the reader is familiar with the basics of nonstandard analysis; for the reader who is not or the reader who requires more detail see [CaCu 95] in which a fuller exposition is given starting with a construction of ${}^*\mathbb{R}$. For additional reading see [HuLo 85], [ACH 97], [Cu 88] and in particular [Li 88].
This chapter is split into two sections. In the first the Hilbert spaces H and V are studied, in particular we are interested in the hyperfinite dimensional subspace H_N of ${}^*H$. Various results concerning the strong and weak topologies and the relationship between them are stated.
In the second chapter we cover Loeb measure theory and Loeb integration theory. Again this is only intended as a brief over view of the terrain and more details can be found in, for example: [CaCu 95] , [AFHL 83] and [Cu 83].

3.1 The Hilbert space setting

Let H be the Hilbert space defined in Chapter 2. Nonstandard analysis provides us with the nonstandard extension ${}^*H$. The transfer principle tells us that ${}^*H$ has a basis $\{{}^*e_K\}_{K\in{}^*\mathbb{N}}$, this is the star of the function $e:\mathbb{N}\to H$.
This is a basis, in the sense that if $U\in{}^*H$ then there exists a unique internal sequence of hyperreals $\{U_K\}_{K\in{}^*\mathbb{N}}$ such that

$$U=\sum_{K=1}^{{}^*\infty} U_K\,{}^*e_K$$

This basis is orthonormal in that for any $K,N\in{}^*\mathbb{N}$ we have

$$(\,{}^*e_N,\,{}^*e_K)=\delta_{N,K}. \tag{3.1.1}$$

Remark 3.1.1 The inner product $(\cdot,\cdot)$ as defined in Chapter 2 is such that

$$(\cdot,\cdot):H\times H\to\mathbb{R}$$

and thus we have the nonstandard extension ${}^*(\cdot,\cdot)$, with

$${}^*(\cdot,\cdot):\ {}^*H\times{}^*H\to{}^*\mathbb{R}.$$

It is this function that is meant in (3.1.1), however for convenience of notation we omit the star, the context should make it clear what is intended. Similarly, really we should say that $({}^*e_K)_{K\in {}^*\mathbb{N}}$ is a star-basis for ${}^*H$.

Let us denote the nonstandard extension ${}^*e$ of the function e by E, thus $\{{}^*e_K\}_{K\in {}^*\mathbb{N}} = \{E_K\}_{K\in {}^*\mathbb{N}}$. A crucial role is played by the hyperfinite dimensional subspace H_N of ${}^*H$, where N is a fixed infinite member of ${}^*\mathbb{N}$. This space is define as one might expect; as the internal span (over ${}^*\mathbb{R}$) of the first N eigenvectors. Thus, we may say that $U \in H_N$ if and only if

$$\text{There exists an internal } N-\text{tuple } (U_K)_{1\leq K\leq N} \text{ such that } U = \sum_{K=1}^{N} U_K E_K,$$

alternatively

$$H_N = \{U \in {}^*H : (U, E_M) = 0 \text{ for } M > N\}.$$

Remark 3.1.2 The important thing here is that the transfer principle tells us that H_N, behaves very much like the standard finite dimensional subspace H_n, with $n \in N$. In particular the standard theory of finite dimensional differential equations carries over (via the transfer principle) to differential equations in H_N. This idea is the very heart of this thesis and was developed by Capiński and Cutland, see [CaCu 95] . Also H_N is rich enough to represent H in some sense.

Let $\Pr_N$ denote the orthogonal projection of ${}^*H$ onto H_N i.e

$$\Pr_N : {}^*H \to H_N \qquad \text{such that} \qquad \Pr_N(U) = \sum_{K=1}^{N} U_K E_K$$

where $U_K = (U, E_K)$.
Usually U and V are used to represent elements of H_N, we have

$$(U, V) = \sum_{K=1}^{N} U_K V_K \qquad \text{and} \qquad |\, U\,|^2 = \sum_{K=1}^{N} U_K^2.$$

More generally for the spaces H^r (with $r \in \mathbb{R}$) we have

$$H_N \subset {}^*H^r \qquad \text{in particular} \qquad H_N \subset {}^*V$$

and

$$|\, U\,|_r^2 = \sum_{K=1}^{N} \lambda_K^r U_K^2.$$

3.2 Topology

Before looking at the topologies of the particular Hilbert space, we recall the important notion of S-continuity:

Definition 3.2.1 *Let X and Y be nonstandard topological spaces; and $F : X \to Y$. Then F is S-continuous if*

$$x_1 \approx x_2 \Longrightarrow F(x_1) \approx F(x_2)$$

We are interested in both the strong and weak topologies of H and V and the relationship between them.
Firstly for some definitions, we note that these definition can be extended to the general space H^r in a natural way.

Definition 3.2.2 *If $U \in H_N$ and $u \in {}^*H$ then $U \approx u$ in the strong topology of H iff*

$$| U - {}^*u | \approx 0.$$

Then we say that u is the standard part of U in the strong topology of H and denote this by $u = st_H(U)$.

Denote by $\mathrm{ns}_H(H_N)$ those elements of H_N which are near standard i.e. those which are infinitely close (in the sense of the strong topology) to a standard element of H.

Definition 3.2.3 *If $U \in H_N$ and $u \in {}^*H$ then $U \approx u$ in the weak topology of H iff for all $v \in H$*

$$(U, {}^*v) \approx (u, v)$$

Then we say that u is the standard part of U in the weak topology of H and denote this by $u = st_{H-W}(U)$. The notation $u \approx_W U$ is used to denote closeness in the weak topology.

Denote by $\mathrm{ns}_{H-W}(H_N)$ those elements of H_N which are weakly near standard i.e. those which are infinitely close (in the sense of the weak topologies) to a standard element of H.
Now for some facts arising from these definition see [CaCu 95] for the proofs.

Proposition 3.2.4 *If $U \in H_N$ is strongly nearstandard then U is weakly nearstandard and the standard parts coincide.*

Proposition 3.2.5 *If $U \in H_N$ and U satisfies any of:*
1) *U is strongly nearstandard in H.*

2) *U is weakly nearstandard in H.*

3) *U is strongly nearstandard in V.*

4) *U is strongly nearstandard in V.*
then the relevant standard part (in all cases) is given by

$$\sum_{k=1}^{\infty} {}^\circ U_k e_k$$

Thus Proposition 3.2.5 prompts us to make the following definition

Definition 3.2.6 If $U \in H_N$ then we define (when it exists)

$${}^\circ U = \sum_{k=1}^{\infty} {}^\circ U_k e_k$$

Proposition 3.2.7 *If $u \in H$ then*

$$Pr_N {}^*u \approx u \qquad \textit{in the strong topology of } H.$$

Proposition 3.2.8 *If $U \in H_N$ then*

$$| U | < \infty \Longrightarrow U \textit{ is weakly nearstandard in } H,$$

and

$$\| U \| < \infty \Longrightarrow U \textit{ is weakly nearstandard in } V.$$

Proposition 3.2.9 *If $U \in H_N$ and $\| U \| < \infty$ then U is strongly nearstandard in H.*

Proposition 3.2.10 *If $U \in H_N$ then (allowing infinity on either side) we have*

$$|{}^\circ U| \leq {}^\circ | U | \qquad \textit{and} \qquad \|{}^\circ U\| \leq {}^\circ \| U \|$$

Finally we have the following trivial, but useful Proposition

Proposition 3.2.11 *If V, U, $Y \in {}^*H$ and $U \approx V$ strongly in H and $| U | < \infty$ then*

$$(U, Y) \approx (V, Y)$$

Operators

When dealing with the stochastic case use will be made of the weak topology of $L(H, H)$.

Definition 3.2.12 *By the weak topology of $L(H, H)$ we mean the topology given by*

$$g_n \to g \Longleftrightarrow (u, g_n v) \to (u, gv) \qquad \textit{for all } u, v \in H.$$

The nonstandard characterisation of this topology, is that if $G \in {}^*L(H, H)$ and $g \in L(H, H)$ then

$$G \approx_W g \Longleftrightarrow ({}^*u, G\,{}^*v) \approx (u, gv) \qquad \text{for all } u, v \in H.$$

Note that the coordinates $g_{ij} = (e_i, ge_j)$ are given by

$$g_{ij} = {}^\circ G_{ij} = {}^\circ(E_i, Ge_j)$$

and hence in line with Definition 3.2.6 we write $g = {}^\circ G$.
Now suppose that $G \in {}^*L(H_N, H_N)$. Then we can define nearstandardness of G and ${}^\circ G$ as follows

Definition 3.2.13 *Let $G \in {}^*L(H_N, H_N)$ and $g \in L(H, H)$. Then*

$$G \approx_W g \Longleftrightarrow GPr_N \approx_W g.$$

We write ${}^\circ G = g$ and say that G is weakly nearstandard in $L(H, H)$.

Note that we have still have $G_{ij} \approx g_{ij}$.
The following Proposition is the counterpart of Proposition 3.2.8 and Proposition 3.2.10

Proposition 3.2.14 *Let $G \in {}^*L(H_N, H_N)$ with $| G |_{H_N, H_N} < \infty$. Then G is weakly nearstandard and*

$$| {}^\circ G |_{H_N, H_N} \leq {}^\circ | G |_{H_N, H_N}$$

3.3 Loeb Measure and Integration Theory

In this section a brief review of Loeb measure and integration theory is given; again no proofs are provided since the material is well known. Proofs can be found in [CaCu 95] .
Given an internal set Ω with an internal algebra $\mathcal{A}$ of subsets of Ω and a finite internally finitely additive measure $\mathcal{M}$ on $\mathcal{A}$ then a standard measure space $(\Omega, L(\mathcal{A}), \mathcal{M}_L)$ can be constructed from the internal measure space $(\Omega, \mathcal{A}, \mathcal{M})$. This standard measure space is called a Loeb space.
The construction was first preformed by Loeb in [Lo 75] using Caratheodory's extension theorem. Alternative constructions exist which are more illuminating, for example see [CaCu 95] .
The upshot is that there exists a unique $\sigma-$additive extension of $^\circ\mathcal{M}$ to the $\sigma-$algebra $\sigma(\mathcal{A})$. The completion of this measure is the Loeb measure $\mathcal{M}_L$ and the completion of $\sigma(\mathcal{A})$ is the Loeb algebra $L(\mathcal{A})$.

Remark 3.3.1 The Loeb space $(\Omega, L(\mathcal{A}), \mathcal{M}_L)$ is a standard measure space even though Ω is a nonstandard set.

Loeb spaces play a fundamental role in this work since it will be a Loeb space that carries the solution to the stochastic equations considered later. Also Loeb measure can be used to represent a standard measure. For instance if $(X, \mathcal{S}, \pi)$ is a suitable standard measure space then we can form the internal measure space $({}^*X,\ {}^*\mathcal{S},\ {}^*\pi)$ and then form the Loeb space $({}^*X, L({}^*\mathcal{S}),\ {}^*\pi_L)$ then this Loeb space can be used to represent the standard space. This idea is used in the following theorem

Theorem 3.3.2 *If X is a separable metric space and μ is a finite Borel measure on the $\sigma-$algebra of Borel sets $\mathcal{B}$ then*

$$st : ns({}^*X) \to X$$

is measurable with respect to $\sigma({}^*\mathcal{B})$ and hence is $L({}^*\mathcal{B})$ measurable. Further

$$\mu(B) = {}^*\mu_L(st^{-1}(B)) \qquad \text{for all } B \in \mathcal{B}$$

and

$${}^*\mu_L({}^*X \backslash ns({}^*X)) = 0 \quad .$$

Loeb Measurable Function and Integration

This section provides us with some tools to handle internal integrals by considering corresponding Loeb integrals.
An important notion is that of a lifting

Definition 3.3.3 An internal $\mathcal{A}$ measurable function $F : \Omega \to {}^*\mathbb{R}$ is a lifting of a function $f : \Omega \to \mathbb{R}$ if for $\mathcal{M}_L - a.a.\omega$

$$F(\omega) \approx f(\omega)$$

Then we have the fundamental theorem

Theorem 3.3.4 *The function f is Loeb measurable if and only if it has a lifting F.*

We now start to look at connections between internal integration and Loeb integration, starting with

Proposition 3.3.5 *If F is a bounded internal measurable function then*

$$^\circ\!\int Fd\mathcal{M} = \int {}^\circ F d\mathcal{M}_L$$

Remark 3.3.6 For an internal measure $\mathcal{M}$ it is sometimes more convenient to denote integration with respect to $\mathcal{M}_L$ by $d_L\mathcal{M}$, rather than $d\mathcal{M}_L$, both notations are used in this work

Proposition 3.3.7 *For any internal $\mathcal{A}$ measurable function F, with $F \geq 0$ then*

$$\int {}^\circ F\mathcal{M}_L \leq {}^\circ\!\int Fd\mathcal{M}$$

where we allow the value ∞ on either side.

We do not necessarily have equality in Proposition 3.3.7. In fact we can easily construct examples of functions F such that the inequality is strict. These examples involve F being very large on very small sets, if we prevent this from happening that we do get equality. This is made precise in the following definition and Theorem involving the important notation of S-integrability.

Definition 3.3.8 *Let a function $F : \Omega \to {}^*\mathbb{R}$ be $\mathcal{A}$ measurable and internal. Then F is S-integrable if*
1) $\int_\Omega \mid F \mid d\mathcal{M}$ *is finite.*

2) *if $A \in \mathcal{A}$ and $\mathcal{M}(A) \approx 0$, then*

$$\int_A \mid F \mid d\mathcal{M} \approx 0.$$

And then we have

Theorem 3.3.9 *Let $F : \Omega \to {}^*\mathbb{R}$ be $\mathcal{A}$ measurable and nonnegative. Then the following two conditions are equivalent:*
1) *F is S-integrable.*

2) *$^\circ F$ is Loeb integrable and*

$$\int {}^\circ F\mathcal{M}_L = {}^\circ\!\int Fd\mathcal{M}$$

The following fact is used repeatedly in the proof of the existence theorem.

Proposition 3.3.10 *If $F : {}^*[0,T] \to {}^*\mathbb{R}$ and $f : [0,T] \to \mathbb{R}$ are such that, for a.a.τ*

$$F(\tau) \approx f({}^\circ\tau) \tag{3.3.2}$$

and F is S-continuous. Then

$$\int_0^T F(\tau)d\tau \approx \int_0^T f(t)dt$$

For the proof use Theorem 3.3.9 and Theorem 3.3.2.

Remark 3.3.11 The property 3.3.2 of F and f is usually refered to, by saying: F is a two-legged lifting (or just a lifting) of f.

The following Theorem is often used to show that a certain function is S-integrable.

Theorem 3.3.12 *If $F : \Omega \to {}^*\mathbb{R}$ is internal, $\mathcal{A}$ measurable, and*

$$\int | F |^p \, d\mathcal{M} < \infty$$

for some $p > 1$, $p \in \mathbb{R}$, then F is S-integrable.

Also the following elementary proposition is useful

Proposition 3.3.13 *If $F : {}^*[0, T] \to {}^*\mathbb{R}$ is S-integrable with respect to the measure ${}^*\lambda$ (λ is Lebesgue measure), then the function*

$$G(\tau) = \int_0^\tau F(\sigma) d\sigma$$

is S-continuous.

Next is Anderson's 'Luzin' Theorem, a very important result regarding liftings that will be used extensively in this work. For more general forms see [An 82].

Theorem 3.3.14 (Anderson's Theorem) *Let X, Y be metric spaces with Y separable and suppose that $f : X \to Y$ is measurable. Suppose μ is a Borel measure on X. Then ${}^*f$ is a lifting of f with respect to ${}^*\mu_L$ i.e. for ${}^*\mu_L$ − a.a.$x \in {}^*X$ we have*

$${}^*f(x) \approx f({}^\circ x).$$

3.4 Construction of a Wiener Process on H

The aim here is to construct an H valued Wiener process on a space (in fact a Loeb space) that is rich enough to carry the solution to the stochastic equations studied in chapter 5. The plan is to first construct an internal Wiener process on H_N and then push this down onto H. Then we can deal with infinite dimensional stochastic Itô integrals by considering hyperfinite Itô integrals that are in some sense close. The ideas here was first used in [Ke 84] and later in [Cu 86], more details can be found in [CaCu 95] .
Let Q be the fixed (but general) covariance operator introduced in section 2.2. and define $Q_n = \mathrm{Pr}_n Q \mathrm{Pr}_n$. Then it is possible to construct a standard Wiener process w^n with covariance Q_n, see [CaCu 95] for details.
Now take a nonstandard internal, filtered probability space

$$(\Omega, \mathcal{A}, (\mathcal{A}_\tau)_{\tau \geq 0}, \Pi)$$

carrying an internal Wiener process $W(\tau) \in H_N$ with covariance Q_N, adapted to $(\mathcal{A}_\tau)_{\tau \geq 0}$.

The corresponding Loeb space $(\Omega, L(\mathcal{A}), \Pi_L)$ can be equipped with a standard filtration and a $W(\tau)$ induced Wiener process in H with covariance Q. Let $P = \Pi_L$ and $\mathcal{F} = L(\mathcal{A})$ and denote by $\mathcal{N}$ the family of $P-$null sets. And then a right continuous filtration is defined by

$$\mathcal{F}_t = \bigcap_{t<^\circ\tau} \sigma(\mathcal{A}_\tau) \vee \mathcal{N}.$$

And then we have that

$$w(t) = \mathrm{st}_H W(\tau) \text{ (for } \tau \approx t)$$

defines a standard $\mathcal{F}_t$ adapted Wiener process on $(\Omega, \mathcal{F}, P)$ with values in H, and with covariance Q. For details see [CaCu 95] Theorem 3.6.1.

3.5 Stochastic Integration

We make use of the following two results, also to be found in [CaCu 95] .
Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered Loeb space carrying a H valued Wiener process $w = {}^\circ W$ with covariance Q, where W is an internal H_N valued Wiener process with covariance Q_N on the corresponding internal space as above. Then we have

Theorem 3.5.1 *Suppose that*

$$G: \ {}^*[0,T] \times \Omega \to L(H_N, H_N)$$

*is internal, *measurable, and adapted to the internal filtration* $(\mathcal{A}_\tau)_{\tau\geq 0}$. *Assume that*
1) $E\left(\int_0^T \mid G(\tau) \mid^2_{H_N,H_N} d\tau\right) < \infty$

2) *for a.a.*ω, $\mid G(\cdot,\omega) \mid^2_{H_N,H_N}$ *is S-integrable on* ${}^*[0,T]$.
Put

$$Y(\tau) = \int_0^\tau G(\sigma) dW(\sigma),$$

the internal Itô integral in H_N.
Then $Y(\tau)$ *is S-continuous in* $|\cdot|$ *for a.a.*ω.

For G and Y as in the above theorem we can put $y(t) = {}^\circ Y(\tau)$ for $\tau \approx t$ and y is a continuous H valued process.
Before the last theorem of this section we need the following definition

Definition 3.5.2 The internal process G is a lifting of a standard process $g : [0,T] \times \Omega \to L(H,H)$ if for $a.a.(\tau,\omega)$ we have

$${}^\circ G(\tau,\omega) = g({}^\circ\tau, \omega)$$

where the standard part here is in the sense of 3.2.12.

Theorem 3.5.3 *Suppose that* G *and* $Y = \int G d\dot{W}$ *are as in Theorem 3.5.1, and* $y = {}^\circ Y$ *as above. If* G *is a lifting of an adapted process* $g : [0,T] \times \Omega \to L(H_N, H_N)$ *then*

$$y(t) = \int_0^t g(s) dw(s).$$

Chapter 4

The Nonhomogeneous Navier-Stokes Equations

4.1 Introduction

The nonhomogeneous Navier-Stokes equations were first solved by Kazhikhov in 1974, see [Ka 74] ; for a more detailed presentation see [AKM 90].
What is presented here is a new proof using the techniques of nonstandard analysis; the new approach simplifies the existing proofs and is based on techniques developed by Capiński and Cutland, see [CaCu 95].
This proof is included as a stepping stone to the more difficult general stochastic case; also it provides us with an opportunity to introduce and elucidate the general methodology.

The nonhomogeneous Navier-Stokes equations will be studied in the following form:

$$\rho[\frac{\partial u}{\partial t} + < u, \nabla > u] = \nu \Delta u - \nabla p + \rho f \tag{4.1.1}$$

$$\frac{\partial \rho}{\partial t} + < u, \nabla > \rho = 0 \tag{4.1.2}$$

$$\text{div } u = 0. \tag{4.1.3}$$

in a bounded, open domain $D \subset \mathbb{R}^3$, with the boundary ∂D of class C^2. Define $S = D \times [0, T]$. The function $u : S \rightarrow \mathbb{R}^3$ represents the velocity, $p : S \rightarrow \mathbb{R}$ is the pressure, and $\rho : S \rightarrow \mathbb{R}$ is the density; equation (4.1.2) will be referred to as the density equation.
We assume that $f : S \rightarrow \mathbb{R}^3$ is a given function representing the external force, with $f \in L^2(0, T; L^2(D))$; note that the more general case where f has feedback is treated in the stochastic case, for simplicity f here has no feedback.
The homogeneous Dirichlet condition $u|_{\partial D} = 0$ is assumed and also that

$$u|_{t=0} = u_0 \qquad \rho|_{t=0} = \rho_0$$

are given functions. Further suppose that this initial data satisfies the following:

$$0 < m \leq \rho_0(x) \leq M \tag{4.1.4}$$

$$\text{div } u_0 = 0. \tag{4.1.5}$$

These equations are to be considered as an evolution equation in V' i.e considering

$$u : [0,T] \to V'$$

Note that if $p \in L^2$ then ∇p is zero in V' since $(\nabla p, u) = -(p, \text{div } u) = 0$ for $u \in V$, thus there is no need to consider the pressure at this stage. Once (4.1.1) is solved there is a well known classical method for recovering p, for example see [Te].
Now to introduce what is meant by a weak solution to the equations, this is sometimes referred to as a generalised solution, and this is taken directly from [AKM 90].

Definition 4.1.1 *Given $u_0 \in H$, $\rho_0 \in L^\infty(D)$ and $f \in L^2(0,T;L^2(D))$ a pair of functions (ρ, u) is a weak solution to the nonhomogeneous Navier-Stokes equations if $u : [0,T] \to H$, $\rho : S \to \mathbb{R}$ are such that:*

(i) $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$

(ii) $\rho \in L^\infty(S)$

(iii) for all $\Phi \in C^1(0,T;V)$ such that $\Phi(T) = 0$

$$\int_0^T [(\rho u, \Phi_t + < u, \nabla > \Phi) - \nu((u, \Phi)) + (\rho f, \Phi)]dt + (\rho_0 u_0, \Phi(0)) = 0 \tag{4.1.6}$$

(iv) for all $\varphi \in C^1(0,T;W_2^1(D))$ such that $\varphi(T) = 0$

$$\int_0^T (\rho, \varphi_t + < u, \nabla > \varphi)dt + (\rho_0, \varphi(0)) = 0 \tag{4.1.7}$$

Remark 4.1.2 Heuristically, the formulation of the weak solution can be derived from the equations (4.1.1), (4.1.2) and (4.1.3). For example suppose that (u, ρ) satisfies these three equations, then taking the innerproduct of, say (4.1.1) with Φ and performing the necessary integration by parts will yield (4.1.6)

In [AKM 90] the starting point for the existence of a weak solution is the formation of finite dimensional approximations to the equations; this is a common approach in such situations. In the proof presented in this work these approximations are also formed, but then the tools of nonstandard analysis are employed to considerably simplify the remaining part of the proof.
The idea is that for each $n \in \mathbb{N}$ an approximate equation is constructed. Solving these equations yields a sequence of approximate solutions. The standard approach is then to prove that a subsequence of this sequence is compact and then to show that the resulting limit is a weak solution. The obvious nonstandard analog is to solve the internal approximate equation of hyperfinite dimension, and take a standard part in an appropriate sense.

4.2 Finite Dimensional Approximations

Fix an arbitrary $n \in \mathbb{N}$, we seek a solution to the finite dimensional approximation of dimension n, the so-called Galerkin approximation.
That is to say, a function

$$y : [0,T] \to \mathbb{R}^n, y(t) = (y_1(t), \dots, y_n(t))$$

is sought, such that $u^n(x,t) = \sum_{k=1}^{n} y_k(t)e_k(x)$ solves:

$$(\rho^n[\frac{\partial u^n}{\partial t} + < u^n, \nabla > u^n], e_k) + \nu((u^n, e_k)) = (\rho^n f, e_k) \qquad k = 1, \dots, n \qquad (4.2.8)$$

$$u^n(0) = \sum_{k=1}^{n}(u_0, e_k)e_k \qquad (4.2.9)$$

where $\rho^n(x,t)$ is sought from the equation

$$\frac{\partial \rho^n}{\partial t} + < u^n, \nabla > \rho^n = 0 \qquad \rho^n(x,0) = \rho_0(x) \qquad (4.2.10)$$

For simplicity, assume here that $\rho_0 \in C^1(D)$. This can be weakened later to the more general $\rho_0 \in L^\infty(D)$, see Remark 4.6.3

In connection with the density equation (4.2.11), the classical result concerning the solvability of such an equation is presented next.

Lemma 4.2.1 *If $u \in C(0,T; C^1(D) \cap H)$ and $\rho_0 \in C^1(D)$ with*

$$0 < m \le \rho_0(x) \le M \qquad \forall x \in D$$

then the equation

$$\begin{cases} \frac{\partial \rho}{\partial t} + < u, \nabla > \rho & = 0 \\ \rho(0,\cdot) & = \rho_0 \end{cases} \qquad (4.2.11)$$

has a solution $\rho \in C^1([0,T] \times D)$.
And the dependence of ρ on u is continuous, in the sense that if

$$r : C(0,T; C^1(D) \cap H) \to C^1([0,T] \times D)$$

is such that $r(u)$ is the solution to the density equation then r is continuous with respect to the uniform topologies on both sides.
Also

$$0 < m \le \rho(t,x) \le M \qquad \forall (t,x) \in [0,T] \times D$$

For a proof of these facts, see for example [Ya 92] Lemma 2.1.
In order to apply this result, note that if

$$v : [0,T] \to \mathbb{R}^n$$

is continuous then $\hat{v}(t) := \sum_{k=1}^{n} v_k(t)e_k$ is such that

$$\hat{v} \in C(0,T; H_n).$$

Then since $H_n \subset C^1_B(D)$, clearly $H_n \subset C^1(D) \cap H$, and therefore we may (by a slight abuse of notation) define $r(v)$ as the unique solution to the density equation (4.2.11) with u^n replaced by $\hat{v}$ i.e.

$$r(v) = r(\hat{v})$$

It is clear that (4.2.8) is the natural projection of the equation (4.1.1) onto H_n. However (4.2.8) and (4.2.11) can also be derived by requiring that (4.1.6) and (4.1.7), with u_0 replaced by $\mathrm{Pr}_n u_0$ are satisfied for u^n and ρ^n on all test functions of the form $\Phi = h(t)e_k(x)$ $k = 1, \ldots, n$ where $h \in C^1[0,T]$ and $h(T) = 0$.

Proof Assume that (ρ^n, u^n) satisfies (4.1.6), with $\mathrm{Pr}_n u_0$ replacing u_0, on all test functions of the above form. That is

$$\int_0^T [(\rho^n u^n, \Phi_t + < u^n, \nabla > \Phi) - \nu((u^n, \Phi)) + (\rho^n f, \Phi)]\, dt + (\rho_0 \mathrm{Pr}_n u_0, \Phi(0)) = 0 \tag{4.2.12}$$

Firstly note that integration by parts, remembering that $u^n|_{\partial D} = 0$ and that $\mathrm{div}\, u^n = 0$, yields the following

$$(\rho^n u^n, < u^n, \nabla > \Phi) = -(< u^n, \nabla > (\rho^n u^n), \Phi). \tag{4.2.13}$$

By using the above fact and the density equation (4.2.11), it can easily be shown that

$$(\rho^n u^n, \Phi_t + < u^n, \nabla > \Phi) = (\frac{d}{dt}(\rho^n u^n, \Phi)) - (\rho^n \frac{du^n}{dt}, \Phi) - (< \rho^n u^n, \nabla > u^n, \Phi). \tag{4.2.14}$$

Now substituting this into (4.2.12), recalling that $\Phi(T) = 0$, $u^n(0) = \mathrm{Pr}_n u_0$ and that $\rho^n(0) = \rho_0$ yields

$$\int_0^T [(\rho u^n f, \Phi) - \nu((u^n, \Phi)) - (< \rho^n u^n, \nabla > u^n, \Phi) - (\rho^n \frac{du^n}{dt}, \Phi)]dt = 0. \tag{4.2.15}$$

But $\Phi(x,t) = H(t)e_k(x)$, so denoting

$$l_k(t) = (\rho^n f, e_k) - \nu^n((u^n, e_k)) - (< \rho^n u^n, \nabla > u^n, e_k) - (\rho^n \frac{du^n}{dt}, e_k)$$

which is continuous, therefore

$$\int_0^T H(t) l_k(t) dt = 0 \qquad \forall H \in C^1[0,T],\ H(T) = 0$$

This is sufficient to conclude that for each $1 \le k \le n$, $l_k \equiv 0$; exactly as required. ♣
In demonstrating the existence of a solutions to the finite dimensional approximations, use will be made of the following well known energy inequality.

Theorem 4.2.2 (The Energy Inequality) *If (ρ^n, u^n) solves the finite dimensional approximation i.e (4.2.8), (4.2.9) and (4.2.11) then the following estimates are valid*

$$\sup_{t \leq T} | u^n(t) |^2 + \int_0^T || u^n ||^2 \, dt \leq c_1 \tag{4.2.16}$$

$$0 < m \leq \rho^n(x,t) \leq M \tag{4.2.17}$$

where c_1 is a constant independent of the dimension n.

Proof We have already discussed (4.2.17), see Lemma 4.2.1. Now for (4.2.16):
It is required to show that if (ρ^n, u^n) solve (4.2.8),(4.2.9) and (4.2.11), then (4.2.16) holds.
For each $k \in \{1, \dots, n\}$ multiply the corresponding equation in the system (4.2.8) by $y_k(t)$ and sum over $k = 1, \dots, n$ to get

$$(\rho^n[u_t^n + < u^n, \nabla > u^n], u^n) + \nu ||u^n||^2 = (\rho^n f, u^n). \tag{4.2.18}$$

Towards establishing (4.2.16), first it is shown that

$$(\rho^n[u_t^n + < u^n, \nabla > u^n], u^n) = \frac{1}{2}\frac{d}{dt}\int_D \rho^n |u^n|^2_{\mathbb{R}^3} dx. \tag{4.2.19}$$

Well

$$\frac{1}{2}\frac{d}{dt}\int_D \rho^n |u^n|^2_{\mathbb{R}^3} dx = \frac{1}{2}\frac{d}{dt}(\rho^n u^n, u^n) = (\rho^n u^n, u_t^n) + \frac{1}{2}(\rho_t^n u^n, u^n)$$

now using (4.2.11) to conclude that

$$\frac{1}{2}\frac{d}{dt}\int_D \rho^n |u^n|^2 dx = (\rho^n u^n, u_t^n) - \frac{1}{2}(u^n < u^n, \nabla > \rho^n, u^n)$$

and therefore it remains to show that

$$-\frac{1}{2}(u^n < u^n, \nabla > \rho^n, u^n) = (\rho^n < u^n, \nabla > u^n, u^n). \tag{4.2.20}$$

It can be seen that (4.2.20) follows easily by an integration by parts, remembering that div $u^n = 0$ and $u^n|_{\partial D} = 0$. Thus (4.2.19) is established.
Substituting (4.2.19) into (4.2.18) gives

$$\frac{1}{2}\frac{d}{dt}\int_D \rho^n |u^n|^2_{\mathbb{R}^3} dx + \nu ||u^n||^2 = (\rho^n f, u^n). \tag{4.2.21}$$

Now using (4.2.17)

$$(\rho^n f, u^n) \leq |(\rho^n f, u^n)| \leq M |f|_{L^2(D)} |u^n|$$

Let $t \in [0, T]$, then integrating (4.2.21) from 0 to t w.r.t s yields

$$\frac{1}{2}\int_D \rho^n(t,x)|u^n(t,x)|^2_{\mathbb{R}^3} dx - \frac{1}{2}\int_D \rho^n(0,x)|u^n(0,x)|^2_{\mathbb{R}^3} dx + \nu \int_0^t ||u^n(s)||^2 ds$$

$$\leq M \int_0^t |f(s)|_{L^2(D)}|u^n(s)|ds$$

which implies(again using (4.2.17)) that

$$\frac{m}{2}|u^n(t)|^2 + \nu \int_0^t ||u^n(s)||^2 ds \leq M \int_0^t |f|_{L^2(D)}|u^n(s)|ds + \frac{1}{2}M|u^n(0)|^2. \qquad (4.2.22)$$

Clearly from (4.2.22), if it is shown that $|u^n(s)| \leq c$ for all s, with $c \in \mathbb{R}^+$ being independent of n, then since $f \in L^1(0,T;L^2(D))$ and $| u^n(0) | \leq | u_0 |$

$$\nu \int_0^T ||u^n(s)||^2 ds \leq k \qquad \text{for some } k \in \mathbb{R}^+$$

again with k independent of n.
So it is sufficient to show that $|u^n(s)| \leq c$ for all s.
Well from (4.2.22) for each $t \in [0,T]$

$$|u^n(t)|^2 \leq \frac{2M}{m} \int_0^t |f(s)|_{L^2(D)}|u^n(s)|ds + \frac{M}{m}|u^n(0)|^2$$

put $h(t) = |u^n(t)|$, using $h(t) \leq 1 + (h(t))^2$ then

$$(h(t))^2 \leq \frac{2M}{m} \int_0^t \left[|f(s)|_{L^2(D)} + |f(s)|_{L^2(D)}(h(s))^2\right] ds + \frac{M}{m}(h(0))^2. \qquad (4.2.23)$$

Now since

$$\int_0^t |f(s)|_{L^2(D)} ds \leq |f|_{L^1(0,T;L^2(D))} < \infty$$

and since

$$(h(0))^2 =| u(0) |^2=| \mathrm{Pr}_n u_0 |\leq| u_0 |$$

then we have

$$(h(t))^2 \leq c_2 \int_0^t |f(s)|_{L^2(D)}(h(s))^2 ds + c_3.$$

with c_2, c_3 positive constants independent of the dimension n.
Thus it then follows from Gronwall's Lemma that for all $t \in [0,T]$ we have

$$(h(t))^2 \leq c_1$$

where c_1 is a constant independent of of n, as was required. And thus

$$\sup_{t \leq T} | u(t) |^2 \leq c_1$$

as required. ♣

4.3 Solving The Finite Dimensional Approximations

As indicated earlier, the solving of these equations relies very much on the energy estimate (4.2.16). The proof presented below is different from the one found in [AKM 90]; and is much more simple. The fact that the Galerkin approximation has a solution is presented in the following theorem.

Theorem 4.3.1 *Given $u_0 \in H$, $f \in L^2(0,T;L^2(D))$ and $\rho_0 \in C^1(D)$, then for each $n \in \mathbb{N}$ there exists a solution to the finite dimensional approximation of dimension n. That is there is a solution to the equations (4.2.8), (4.2.9) and (4.2.11)*

Proof The Galerkin approximation is presented here in vector notation, where the unknown is a function $y : [0,T] \to \mathbb{R}^n$.
Our aim here is to show that the approximations can be solved by appealing to the well known standard existence theory of finite dimensional differential equations; to this end some notation is introduced.
Let $\Theta(t,y)$ be the symmetric $n \times n$ matrix given by $\Theta_{ij} = ((r(y))(t)e_j, e_i)$, let Γ denote the fixed $n \times n$ matrix $\Gamma_{ij} = \nu((e_j, e_i))$. Define $F(t,y) = (F_1(t,y), \ldots, F_n(t,y))$ by stating that $F_k(t,y) = ((r(y))(t)f(t), e_k)$.
Finally for any $\theta \in L^\infty(D), w \in \mathbb{R}^n, z \in \mathbb{R}^n$ define $\beta(\theta, w, z) \in \mathbb{R}^n$ such that for any $v \in \mathbb{R}^n$

$$< \beta(\theta, w, z), v >= b(\theta\hat{w}, \hat{z}, \hat{v})$$

where $\hat{w} = \sum_{k=1}^{n} w_k e_k$ etc, and b is the trilinear form introduced earlier.
Now it is possible to rewrite the system of equations (4.2.8) and (4.2.9) in the following form

$$\Theta(t,y)\frac{dy}{dt} = F(t,y) - \beta(r(y)(t), y(t), y(t)) - \Gamma y(t) =: \tilde{h}(t,y) \tag{4.3.24}$$

$$y(0) = ((u_0, e_1), \ldots, (u_0, e_k)). \tag{4.3.25}$$

Note that $\tilde{h}$ is continuous with respect to the first and second coordinates; also note that $\tilde{h}$ is adapted in the sense that if $y|_{[0,s]} = z|_{[0,s]}$ then $\tilde{h}(t,y)|_{[0,s]} = \tilde{h}(t,z)|_{[0,s]}$.
Further it is clear that

$$|F(t,y)| \leq M|f(t)|_{L^2(D)},$$

note that by our assumption on f it follows that $|f|_{L^2(D)} \in L^1[0,T]$.
Now it is shown that for any y the matrix Θ is nonsingular.
For any $v \in \mathbb{R}^n$ $v \neq 0$

$$v^T\Theta v = \sum_{l,j=1}^{n} \Theta_{lj} v_l v_j = (\rho^n(x,t)\hat{v}(x), \hat{v}(x)) \tag{4.3.26}$$

where $\hat{v}(x) = \sum_{k=1}^{n} v_k e_k(x)$ and v^T is the transpose of v.
So

$$0 < m \mid v \mid^2 \leq \mid v^T\Theta v \mid \leq M \mid v \mid^2 . \tag{4.3.27}$$

This shows that Θ is invertible and enables us to rewrite equation (4.3.24) in the following form.

$$\frac{dy}{dt} = (\Theta(t,y))^{-1}(\tilde{h}(t,y)) =: h(t,y) \tag{4.3.28}$$

In order to show that h inherits the properties of $\tilde{h}$ it is necessary to show that Θ^{-1} is bounded.
From (4.3.27) for any $v \in \mathbb{R}^n, v \neq 0$

$$m \mid v \mid \leq \mid \Theta v \mid$$

and this implies that $\mid \Theta^{-1} \mid \leq m^{-1}$ and therefore Θ^{-1} is bounded.
In order to solve (4.3.28) by employing the standard existence theorems for ordinary differential equations the β term must be truncated, so that it has linear growth.
Therefore let us consider a truncated version of (4.3.28).
Let $y : [0,T] \to \mathbb{R}^n$ then define $\overline{y} : [0,T] \to \mathbb{R}^n$ by

$$\overline{y}(t) = \begin{cases} y(t) & \text{if } |y(t)|_{\mathbb{R}^n} \leq c_1 \\ \frac{c_1 y(t)}{|y(t)|_{\mathbb{R}^n}} & \text{otherwise} \end{cases}$$

where c_1 is the constant from the energy inequality (4.2.16).
The truncated version of (4.3.28) is presented in the following form

$$\frac{dy}{dt} = (\Theta(y,\overline{y}))^{-1}\left[F(t,y) - \beta(r(\overline{y})(t), \overline{y}(t), y(t)) - \Gamma y(t)\right]. \tag{4.3.29}$$

Now the matrix Θ^{-1} is bounded, and $\beta(r(\overline{y})(t), \overline{y}(t), y(t))$ has linear growth, therefore by the theory of finite dimensional ODE's there exists a unique solution to (4.3.29), denote such a solution by y, so

$$y : [0,T] \to \mathbb{R}^n \qquad \text{i.e.} \qquad y(t) = (y_1(t), y_2(t) \ldots, y_k(t)).$$

Now define $u(x,t) = \sum_{k=1}^{n} y_k(t)e_k(x)$ and $\overline{u} = \sum_{k=1}^{n} \overline{y_k}e_k(x)$, for simplicity of notation, write ρ for $r(y)$ and $\overline{\rho}$ for $r(\overline{y})$.
So y satisfies (4.3.29) i.e.

$$\Theta(t,\overline{y})\frac{dy}{dt} = F(t,y) - \beta(\overline{\rho}(t), \overline{y}(t), y(t)) - \Gamma y(t) \tag{4.3.30}$$

Now the aim to show that

$$|y(t)| \leq c_1 \qquad \text{for all } t \in [0,T] \tag{4.3.31}$$

to this end, rewrite equation (4.3.30) in functional form, as follows

$$(\overline{\rho}[\frac{du}{dt} + < \overline{u}, \nabla > u], e_k) + \nu((u, e_k)) = (\rho f, e_k) \qquad k = 1, \ldots, n \tag{4.3.32}$$

multiplying each corresponding equation by $(y)_k$ and summing over $k = 1$ to n, gives

$$(\overline{\rho}\frac{\partial u}{\partial t} + \overline{\rho} < \overline{u}, \nabla > u, u) + \nu||u||^2 = (\rho f, u) \tag{4.3.33}$$

Now clearly, using the density equation for $\overline{\rho}$

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}(\overline{\rho}u,u) &= (\overline{\rho}\tfrac{\partial u}{\partial t},u)+\tfrac{1}{2}(\tfrac{\partial\overline{\rho}}{\partial t}u,u)\\ &= (\overline{\rho}\tfrac{\partial u}{\partial t},u)-\tfrac{1}{2}(u<\overline{u},\nabla>\overline{\rho},u)\end{aligned}$$

but an integration by parts gives

$$(\overline{\rho}<\overline{u},\nabla>u,u)=-\frac{1}{2}(u<\overline{u},\nabla>\overline{\rho},u) \tag{4.3.34}$$

therefore it is possible to rewrite (4.3.33) as

$$\frac{1}{2}\frac{d}{dt}(\overline{\rho}u,u)+\nu||u||^2=(\rho f,u). \tag{4.3.35}$$

From this point onwards, the proof of the energy inequality can be repeated, starting at (4.2.21), this will result in concluding that

$$|u(t)|\leq c_1 \qquad \text{for all } t\in[0,T].$$

but $|u(t)|=|y(t)|$, therefore (4.3.31) is demonstrated.
Thus $\overline{y}=y$, and so $\overline{\rho}=\rho$ and therefore from (4.3.30)

$$\Theta(t,y)\frac{dy}{dt}=F(t,y)-\beta(\rho(t),y(t),y(t))-\Gamma y(t). \tag{4.3.36}$$

That is to say y satisfies (4.3.24), and therefore (ρ,u) is a solution to the Galerkin approximation. ♣

Remark 4.3.2 It has been seen that for each finite $n\in\mathbb{N}$ there is a solution (ρ^n,u^n) to the finite dimensional approximation, and that the solution satisfies the energy inequality (4.2.17) and (4.2.16). Therefore it can be deduced, by transfer, that there exists a pair of functions (R,U) satisfying the internal approximation of dimension N, where N is an infinite nonstandard integer.
That is to say, there exists (R,U) such that:

$$U:{}^*[0,T]\longrightarrow H_N \qquad R:{}^*D\times{}^*[0,T]\longrightarrow{}^*\mathbb{R}$$

and (R,U) satisfy the following internal equations:

$$(R[\frac{dU}{d\tau}+<U,\nabla>U],E_k)+\nu((U,E_k))=(R^*f,E_k) \qquad k=1,\ldots,N \tag{4.3.37}$$

$$\frac{\partial R}{\partial\tau}+<U,\nabla>R=0 \tag{4.3.38}$$

with $U(0)=\sum\limits_{k=1}^{N}({}^*u_0,E_k)E_k$ and $R(0)={}^*\rho_0$
further the following estimates are valid:

$$0<m\leq R(\xi,\tau)\leq M \tag{4.3.39}$$

$$\sup_{\tau\leq T}|\,U(\tau)\,|^2+\int_0^T||\,U(\tau)\,||^2\,d\tau\leq c_1 \tag{4.3.40}$$

where c_1 is the finite constant of the energy inequality (4.2.16).

The goal is to show that the internal pair (R,U) is close in some sense to a standard weak solution (ρ,u). This is to be achieved by showing that R and U possess sufficient regularity to be close to standard functions. This will be achieved using the following Lemmas.

Lemma 4.3.3 *Let* Pr_N *be the projection from* $^*(L^2(D))$ *onto* H_N*; i.e* $\mathrm{Pr}_N(X) = \sum_{k=1}^{N}(X,E_k)E_k$. *Then* $\mathrm{Pr}_N(RU)$ *is weakly S-continuous, that is if* $\sigma,\tau \in^* [0,T]$ *and* $\sigma \approx \tau$ *then*

$$\mathrm{Pr}_N(R(\sigma)U(\sigma)) \approx_W \mathrm{Pr}_N(R(\tau)U(\tau)).$$

Proof Firstly, using (4.3.38) rewrite (4.3.37) in the following way

$$(\frac{d(RU)}{d\tau},E_k)+(<U,\nabla>RU,E_k)+\nu((U,E_k)) = (R^*f,E_k) \quad k=1,\ldots,N \quad (4.3.41)$$

noting that

$$((U,E_k)) = (U,AE_k) = \lambda_k U_k \quad \text{and} \quad \frac{d}{dt}(RU,E_k) = (\frac{d(RU)}{dt},E_k)$$

then an integration by parts yields, analogous to (4.2.13)

$$\frac{d}{dt}(RU,E_k) = (R^*f,E_k) - \nu\lambda_k U_k +^* b(U,E_k,RU) \quad (4.3.42)$$

Now we show that the right hand side of (4.3.42) is S-integrable on $^*[0,T]$, when k is finite.
Now since R is bounded then by the inequality (2.1.10) we have

$$\begin{aligned}\int_0^T {}^*b(U(\tau),E_k,U(\tau))^2 d\tau &= \int_0^T {}^*b(R(\tau)U(\tau),E_k,U(\tau))^2 d\tau \\ &\leq M^2\lambda^2\int_0^T \mid U(\tau)\mid^2 \| U(\tau)\|^2 d\tau \\ &\leq c(\sup_{\tau\leq T}\mid U(\tau)\mid^2)\int_0^T \| U(\tau)\|^2 d\tau < \infty\end{aligned}$$

since the energy inequality (4.3.40) ensures that the right hand side is finite.
Therefore by Theorem 3.3.12 – $^*b(R(\tau)U(\tau),U(\tau),E_k)$ is S-integrable on $^*[0,T]$ when k is finite.
Now for the remaining terms in (4.3.42);

$$\int_0^T \nu^2\lambda_k^2 U_k^2 d\tau \leq \nu^2\lambda_k^2\int_0^T (U(\tau),E_k)^2 \leq \nu^2\lambda_k^2\int_0^T \mid U(\tau)\mid^2 d\tau < \infty$$

again by using the estimate (4.3.40).
Finally

$$\int_0^T (R(\tau)^*f(\tau),E_k)^2 d\tau \leq \int_0^T \mid R(\tau)^*f(\tau)\mid^2_{L^2(^*D)} d\tau \leq M^2\int_0^T \mid^* f(\tau)\mid^2 d\tau < \infty$$

which is finite due to the assumptions on f.
So if k is finite then $(RU)_k(\tau)$ is by (4.3.42) the integral of an S-integrable integrand, and so by Proposition 3.3.13 is S-continuous; that is

$$\sigma, \tau \in^* [0,T] \text{ with } \sigma \approx \tau \Longrightarrow (R(\sigma)U(\sigma))_k \approx (R(\tau)U(\tau))_k. \tag{4.3.43}$$

Now $| \Pr_N(R(\tau)U(\tau)) | \leq | R(\tau)U(\tau) | \leq M \mid U(\tau) | < \infty$, so $\Pr_N(R(\tau)U(\tau)) \in H_N$ is weakly nearstandard by Proposition 3.2.8
And so for $\sigma \approx \tau$.

$$\Pr_N(R(\tau)U(\tau)) \approx_W \mathrm{st}_{W-H}(\Pr_N(R(\tau)U(\tau))) = \sum_{k=1}^{\infty} {}^\circ(\Pr_N(R(\tau)U(\tau)), E_k)e_k$$

$$\Pr_N(R(\sigma)U(\sigma)) \approx_W \mathrm{st}_{W-H}(\Pr_N(R(\sigma)U(\sigma))) = \sum_{k=1}^{\infty} {}^\circ(\Pr_N(R(\sigma)U(\sigma)), E_k)e_k$$

but from (4.3.43)

$$(\Pr_N(R(\tau)U(\tau)), E_k) = (R(\tau)U(\tau), E_k) \approx (R(\sigma)U(\sigma), E_k) = (\Pr_N(R(\sigma)U(\sigma)), E_k)$$

so $\Pr_N(R(\sigma)U(\sigma)) \approx_W \Pr_N(R(\sigma)U(\sigma))$
and hence, weak S-continuity of $\Pr_N(RU)$ is established. ♣

The following Lemma is crucial since it provides conditions under which U is well behaved. The idea behind the proof is distilled from Lemma 1.2 of [AKM 90], which is much more complicated than the one found below.

Lemma 4.3.4 *If $\sigma, \tau \in^* [0,T]$, with $\sigma \approx \tau$ and $\| U(\sigma) \|, \| U(\tau) \| < \infty$ then $U(\sigma) \approx U(\tau)$ strongly in H.*

Proof Let $\sigma \approx \tau$ with say $\sigma > \tau$, put $V(\theta) = R(\theta)U(\theta)$ for any $\theta \in^* [0,T]$.
Since $\| U(\sigma) \|, \| U(\tau) \| < \infty$ then by [CaCu 95] Prop 2.7.1 (e) $U(\sigma), U(\tau)$ are both strongly nearstandard in H, and it is possible to define

$$u(\sigma) = \mathrm{st}_H(U(\sigma)) = {}^\circ(U(\sigma)) \quad \text{and} \quad u(\tau) = \mathrm{st}_H(U(\tau)) = {}^\circ(U(\tau)).$$

Clearly for any $X \in^* H$

$$(V(\sigma) - V(\tau), X) = (R(\sigma)[U(\sigma) - U(\tau)], X) + ([R(\sigma) - R(\tau)]U(\tau), X) \tag{4.3.44}$$

in particular consider (4.3.44) with $X = U(\sigma) - U(\tau)$, first the left hand side; since $U(\sigma) - U(\tau) \in H_N$ and since

$$| \Pr_N(V(\sigma)) - \Pr_N(V(\tau)) | \leq M(| U(\sigma) | + | U(\tau) |) \leq Mc\,(\| U(\sigma) \| + \| U(\tau) \|)$$

then using Proposition 3.2.11

$$\begin{aligned} (V(\sigma) - V(\tau), U(\sigma) - U(\tau)) &= (\Pr_N(V(\sigma)) - \Pr_N(V(\tau)), U(\sigma) - U(\tau)) \\ &\approx (\Pr_N(V(\sigma)) - \Pr_N(V(\tau)), {}^*(u(\sigma)) - {}^*(u(\tau))) \\ &\approx 0 \end{aligned}$$

since by Lemma 4.3.3 $\mathrm{Pr}_N(V(\sigma)) \approx_W \mathrm{Pr}_N(V(\tau))$.
Now considering the first addend of the right hand side of (4.3.44); and by using the estimate (4.3.39) we have

$$(R(\sigma)[U(\sigma)-U(\tau)], U(\sigma)-U(\tau)) \geq m \mid U(\sigma)-U(\tau) \mid^2$$

Therefore if the remaining addend of (4.3.44) is infinitesimal then

$$\mid U(\sigma)-U(\tau) \mid \approx 0$$

thus $U(\sigma) \approx U(\tau)$ and the proof will be complete.
Therefore it is sufficient to show the following

$$([R(\sigma)-R(\tau)]U(\tau), U(\sigma)-U(\tau)) \approx 0. \tag{4.3.45}$$

To prove (4.3.45) we note that

$$([R(\sigma)-R(\tau)]U(\tau), U(\sigma)-U(\tau))$$

$$= \int_\tau^\sigma \frac{d}{d\delta}(R(\delta)U(\tau), U(\sigma)-U(\tau))d\delta$$

$$= \int_\tau^\sigma (R(\delta)U(\delta), \nabla < U(\tau), U(\sigma)-U(\tau) >)d\delta$$

$$= \int_\tau^\sigma {}^*b(R(\delta)U(\delta), U(\tau), U(\sigma)-U(\tau)) + {}^*b(R(\delta)U(\delta), U(\tau)-U(\sigma), U(\tau))ds.$$

Hence

$$\mid ([R(\sigma)-R(\tau)]U(\tau), U(\sigma)-U(\tau)) \mid$$

$$\leq \int_\tau^\sigma \mid {}^*b(R(\delta)U(\delta), U(\tau), U(\sigma)-U(\tau)) \mid + \mid {}^*b(R(\delta)U(\delta), U(\tau)-U(\sigma), U(\tau)) \mid d\delta.$$

Now by using (2.1.8) twice gives

$$\mid ([R(\sigma)-R(\tau)]U(\tau), U(\sigma)-U(\tau)) \mid$$

$$\leq 2c \parallel U(\tau) \parallel \parallel U(\sigma)-U(\tau) \parallel \int_\tau^\sigma \parallel U(\delta) \parallel d\delta$$

$$\leq 2c \parallel U(\tau) \parallel \parallel U(\sigma)-U(\tau) \parallel \left(\int_0^T \parallel U(\delta) \parallel^2 d\delta \right)^{\frac{1}{2}} (\sigma-\tau)^{\frac{1}{2}} \approx 0$$

since $\parallel U(\tau) \parallel$ and $\parallel U(\sigma) \parallel$ are finite by assumption, by the estimate (4.3.40) and since $\sigma \approx \tau$.
Thus (4.3.45) has been established and hence $U(\tau) \approx U(\sigma)$ as required. ♣

4.4 The Definition of u

The previous Lemma makes it possible to define a standard measurable function u that is close to U, more precisely, U is shown to be a lifting of a standard u.
The energy inequality (4.3.40) implies that for $\Lambda_L - a.a.\tau \in^* [0,T]$

$$\| U(\tau) \| < \infty \tag{4.4.46}$$

Let this full Loeb subset be denoted by $\mathcal{E}$, then for each $\tau \in \mathcal{E}$ by Proposition 3.2.9

$$\mathrm{st}_H(U(\tau)) \text{ exists in } H$$

Now consider $\mathrm{st}(\mathcal{E}) := \{{}^\circ\tau : \tau \in \mathcal{E}\}$, clearly $\mathcal{E} \subseteq \mathrm{st}^{-1}(\mathrm{st}(\mathcal{E}))$ and since the Loeb σ-algebra is complete, then $\mathrm{st}^{-1}(\mathrm{st}(\mathcal{E}))$ is Loeb measurable and therefore $\mathrm{st}(\mathcal{E})$ is Lebesgue measurable, also by Theorem 3.3.2

$$\lambda(\mathrm{st}(\mathcal{E})) = \Lambda_L(\mathrm{st}^{-1}(\mathrm{st}(\mathcal{E})) = T$$

And so it is possible to define a standard measurable function $u : [0,T] \longrightarrow H$ by

$$u(t) = \mathrm{st}_H(U(\tau)) = {}^\circ U(\tau) \tag{4.4.47}$$

for all $t \in \mathrm{st}(\mathcal{E})$, where $t \approx \tau \in \mathcal{E}$
The previous Lemma ensuring that u is well defined, it is clear that U lifts u, since $\Lambda_L(\mathcal{E}) = T$ and for all $\tau \in \mathcal{E}$

$$u({}^\circ\tau) \approx U(\tau). \tag{4.4.48}$$

Before defining a standard ρ from R, some properties of the new function u are demonstrated.
In particular

$$u \in L^\infty(0,T;H). \tag{4.4.49}$$

$$u \in L^2(0,T;V). \tag{4.4.50}$$

$$u \approx U \text{ in } L^2(S). \tag{4.4.51}$$

Note that (4.4.49) and (4.4.50) constitute requirement (i) of Definition 4.1.1.
To show (4.4.49), note that by Proposition 3.2.10 , and the energy inequality (4.3.40), for almost all $t \in [0,T]$

$$|u(t)| = |{}^\circ U(\tau)| \leq^\circ |U(\tau)| \leq c_1 < \infty$$

Now for property (4.4.50).
Well

$$\begin{aligned}
\int_0^T \|u(t)\|^2 dt &= \int_0^T \|u({}^\circ\tau)\|^2 d\Lambda_L && \text{Theorem 3.3.2} \\
&= \int_0^T \|{}^\circ U(\tau)\|^2 d\Lambda_L && \text{by definition of } u \\
&\leq \int_0^T {}^\circ\|U(\tau)\|^2 d\Lambda_L && \text{Proposition 3.2.10} \\
&\leq {}^\circ\left(\int_0^T \|U(\tau)\|^2 d\Lambda\right) && \text{Proposition 3.3.7} \\
&< \infty && \text{by estimate (4.3.40)}
\end{aligned}$$

Finally to show (4.4.51) it is sufficient to show that $|U - {}^*u|_{^*L^2(S)} \approx 0$
Well (4.3.40) and the definition of u gives that $|U(\cdot) - {}^*u(\cdot)|^2$ is bounded and hence S-integrable, and so

$$\begin{aligned} |U - {}^*u|_{^*L^2(S)} &= \int_0^T |U(\tau) - {}^*u(\tau)|^2 d\Lambda \\ &\approx \int_0^T {}^\circ|U(\tau) - {}^*u(\tau)|^2 d\Lambda_L \end{aligned}$$

Now by the definition of u and by Anderson's Theorem for $\Lambda_L.a.a.\tau$

$${}^*u(\tau) \approx u({}^\circ\tau) \approx U(\tau) \text{ in } H.$$

Therefore for $\Lambda_L.a.a.\tau$

$${}^\circ|U(\tau) -{}^* u(\tau)|^2 = 0$$

and thus

$$|U -{}^* u|_{^*L^2(S)} \approx 0$$

and so $u \approx U$ in $L^2(S)$ as required.

4.5 Defining ρ from R

In this section the aim is to define a standard function ρ from the function R. The goal is to produce a ρ such that for certain z if Z is a suitable lifting of z then for all τ

$$\int_{^*D} R(\tau)Zd\xi \approx \int_D \rho({}^\circ\tau)zdx \tag{4.5.52}$$

this will be made precise in the Lemma following the construction of ρ from R.
Well we have a function $R: {}^*[0,T] \times {}^*D \to {}^*\mathbb{R}$ satisfying

$$\frac{\partial R}{\partial \tau} + < U, \nabla > R = 0 \tag{4.5.53}$$

with

$$R(0) = {}^*\rho_0$$

and further

$$0 < m \le R(\tau, \xi) \le M < \infty.$$

Reconsider this as

$$R :{}^* [0,T] \to L^\infty({}^*D,{}^*\mathbb{R})$$

now define

$${}^\circ R :{}^* [0,T] \to L^\infty({}^*D, \mathbb{R}) \quad \text{by} \quad ({}^\circ R(\tau))(\xi) ={}^\circ ((R(\tau))(\xi))$$

Now if $\tau \in^* [0,T]$ then $R(\tau)$ is ${}^*\mathcal{B}$ measurable , thus ${}^\circ R(\tau)$ is $L({}^*\mathcal{B})$ measurable, here $\mathcal{B} = \mathcal{B}(D)$ i.e the Borel σ-algebra on D.
Now the standard part map is such that

$$\text{st} : \text{ns}({}^*D) \to D \text{ and } \text{st}(\xi) = {}^\circ\xi$$

note that by Theorem 3.3.2 st is $(\sigma(\,^*\mathcal{B}),\ \mathcal{B})$ measurable.
Let us define a σ-algebra on $^*D$ by

$$\mathcal{G} = \sigma(\mathrm{st}^{-1}(E) : E \in \mathcal{B})$$

and therefore

$$\mathcal{G} \subseteq \sigma(\,^*\mathcal{B}) \subseteq L(\,^*\mathcal{B})$$

Clearly for all $\tau \in^* [0,T]$

$$^\circ R(\tau) \in L^1(\,^*D, L(\,^*\mathcal{B}), \xi_L).$$

then define

$$\tilde{\rho} :^* [0,T] \to L^\infty(\,^*D\,, \mathbb{R}) \text{ by } \tilde{\rho}(\tau) = \mathbb{E}(\,^\circ R(\tau) \mid \mathcal{G})$$

and thus

$$0 < m \leq \tilde{\rho}(\tau)(\xi) \leq M$$

Now since the map st is measurable and onto, and since for each τ (by definition) $\tilde{\rho}(\tau)$ is $\mathcal{G}$ measurable then there exists for each τ a unique $\mathcal{B}$ measurable function

$$\hat{\rho}(\tau) : D \to \mathbb{R}$$

such that

$$\tilde{\rho}(\tau) = \hat{\rho}(\tau) \circ \mathrm{st}$$

i.e for all τ and ξ

$$\tilde{\rho}(\tau)(\xi) = \hat{\rho}(\,^\circ\xi)$$

and thus

$$0 < m \leq (\hat{\rho}(\tau))(x) \leq M$$

therefore it is possible to consider $\hat{\rho}$ as

$$\hat{\rho} :^* [0,T] \to L^\infty(D, \mathbb{R})$$

with $\hat{\rho}(\tau)$ being $\mathcal{B}$ measurable for all $\tau \in^* [0,T]$.
The following Lemma makes precise the formula (4.5.52), this Lemma has many possible forms, the one presented here being quite natural and suitable.

Lemma 4.5.1 *If $z : D \to \mathbb{R}$ is $\mathcal{B}$ measurable and has an S-integrable lifting*

$$Z : {}^*D \to {}^*\mathbb{R}$$

i.e for a.a.ξ

$$Z(\xi) \approx z(\,^\circ\xi)$$

Then for any τ

$$\int_{^*D} R(\tau) Z d\xi \approx \int_D \hat{\rho}(\tau) z dx$$

Proof Fix $\tau \in^* [0,T]$, now Z is S-integrable, and R is bounded thus

$$(R(\tau))(\cdot)Z(\cdot) :^* D \to^* \mathbb{R}$$

is S-integrable.
Thus

$$\int_{{}^*D} (R(\tau))(\xi)Z(\xi)d\xi$$

$$\approx \int_{{}^*D} {}^\circ((R(\tau))(\xi))^\circ(Z(\xi))d\xi_L$$

$$= \int_{\mathrm{ns}({}^*D)} {}^\circ((R(\tau))(\xi)z({}^\circ\xi)d\xi_L \qquad \text{Z lifts z and } {}^*D\backslash \mathrm{ns}({}^*D) \text{ is null(Theorem 3.3.2)}$$

$$= \int_{\mathrm{ns}({}^*D)} E\left({}^\circ((R(\tau))(\xi)z({}^\circ\xi)|\mathcal{G}\right) d\xi_L \qquad \mathrm{ns}({}^*D) \in \mathcal{G}$$

$$= \int_{\mathrm{ns}({}^*D)} z({}^\circ\xi)E\left({}^\circ((R(\tau))(\xi)|\mathcal{G}\right) d\xi_L \qquad \text{since } z({}^\circ\xi) \text{ is } \mathcal{G} measurable$$

$$= \int_{{}^*D} z({}^\circ\xi)(\tilde{\rho}(\tau))(\xi)d\xi_L \qquad \text{by def of } \tilde{\rho} \text{ and by Theorem 3.3.2}$$

$$= \int_{{}^*D} z({}^\circ\xi)(\hat{\rho}(\tau))({}^\circ\xi)d\xi_L \qquad \text{by def of } \hat{\rho}$$

$$= \int_{D} z(x)(\hat{\rho}(\tau))(x)dx \qquad \text{by Theorem 3.3.2.}$$

as required. ♣
Next an immediate consequence of the above Lemma.

Corollary 4.5.2 *For all* $\tau \in^* [0,T]$

$$R(\tau) \approx \hat{\rho}(\tau) \text{ weakly in } L^p(D,\mathbb{R})$$

for any $p \geq 1$.

Proof Let q be such that $\frac{1}{q} + \frac{1}{p} = 1$.
If $y \in L^q(D,\mathbb{R})$ then Anderson's Theorem tells us that ${}^*y$ lifts y. Further, clearly ${}^*y$ is S-integrable and so applying the above Lemma shows that the required integrals are close. ♣

Now an important theorem, this will be used repeatedly in the main existence theorem.

Theorem 4.5.3 *Let* $u, v \in H$ *and* $U, V \in^* H$ *be such that* $u \approx U$ *and* $v \approx V$. *Then for all* $\tau \in {}^*[0,T]$ *we have*

$$(R(\tau)U, V) \approx (\hat{\rho}(\tau)u, v) \tag{4.5.54}$$

Proof Firstly, rewrite (4.5.54) in the following way

$$\int_{^*D} R(\tau) < U,V > d\xi \approx \int_D \hat{\rho}(\tau) < u,v > dx. \tag{4.5.55}$$

Therefore by 4.5.1, it is sufficient to prove that $< U,V >$ is an S-integrable lifting of $< u,v >$. Note we clearly have that $< {}^*u, {}^*v >$ is an S-integrable lifting of $< u,v >$; and so for $a.a.\xi \in {}^*D$ we have

$$< {}^*u(\xi), {}^*v(\xi) > \approx < u({}^\circ\xi), v({}^\circ\xi) > . \tag{4.5.56}$$

Now

$$\begin{aligned} |< U,V > - < {}^*u, {}^*v >| &\leq |< U - {}^*u, V >| + |< {}^*u, V - {}^*v >| \\ &\leq | U - {}^*u |_{\mathbb{R}^3} | V |_{\mathbb{R}^3} + | {}^*u |_{\mathbb{R}^3} | V - {}^*v |_{\mathbb{R}^3} . \end{aligned}$$

Hence by Hölderwe have

$$\int_{^*D} |< U,V > - < {}^*u, {}^*v >|_{\mathbb{R}^3} \, d\xi \leq | U - {}^*u | \, | V | + | {}^*u | \, | V - {}^*v | \approx 0$$

since $| V |$, $|^* u | = | u | < \infty$ and $u \approx U$, $v \approx V$.
Thus $< U,V > - < {}^*u, {}^*v >$ is S-integrable. Also by Proposition 3.3.7 we have that for $a.a.\xi$

$$< U(\xi), V(\xi) > \approx < {}^*u(\xi), {}^*v(\xi) >$$

and hence this along with (4.5.56) yields

$$< U(\xi), V(\xi) > \approx < u({}^\circ\xi), v({}^\circ\xi) > .$$

Thus $< U,V >$ lifts $< u,v >$. Finally, since both $< U,V > - < {}^*u, {}^*v >$ and $< {}^*u, {}^*v >$ are S-integrable then so is there sum. ♣

Next it is shown that $\hat{\rho}$ is well behaved as a function of τ, to this end it is necessary to prove the following Lemma.

Lemma 4.5.4 *For any $z \in W_2^1(D, \mathbb{R})$ the function*

$$^*[0,T] \ni \tau \longmapsto \int_{^*D} R(\tau)^* z d\xi \in {}^* \mathbb{R} \tag{4.5.57}$$

is S-continuous on $^*[0,T]$.

Proof Let $z \in W_2^1(D, \mathbb{R})$ then using (4.5.53) and an application of integration by parts, gives

$$\frac{d}{d\tau}(R,{}^* z) = (\frac{\partial R}{d\tau},{}^* z) = (- < U, \nabla > R,{}^* z) = (R, < U, \nabla >^* z)$$

And thus for any $\tau \in [0,T]$

$$(R(\tau), {}^*z) = (R(0), {}^*z) + \int_0^\tau (R(\sigma), < U(\sigma), \nabla > {}^*z) d\sigma \tag{4.5.58}$$

Therefore it is sufficient to show that

$$^*[0,T] \ni \sigma \longmapsto (R(\sigma), < U(\sigma), \nabla > \ ^*z) \tag{4.5.59}$$

is S-integrable on $^*[0,T]$, S-continuity of $(R(\tau),\ ^*z)$ then follows by Proposition 3.3.13.
Denote by η the function defined in (4.5.59), thus for any $\sigma \in^* [0,T]$

$$| \eta(\sigma) | \leq M \mid U(\sigma) \mid \mid \nabla z \mid \leq M \sup_{\tau \leq T} \mid U(\tau) \mid \mid \nabla z \mid < \infty$$

by the estimate (4.3.40) and the fact that $z \in W_2^1(D, \mathbb{R})$.
Thus, since η is bounded it is clearly S-integrable on $^*[0,T]$ and hence the proof is complete. ♣.
Now an immediate consequence is the following lemma

Lemma 4.5.5 *If $\tau \approx \sigma$ then*

$$\hat{\rho}(\tau) = \hat{\rho}(\sigma) \ in \ L^\infty(D, \mathbb{R}).$$

Proof Well by Corollary 4.5.2 and by Lemma 4.5.4 it easily follows that if $z \in W_2^1(D, \mathbb{R})$ then

$$\int_D \hat{\rho}(\tau) z dx \approx \int_{^*D} R(\tau)^* z d\xi \approx \int_{^*D} R(\sigma)^* z \ d\xi \approx \int_D \hat{\rho}(\sigma) z dx$$

and hence, since $W_2^1(D, \mathbb{R})$ is dense in $L^1(D, \mathbb{R})$ then for *a.a.ω*

$$\hat{\rho}(\tau) = \hat{\rho}(\sigma) \text{ in } L^\infty(D)$$

as required. ♣
This control in time makes it possible to define a standard

$$\rho : [0,T] \to L^\infty(D), \mathbb{R})$$

via the following definition.
Definition of ρ

$$\rho(t) := \hat{\rho}(\tau) =^\circ R(\tau) \text{ for any } \tau \approx t$$

The previous Lemma ensuring that this is well defined. (The standard part being in the sense of the weak topology on $L^p(D, \mathbb{R})$ with $p \geq 1$)
It is clear that

$$0 < m \leq \rho(t,x) \leq M \qquad \text{for all } t \in [0,T], \ x \in D$$

note also that ρ is weakly continuous.
It follows from this definition and the corollary to Lemma 3 that

$$R(\tau) \approx \rho(\ ^\circ\tau)$$

weakly in $L^p(D, \mathbb{R})$.

4.6 Existence Theorem

Now a position has been reached, where it is possible to state and prove the main theorem of this Chapter.

Theorem 4.6.1 *For any $u_0 \in H$, $\rho_0 \in C^1(D)$ and $f \in L^2(0,T;L^2(D))$ there exists a weak solution to the nonhomogeneous Navier-Stokes equations.*

Proof It will be shown that the pair (ρ, u) defined from the internal pair (R, U) is such a weak solution.
Firstly, note that it has been already shown that (ρ, u) satisfies conditions (i) and (ii) of Definition 1.
Next it is shown that condition (iii) of the definition is satisfied. It is sufficient to consider to consider test functions of the following form, where k is any fixed natural number.

$$\Phi(x,t) = z(t)e_k(x)$$

with $z \in C^1[0,T]$ and $z(T) = 0$.
Thus it is required to show that

$$\int_0^T [(\rho u, < u, \nabla > \Phi + \Phi_t) - \nu((u, \Phi)) + (\rho f, \Phi)]dt + (\rho_0 u_0, \Phi(0)) = 0 \qquad (4.6.60)$$

It is clear that for such a Φ, the internal pair (R, U) satisfy the following

$$\int_0^T [(RU, < U, \nabla > \ {}^*\Phi + {}^*\Phi_\tau) - \nu((U, {}^*\Phi)) + (R^* f, {}^*\Phi)]d\tau + (R(0)U(0), {}^*\Phi(0)) = 0 \qquad (4.6.61)$$

This can be derived from the internal Galerkin approximation of dimension N by multiplying the kth equation by ${}^*z$, summing, and performing the integration by parts.
The aim is to apply the standard part map to (4.6.61) to produce the integral equality (4.6.60). Therefore it is sufficient to show each of the following:

$$(I) \qquad \int_0^T \nu((U(\tau), {}^*\Phi(\tau)))d\tau \approx \int_0^T \nu((u(t), \Phi(t)))dt.$$

$$(II) \qquad \int_0^T (R(\sigma)U(\sigma), {}^*\Phi_\tau(\sigma))d\sigma \approx \int_0^T (\rho(s)u(s), \Phi_t(s))ds.$$

$$(III) \qquad \int_0^T (R(\tau)U(\tau), < U(\tau), \nabla > {}^*\Phi(\tau))d\tau \approx \int_0^T (\rho(t)u(t), < u(t), \nabla > \Phi(t))dt.$$

$$(IV) \qquad \int_0^T (R(\tau)^* f(\tau), {}^*\Phi(\tau))d\tau \approx \int_0^T (\rho(t)f(t), \Phi(t))dt.$$

$$(V) \qquad (R(0)U(0), {}^*\Phi(0)) \approx (\rho_0 u_0, \Phi(0)).$$

Once the above is shown, since (4.6.61) holds, then it is clear that (4.6.60) holds.
Starting with (I) .

Well

$$\int_0^T \nu((U(\tau),\ {}^*\Phi(\tau)))d\tau = \int_0^T \nu\ {}^*z(\tau)((U(\tau),\ {}^*e_k))d\tau = \int_0^T \nu\ {}^*z(\tau)U_k(\tau)\lambda_k d\tau$$

now since $z \in C^1[0,T]$ and $|U(\tau)| \leq c_1$ then ${}^*z(\tau)U_k(\tau)$ is bounded and therefore S-integrable.
Noting that ${}^\circ U_k(\tau) = u_k({}^\circ\tau)$ for $\Lambda_L - a.a.\tau$, and since z is continuous we have for all τ

$$^\circ({}^*z(\tau)) = z({}^\circ\tau). \tag{4.6.62}$$

Then by Theorem 3.3.2 we have

$$\begin{aligned}\int_0^T \nu\ {}^*z(\tau)U_k(\tau)\lambda_k d\tau &\approx \int_0^T \nu\ {}^\circ({}^*z(\tau))\ {}^\circ U_k(\tau)\lambda_k d_L\tau = \int_0^T \nu\lambda_k z({}^\circ\tau)u_k({}^\circ\tau)d_L\tau \\ &= \int_0^T \nu\lambda_k z(t)u_k(t)dt = \int_0^T \nu((u(t),\Phi(t)))dt.\end{aligned}$$

Thus (I) is established.
Now For (II). It is required to show that

$$\int_0^T (R(\sigma)U(\sigma),\ {}^*\Phi_\tau(\sigma))d\sigma \approx \int_0^T (\rho(s)u(s),\Phi_t(s))ds$$

Well by the definition of u, for $a.a.\sigma$ we have

$$U(\sigma) \approx u({}^\circ\sigma).$$

Since Φ_t is continuous, then for all σ

$${}^*\Phi_\tau(\sigma) \approx \Phi_t({}^\circ\sigma)$$

Thus by Theorem 4.5.3 , we have that for $a.a.\sigma$

$$(R(\sigma)U(\sigma),\ {}^*\Phi_\tau(\sigma))) \approx (\rho({}^\circ\sigma)u({}^\circ\sigma),\Phi_t({}^\circ\sigma)). \tag{4.6.63}$$

Now $\Phi_t = z_t e_k$, but $z \in C^1[0,T]$ and hence Φ_t is bounded in H. This fact along with the boundedness of R and the estimate (4.3.40), ensures that the function

$${}^*[0,T] \ni \sigma \longmapsto (R(\sigma)U(\sigma),\ {}^*\Phi_\tau(\sigma))$$

is bounded and hence S-integrable.
Thus by Proposition 3.3.10 we are done.
Now for (III)
Firstly, it is claimed that for $a.a.\tau$

$$(R(\tau)U(\tau), < U(\tau),\nabla > E_k) \approx (\rho({}^\circ\tau)u({}^\circ\tau), < u({}^\circ\tau),\nabla > e_k) \tag{4.6.64}$$

i.e. that

$${}^*b(U(\tau),E_k,R(\tau)U(\tau)) \approx b(u({}^\circ\tau),e_k,\rho({}^\circ\tau)u({}^\circ\tau))$$

Well by the definition of u, for $a.a.\tau$

$$U(\tau) \approx u(\,^\circ\tau) \quad \text{in } H \tag{4.6.65}$$

so Theorem 4.5.3 and the definition of ρ imply that for $a.a.\tau$

$$^*b(^*(u(\,^\circ\tau)), E_k, R(\tau)U(\tau)) \approx b(u(^\circ\tau), e_k, \rho(\,^\circ\tau)u(\,^\circ\tau)) \tag{4.6.66}$$

Hence in order to prove (4.6.64) it is sufficient to show that for $a.a.\tau$

$$^*b(U(\tau), E_k, R(\tau)U(\tau)) \approx b((^*u(\,^\circ\tau)), E_k, R(\tau)U(\tau))$$

i.e. that

$$^*b(U(\tau) - \,^*u(\tau), E_k, R(\tau)U(\tau)) \approx 0.$$

But by (2.1.10) and (4.3.40) for $a.a.\tau$

$$\begin{aligned} |^*\, b(U(\tau) - \,^*u(\tau), E_k, R(\tau)U(\tau)) \mid &= \mid \,^*b(R(\tau)[U(\tau) - \,^*u(\tau)], E_k, U(\tau)) \mid \\ &\le c \mid U(\tau) - \,^*u(\tau) \mid \mid AE_k \mid \| U(\tau) \| \approx 0 \end{aligned}$$

and so (4.6.64) is established.
Thus, using (4.6.62) we have for $a.a.\tau$

$$(R(\tau)U(\tau), < U(\tau), \nabla > \,^*\Phi(\tau)) \approx (\rho(\,^\circ\tau)u(\,^\circ\tau), < u(\,^\circ\tau), \nabla > \Phi(\,^\circ\tau)) \tag{4.6.67}$$

Now by (2.1.10) and since z is bounded

$$\begin{aligned} \int_0^T \,^*b(R(\tau)U(\tau), \,^*\Phi(\tau), U(\tau))^2 d\tau &\le c\int_0^T \mid U(\tau) \mid^2 \mid A \,^*\Phi(\tau) \mid^2 \| U(\tau) \|^2 \, d\tau \\ &\le c_1 (\sup_{\tau \le T} \mid U(\tau) \mid^2) \int_0^T \| U(\tau) \|^2 \, d\tau < \infty \end{aligned}$$

Thus Proposition 3.3.10 implies that (III) is true.
Next for the proof of (IV).
Well, by Anderson's Theorem for $a.a.\tau$

$$^*f(\tau) \approx f(\,^\circ\tau)$$

Also the continuity of Φ implies that for all τ

$$^*\Phi(\tau) \approx \Phi(\,^\circ\tau)$$

Hence Theorem 4.5.3 implies that for $a.a.\tau$

$$(R(\tau)^*f(\tau), ^*\Phi(\tau)) \approx (\rho(\,^\circ\tau)f(\,^\circ\tau), \Phi(\,^\circ\tau)). \tag{4.6.68}$$

Now since Φ is bounded, by the assumptions on f and by 4.3.40, we have

$$\begin{aligned} \int_0^T (R(\tau)^*f(\tau, U(\tau), \,^*\Phi(\tau))^2 d\tau &\le c\int_0^T \mid R(\tau)^*f(\tau) \mid^2 d\tau \\ &\le c_1 M^2 < \infty. \end{aligned}$$

since $a \in L^2[0,T]$.
Thus by Theorem 3.3.12, the function

$$\tau \longmapsto (R(\tau)^* f(\tau, U(\tau), \ ^*\Phi(\tau))$$

is S-integrable on $^*[0,T]$. Thus, Proposition 3.3.10 implies that (IV) is true.

Finally for (V) it is sufficient to show that

$$(R(0)U(0), \ ^*\Phi(0)) \approx (\rho_0 u_0, \Phi(0)).$$

Recall that $R(0) = \ ^*\rho_0$ and $U(0) = \mathrm{Pr}_N(\ ^*u_0)$, therefore by Proposition 3.2.7

$$| \ ^*\rho_0(Pr_N(\ ^*u_0) - \ ^*u_0)| \leq M|Pr_N(\ ^*u_0) - \ ^*u_0| \approx 0$$

and therefore (V) is demonstrated .
So each of (I), (II), (III), (IV) and (V) have been established , therefore it has been shown that (ρ, u) satisfies (4.6.60), now since the test function considered in (4.6.60) are dense in those found in the definition of a weak solution then (ρ, u) satisfies condition (iii) of Definition 4.1.1.

Now to show that the pair (ρ, u) satisfies condition (iv) of Definition 4.1.1. for all test functions φ such that i.e. that

$$\int_0^T (\rho, \varphi_t + < u, \nabla > \varphi)dt + (\rho_0, \varphi(0)) = 0. \tag{4.6.69}$$

Well

$$\frac{\partial R}{\partial t} + < U, \nabla > R = 0 \tag{4.6.70}$$

$$R(0) = \ ^*\rho_0 \tag{4.6.71}$$

now for φ as above using (4.6.70) gives

$$\frac{d}{dt}(R, \ ^*\varphi) = (\frac{\partial R}{\partial t}, \ ^*\varphi) + (R, \ ^*\varphi_t) = -(< U, \nabla > R, ^*\varphi) + (R, \ ^*\varphi_t) \tag{4.6.72}$$

then an integration by parts gives

$$-(< U, \nabla > R, ^*\varphi) = (R, < U, \nabla > \ ^*\varphi)$$

substituting this into (4.6.72) and integrating w.r.t s from 0 to T yields

$$\int_0^T (R(s), < U(s), \nabla > \ ^*\varphi(s) + \ ^*\varphi_t(s))ds + (R(0), \ ^*\varphi(0)) = 0. \tag{4.6.73}$$

The aim is now to take standard parts of (4.6.73) to produce the required (5.6.65).
Well clearly

$$(\ ^*\rho_0, \ ^*\varphi(0)) = (\rho_0, \varphi(0)).$$

It is claimed that we have

$$\int_0^T (R(\sigma),\ {}^*\varphi_\tau(\sigma))d\sigma \approx \int_0^T (\rho(s), \varphi_t(s))ds \tag{4.6.74}$$

and

$$\int_0^T (R(\sigma), < U(\sigma), \nabla >^* \varphi(\sigma))d\sigma \approx \int_0^T (\rho(s), < u(s), \nabla > \varphi(s))ds \tag{4.6.75}$$

Firstly we show (5.6.70) is true.
Well $\varphi_t \in C[0,T; W_2^1(D)]$, therefore for all $\sigma \in^* [0,T]$

$${}^*\varphi_\tau(\sigma) \approx \varphi_t({}^\circ\sigma) \qquad \text{in } L^2(D). \tag{4.6.76}$$

Now by Lemma 4.5.1, since ${}^*\varphi_\tau$ is S-integrable, then for all σ

$$(R(\sigma),\ {}^*\varphi_\tau(\sigma)) \approx (\rho({}^\circ\sigma), \varphi_t({}^\circ\sigma)) \tag{4.6.77}$$

Now the function

$${}^*[0,T] \ni \sigma \longmapsto (R(\sigma),\ {}^*\varphi_\tau(\sigma))$$

is clearly S-integrable, and hence by Proposition 3.3.10 we are done.
Now for (4.6.75). It is claimed that for $a.a.\sigma$ we have

$$(R(\sigma), < U(\sigma), \nabla >^* \varphi(\sigma)) \approx (\rho({}^\circ\sigma), < u({}^\circ\sigma), \nabla > \varphi({}^\circ\sigma)). \tag{4.6.78}$$

Well rewrite this as

$$(R(\sigma)U(\sigma), \nabla\, {}^*\varphi(\sigma)) \approx (\rho({}^\circ\sigma)u({}^\circ\sigma), \nabla\varphi({}^\circ\sigma)). \tag{4.6.79}$$

Well by the definition of u, for $a.a.\sigma$ we have

$$U(\tau) \approx u(\,{}^\circ\tau)$$

and since Φ is continuous into $W_2^1(D)$ we have for any τ

$$\nabla\, {}^*\Phi(\tau) \approx \nabla\Phi(\,{}^\circ\tau).$$

Hence Theorem 4.5.3 proves (4.6.79), for $a.a.\tau$.
Note that the function

$${}^*[0,T] \ni \sigma \longmapsto (R(\sigma), < U(\sigma), \nabla >^* \varphi(\sigma))$$

is S-integrable, and thus by using Proposition 3.3.10 we arrive at (4.6.75).

Since the last condition has been verified then the pair (ρ, u) is a weak solution to the nonhomogeneous Navier-Stokes equations, as defined in Definition 4.1.1. ♣

Remark 4.6.2 It is possible to reformulate this chapter so that the solution u is defined for all times i.e

$$u : [0, \infty) \to H.$$

This is done in [CaCu 95] for the homogeneous case.

Remark 4.6.3 The more general case of $\rho_0 \in L^\infty(D)$ can be dealt with by approximating ρ_0 by a sequence ρ_0^n of elements in $C^1(D)$.

Chapter 5

The Stochastic Nonhomogeneous Navier-Stokes Equations

5.1 Introduction

The general stochastic nonhomogeneous Navier-Stokes equations will be studied in the following form:

$$\rho du = [\nu\Delta u - < \rho u, \nabla > u - \nabla p + \rho f(t,u)]\, dt + [\rho g(t,u)]\, dw_t \tag{5.1.1}$$

$$\frac{\partial \rho}{\partial t} + < u, \nabla > \rho = 0 \tag{5.1.2}$$

$$\text{div } u = 0 \tag{5.1.3}$$

These equations are obtained from from the deterministic equation by adding a random force; essentially fdt is replaced by $f(t,u)dt + g(t,u)dw_t$, here w is a Wiener process in H with covariance Q, as in Section 3.4.
Note that feedback occurs in both f and g. Note that this generalises the deterministic case ($g = 0$) studied in Chapter 4 and that studied in [AKM 90].

Definition of solution

A definition of a weak solution to (5.1.1),(5.1.2) and (5.1.3) is presented; as before $D \subseteq \mathbb{R}^3$ is open, bounded and of class C^2. Note again that the pressure term p is equal to 0 in V'.

Definition 5.1.1 *Given $u_0 \in H$, $\rho_0 \in C^1(D)$ with $0 < m \leq \rho_0 \leq M$, $f : [0,T] \times H \rightarrow H$ and $g : [0,T] \times H \rightarrow L(H,H)$; then a pair of processes (ρ, u) is a weak solution to the stochastic nonhomogeneous Navier-Stokes equations if each of the following four conditions is satisfied.*

1. $u : [0,T] \times \Omega \rightarrow H$ *and* $\rho : [0,T] \times D \times \Omega \rightarrow \mathbb{R}$.

2. *For a.a.ω* $u(\cdot,\omega) \in L^\infty(0,T;H) \cap L^2(0,T;V)$ *and* $\rho(\cdot,\cdot,\omega) \in L^\infty(Q)$.

3. *For a.a.ω,* $\forall \Phi \in C^1[0,T;V]$ *such that* $\Phi(T) = 0$

$$
\begin{aligned}
-(\rho_0 u_0, e_k) = & \int_0^T (\rho(s)u(s), < u(s), \nabla > \Phi(s))ds \\
& -\nu \int_0^T ((u(s), \Phi(s)))ds \\
& + \int_0^T (\rho(s)f(s, u(s)), \Phi(s))ds \\
& + \int_0^T (\Phi(s), \rho(s)g(s, u(s))dw(s).
\end{aligned}
$$

4. *For a.a.ω,* $\forall \varphi \in C^1[0, T; W_2^1(D, \mathbb{R})]$ *such that* $\varphi(T) = 0$

$$
\int_0^T (\rho(s), \varphi_t + < u(s), \nabla > \varphi)dt + (\rho_0, \varphi(0)) = 0.
$$

The method employed to find such a solution is analogous to the method found in the deterministic case, that is a hyperfinite approximation to the equations is formed and solved; the aim is then to show that this solution is 'close' in some sense to a standard solution of the stochastic nonhomogeneous Navier-Stokes equations. The fact that a solution to the Galerkin approximation exists is stated in following Theorem. Before this, for purpose of clarity and notation the form of such an approximation is presented.

5.2 The Hyperfinite Approximation of Dimension N

Fix $N \in {}^*\mathrm{N} \setminus \mathrm{N}$, a pair of processes (R, Y) is sought such that:

$$
Y :^* [0, T] \times \Omega \to {}^*\mathbb{R}^N \qquad \text{and } R : {}^*[0, T] \times^* D \times \Omega \to {}^*\mathbb{R}
$$

and such that with $U : [0, T] \times \Omega \to H_N$ defined by

$$
U(\tau, \omega) = \sum_{k=1}^{N} Y_k(\tau, \omega) E_k
$$

the following are satisfied:

$$
\Theta(\tau, Y)dY = [F(\tau, Y) - {}^*\beta(r(Y)(\tau), Y(\tau), Y(\tau)) - {}^*\Gamma Y(\tau)]\, d\tau + [G(\tau, Y)]\, dW^N(\tau) \tag{5.2.4}
$$

with F and G defined below and

$$
Y(0) = (({}^*u_0, E_1), \ldots, ({}^*u_0, E_N))
$$

and

$$
\frac{\partial R}{\partial \tau} + < U, \nabla > R = 0 \tag{5.2.5}
$$

with

$$R(0) = {}^*\rho_0$$

Then the pair (R, U) is called a solution to the hyperfinite approximation of dimension N.
The notation Θ, β and Γ are as defined in the deterministic case, however note that here the dimension is N, and thus for example Θ is an internal $N \times N$ matrix. Analogously W^N is an internal ${}^*\mathbb{R}^N$ valued Wiener process with covariance Q_N.
Let $\mathcal{C}$ be the space of all internal *continuous ${}^*\mathbb{R}^N$ valued processes.
The function $F: {}^*[0,T] \times \mathcal{C} \to {}^*\mathbb{R}^N$ is redefined to incorporate feedback, that is

$$F(\tau, Y) = (({}^*r(Y)\ {}^*f(\tau, U(\tau)), E_1), \ldots, ({}^*r(Y)\ {}^*f(\tau, U(\tau)), E_N))$$

therefore

$$F(\tau, Y) = \mathrm{Pr}_N \left[{}^*r(Y)\ {}^*f(\tau, U(\tau))\right] \in H_N \cong {}^*\mathbb{R}^N$$

with the identification being trivial.
The new function $G(\tau, Y): {}^*[0,T] \times \mathcal{C} \to L({}^*\mathbb{R}^N, {}^*\mathbb{R}^N)$ is such that for any $X \in {}^*\mathbb{R}^N$

$$G(\tau, Y)X = \mathrm{Pr}_N \left[(r(Y))(\tau)\ {}^*g(\tau, U(\tau))\hat{X}\right] \in H_N \cong {}^*\mathbb{R}^N$$

recall that for $X \in {}^*\mathbb{R}^N$ define $\hat{X} = \sum_{k=1}^{N} X_k E_k \in H_N$, also if say F is a ${}^*\mathbb{R}^N$ valued function then $\hat{F}$ will denote the corresponding H_N valued function.
Note that on occasions we will drop the stars, for example ${}^*\beta$ becomes β, the context will make clear what is intended.
The fact that under certain conditions on $f$and g such an equation has a solution is expressed in the following Theorem.

5.3 Solving the Approximation

Theorem 5.3.1 *Suppose that $u_0 \in H$, $\rho_0 \in C^1(D)$ with $0 < m \leq \rho_0(x) \leq M$ and that*

$$f : [0,T] \times H \to H \text{ and } g : [0,T] \times H \to L(H, H)$$

are jointly measurable functions satisfying the following properties

$$i) f(t, \cdot) \in C(K_m, H) \qquad \forall m \in \mathbb{N}.$$

$$ii) g(t, \cdot) \in C(K_m, L(H, H)) \qquad \forall m \in \mathbb{N}.$$

$$iii) \mid f(t,u) \mid + \mid g(t,u) \mid_{H,H} \leq a(t)(1 + \mid u \mid) \quad \forall u \in H, \text{ where } a \in L^2[0,T].$$

Then there exists a solution (R, U) to the hyperfinite approximation of dimension N, that is (R, U) satisfy (5.2.4) and (5.2.5).

Proof Consider a truncated version of (5.2.4) in the following form:

$$\Theta(\tau, \overline{Y})dY = \left[F(\tau, Y) - \beta(r(\overline{Y})(\tau), \overline{Y}(\tau), Y(\tau)) - \Gamma Y(\tau)\right] d\tau + \left[G(\tau, Y)\right] dW^N(\tau) \tag{5.3.6}$$

with

$$Y(0) = (({}^*u_0, E_1), \ldots, ({}^*u_0, E_N))$$

where $I \in^* \mathbb{N} \setminus \mathbb{N}$ and

$$\overline{Y}(\tau) = \begin{cases} Y(\tau) & \text{if} \mid Y(\tau) \mid \leq I \\ \frac{Y(\tau)}{|Y(\tau)|} & \text{otherwise.} \end{cases}$$

Now rewrite (5.3.6) in the following manner

$$dY = \left[\Theta^{-1}(\tau, \overline{Y}) \left[F(\tau, U) - \beta(r(\overline{Y})(\tau), \overline{Y}(\tau), Y(\tau)))\right]\right] d\tau + \left[\Theta^{-1}(\tau, Y)G(\tau, Y)\right] dW^N(\tau). \tag{5.3.7}$$

Since (5.3.7) is a ${}^*\mathbb{R}^N$ valued stochastic differential equation and the β term now has linear growth; considering the conditions imposed on f, g then (by the transfer of the standard theory of SDE's) there exists a internal adapted solution Y. In fact with $R = r(Y)$, then the pair (R, U) is a solution to the truncated Galerkin approximation.
The aim is now to show that U satisfies the stochastic energy inequality

$$\mathbb{E}\left(\sup_{\tau \leq T} \mid U(\tau) \mid^2 + \nu \int_0^T \parallel U(\sigma) \parallel^2 d\sigma\right) < c \tag{5.3.8}$$

with c a finite constant that is independent of N.
A consequence of this is that for $a.a.\omega$ and all $\tau \in^* [0, T]$

$$\mid U(\tau) \mid < \infty$$

so that $\overline{Y} = Y$, and therefore (5.3.6) is actually (5.2.4) and thus the pair (R, U) will be a solution to the hyperfinite approximation of dimension N.
For ease of notation denote $\overline{R} = r(\overline{Y})$.
Now by applying the transfer of the Itô formula and noting the cancelation that is the analog of (4.3.34) in chapter 4 gives

$$\begin{aligned} d(\overline{R}U, U) = \; & \left[2(U, \hat{F}(\tau, Y) - \hat{\Gamma}Y) + \operatorname{tr}[Q_N(\Theta^{-1}(\tau, \overline{Y})G)^T G]\right] d\tau \\ & +2(U, \hat{G})dW^N(\tau). \end{aligned}$$

Now clearly

$$2(U, \hat{\Gamma}Y) = 2(U, \nu AU) = 2\nu \parallel U \parallel^2$$

thus for $a.a.\omega$, for any $\tau \in {}^*[0, T]$

$$\begin{aligned} (\overline{R}(\tau)U(\tau), U(\tau)) + 2\nu \int_0^T \parallel U(\sigma) \parallel^2 d\sigma = \; & 2\int_0^T (U(\sigma), \hat{F}(\sigma, Y))d\sigma \\ & + \int_0^T \operatorname{tr}\left[Q_N(\Theta^{-1}(\sigma, Y)G(\sigma, Y))^T G(\sigma, Y)\right] d\sigma \\ & +2\int_0^T (U(\sigma), \hat{G}(\sigma, Y))dW^N(\sigma) \\ & +(\overline{R}(0)U(0), U(0)). \end{aligned} \tag{5.3.9}$$

Now by using the fact that $0 < m \leq \overline{R}(\xi, \tau, \omega) \leq M$ it follows from (5.3.9) that for any $\tilde{\tau} \in {}^*[0, T]$

$$m \mid U(\tilde{\tau}) \mid^2 + 2\nu \int_0^{\tilde{\tau}} \| U(\sigma) \|^2 \, d\sigma \leq \; M \mid U(0) \mid^2 + \int_0^{\tilde{\tau}} \mid 2(U(\sigma), \hat{F}(\sigma, Y)) \mid d\sigma$$

$$+ \int_0^T \mathrm{tr} \left[Q_N(\Theta^{-1}(\sigma, Y) G(\sigma, Y))^T G(\sigma, Y) \right] d\sigma$$

$$+2 \sup_{\tau \leq T} \mid I(\tau) \mid \tag{5.3.10}$$

where $I(\tau)$ is the internal martingale

$$I(\tau) = \int_0^\tau (U(\sigma), \hat{G}(\sigma, Y)) dW^N(\sigma).$$

The aim now is to find an upper bound for the right hand side of (5.3.10). In the following c_n where $n \in \mathrm{N}$ will represent various finite positive constants, all of which are independent of the dimension N.
Firstly since Θ is symmetric then

$$\mathrm{tr} \left[Q_N(\Theta^{-1}(\sigma, Y) G(\sigma, Y))^T G(\sigma, Y) \right] \leq (\mathrm{tr}\; Q_N) \mid G(\sigma, Y) \mid^2 \mid \Theta^{-1}(\sigma, \overline{Y}) \mid$$

Now as in the deterministic case $\mid \Theta^{-1}(\sigma, \overline{R}) \mid \leq m^{-1}$ and by considering the definition of G and the growth condition on g it follows that

$$\mid G(\sigma, Y) \mid_{{}^*\mathrm{R}^N,\, {}^*\mathrm{R}^N} \leq M \; {}^*a(\sigma)(1 + \mid U(\sigma) \mid) \tag{5.3.11}$$

Further using the facts that

$$\mathrm{tr}\; Q_N \leq \mathrm{tr}\; Q < \infty \quad \text{and} \quad (1 + \mid U(\sigma) \mid)^2 \leq 3(1 + \mid U(\sigma) \mid^2)$$

then

$$\mathrm{tr} \left[Q_N(\Theta^{-1}(\sigma, Y) G(\sigma, Y))^T G(\sigma, Y) \right] \leq c_1 \; {}^*a^2(\sigma)(1 + \mid U(\sigma) \mid^2) \tag{5.3.12}$$

Next consider the force term, well by (2.1.7)

$$\mid 2(U(\sigma), \hat{F}(\sigma, Y)) \mid \leq 2 \mid U(\sigma) \mid \mid \hat{F}(\sigma, Y) \mid \leq 2c \| U(\sigma) \| \mid \hat{F}(\sigma, Y) \mid$$

now applying young's inequality with $p = q = 2$ and $\epsilon = 2\nu$ gives

$$\mid 2(U(\sigma), \hat{F}(\sigma, Y)) \mid \leq \nu \| U(\sigma) \|^2 + c_2 \mid \hat{F}(\sigma, Y) \mid^2$$

the assumed growth conditions on f implies that

$$\mid F(\sigma, Y) \mid \leq M \; {}^*a(\sigma)(1 + \mid U(\sigma) \mid)$$

and so

$$\mid 2(U(\sigma), \hat{F}(\sigma, Y)) \mid \leq \nu \| U(\sigma) \|^2 + c_3 \; {}^*a^2(\sigma)(1 + \mid U(\sigma) \mid^2). \tag{5.3.13}$$

Thus, substituting (5.3.12) and (5.3.13) into (5.3.10) yields that for any $\tilde{\tau} \in {}^*[0,T]$

$$m \mid U(\tilde{\tau}) \mid^2 + \nu \int_0^{\tilde{\tau}} \| U(\sigma) \|^2 \, d\sigma \leq \ M \mid U(0) \mid^2 + c_4 \int_0^{\tilde{\tau}} {}^*a^2(\sigma)(1+ \mid U(\sigma) \mid^2) d\sigma$$

$$+2 \sup_{\tau \leq T} \mid I(\tau) \mid . \tag{5.3.14}$$

Now to consider the internal martingale I. Well I has quadratic variation $[I]$ given by

$$[I](\tau) = \int_0^{\tau} \mathrm{tr}[Z(\sigma)^T Q_N Z(\sigma)] d\sigma$$

where $Z(\sigma) = (U(\sigma), \hat{G}(\sigma, Y)) \in H_N$, now by (5.3.11)

$$\mid Z(\sigma) \mid \leq \mid U(\sigma) \mid M \ {}^*a(\sigma)(1+ \mid U(\sigma) \mid)$$

so

$$([I](\tilde{\tau}))^{\frac{1}{2}} \leq \left(\sup_{\tau \leq \tilde{\tau}} \mid U(\tau) \mid^2 \right)^{\frac{1}{2}} \left(c_5 \int_0^{\tilde{\tau}} {}^*a^2(\sigma)(1+ \mid U(\sigma) \mid^2) d\sigma \right)^{\frac{1}{2}} .$$

Let k be the positive constant found in the Burkholder-Davis-Gundy inequality; then apply Young's inequality with $p = q = 2$ and $\epsilon = \frac{m}{2k}$ to get

$$([I](\tilde{\tau}))^{\frac{1}{2}} \leq \frac{m}{4k} \sup_{\tau \leq \tilde{\tau}} \mid \ U(\tau) \mid^2 + c_6 \int_0^{\tilde{\tau}} {}^*a^2(\sigma)(1+ \mid U(\sigma) \mid^2) d\sigma$$

and finally by the Burkholder-Davis-Gundy inequality

$$2 \ \mathbb{E}(\sup_{\tau \leq \tilde{\tau}} \mid I(\tau) \mid) \leq \frac{m}{2} \mathbb{E} \left(\sup_{\tau \leq \tilde{\tau}} \mid \ U(\tau) \mid^2 \right) + c_7 \mathbb{E} \left(\int_0^{\tilde{\tau}} {}^*a^2(\sigma)(1+ \mid U(\sigma) \mid^2) d\sigma \right) \tag{5.3.15}$$

Now let $\tilde{\tau} = T$ in (5.3.14), taking expectations and using (5.3.15) (with $\tilde{\tau} = T$) yields

$$\mathbb{E} \left(\nu \int_0^T \| U(\sigma) \|^2 \, d\sigma \right) \leq \ M \mid U(0) \mid^2 + \tfrac{m}{2} \mathbb{E}(\sup_{\tau \leq T} \mid U(\tau) \mid^2)$$

$$+ c_8 \mathbb{E} \left(\int_0^T {}^*a^2(\sigma)(1+ \mid U(\sigma) \mid^2) d\sigma \right)$$

Now if

$$\mathbb{E}(\sup_{\tau \leq T} \mid U(\tau) \mid^2) < c_9$$

say, with c_9 independent of N then clearly

$$\mathbb{E} \left(\nu \int_0^T \| U(\sigma) \|^2 \, d\sigma \right) \leq M \mid \mathrm{Pr}_N \ {}^*u_0 \mid + c_{11} \int_0^T {}^*a^2(\sigma) d\sigma + c_{12}$$

since $\mid \mathrm{Pr}_N \ {}^*u_0 \mid \leq \mid u_0 \mid$ and $a \in L^2[0,T]$ then

$$\mathbb{E} \left(\nu \int_0^T \| U(\sigma) \|^2 \, d\sigma \right) < c_{13}$$

again with c_{13} independent of N.
Thus to establish (5.3.8) it is sufficient to show that

$$\mathbb{E}(\sup_{\tau\leq T} \mid U(\tau) \mid)^2) < \tilde{c}$$

with $\tilde{c}$ independent of N .
Towards this end fix $\tilde{\tau} \in {}^*[0,T]$, now by (5.3.14) for all $\tau \in {}^*[0,\tilde{\tau}]$

$$m \mid U(\tau) \mid^2 \leq M \mid U(0) \mid^2 + c_4 \int_0^{\tilde{\tau}} {}^*a^2(\sigma)(1+ \mid U(\sigma) \mid^2)d\sigma + 2\sup_{\tau\leq\tilde{\tau}} \mid I(\tau) \mid .$$

thus

$$m \sup_{\tau\leq\tilde{\tau}} \mid U(\tau) \mid^2 \leq M \mid U(0) \mid^2 + c_4 \int_0^{\tilde{\tau}} {}^*a^2(\sigma)(1+ \mid U(\sigma) \mid^2)d\sigma + 2\sup_{\tau\leq\tilde{\tau}} \mid I(\tau) \mid$$

then taking expectations and using (5.3.15) yields

$$\frac{m}{2}\mathbb{E}\left(\sup_{\tau\leq\tilde{\tau}} \mid U(\tau) \mid^2\right) \leq M \mid U(0) \mid^2 + c_{14} \int_0^{\tilde{\tau}} {}^*a^2(\sigma)(1+\mathbb{E} \mid U(\sigma) \mid^2)d\sigma \qquad (5.3.16)$$

therefore for any $\tilde{\tau} \in^* [0,T]$

$$\frac{m}{2}\mathbb{E} \mid U(\tilde{\tau}) \mid^2 \leq M \mid U(0) \mid^2 + c_{14} \int_0^{\tilde{\tau}} {}^*a^2(\sigma)\mathbb{E} \mid U(\sigma) \mid^2 d\sigma + c_{15}$$

Now putting $y(\tau) = \mathbb{E} \mid U(\tau) \mid^2$ then for any $\tau \in {}^*[0,T]$

$$\frac{m}{2}y(\tau) \leq c_{14} \int_0^{\tau} {}^*a^2(\sigma)y(\sigma)d\sigma + c_{16}$$

then since $a \in L^2[0,T]$ an application of Gronwall's Lemma gives that for all $\tau \in^*$ $[0,T]$

$$y(\tau) \leq c_{17} \qquad (5.3.17)$$

substituting this into (5.3.16) with $\tilde{\tau} = T$ gives

$$\frac{m}{2}\mathbb{E}(\sup_{\tau\leq T} \mid U(\tau) \mid^2) \leq M \mid u_0 \mid^2 + c_{14}(1+c_{17}) \int_0^T {}^*a^2(\sigma)d\sigma$$

and thus

$$\mathbb{E}(\sup_{\tau\leq T} \mid U(\tau) \mid^2) \leq \tilde{c}$$

as required.
Hence

$$\mathbb{E}\left(\nu \int_0^T \| U(\sigma) \|^2 d\sigma + \sup_{\tau\leq T} \mid U(\tau) \mid^2\right) < c \qquad (5.3.18)$$

with c independent of N, and thus for *a.a.ω*

$$\sup_{\tau \leq T} | U(\tau) |^2 < \infty \tag{5.3.19}$$

therefore for *a.a.ω* and all $\tau \in {}^*[0,T]$

$$\overline{Y}(\tau) = Y(\tau)$$

consequently

$$\overline{R} = R.$$

Therefore (R, U) is a solution to Galerkin approximation of dimension N. ♣

The aim now is to show that the R is close to a standard process ρ and that U is close to a standard process u, in order to do this it is not surprising that some regularity is required on U.
This is presented in the form of two Lemmas, which are the counterparts of Lemma 4.3.3 and Lemma 4.3.4 in Chapter 4.

Lemma 5.3.2 *With* Pr_N *being the projection from* $L^2({}^*D)$ *onto* H_N *then for a.a.ω*

$$\mathrm{Pr}_N(RU)$$

is weakly S-continuous on ${}^*[0,T]$, *that is for a.a.ω if* $\tau \approx \sigma$ *then*

$$\mathrm{Pr}_N(R(\tau)U(\tau)) \approx_W \; \mathrm{Pr}_N(R(\sigma)U(\sigma)).$$

Proof Since the pair (R, U) is a solution to the Galerkin approximation then

$$d(RU, E_k) = (\hat{F}(\tau, Y) - \Gamma Y(\tau), E_k) + {}^*b(U(\tau), E_k, R(\tau)U(\tau))d\tau + (E_k, G(\tau, Y))dW^N$$

This is analogous to (4.3.42) from the deterministic case plus the stochastic term (E_k, G), and can be derived in an analogous way.
Thus for any $\tau \in {}^*[0,T]$ and any $k \in \mathrm{N}$

$$(RU)_k(\tau) = (RU)_k(0) \quad + \int_0^\tau [(F(\sigma, Y), E_k) + \; {}^*b(U(\sigma), E_k, R(\sigma)U(\sigma)) - \nu\lambda U_k(\sigma)]\, d\sigma$$

$$+ \int_0^\tau (E_k, G(\sigma, Y))dW^N(\sigma)$$

The aim is to show that $(RU)_k$ is S-continuous on ${}^*[0,T]$.
Now by the assumed growth condition on f and the estimate (5.3.19) for *a.a.ω*

$$\int_0^T (\hat{F}(\sigma, Y), E_k)^2 d\sigma \leq M^2 \int_0^T {}^*a^2(\sigma)(1 + | U(\sigma) |)^2 \leq c(1 + \sup_{\tau \leq T} | U(\tau) |^2) < \infty$$

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from the internal process U, such that u is close in some sense to U.
It will be shown that this u, along with a process ρ to be defined in the next section, is a weak solution to the stochastic nonhomogeneous Navier-Stokes equation.
Firstly note that estimate (5.3.18) implies that for $a.a.(\tau,\omega)$

$$\| U(\tau,\omega) \| < \infty$$

therefore define

$$\mathcal{E} = \{(\tau,\omega) : \| U(\tau,\omega) \| < \infty\}$$

then $\mathcal{E}$ is Loeb measurable and of full measure in the product space i.e.

$$(\Lambda_L \times P)(\mathcal{E}) = T$$

recall that $P = \Pi_L$.
Next define the standard part map

$$\text{st} : \ {}^*[0,T] \times \Omega \to [0,T] \times \Omega$$

in the obvious way i.e.

$$\text{st}(\tau,\omega) = ({}^\circ\tau,\omega)$$

further define

$$\text{st}(\mathcal{E}) = \{({}^\circ\tau, w) : (\tau,\omega) \in \mathcal{E}\}$$

then clearly

$$\mathcal{E} \subseteq \text{st}^{-1}(\text{st}(\mathcal{E})).$$

Now since the Loeb σ-algebra is complete then $\text{st}^{-1}(\text{st}(\mathcal{E}))$ is Loeb measurable with

$$(\Lambda_L \times P)(\text{st}^{-1}(\mathcal{E})) = T$$

and hence by elementary Loeb theory $\text{st}(\mathcal{E})$ is measurable and that

$$(\lambda \times P)(\text{st}(\mathcal{E})) = T.$$

5.4 The Definition of u

It is now possible to define a standard process u from the internal process U, this is done in an analogous manner to that found in the deterministic case.

$$u(t,\omega) = \text{st}_H(U(\tau,\omega)) \tag{5.4.24}$$

for all $(t,\omega) \in \text{st}(\mathcal{E})$ such that $\tau \approx t$ with $(\tau,\omega) \in \mathcal{E}$.
It is clear that, U is a lifting of u in the sense that for $a.a.(\tau,\omega)$

$$u({}^\circ\tau,\omega) = {}^\circ U(\tau,\omega). \tag{5.4.25}$$

and that u is $\mathcal{F}_t$- adapted.
Also it is shown below that for $a.a.\omega$

$$u(\cdot,\omega) \in L^\infty(0,T;H) \cap L^2(0,T;V) \tag{5.4.26}$$

Note that (5.4.26) is the first part of condition 2 of Definition 5.1.1.
Now to demonstrate that this condition is satisfied. Firstly $a.a.\omega$ the estimate (5.3.19) and [Cacu] Proposition 3.2.10 gives

$$| u(t,\omega) | = | \ {}^\circ U(\tau,\omega) | \leq {}^\circ | U(\tau,\omega) | < \infty \qquad \text{for all } \tau \leq T$$

and hence $u(\cdot,\omega) \in L^\infty(0,T;H)$ almost surely; as required.
Now for $a.a.\omega$

$$\begin{aligned}
\int_0^T ||u(t,\omega)||^2 dt &= \int_0^T ||u({}^\circ\tau,\omega)||^2 d\Lambda_L && \text{Theorem 3.3.2} \\
&= \int_0^T ||{}^\circ U(\tau,\omega)||^2 d\Lambda_L && \text{by definition of } u \\
&\leq \int_0^T {}^\circ||U(\tau,\omega)||^2 d\Lambda_L && \text{Prop 3.2.10} \\
&\leq {}^\circ\left(\int_0^T ||U(\tau,\omega)||^2 d\Lambda\right) && \text{Proposition 3.3.7} \\
&< \infty && \text{by (5.3.18)}
\end{aligned}$$

and so $u(\cdot,\omega) \in L^2(0,T;V)$ almost surely; and hence (5.4.26) is established.

5.5 Defining a ρ from R

In this section the aim is to define a standard process ρ from the internal process R the method employed is basically the same as in the deterministic case.
The goal is to produce a ρ such that for suitable z if Z is a suitable lifting of z then for $a.a.\omega$, for all τ

$$\int_{{}^*D} R(\tau) Z d\xi \approx \int_D \rho({}^\circ\tau) z dx \tag{5.5.27}$$

this will be made precise in the Lemma following the construction of ρ from R.

Construction of ρ

We have

$$\begin{aligned}
&R :^* [0,T] \times^* D \times \Omega \to^* \mathbb{R} \\
&\frac{\partial R}{\partial \tau} + < U, \nabla > R = 0
\end{aligned} \tag{5.5.28}$$

with

$$R(0) = {}^*\rho_0$$

and further

$$0 < m \leq R(\tau,\xi,\omega) \leq M < \infty.$$

Reconsider this as

$$R :^* [0,T] \to L^\infty({}^*D \times \Omega, {}^*\mathbb{R})$$

now define

$${}^\circ R :^* [0,T] \to L^\infty({}^*D \times \Omega, \mathbb{R})$$

by

$$({}^{\circ}R(\tau))(X,\omega) = {}^{\circ}((R(\tau))(X,\omega))$$

Now if $\tau \in^{*} [0,T]$ then $R(\tau)$ is ${}^{*}\mathcal{B} \times \mathcal{A}$ measurable , thus ${}^{\circ}R(\tau)$ is $L({}^{*}\mathcal{B} \times \mathcal{A})$ measurable, here $\mathcal{B} = \mathcal{B}(D)$ i.e the Borel σ-algebra on D.
Now define

$$\mathrm{st} : \mathrm{ns}({}^{*}D) \times \Omega \to D \times \Omega$$

by

$$\mathrm{st}(\xi,\omega) = ({}^{\circ}\xi,\omega)$$

note that st is $(\sigma({}^{*}\mathcal{B}) \times \mathcal{F}, \mathcal{B} \times \mathcal{F})$ measurable.
Let us define a σ-algebra on ${}^{*}D \times \Omega$ by

$$\mathcal{G} = \sigma(\mathrm{st}^{-1}(E) : E \in \mathcal{B} \times \mathcal{F})$$

and therefore

$$\mathcal{G} \subseteq \sigma({}^{*}\mathcal{B}) \times \mathcal{F} \subseteq L({}^{*}\mathcal{B}) \times \mathcal{F} \subseteq L({}^{*}\mathcal{B} \times \mathcal{A}).$$

Recall that $L(\mathcal{A}) = \mathcal{F}$.
Clearly for all $\tau \in^{*} [0,T]$, we have that

$${}^{\circ}R(\tau) \in L^{1}({}^{*}D \times \Omega, L({}^{*}\mathcal{B} \times \mathcal{A}), L(\xi \times \Pi)).$$

So we may define

$$\tilde{\rho} :^{*} [0,T] \to L^{\infty}({}^{*}D \times \Omega, \mathbb{R})$$

by

$$\tilde{\rho}(\tau) = \mathbb{E}({}^{\circ}R(\tau) \mid \mathcal{G})$$

and thus

$$0 < m \leq (\tilde{\rho}(\tau))(\xi,\omega) \leq M$$

Now since the map st is measurable and onto, and since for each τ (by definition) $\tilde{\rho}(\tau)$ is $\mathcal{G}$ measurable then there exists for each τ a unique $\mathcal{B}\times\mathcal{F}$ measurable function

$$\hat{\rho}(\tau) : D \times \Omega \to \mathbb{R}$$

such that

$$\tilde{\rho}(\tau) = \hat{\rho}(\tau) \circ \mathrm{st}$$

i.e for all τ, ξ, and ω

$$\tilde{\rho}(\tau)(\xi,\omega) = \hat{\rho}({}^{\circ}\xi,\omega)$$

and thus

$$0 < m \leq (\hat{\rho}(\tau))(x,\omega) \leq M$$

therefore it is possible to consider $\hat{\rho}$ as

$$\hat{\rho} :^{*} [0,T] \to L^{\infty}(D \times \Omega, \mathbb{R})$$

with $\hat{\rho}(\tau)$ being $\mathcal{B} \times \mathcal{F}$ measurable for all $\tau \in^{*} [0,T]$.
The following Lemma makes precise the formula (5.5.27), this Lemma has many possible forms, the one presented here being quite natural and suitable.

Lemma 5.5.1 *If $z : D \times \Omega \to \mathbb{R}$ is $\mathcal{B} \times \mathcal{F}$ measurable and has an S-integrable lifting $Z :^* D \times \Omega \to^* \mathbb{R}$, i.e for a.a.$(\xi, \omega)$*

$$Z(\xi,\omega) \approx z(\ {}^\circ\xi,\omega)$$

and for a.a.ω

$$Z(\cdot,\omega) :^* D \to^* \mathbb{R} \text{ is S-integrable.}$$

Then for a.a.ω, any τ

$$\int_{^*D} R(\tau) Z d\xi \approx \int_D \hat{\rho}(\tau) z dx$$

Proof Fix $\tau \in^* [0,T]$ and take any $F \in \mathcal{F}$ and consider

$$\int_F \left[{}^\circ \int_{^*D} (R(\tau))(\xi,\omega) Z(\xi,\omega) d\xi \right] d\Pi_L. \tag{5.5.29}$$

Now Z is S-integrable, and R is bounded thus for $a.a.\omega$

$$(R(\tau))(\cdot,\omega) Z(\cdot,\omega) :^* D \to^* \mathbb{R}$$

is S-integrable.
Thus for $a.a.\omega$

$${}^\circ \int_{^*D} (R(\tau))(\xi,\omega) Z(\xi,\omega) d\xi = \int_{^*D} {}^\circ((R(\tau))(\xi,\omega))^\circ(Z(\xi,\omega)) d\xi_L$$

Thus rewrite (5.5.29) as

$$\int_F \left[\int_{^*D} {}^\circ((R(\tau))(\xi,\omega))^\circ(Z(\xi,\omega)) d\xi_L \right] d\Pi_L.$$

Since

$$\xi_L(\ {}^*D \backslash \mathrm{ns}({}^*D)) = 0 \tag{5.5.30}$$

then the integral above is equal to

$$\int_F \left[\int_{\mathrm{ns}({}^*D)} {}^\circ((R(\tau))(\xi,\omega))^\circ(Z(\xi,\omega)) d\xi_L \right] d\Pi_L.$$

Now by applying Keisler's Fubini Theorem (see, for example [AFHL86]) then (5.5.29) is equal to

$$\int_{F \times \mathrm{ns}({}^*D)} {}^\circ((R(\tau))(\xi,\omega))^\circ(Z(\xi,\omega)) d_L(\xi \times \Pi) \tag{5.5.31}$$

Now for $a.a.(\xi,\omega)$

$${}^\circ(Z(\xi,\omega)) = z(\ {}^\circ\xi,\omega)$$

So the integral (5.5.31) is equal to

$$\int_{F\times \mathrm{ns}(^*D)} {}^\circ((R(\tau))(\xi,\omega))z({}^\circ\xi,\omega)d_L(\Pi\times\xi). \tag{5.5.32}$$

Recall that $\tilde{\rho}(\tau) = \mathbb{E}({}^\circ R(\tau) \mid \mathcal{G})$, now define

$$\overline{z} :^* D \times \Omega \to \mathbb{R}$$

by

$$\overline{z}(\xi,\omega) = z({}^\circ\xi,\omega)$$

i.e

$$\overline{z} = z \circ \mathrm{st}$$

and so $\overline{z}$ is $\mathcal{G}$ measurable, and hence

$$\mathbb{E}({}^\circ R(\tau)\overline{z} \mid \mathcal{G}) = \overline{z}\mathbb{E}({}^\circ R(\tau) \mid \mathcal{G}) = \overline{z}\tilde{\rho}(\tau). \tag{5.5.33}$$

Since $\mathrm{ns}(^*D) \times F \in \mathcal{G}$ then rewriting (5.5.32) and using (5.5.33) gives

$$\int_{\mathrm{ns}(^*D)\times F} {}^\circ(R(\tau))(\xi,\omega)\overline{z}(\xi,\omega)d_L(\xi\times\Pi) = \int_{\mathrm{ns}(^*D)\times F} ((\tilde{\rho})(\tau))(\xi,\omega)\overline{z}(\xi,\omega)d_L(\xi\times\Pi).$$

Now by the definitions of $\hat{\rho}(\tau)$ and $\overline{z}$ the integral on the right is equal to

$$\int_{\mathrm{ns}(^*D)\times F} ((\hat{\rho})(\tau))({}^\circ\xi,\omega)z({}^\circ\xi,\omega)d_L(\xi\times\Pi).$$

Now finally by using (5.5.30) and Keisler's Fubini Theorem this integral is equal to

$$\int_F \left[\int_{^*D} ((\hat{\rho})(\tau))({}^\circ\xi,\omega)z({}^\circ\xi,\omega)d\xi_L\right] d_L\Pi.$$

$$= \int_F \left[\int_D ((\hat{\rho})(\tau))(x,\omega)z(x,\omega)dx\right] d_L\Pi.$$

and thus for any $F \in \mathcal{F}$

$$\int_F \left[{}^\circ\int_{^*D} R(\tau)Z\ d\xi\right] d_L\Pi = \int_F \left[\int_D \hat{\rho}(\tau)z\ dx\right] d_L\Pi$$

and so since $F \in \mathcal{F}$ was generic then for $a.a.\omega$

$$\int_{^*D} R(\tau)Z\ d\xi \approx \int_D \hat{\rho}(\tau)z\ dx$$

as required. ♣

Next an immediate consequence of the above Lemma.

Corollary 5.5.2 *For a.a.ω, for all $\tau \in^* [0,T]$*

$$R(\tau) \approx \hat{\rho}(\tau) \text{ weakly in } L^p(D,\mathbb{R})$$

for any $p \geq 1$.

Proof Let q be such that $\frac{1}{q}+\frac{1}{p}=1$.
If $y \in L^q(D,\mathbb{R})$ then define $z : D \times \Omega \to \mathbb{R}$ by

$$z(x,\omega) = y(x)$$

then clearly z is $\mathcal{B} \times \mathcal{F}$ measurable , and $Z : {}^*D \times \Omega \to {}^*\mathbb{R}$ defined by

$$Z(\xi,\omega) = {}^*y(\xi)$$

is a lifting (since Anderson's Theorem tells us that ${}^*y$ lifts y).
Further, clearly Z is S-integrable and so applying the above Lemma shows that the required integrals are close. ♣

Now an important theorem, this will be used repeatedly in the main existence theorem.

Theorem 5.5.3 *Let $u,v \in H$ and $U,V \in {}^*H$ be such that $u \approx U$ and $v \approx V$. Then for a.a.ω all $\tau \in {}^*[0,T]$ we have*

$$(R(\tau)U,V) \approx (\hat{\rho}(\tau)u,v) \tag{5.5.34}$$

Proof Firstly, rewrite (5.5.34) in the following way

$$\int_{{}^*D} R(\tau) < U,V > d\xi \approx \int_D \hat{\rho}(\tau) < u,v > dx. \tag{5.5.35}$$

Define

$$z : D \times \Omega \to \mathbb{R} \text{ by } z(x,\omega) =< u(x),v(x) >$$

and

$$Z : {}^*D \times \Omega \to {}^*\mathbb{R} \text{ by } Z(\xi,\omega) =< U(\xi),V(\xi) >$$

then by 5.5.1, it is sufficient to prove that $< U,V >$ is an S-integrable lifting of $< u,v >$.
We can now employ the proof of Theorem 4.5.3 to complete this proof. ♣

Next it is shown that $\hat{\rho}$ is well behaved as a function of τ, to this end it is necessary to prove the following Lemma.

Lemma 5.5.4 *For a.a.ω and for any $z \in W_2^1(D,\mathbb{R})$ the function*

$$^*[0,T] \ni \tau \longmapsto \int_{*D} R(\tau)^*z d\xi \ \in^* \mathbb{R} \tag{5.5.36}$$

is S-continuous on ${}^*[0,T]$.

Proof Let $z \in W_2^1(D,\mathbb{R})$ then by performing an integration by parts, and using (5.5.28), gives that for $a.a.\omega$

$$\frac{d}{d\tau}(R,^* z) = (\frac{\partial R}{d\tau},^* z) = (- < U, \nabla > R,^* z) = (R, < U, \nabla > {}^*z)$$

And thus for any $\tau \in [0,T]$

$$(R(\tau),\ {}^*z) = (R(0),\ {}^*z) + \int_0^\tau (R(\sigma), < U(\sigma), \nabla > \ {}^*z) d\sigma \tag{5.5.37}$$

Therefore it is sufficient to show that for $a.a.\omega$

$${}^*[0,T] \ni \sigma \longmapsto (R(\sigma), < U(\sigma), \nabla > \ {}^*z) \tag{5.5.38}$$

is S-integrable on ${}^*[0,T]$, S-continuity of $(R(\tau),\ {}^*z)$ then follows by Proposition 3.3.13.
Denote by η the function defined in (5.5.38), thus for $a.a.\omega$, any $\sigma \in^* [0,T]$

$$| \eta(\sigma) | \leq M \mid U(\sigma) \mid | \nabla z | \leq M \sup_{\tau \leq T} | U(\tau) | \mid \nabla z | < \infty$$

by (5.3.19) and the fact that $z \in W_2^1(D,\mathbb{R})$.
Thus, since η is bounded it is clearly S-integrable on ${}^*[0,T]$ and hence the proof is complete. ♣.
Now an immediate consequence is the following lemma

Lemma 5.5.5 *For $a.a.\omega$, if $\tau \approx \sigma$ then*

$$\hat\rho(\tau) = \hat\rho(\sigma) \ in\ L^\infty(D,\mathbb{R})$$

Proof By Corollary 5.5.2 and by Lemma 5.5.4 it easily follows that if $z \in W_2^1(D,\mathbb{R})$ then for $a.a.\omega$

$$\int_D \hat\rho(\tau) z dx \approx \int_{^*D} R(\tau)^* z d\xi \approx \int_{^*D} R(\sigma)^* z \ d\xi \approx \int_D \hat\rho(\sigma) z dx$$

and hence since $W_2^1(D,\mathbb{R})$ is dense in $L^1(D,\mathbb{R})$ then for $a.a.\omega$ we have

$$\hat\rho(\tau) = \hat\rho(\sigma) \text{ in } L^\infty(D)$$

as required. ♣
This control in time makes it possible to define a standard

$$\rho : [0,T] \to L^\infty(D \times \Omega, \mathbb{R})$$

via the following definition.
Definition of ρ

$$\rho(t) := \hat\rho(\tau) =^\circ R(\tau) \text{ for any } \tau \approx t$$

The previous Lemma ensuring that this is well defined. (The standard part being in the sense of the weak topology on $L^p(D,\mathbb{R})$ with $p \geq 1$)
It is clear that

$$0 < m \leq \rho(t,x,\omega) \leq M \qquad \text{for all } t \in [0,T],\ x \in D,\ \omega \in \Omega.$$

note also that ρ is weakly continuous.
It follows from this definition and the corollary to Lemma 5.5.1 that

$$R(\tau) \approx \rho(\,^\circ\tau)$$

weakly in $L^p(D, \mathbb{R})$.
Lifting Lemmas

Next is the presentation of two lifting Lemma's that will be needed to prove the main Theorem. Again there are many possible formulations.
Firstly a lifting Lemma for the force term, recall the assumptions made on f i.e. that

$$f : [0,T] \times H \to H$$

is jointly measurable and

$$i) \mid f(t,u) \mid \leq a(t)(1+ \mid u \mid) \qquad \forall u \in H \text{ where } a \in L^2[0,T].$$

$$ii) f(t,\cdot) \in C(K_m, H) \qquad \forall m \in \mathbb{N}.$$

recall that

$$K_m = \{u \in H : \parallel u \parallel \leq m\}$$

and that K_m is endowed with the strong topology of H.
Now for the Lemma.

Lemma 5.5.6 *For a.a.(τ,ω)*

$$^*f(\tau, U(\tau,\omega)) \approx f(\,^\circ\tau, u(\,^\circ\tau,\omega)) \textit{ strongly in } H$$

Proof Firstly it is claimed that for $a.a.\tau$, for all U such that $\parallel U \parallel < \infty$

$$^*f(\tau, U) \approx f(\,^\circ\tau, ^\circ U) \text{ strongly in } H \tag{5.5.39}$$

Well for each $m \in \mathbb{N}$ define

$$\tilde{f} : [0,T] \to C(K_m, H) \text{ by } \tilde{f}(t) = f(t,\cdot)$$

now by Anderson's Theorem, for $a.a.\tau$

$$^*\tilde{f}(\tau) \approx \tilde{f}(\,^\circ\tau) \text{ in } C(K_m, H)$$

therefore if $\parallel U \parallel < \infty$ then $U \in K_m$ for some $m \in \mathbb{N}$, and thus

$$(\,^*\tilde{f}(\tau))(U) \approx (\tilde{f}(\,^\circ\tau))(\,^\circ U) \text{ strongly in } H$$

that is

$$^*f(\tau, U) \approx f(\,^\circ\tau, ^\circ U) \text{ strongly in } H$$

and thus (5.5.39) is proven.
Now to complete the proof recall that for $a.a.(\tau,\omega)$

$$\parallel U(\tau,\omega) \parallel < \infty \quad \text{and} \quad ^\circ U(\tau,\omega) = u(\,^\circ\tau,\omega)$$

and thus by using (5.5.39), for $a.a.(\tau,\omega)$

$$^*f(\tau, U(\tau,\omega)) \approx f(\,^\circ\tau, u(\,^\circ\tau,\omega)) \text{ strongly in } H$$

as required. ♣

Now for a similar lifting theorem for the stochastic term.
Again let us recall the assumptions made on g, i.e. that

$$g : [0,T] \times H \to L(H,H)$$

is jointly measurable and

$$i) \mid g(t,u) \mid_{H,H} \ \leq a(t)(1+ \mid u \mid) \qquad \forall u \in H \text{ where } a \in L^2[0,T].$$

$$ii) g(t,\cdot) \in C(K_m, L(H,H)) \qquad \forall m \in \mathbf{N}.$$

Now for the lifting Lemma.

Lemma 5.5.7 *For $a.a.(\tau,\omega)$, and any $k \in \mathbf{N}$*

$$^*g(\tau, U(\tau,\omega))E_k \approx g(\,^\circ\tau, u(\,^\circ\tau,\omega))e_k \text{ strongly in } H.$$

Proof Firstly the claim is that for $a.a.\tau$, for all U such that $\| U \| < \infty$

$$^*g(\tau, U) \approx g(\,^\circ\tau, ^\circ U) \text{ strongly in } L(H,H) \tag{5.5.40}$$

For each $m \in \mathbf{N}$ define

$$\tilde{g} : [0,T] \to C(K_m, L(H,H)) \text{ by } \tilde{g}(t) = g(t,\cdot)$$

Now by Anderson's Theorem, for $a.a.\tau$

$$^*\tilde{g}(\tau) \approx \tilde{g}(\,^\circ\tau) \text{ in } C(K_m, L(H,H))$$

Therefore if $\| U \| < \infty$ then $U \in K_m$ for some $m \in \mathbf{N}$, and thus

$$(\,^*\tilde{g}(\tau))(U) \approx (\tilde{g}(\,^\circ\tau))(\,^\circ U) \text{ strongly in } L(H,H)$$

that is

$$^*g(\tau, U) \approx g(\,^\circ\tau, ^\circ U) \text{ strongly in } L(H,H)$$

and thus (5.5.40) is proven.
Recall that for $a.a.(\tau,\omega)$

$$\| U(\tau,\omega) \| < \infty \text{ and } ^\circ U(\tau,\omega) = u(\,^\circ\tau,\omega)$$

and thus using (5.5.40) for $a.a.(\tau,\omega)$

$$^*g(\tau, U(\tau,\omega)) \approx g(\,^\circ\tau, u(\,^\circ\tau,\omega)) \text{ strongly in } L(H,H). \tag{5.5.41}$$

Now clearly since for any $k \in \mathbf{N}$

$$\mid E_k \mid = \mid e_k \mid = 1 < \infty$$

and thus by using (5.5.41), for $a.a.(\tau,\omega)$

$$^*g(\tau, U(\tau,\omega))E_k \approx g(\,^\circ\tau, u(\,^\circ\tau,\omega))e_k \text{ strongly in } H.$$

as required. ♣

5.6 Existence Theorem

Now a position has been reached where it is possible to state and prove the main theorem of this work.
The claim is that the pair (ρ, u) constructed from the internal pair (R, U) is a solution to the stochastic nonhomogeneous Navier-Stokes equations.

Theorem 5.6.1 *Suppose that $u_0 \in H$, $\rho_0 \in C^1(D)$ and that*

$$f : [0,T] \times H \to H \text{ and } g : [0,T] \times H \to L(H,H)$$

are jointly measurable functions satisfying the following properties

$i) f(t,\cdot) \in C(K_m, H) \qquad \forall m \in \mathbb{N}.$

$ii) g(t,\cdot) \in C(K_m, L(H,H)) \qquad \forall m \in \mathbb{N}.$

$iii) \mid f(t,u) \mid + \mid g(t,u) \mid_{H,H} \ \leq a(t)(1+ \mid u \mid) \quad \forall u \in H,$ *where* $a \in L^2[0,T]$.

Then there exists a solution on a filtered Loeb space to the stochastic nonhomogeneous Navier-Stokes equations as defined in Definition 5.1.1.

Proof Again the claim is the pair of standard processes (ρ, u) constructed from the internal pair of processes (R, U) is such a solution.
Note that to satisfy condition 3 of the definition of a weak solution it is sufficient to prove the integral equality 3 in Definition 5.1.1 for any $\Phi = ze_k$; where $z \in [0,T]$, with $z(T) = 0$.
It is clear that the pair (R, U) satisfies the following equality, this can be derived in an analogous manner to (4.6.61) in the deterministic case.

$$\begin{aligned} -(R(0)U(0),{}^*\Phi(0)) = & \int_0^T (R(\sigma)U(\sigma), < U(\sigma), \nabla > \ {}^*\Phi(\sigma))d\sigma \\ & -\nu \int_0^T \lambda_k U_k(\sigma) \ {}^*z(\sigma)d\sigma \\ & + \int_0^T (R(\sigma){}^*f(\sigma.U(\sigma)), \ {}^*\Phi(\sigma))d\sigma \\ & + \int_0^T (R(\sigma)U(\sigma), \ {}^*\Phi_\tau(\sigma))d\sigma \\ & + \int_0^T ({}^*\Phi(\tau), \Psi(\sigma,\omega))dW(\sigma). \end{aligned} \tag{5.6.42}$$

The aim is to take standard parts of (5.6.42) to produce the following integral equation, which will then clearly be true for $a.a.\omega$

$$\begin{aligned}-(\rho_0 u_0, e_k) = &\int_0^T (\rho(s)u(s), < u(s), \nabla > \Phi(s))ds \\ &-\nu \int_0^T ((u(s), \Phi(s)))ds \\ &+\int_0^T (\rho(s)f(s,u(s)),\ \Phi(s))ds \\ &+\int_0^T (\rho(s)u(s), \Phi_t(s))ds \\ &+\int_0^T (\Phi(s), \rho(s)g(s,u(s))dw(s).\end{aligned} \tag{5.6.43}$$

Once this aim is achieved, it can be concluded that the pair (ρ, u) satisfies condition 3 of Definition 5.1.1 of solution. Recall that it has already been shown that (ρ, u) satisfies condition (1) of the definition.
Therefore it will be necessary to show that the following are true for $a.a.\omega$.

$$(R(0)U(0),\ {}^*\Phi(0)) \approx (\rho_0 u_0, \Phi(0)) \tag{5.6.44}$$

$$\int_0^T (R(\sigma)U(\sigma), < U(\sigma), \nabla >\ {}^*\Phi(\sigma))d\sigma \approx \int_0^T (\rho(s)u(s), < u(s), \nabla > \Phi(s))ds \tag{5.6.45}$$

$$\nu \int_0^T \lambda_k U_k(\sigma)\ {}^*z(\sigma)d\sigma \approx \nu \int_0^T ((u(s), \Phi(s)))ds \tag{5.6.46}$$

$$\int_0^T (R(\sigma)^* f(\sigma, U(\sigma)),\ {}^*\Phi(\sigma))d\sigma \approx \int_0^T (\rho(s)f(s,u(s)), \Phi(s))ds \tag{5.6.47}$$

$$\int_0^T (R(\sigma)U(\sigma),\ {}^*\Phi_\tau(\sigma))d\sigma \approx \int_0^T (\rho(s)u(s), \Phi_t(s))ds \tag{5.6.48}$$

$$\int_0^T (\ {}^*\Phi(\sigma), \Psi(\sigma,\omega))dW(\sigma) \approx \int_0^T (\Phi(s), \rho(s)g(s,u(s))dw(s) \tag{5.6.49}$$

Firstly the proof of (5.6.44).
Recall that

$$R(0) =\ {}^*\rho_0 \text{ and } U(0) = Pr_N(\ {}^*u_0) \text{ for all } \omega$$

Thus

$$\begin{aligned}|\ (R(0)U(0),\ {}^*\Phi(0)) - (\ {}^*\rho_0\, {}^*u_0,\ {}^*\Phi(0))\ | &= |\ (\ {}^*\rho_0(Pr_N(\ {}^*u_0) -{}^* u_0),\ {}^*\Phi(0))\ | \\ &\leq M \mid Pr_N\ {}^*u_0 -{}^* u_0 \mid \mid \Phi(0) \mid \\ &\approx 0\end{aligned}$$

by Proposition 3.2.7 and since $\Phi(0)$ is finite, thus (5.6.44) is proven.
Next for the proof of (5.6.46).
Well for $a.a.\omega$ 5.3.19 implies that $U_k(\tau)$ is bounded, also $z \in C[0,T]$ and so ${}^*zU_k$ is S-integrable.
Further since z is continuous then ${}^*z$ lifts z, i.e. that for $a.a.\tau$

$$^*z(\tau) \approx z(\,^\circ\tau). \tag{5.6.50}$$

Thus for $a.a.w$

$$\begin{aligned}
\nu\lambda_k \int_0^T {}^*z(\tau)U_k(\tau)d\tau &\approx \nu\lambda_k \int_0^T {}^\circ(\,{}^*z(\tau))^\circ U_k(\tau)d_L\tau && {}^*zU_k \text{ is S-integrable} \\
&= \nu\lambda_k \int_0^T (z(\,^\circ\tau))u_k(\,^\circ\tau)d_L\tau && \text{by definition of } u \text{ and by (5.6.50)} \\
&= \nu\lambda_k \int_0^T z(t)u_k(t)dt && \text{Theorem 3.3.2} \\
&= \nu\lambda_k \int_0^T (u(t), \Phi(t))dt && \text{by definition of } u_k \text{ and } \Phi \\
&= \nu \int_0^T (u(t), A\Phi(t))dt && \text{recall } Ae_k = \lambda_k e_k \\
&= \nu \int_0^T ((u(t), \Phi(t)))dt. &&
\end{aligned}$$

And thus (5.6.46) is proven.

Next for the proof of (5.6.47).
The Lifting Lemma, Lemma 5.5.4 is employed here, which stated that for $a.a.(\tau,\omega)$

$$^*f(\tau, U(\tau,\omega)) \approx f(\,^\circ\tau, u(\,^\circ\tau,\omega)).$$

The continuity of Φ implies that for all τ

$$^*\Phi(\tau) \approx \Phi(\,^\circ\tau)$$

Hence Theorem 5.5.3 implies that for $a.a.(\tau,\omega)$

$$(R(\tau)^*f(\tau,U(\tau), {}^*\Phi(\tau)) \approx (\rho(\,^\circ\tau)f(\,^\circ\tau, u(\,^\circ\tau)), \Phi(\,^\circ\tau)). \tag{5.6.51}$$

Now since Φ is bounded, by the growth conditions on f and by the internal energy inequality, we have

$$\begin{aligned}
\int_0^T (R(\tau)^*f(\tau,U(\tau), {}^*\Phi(\tau))^2 d\tau &\leq c\int_0^T \mid R(\tau)^*f(\tau,U(\tau)) \mid^2 d\tau \\
&\leq cM^2 \sup_{\tau\leq T}(1+ \mid U(\tau,\omega) \mid^2) \int_0^T {}^*a^2(\tau)d\tau \\
&< \infty.
\end{aligned}$$

since $a \in L^2[0,T]$.
Thus by Theorem 3.3.12, the function

$$\tau \longmapsto (R(\tau)^* f(\tau, U(\tau), \ {}^*\Phi(\tau))$$

is S-integrable on ${}^*[0,T]$. Thus, (5.6.51) along with Proposition 3.3.10 implies that (IV) is true.

Now an appeal to Theorem 5.5.3, with

$$h = e_k \ x = f(\ {}^\circ\tau, u(\ {}^\circ\tau, \omega)) \text{ and } X =^* f(\tau, U(\tau, \omega)$$

then for $a.a.(\tau, \omega)$

$$(R(\tau)^* f(\tau, U(\tau,\omega), E_k) \approx (\rho(\ {}^\circ\tau) f(\ {}^\circ\tau, u(\ {}^\circ\tau, \omega)), e_k). \tag{5.6.52}$$

Now from (5.6.52) using (5.6.50) it can easily be seen that for $a.a.(\tau, \omega)$

$$(R(\tau)^* f(\tau, U(\tau,\omega), \ {}^*\Phi(\tau)) \approx (\rho(\ {}^\circ\tau) f(\ {}^\circ\tau, u(\ {}^\circ\tau, \omega)), \Phi(\ {}^\circ\tau)). \tag{5.6.53}$$

Now using the growth conditions on f, the estimate (5.3.19), and the fact that Φ is bounded in H; then for $a.a.\omega$

$$\begin{aligned}\int_0^T (R(\tau)^* f(\tau, U(\tau,\omega), \ {}^*\Phi(\tau))^2 d\tau \ &\leq c \int_0^T \mid R(\tau)^* f(\tau, U(\tau,\omega)) \mid^2 d\tau \\ &\leq cM^2 \sup_{\tau \leq T}(1+ \mid U(\tau,\omega) \mid^2) \int_0^T {}^*a^2(\tau) d\tau \\ &< \infty.\end{aligned}$$

since $a \in L^2[0,T]$.
Thus by Theorem 3.3.12, for $a.a.\omega$ the function

$$\tau \longmapsto (R(\tau)^* f(\tau, U(\tau,\omega), \ {}^*\Phi(\tau))$$

is S-integrable on ${}^*[0,T]$ and so

$$\int_0^T (R(\tau)^* f(\tau, U(\tau,\omega), \ {}^*\Phi(\tau)) d\tau = \int_0^T {}^\circ(R(\tau)^* f(\tau, U(\tau,\omega), \ {}^*\Phi(\tau)) d_L\tau$$

Now by (5.6.53) the above is equal to

$$\int_0^T (\rho(\ {}^\circ\tau) f(\ {}^\circ\tau, u(\ {}^\circ\tau, \omega)), \Phi(\ {}^\circ\tau)) d_L\tau$$

which in turn, by Theorem 3.3.2 is equal to

$$\int_0^T (\rho(t) f(t, u(t,\omega)), \Phi(t)) dt$$

as required, thus (5.6.47) is proven.

Now for the proof of (5.6.49).
It is required to prove that for $a.a.\omega$

$$\int_0^T ({}^*\Phi(\sigma), \Psi(\sigma,\omega))dW(\sigma) \approx \int_0^T (\Phi(s), \rho(s)g(s,u(s))dw(s) \tag{5.6.54}$$

where

$$\Psi :^* [0,T] \times \Omega \to L(H_N, H_N)$$

is defined by

$$\Psi(\tau,\omega) = \mathrm{Pr}_N\left[(R(\tau)(\cdot,\omega)^* g(\tau, U(\tau,\omega))v\right] \text{ for all } v \in H_N.$$

Recall it has already shown that

$$\mathbb{E}\left(\int_0^T \mid \Psi(\tau,\omega) \mid_{H_N,H_N} d\tau\right) < \infty \tag{5.6.55}$$

and that for $a.a.\omega$

$$\mid \Psi(\cdot,\omega) \mid_{H_N,H_N} \text{ is S-integrable on } {}^*[0,T].$$

Note that (5.6.54) is equivalent to

$$\int_0^T (E_k, {}^*z(\sigma)\Psi(\sigma,\omega))dW(\sigma) \approx \int_0^T (e_k, z(s)\rho(s)g(s,u(s))dw(s). \tag{5.6.56}$$

Since z is bounded it follows from (5.6.55) that

$$\mathbb{E}\left(\int_0^T \mid {}^*z(\tau)\Psi(\tau,\omega) \mid_{H_N,H_N} d\tau\right) < \infty \tag{5.6.57}$$

and that for $a.a.\omega$

$$\mid {}^*z(\cdot)\Psi(\cdot,\omega) \mid_{H_N,H_N} \text{ is S-integrable on } {}^*[0,T].$$

And therefore by Theorem 3.5.3 to prove (5.6.54) it is necessary only to prove that ${}^*z\Psi$ lifts $z\rho g$.
To be more precise it must be shown that for $a.a.(\tau,\omega)$

$${}^*z(\tau)\Psi(\tau,\omega) \approx \mathrm{Pr}_H\left[z({}^\circ\tau)\rho({}^\circ\tau)g({}^\circ\tau, u({}^\circ\tau,\omega))\right].$$

here closeness is in the sense of the weak operator topology, see Definition 3.2.12.
Thus it must be shown that for any $i,k \in \mathrm{N}$, for $a.a.\omega$

$$(E_k, {}^*z(\tau)R(\tau)^* g(\tau, U(\tau,\omega))E_i) \approx (e_k, z({}^\circ\tau)\rho({}^\circ\tau)g({}^\circ\tau, u({}^\circ\tau,\omega))e_i) \tag{5.6.58}$$

Towards this end, we first prove the following

$$(E_k, R(\tau)^* g(\tau, U(\tau,\omega))E_i) \approx (e_k, \rho(\,^\circ\tau) g(^\circ\tau, u(\,^\circ\tau,\omega))e_i). \tag{5.6.59}$$

Lemma 5.5.7 gives us that for $a.a.(\tau,\omega)$

$$^*g(\tau, U(\tau,\omega))E_i \approx g(^\circ\tau, u(\,^\circ\tau,\omega))e_i \text{ strongly in } H$$

therefore appealing to Theorem 5.5.3 with

$$U =^* g(\tau, U(\tau,\omega))E_i,\; u = g(^\circ\tau, u(\,^\circ\tau,\omega))e_i \text{ and } V = E_k, h = e_k$$

results in proving (5.6.58). Now (5.6.50) and (5.6.58), together proves (5.6.59) and hence (5.6.49) is established.

Now for (5.6.48).
It is required to show that for $a.a.\omega$ we have

$$\int_0^T (R(\sigma)U(\sigma),\; ^*\Phi_\tau(\sigma))d\sigma \approx \int_0^T (\rho(s)u(s), \Phi_t(s))ds$$

Now, for $a.a.(\sigma,\omega)$ by the definition of u

$$U(\sigma,\omega) \approx u(\,^\circ\sigma,\omega)$$

and since Φ_t is continuous, then for all σ

$$^*\Phi_\tau(\sigma) \approx \Phi_t(\,^\circ\sigma)$$

Thus by Theorem 5.5.3 , we have that for $a.a.(\sigma,\omega)$

$$(R(\sigma,\omega)U(\sigma,\omega),\; ^*\Phi_\tau(\sigma))) \approx (\rho(\,^\circ\sigma,\omega)u(\,^\circ\sigma,\omega), \Phi_t(\,^\circ\sigma)). \tag{5.6.60}$$

Now $\Phi_t = z_t e_k$, but $z \in C^1[0,T]$ and hence Φ_t is bounded in H. This fact along with the boundedness of R and the estimate (5.3.19), ensures that for $a.a.\omega$,the function

$$^*[0,T] \ni \sigma \longmapsto (R(\sigma,\omega)U(\sigma,\omega),\; ^*\Phi_\tau(\sigma))$$

is bounded and hence S-integrable.
Thus by Proposition 3.3.10 we are done.

Next for the proof of (5.6.45).
Firstly, it is claimed that for $a.a.(\tau,\omega)$

$$(R(\tau)U(\tau), < U(\tau), \nabla > E_k) \approx (\rho(\,^\circ\tau)u(\,^\circ\tau), < u(\,^\circ\tau), \nabla > e_k) \tag{5.6.61}$$

By the definition of u, for $a.a.(\tau,\omega)$

$$U(\tau,\omega) \approx u(\,^\circ\tau,\omega) \quad \text{in } H \tag{5.6.62}$$

so Theorem 1 and the definition of ρ imply that for $a.a.(\tau,\omega)$

$$(R(\tau)U(\tau), <\; ^*u(\,^\circ\tau), \nabla > E_k) \approx (\rho(\,^\circ\tau)u(\,^\circ\tau), < u(\,^\circ\tau), \nabla > e_k) \tag{5.6.63}$$

Hence in order to prove (5.6.61) it is sufficient to show that for $a.a.(\tau,\omega)$

$$(R(\tau)U(\tau), < U(\tau), \nabla > E_k) \approx (R(\tau)U(\tau), < \ ^*u(\ ^\circ\tau), \nabla > E_k)$$

i.e. that

$$^*b(U(\tau) - \ ^*u(\tau), E_k, R(\tau)U(\tau)) \approx 0.$$

But by (2.1.10) for $a.a.(\tau,\omega)$

$$\begin{aligned} |^* b(U(\tau) - \ ^*u(\tau), E_k, R(\tau)U(\tau)) \mid &= \mid \ ^*b(R(\tau)[U(\tau) - \ ^*u(\tau)], E_k, U(\tau)) \mid \\ &\leq c \mid U(\tau) - \ ^*u(\tau) \mid \mid AE_k \mid || U(\tau) || \approx 0 \end{aligned}$$

and so (5.6.61) is established.
Thus, using (5.6.50) we have for $a.a.(\tau,\omega)$

$$(R(\tau)U(\tau), < U(\tau), \nabla > \ ^*\Phi(\tau)) \approx (\rho(\ ^\circ\tau)u(\ ^\circ\tau), < u(\ ^\circ\tau), \nabla > \Phi(\ ^\circ\tau)) \quad (5.6.64)$$

Now by (2.1.10) and since z is bounded

$$\begin{aligned} \int_0^T \ ^*b(R(\tau)U(\tau), \ ^*\Phi(\tau), U(\tau))^2 d\tau &\leq c\int_0^T \mid U(\tau) \mid^2 \mid A \ ^*\Phi(\tau) \mid^2 || U(\tau) ||^2 \, d\tau \\ &\leq c_1(\sup_{\tau\leq T} \mid U(\tau) \mid^2) \int_0^T || U(\tau) ||^2 \, d\tau < \infty \end{aligned}$$

and thus by Theorem 3.3.12, for $a.a.\omega$, $b(RU, \ ^*\Phi, U)$ as a function of τ is S-integrable on $^*[0,T]$.
Thus by Proposition 3.3.10 and using (5.6.64) we arrive at (5.6.45).

Finally a verification of condition 4 of Definition 5.1.1. That is the integral equality for the density equation. It must be shown that for $a.a.\omega$, for any $\varphi \in C^1[0,T; W_2^1(D,\mathbb{R})]$, with $\varphi(T) = 0$ we have

$$\int_0^T (\rho(s), \varphi_t + < u(s), \nabla > dt + (\rho_0, \varphi(0)) = 0. \quad (5.6.65)$$

Well

$$\frac{\partial R}{\partial t} + < U, \nabla > R = 0 \quad (5.6.66)$$

$$R(0) = \ ^*\rho_0 \quad (5.6.67)$$

now for φ as above using (5.6.66) gives

$$\frac{d}{dt}(R, \ ^*\varphi) = (\frac{\partial R}{\partial t}, \ ^*\varphi) + (R, \ ^*\varphi_t) = -(< U, \nabla > R, ^*\varphi) + (R, \ ^*\varphi_t) \quad (5.6.68)$$

then parts gives

$$-(< U, \nabla > R, ^*\varphi) = (R, < U, \nabla > \ ^*\varphi)$$

substituting this into (5.6.68) and integrating w.r.t s from 0 to T yields

$$\int_0^T (R(\sigma), < U(\sigma), \nabla > \ ^*\varphi(\sigma) + \ ^*\varphi_\tau(\sigma))d\sigma + (R(0), \ ^*\varphi(0)) = 0. \quad (5.6.69)$$

The aim is now to take standard parts of (5.6.69) to produce the required (5.6.65). Clearly

$$({}^*\rho_0,\ {}^*\varphi(0)) = (\rho_0, \varphi(0)).$$

It is claimed that for $a.a.\omega$ we have

$$\int_0^T (R(\sigma),\ {}^*\varphi_\tau(\sigma))d\sigma \approx \int_0^T (\rho(s), \varphi_t(s))ds \tag{5.6.70}$$

and

$$\int_0^T (R(\sigma), < U(\sigma), \nabla >^* \varphi(\sigma))d\sigma \approx \int_0^T (\rho(s), < u(s), \nabla > \varphi(s))ds \tag{5.6.71}$$

Firstly we show (5.6.70) is true.
Now $\varphi_t \in C[0,t; W_2^1(D)]$, therefore for $a.a.\sigma \in^* [0,T]$

$${}^*\varphi_\tau(\sigma) \approx \varphi({}^\circ\sigma) \qquad \text{in } L^2(D). \tag{5.6.72}$$

Now by Theorem 5.5.3, for $a.a.\omega$ since $\varphi_t \in L^2(D)$ we have

$$(R(\sigma),\ {}^*(\varphi_t({}^\circ\sigma))) \approx (\rho({}^\circ\sigma), \varphi_t({}^\circ\sigma)).$$

But

$$| (R(\sigma),\ {}^*(\varphi_t({}^\circ\sigma))) - (R(\sigma),\ {}^*\varphi_\tau(\sigma)) | \leq M \mid {}^*\varphi_\tau(\sigma) - \varphi_t({}^\circ\sigma) \mid \approx 0$$

by (5.6.72). Hence for $a.a.\omega$, for $a.a.\sigma$ we have

$$(R(\sigma),\ {}^*\varphi_\tau(\sigma)) \approx (\rho({}^\circ\sigma), \varphi_t({}^\circ\sigma)) \tag{5.6.73}$$

Now the function

$${}^*[0,T] \ni \sigma \longmapsto (R(\sigma),\ {}^*\varphi_\tau(\sigma))$$

is clearly S-integrable, and hence (5.6.70) follows from (5.6.73) and Theorem 3.3.2.
Now for (5.6.71). It is claimed that for $a.a.\omega$, for $a.a.\sigma$ we have

$$(R(\sigma), < U(\sigma), \nabla >^* \varphi(\sigma)) \approx (\rho({}^\circ\sigma), < u({}^\circ\sigma), \nabla > \varphi({}^\circ\sigma)). \tag{5.6.74}$$

By continuity of $\nabla\varphi$ and Theorem 5.5.3 we have for $a.a.\omega$, for $a.a.\sigma$ that

$$(R(\sigma), <^* u({}^\circ\sigma), \nabla >^* \varphi({}^\circ\sigma)) \approx (\rho({}^\circ\sigma), < u({}^\circ\sigma), \nabla > \varphi({}^\circ\sigma)). \tag{5.6.75}$$

But for $a.a.\omega$, for $a.a.\sigma$ we have

$$| (R(\sigma), < U(\sigma), \nabla >^* \varphi(\sigma)) - (R(\sigma), < u({}^\circ\sigma), \nabla > \varphi({}^\circ\sigma)) |$$

$$\leq M \mid U(\sigma) \mid |^* \varphi(\sigma) - \varphi({}^\circ\sigma) |_{W_2^1} + M \mid U(\sigma) - u({}^\circ\sigma) \mid | \varphi({}^\circ\sigma) |_{W_2^1} \approx 0$$

by the estimate (5.3.19), by continuity of φ and by the definition of u.
Hence (5.6.74) is established.
Note also that for $a.a.\omega$ the function

$${}^*[0,T] \ni \sigma \longmapsto (R(\sigma), < U(\sigma), \nabla >^* \varphi(\sigma))$$

is, for $a.a.\omega$, S-integrable, and thus by using (5.6.74) and Theorem 3.3.2 we arrive at (5.6.71). Therefore all the conditions of the definition have been verified, and thus it has been shown that the pair (ρ, u) is indeed a weak solution to the stochastic nonhomogeneous Navier-Stokes equation as defined in Definition 5.1.1. ♣

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