The University of Hull
Ambitwistor Strings in Ambitwistor space

by

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Contents

1 Introduction

This thesis is a recount of the research I undertook in the academic year 2016-2017 and the background work I learned from reading and attending lecture courses. Sections 2 and 3 are background sections providing an introduction to the concepts and methods that will be encountered in the research sections later on in the thesis.

The research covered in this thesis is the process of producing the BRST charge for a String theory in Ambitwistor space. The sstring theory being investigated originates from [1] theory which is a particle theory and [2] which is a string theory in Ambitwistor space. Whilst the link between these two papers may not seem obvious at first but the particle actions these papers are based from are the same. However for the sake of simplicity supersymmetry, which appears in both [1] and [2], is not covered.

The reason for the investigation of Ambitwistor string theory stems from Witten's initial paper on the subject [3] which showed led to a formula for tree-level 4 dimensional Yang-Mills amplitudes. And more recently Ambitwistor string theories of the form in [2] provide a way of producing amplitudes for Yang-Mills and gravitational amplitudes, known as the CHY Amplitudes,which were first shown in[4], [5] [6] and [7] which are a compact formulae for tree-level scattering amplitudes. It has been shown that the Ambitwistor String theory in [2] is able to produce expressions for higher order loop amplitudes [8], [9], [10], [11] and [12] to name just a few examples. From this point we seek to expand upon a theory that should be physically identical to the Ambitwistor string in [2] but representing it in Ambitwistor variables as has been done for the superparticle in [1].

The difficulty with representing the string theory in this way comes from the gauge symmetry being reducible. This means that the gauge constraint is not independent and there are additional constraints that affect the degrees of freedom of the theory as ignoring the reducibility would lead to over fixing of the degrees of freedom of the theory. So the standard process of gauge fixing and introducing ghosts to reduce the degrees of freedom is followed, but we have to introduce another ghost system. The statistics of these ghosts are the same as those of the matter fields (in this case bosonic) and can be thought of as the reintroduction of those degrees of freedom that have been over fixed. However, in the case of this theory, it doesn't stop there. There is another reducibility constraint that shows that all of the degrees of freedom in the previous reducibility constraint aren't independent either necessitating the introduction of another system of ghosts.

The goal of this thesis is to produce the BRST charge for the Bosonic Ambitwistor string represented in the Ambitwistor variables shown in [1] for future study. The procedure for doing this is outlined in [13] and is followed through the insertion of the constraints of the string theory. Once this has been done the nilpotency of the BRST charge is tested on several fields and constraints are obtained that fix the BRST charge to be nilpotent. After that following a similar procedure to Ohmori [11] A simple gauge fixed action for a free topological theory is produced.

2 Quantum Field theory and Gauge theory

2.1 Motivation for Quantum field Theory

Quantum Field theory is the basis for the current understanding of how matter and forces interact with each other. As a theory it has existed for many years and is formed from the combination of quantum mechanics with classical field theories and special relativity. The quantisation of the electromagnetic field resulted in an a lack of distinction between a particle and a field, meaning that the electromagnetic field had particle like nature. As such Photons can be seen to arise as the quanta for the electromagnetic field. This led to the realisation that matter can be described as a spectrum of states arising from matter fields and the forces can be described as separate fields which interact with matter fields. We know that the Electromagnetic force (Quantum Electrodynamics), the Strong force (Quantum Chromodynamics) and Weak nuclear force can be described in this framework and even more recently with the discovery of the Higgs mechanism the masses of matter can be described [14]. The only force that fails to be described in this framework is gravity. This is due not only to the difficult, but impossible due to the fact that the approach of perturbation theory doesn't work in the case of gravity and is non renormalisable [15].

2.2 Second Quantisation and Scalar Field Theories

Now that I have very briefly introduced the idea of matter fields and the motivation of their existence it is important to know how we go about constructing these theories. If we work from the simplest example, a scalar field theory known as the Klein-Gordon equation:

$$\Box\phi(x) + m^2\phi(x) = 0 \tag{2.2.1}$$

where the $x = (x, y, z, c\tau)$, with τ being the proper time. With the Scalar fields being denoted by $\phi(x)$ which will be written simply as ϕ unless a distinction between fields needs to be made. The Klein-Gordon Equation has a Hamiltonian density and

Lagrangian density of

$$\mathcal{H} = (\partial_0\phi)^2 + \nabla\phi \cdot \nabla\phi + m^2\phi^2 \tag{2.2.2}$$

$$\mathcal{L} = \phi\Box\phi - m^2\phi^2 \tag{2.2.3}$$

To change this into a complex scalar field theory it is a simple matter of replacing and ϕ^2 with $\phi\phi^*$. A way of constructing a quantum theory of these fields is by promoting them to operators that act on the vacuum.

$$\phi|0\rangle = |\phi\rangle \tag{2.2.4}$$

As $\phi(x)$ is not a wave function the position basis is outlined as such

$$\phi(x) = \langle x|\phi\rangle = \langle x|\phi|0\rangle \tag{2.2.5}$$

This makes sense when the field is written in it's Fourier expansion in terms of ladder operators a and $a^\dagger$ which are the annihilation and creation operators.

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left[a(k)e^{-ikx} + a^\dagger(k)e^{ikx}\right] \tag{2.2.6}$$

The denominator in this is a consequence of maintaining Lorentz invariance and doesn't affect the operator in any significant way but it is worth noting $(\omega_k)^2 = k^2 + m^2 = {E_k}^2$. From this representation it is clear to see how ϕ acting on a vacuum state, is actually a and $a^\dagger$ acting on the vacuum results in a momentum eigenstate (see 2.2.13 and 2.2.14). As is the case with Classical field theory there exists a momentum conjugate to the field defined as

$$\Pi(\mathbf{x}, t) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(\mathbf{x}, t)} \tag{2.2.7}$$

Where $\dot{\phi}(\mathbf{x}, t)$ is $\partial_0\phi(\mathbf{x}, t)$ or more simply the derivative with respect to time. Now that we have both the field and the conjugate momentum we can begin to apply canonical commutation relations.

$$[\phi(\mathbf{x}, t), \Pi(\mathbf{x'}, t)] = i\delta(\mathbf{x} - \mathbf{x'}) \tag{2.2.8}$$

$$[\phi(\mathbf{x}, t), \phi(\mathbf{x'}, t)] = [\Pi(\mathbf{x}, t), \Pi(\mathbf{x'}, t)] = 0 \tag{2.2.9}$$

These are known as the equal time commutation relations, named as such as they are defined at equal times. From these relations, the Fourier expansion of ϕ and knowing that

$$\Pi(x) = \dot{\phi}(x) \tag{2.2.10}$$

The commutation relations for the creation and annihilation operators can be derived as [16]

$$[a(k), a^\dagger(k')] = (2\pi)^3\sqrt{2\omega_k}\delta^3(k - k') \tag{2.2.11}$$

$$[a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0 \tag{2.2.12}$$

The creation and annihilation operators are used to define the basis of the quantum system through the production of momentum eigenstates.

$$\sqrt{2E_k}a^\dagger(k)|0\rangle = |k\rangle \tag{2.2.13}$$
$$a(k)|0\rangle = 0 \tag{2.2.14}$$

The Hamiltonian can be written in terms of the creation and annihilation operators and ${\omega_k}^2 = (E_k)^2$

$$H = \int \frac{d^3k}{2\pi^3} E_k \left(a_k^\dagger a_k + \frac{1}{2} \right) \tag{2.2.15}$$

It is now useful to define the Number operator $N_k = a_k^\dagger a_k$ and we have the Hamiltonian for the simple harmonic oscillator. One of the primary observables in quantum field theory is known as the propagator which for the Klein-Gordon field is given by

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) \tag{2.2.16}$$

For an interacting theory there exist more terms but they will be covered in the path integral section. An easy way of producing these results is to use a process known as Wick's theorem uses the generalisation of n-point amplitudes to a sum of all "normal ordered"[1] field operators and "contraction"[2] of fields. The contraction of 2 fields produces the propagator between the two points.

2.3 Path Integral quantisation

The modern method used for quantising field theories is known as path integral quantisation and has been used in the context of quantum field theory for decades [17]and the method was developed and used by Richard Feynman. The core concept of path integral quantisation of a theory lies in the initial idea that a particle moving through time takes all possible paths. This section will only cover a small part of the process of the procedure as it will be enough to inform the process covered in section 3.1. This quantisation originates from the definition of a propagator, which is that it is the probability amplitude for a transition in position in time from $\psi(q_i, t_i)$ to $\psi(q_f, t_f)$, where ψ is the wave function of the particle for that position q and time t . This will be represented as

$$K(q_f, t_f; q_i, t_i) = \langle q_f, t_f | q_i, t_i \rangle \tag{2.3.1}$$

This in turn can be split up into smaller transitions and integrating over all of those small steps.

[1]Normal ordering is where all creation operators contained within an operator are on the left of the annihilation operators

[2]Contractions are a way of automatically time ordering operators through Wick's theorem (see [16]

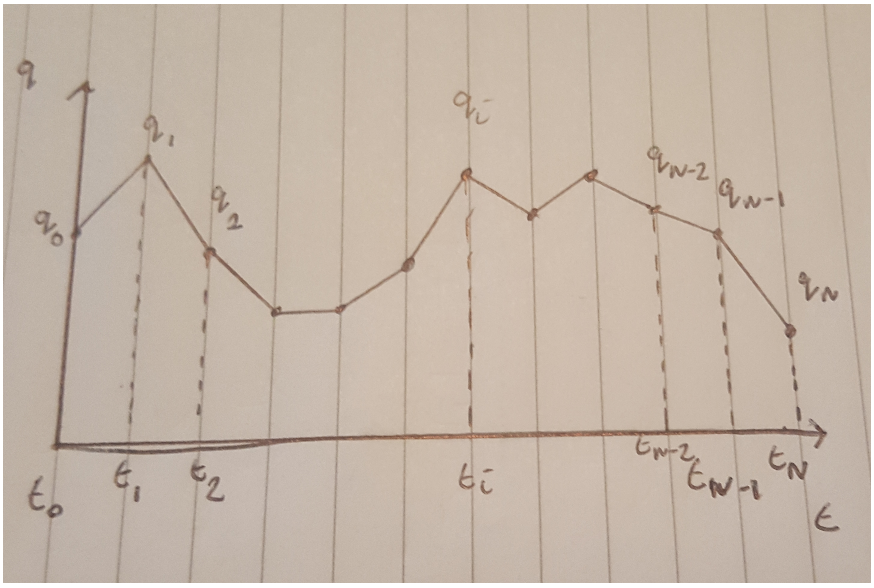

Figure 1. A diagrammatic representation of splitting up paths into discrete segments and states.

The representation of figure 1 as an integral can be given as the propagators between the states.

$$K(q_f, t_f; q_i, t_i) = \int \mathrm{d}q_1 \, \mathrm{d}q_2 \ldots \mathrm{d}q_N \, \langle q_f, t_f | q_N, t_N \rangle \langle q_N, t_N | q_{N-1}, t_{N-1} \rangle \ldots \langle q_1, t_1 | q_i, t_i \rangle \tag{2.3.2}$$

By investigation of one of these steps the propagator between two intermediary points and removing the time dependence from the states into the operator

$$\langle q_{n+1}, t_{n+1} | q_n, t_n \rangle = \langle q_n | e^{-iH\tau} | q_n \rangle \tag{2.3.3}$$

The terms in the exponential are τ, which is the difference in time between the states and H is the Hamiltonian operator. After a series expansion around τ the result is a delta function and a Hamiltonian operation. Taking the Fourier transformation of the delta function the propagator is.

$$\langle q_{n+1}, t_{n+1} | q_n, t_n \rangle = \frac{1}{2\pi} \int \mathrm{d}k \; e^{ik(q_n+1-q_n)} - i\tau \langle q_{n+1}, |H| q_n, \rangle \tag{2.3.4}$$

where H is given it's usual definition of $\frac{k^2}{2m} + V$ and V is some position dependent potential. The second term in 2.3.4 is done by looking at both parts of the Hamiltonian. The operation of the potential is the most straight forward.

$$\langle q_{n+1} | V(q) | q_n \rangle = V(\bar{q}) \delta(q_{n+1} - q_n) = \int \frac{\mathrm{d}k}{2\pi} e^{ik(q_{n+1}-q_n)} V(\bar{q}) \tag{2.3.5}$$

And$\bar{q} = \frac{q_{n+1}-q_n}{2}$. The Momentum operation result is as follows

$$\langle q_{n+1} | \frac{k^2}{2m} | q_n \rangle = \int \frac{\mathrm{d}k}{2\pi} e^{ik(q_n-1-q_n)} \frac{k^2}{2m} \tag{2.3.6}$$

The two operations above give the Hamiltonian term and in conjunction with 2.3.4 the transition is

$$\langle q_{n+1}, t_{n+1} | q_n, t_n \rangle = \int \frac{\mathrm{d}k_n}{2\pi} e^{i\left[k_n(q_n - 1 - q_n) - \tau H\right]} \tag{2.3.7}$$

When taken to the continuum limit the full propagator can be returned by taking the difference between initial position and final position the exponent becomes

$$i\left[\int_{t_i}^{t_f} \mathrm{d}t\left[k\dot{q} - H\right]\right] = i\left[\int_{t_i}^{t_f} \mathrm{d}t\left[L\right]\right] = iS \tag{2.3.8}$$

where L is the Lagrangian of the theory and S is the action. This gives one segment of one possible path taken [14]. To consider all paths all possible positions must be considered and as a consequence of the aforementioned continuum limit where the number of intermediary steps between thin initial and final states tends towards infinity results in the need for a redefinition of the measure these become $\mathcal{D}q$ and $\mathcal{D}k$ which represents all possible positions and all possible momenta. However, if the Hamiltonian is of the form $\frac{p^2}{2m} - V$ then the continuum limit integration can be done for momentum and the expression for the propagator becomes.

$$\langle q_{n+1}, t_{n+1} | q_n, t_n \rangle = \int \mathcal{D}q \; e^{is} \tag{2.3.9}$$

This represents the propagator of a free theory with no interactions. However, to generalise this to an expression from a state at $t = -\infty$ to a state at $t = \infty$ to obtain the vacuum to vacuum amplitude and one of the core things we want to obtain is the expectation for field operators as has been done in 2.2.16. The general way of writing an expectation of an operator $\mathcal{O}$ is

$$\langle \mathcal{O}(q) \rangle = \int \mathcal{D}q \left(e^{is} \mathcal{O}(q) \right) \tag{2.3.10}$$

A way of doing this is through the altering of the Lagrangian by inserting some source term $j(t)$ so that $L \to L + jq$. So the expression is now a functional of the source j. The motivation for this will become clearer when discussing how n-point functions are obtained through functional differentiation 2.3.19.

$$Z[j] = \int \mathcal{D}q \; \exp\left[i \int_{-\infty}^{\infty} \mathrm{d}t \; (L + jq)\right] \tag{2.3.11}$$

The expression above is almost the final form for the vacuum to vacuum amplitude for a free field theory however there needs to be an imaginary term added to the Lagrangian to isolate the ground state energy contributions[3]. So the result is

$$Z[j] = \int \mathcal{D}q \; \exp\left[i \int_{-\infty}^{\infty} \mathrm{d}\; t(L + jq + \frac{1}{2} i\epsilon q^2)\right] \tag{2.3.12}$$

[3]The full reasoning for this involves the damping of higher order terms through a series expansion and the full rational can be found in [14]

The expression can be altered to be relevant to a field theory simply by the replacing the position $q(t)$ with the field $\phi(x^\mu)$ that has been used in the cannonical quantisation process in section 2.2. Using the Klein Gordon equation for a scalar field the generating functional is.

$$Z[j] = \int \mathcal{D}\phi \; \exp\Big[-i \int \mathrm{d}^4x \; \frac{1}{2}(\phi(\Box + m^2 - i\epsilon)\phi - \phi j)\Big] \tag{2.3.13}$$

2.3.13 is not the easiest to work with and it would be better to be able to write this in a form that contains the propagator. The expression can be obtained through functional integration [14]. As the expression is analogous to

$$\int \mathrm{d}x \; \exp\big[-\frac{1}{2}(x, Ax) + (b, x) + c\big] = \exp\big[(b, A^{-1}b) - c\big](\det A)^{-\frac{1}{2}} \tag{2.3.14}$$

making the assumption that this identity holds for the case of infinite dimensional function where the inner product $(\phi, \phi) = \int \phi^2 \,\mathrm{d}^4x$. The x is replaced by ϕ, $A = i(\Box + m^2 - i\epsilon)$, $b = -ij$ and there are no c terms. As such, the integral produces

$$Z[j] = \exp\left[\frac{i}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y \; j(x)(\Box + m^2 - i\epsilon)^{-1} j(y)\right][\det(i(\Box + m^2 - i\epsilon))]^{-\frac{1}{2}} \tag{2.3.15}$$

And using the generalisation that,

$$\int \mathcal{D}\phi \; \exp\Big[\phi A \phi\Big] = \det(A)^{-\frac{1}{2}} \tag{2.3.16}$$

and defining $(\Box + m^2 - i\epsilon)^{-1}$ to be the propagator $-D_f(x - y)$ then the generating functional:

$$Z_0[j] = \int \mathcal{D}\phi \; \exp\left[-\frac{i}{2}\int \mathrm{d}^4x \; \phi(\Box + m^2 - i\epsilon)\phi\right] \times \exp\left[-\frac{i}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y \; j(x)D_f(x-y)j(y)\right] \tag{2.3.17}$$

The first integral above can be carried out and just returns a numerical factor, N, so the generating functional.

$$Z[J] = N \exp\left[-\frac{i}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y \; j(x)D_F(x-y)j(y)\right] \tag{2.3.18}$$

The numerical factor N can be normalised out and from this expression the n-point functions can be found through functional differentiation with respect to j and then setting $j = 0$. For example, the 2- point function for the free theory simply produces the propagator between the two points. The general formula for n point functions is.

$$\left(\frac{1}{i}\right)^n \left[\frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_n)}\right] Z_0[j] = \langle 0|\phi(x_1)\ldots\phi(x_n)|0\rangle \tag{2.3.19}$$

The next thing to do is to introduce the interaction terms, one of the simplest ways to do this is to add a polynomial term into the Lagrangian $\mathcal{L} \to \mathcal{L} + \mathcal{L}_{int}$(for this example the theory considered will be ϕ^4). The interaction term is $-\frac{g}{4!}\phi^4$ Now the normalised generating functional is

$$Z[J] = \exp\left[-i\frac{g}{4!}\int \mathrm{d}^4z\ \phi^4(z)\right]\exp\left[-\frac{i}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\ j(x)D_F(x-y)j(y)\right] \quad (2.3.20)$$

By treating the interaction term as series expansion with respect to the coupling constant$g$ as we assume it is small [18], otherwise this process doesn't work as well and replacing $\phi(z)$ with $\left[\frac{1}{i}\frac{\delta}{\delta j(z)}\right]$. To first order in g the generating functional is,

$$Z[J] = \left[1-\frac{ig}{4!}\int \mathrm{d}^4z\left\{\frac{1}{i}\frac{\delta}{\delta j(z)}\right\}^4\right]\exp\left[-\frac{i}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\ j(x)D_F(x-y)j(y)\right] \quad (2.3.21)$$

It is easy to see that if $g = 0$ then this is simply the generating functional for the free theory. The way to then produce n-point functions is the same as before but carrying out the functional differentiation that results from the interaction term first. As such it is easy to see that the interaction term plays an important role in defining the resulting scattering matrix elements. The interaction term in the theory describes the types of vertices in a Feynman diagram, in the case of ϕ^4 the vertices are formed from 4 scalar fields meeting at a single point, whether it forms what is known as a fully connected diagrams, which represent the interesting parts of the scattering matrix or a disconnected diagram, which represent the vacuum corrections or free propagation. To create more complex theories with different kinds of fields then The Lagrangian can be modified to include the free Lagrangian of the free theory and any interacting terms that describe self interaction through perturbation and interaction of the multiple fields. A common example of this is Yukawa theory which describes interaction between Dirac fields (Spinor, fermionic fields) and scalar fields.

2.4 Gauge Theories and Faddeev-Popov

A specific type of field theory is a Gauge theory, so called as the fields are invariant under transformations, known as gauge transformations. These transformations occur on a local scale rather than a global transformation. For example;

$$\phi(x) \to \phi(x)e^{i\lambda(x)} \quad (2.4.1)$$

An example of where this occurs is in Quantum electrodynamics. Where the Lagrangian is given by.

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + iej^{\mu}A_{\mu} \quad (2.4.2)$$

Where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, is the Faraday tensor and the second term is the a source term. The vector field A_μ is invariant under the transformation.

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\lambda(x) \quad (2.4.3)$$

This requires the existence of a Covariant Derivative that acting on a field is,

$$D_\mu \phi = \partial_\mu \phi + icA_\mu \phi \tag{2.4.4}$$

where c is some constant. The path integral quantisation method now presents some difficulties. The generating functional for the theory is

$$Z = \int \mathcal{D}A^\mu \; \exp\left[i \int \mathrm{d}^4x \; \mathcal{L}_{QED} \right] \tag{2.4.5}$$

The problem arises from the measure $\mathcal{D}A^\mu$ as this integration over all A^μ includes those which are identical due to the invariance of the field shown in 2.4.3. This requires a term to be inserted into the Lagrangian to remove this over counting. This term is known as gauge fixing term and fixes the integral to not include the values related by the gauge transformation. The Gauge constraint imposed is $\partial_\mu A^\mu = 0$. Resulting in the new Lagrangian.

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \big(\partial_\mu A^\mu \big) \tag{2.4.6}$$

The symmetry in Quantum electrodynamics is a U(1) symmetry [19], which is an abelian group. However, there are other theories known as non-abelian Gauge theories, Or Yang-Mills Theories. The starting point is much the same, there is a vector field that is invariant under a transformation $\phi(x) \to e^{i\lambda^\alpha(x)(T_\alpha)} \phi(x)$. Which when taken to a power series expansion gives the infinitesimal transformation in the form $\phi(x) \to \phi(x) + \delta\phi(x)$, where $\delta\phi(x)$ is given by:

$$\delta\phi(x) = i\lambda^\alpha(x)(T_\alpha)\phi(x) \tag{2.4.7}$$

with the term T_α representing matrices which are the generators of the gauge group and produce the structure functions.

$$\big[T_\alpha, T_\beta \big] = f^\gamma_{\alpha\beta} T_\gamma \tag{2.4.8}$$

And λ^α are the infinitesimal parameters that likewise appear in QED. As such for some gauge field $A_\mu(x)$ the transformation given as.

$$\delta A^\alpha_\mu = \partial_\mu \lambda^\alpha + f^\alpha_{\gamma\beta} \lambda^\beta A^\gamma_\mu \tag{2.4.9}$$

This allows for the construction of a covariant derivative.

$$D_\mu \phi(x) = \partial_\mu \phi(x) - iA^\alpha_\mu T_\alpha \phi(x) \tag{2.4.10}$$

This is so that the transformation of the covariant derivative becomes a a simple gauge transformation on the whole derivative as extra terms cancel as follows.

$$\delta\big(D_\mu \phi \big) = \partial_\mu(\delta\phi) - i(\delta A^\alpha_\mu) T_\alpha \phi(x) - iA^\alpha_\mu T_\alpha (\delta\phi(x)) \tag{2.4.11}$$

The inserted transformations cancel down to simply.

$$\delta\big(D_\mu\phi\big) = i\lambda^\alpha(T_\alpha)(D_\mu\phi) \tag{2.4.12}$$

Now the commutation relation for these covariant derivatives gives:

$$[D_\mu, D_\nu] = -iG^\gamma_{\mu\nu}T_\gamma \tag{2.4.13}$$

Where the tensor $G^\gamma_{\mu\nu}$ is analogous to the Faraday tensor but includes a term that arises due to this being built from a non-abelian gauge field [14]. The tensor's general form is,

$$G^\gamma_{\mu\nu} \equiv \partial_\mu A^\gamma_\nu - \partial_\nu A^\gamma_\mu + f^\gamma_{\alpha\beta}A^\alpha_\mu A^\beta_\nu \tag{2.4.14}$$

It transforms as $\delta G^\gamma_{\mu\nu} = \lambda^\alpha f^\gamma_{\alpha\beta}G^\beta_{\mu\nu}$. This tensor is what is needed to construct a Yang-Mills theory which has the Lagrangian.

$$\mathcal{L}_{YM} = -\frac{1}{2}\mathrm{Tr}(G^2) = -\frac{1}{4}G^\alpha_{\mu\nu}G^{\alpha\mu\nu} \tag{2.4.15}$$

The quantisation of a theory like this is not as simple as QED as there needs to be a term included to fix the gauge and some extra terms that arise from the Faddeev-Popov method [20], known as ghosts which are a requirement due to the the non-abelian nature of the theory The ghosts can be thought of as negative degrees of freedom which are inserted into the theory to account for the non-physical degrees of freedom that arise from the non-abelian gauge symmetry. These ghosts are not physical fields as they violate spin statistics as they obey Fermi-Dirac statistics but are complex scalar fields, in this case [19]. The generating functional for the theory without this consideration is.

$$Z = \int \mathcal{D}A_\mu\ \exp\left[i\int \mathrm{d}^4x\Big(-\frac{1}{2}\mathrm{Tr}(G^\alpha_{\mu\nu}G^{\alpha\mu\nu}) + j^\alpha_\mu A^\alpha_\mu\Big)\right] \tag{2.4.16}$$

The choice of gauge fixing term is selected dependent on the theory. In this case a simple gauge fixing term can be used to fix $\partial_\mu A^\mu = 0$

$$\mathcal{L}_{gauge} = -e\frac{1}{2}(\partial_\mu A^\alpha\mu)^2 \tag{2.4.17}$$

Where e has no independent degrees of freedom and solving its equations of motion produces the desired constraint. The Lagrangian for the ghosts in a Yang-Mills theory follow the general form of:

$$\mathcal{L}_{ghost} = -\partial_\mu\bar{c}_\alpha\partial^\mu c^\alpha - Mf^{\alpha\beta\gamma}(\partial^\mu\bar{c}^\alpha)A^\beta_\mu c^\gamma \tag{2.4.18}$$

Where the $\bar{c}$ denotes an anti-ghost conjugate to the normal ghost. The first term in the Lagrangian acts as the kinetic term and the second describes the ghost interactions with the gauge fields through some coupling constant M. As a result of these ghosts they must also have some source terms so the full generating functional becomes.

$$Z = \int \mathcal{D}A^{\mu}\mathcal{D}c\mathcal{D}\bar{c}\ \exp\left[\int \mathrm{d}^4x\ \mathcal{L}_{YM} + \mathcal{L}_{gauge} + \mathcal{L}_{ghost}\right] \tag{2.4.19}$$

$$\times \exp\left[i\Big(\int \mathrm{d}^4x\ j^{\alpha}\mu A^{\alpha\mu} + c^{\alpha}\bar{\xi}^{\alpha} + \bar{c}^{\alpha}\xi^{\alpha}\Big)\right] \tag{2.4.20}$$

The ξ terms represent the sources for the ghosts and anti-ghosts. This approach will be needed again, and covered in more detail, when encountering the path-integral formulation of String Theory.

3 String Theory

String theory was originally a theory proposed to describe the physics of the strong nuclear force. Now Quantum Chromodynamics[4] is the more widely accepted and String Theory now serves as a way of producing scattering amplitudes that have the same form as supersymmetric [5] Yang-mills theories [23] and that supersymmetric string theories are dual to Supersymmetric Yang-Mills theory through Ads/CFT correspondance [24]. The starting point for studying String theory is to extend the picture of a particle to that of a string. Which is to say if you start with a relativistic point particle travelling through space-time, said particle traces out a path. The action of that particle can then be seen to be an integration over the line element and by parametrisation of the line element with respect to the proper time the action becomes.

$$S_p = -m\int \mathrm{d}\tau\ \sqrt{-\eta_{\mu\nu}\dot{x^{\mu}}\dot{x^{\nu}}} = -m\int \mathrm{d}\tau\ \sqrt{-\dot{x}^2} \tag{3.0.1}$$

Where $\dot{x}^2$ is the derivative with respect to proper time and x^{μ} is parametrised by τ This action is insufficient as it doesn't allow for massless particles so there is the need for the introduction of an auxiliary field in this case known as an einbien $e(\tau)$.

$$S = \frac{1}{2}\int \mathrm{d}\tau\ \left(e^{-1}\dot{x}^2 - em^2\right) \tag{3.0.2}$$

[4]QCD is a Yang-Mills Theory. Yang-Mills theories currently underpin our understanding of the standard model and provides unification of the Strong force, Weak force and electromagnetism given by the symmetry group SU(2) x U(1) x SU(3)

[5]Supersymmetry is a requirement for string theory to describe any theory with fermions in them as the fields that form the basis of string theory are scalar fieldss with Bosonic statistics so when quantised only represent Bosonic states. However, as supersymmetry is not discussed in the research of this thesis, it will not be discussed here, for a comprehensive review of supersymmetry see [21] and for its application to string theory see [22]

Solving the equation of motion for the einbien leads to the solution that $e = \frac{\sqrt{-\dot{x}^2}}{m}$ if $m \neq 0$ By fixing this gauge and when $m = 0$, $e = 1$ the einbien has no indipendent degrees of freedom [22]. To make this into a string theory some alterations must be made. First it is important to outline the coordinates that will be used as this will no longer be a world line theory, rather a Worldsheet theory the parametrisation with τ remains but there needs to be an extra degree of freedom σ that for an open string is constrained by $\sigma = 0, \ldots, l$ .l being the string length, and for a closed string $\sigma = \sigma + 2\pi$. Now the integrand of the theory must represent the area element of the worldsheet. To do so impose the metric g_{ab} such that.

$$g_{ab} = \eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \tag{3.0.3}$$

$$g = \det(g_{ab}) \tag{3.0.4}$$

The differentials ∂_a represent differentiation with respect to σ^a where $a = 0, 1$ and $\sigma^0 = \tau$ and $\sigma^1 = \sigma$. $X^\mu(\sigma^a)$ is the function representing the embedding of the worldsheet onto space-time. The action for this string is.[6]

$$S = \frac{-T}{2} \int \mathrm{d}\sigma^a \sqrt{-g} \tag{3.0.5}$$

The factor T represents the String Tension and is equal to $\frac{1}{2\pi\alpha'}$ [25]. The significance of this lies in the term α' which corresponds to the Regge slope. [7]The action is known as the Nambu-Goto action for the string and from it the equations of motion of the string can be found. There are some things that can be done to make the action easier to deal with, by removing the square root, in a similar method to how the action for a relativistic point particle is altered, by introducing an auxiliary field. However, in this case the term introduced is the worldsheet metric $h^{\alpha\beta}(\tau, \sigma)$ and the action becomes.

$$S = \frac{-T}{2} \int \mathrm{d}^2\sigma \sqrt{-h} h^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \tag{3.0.6}$$

Where $h = det(h_{\alpha\beta})$. As would be expected the Polyakov action has some symmetries that have to be considered. The first of which is Poincaré invariance which is a global symmetry.

$$X^\mu = L^\mu_\nu X^\nu + a^\mu \tag{3.0.7}$$

L^μ_ν is the tensor for the Lorentz transformation. There is also the Diffeomorphism invariance, which means that the fields are invariant under the redefinition of coordinates. For X^μ the transformation is.

$$X^\mu(\sigma) \Rightarrow X'^\mu(\sigma') \equiv X^\mu(\sigma) \tag{3.0.8}$$

[6]It would be possible to further extend this action for higher dimensional objects by increasing the index $a = 0, \ldots, n$ where n is the dimensionality of the object in question.

[7]The Regge slope is obtained from the plotting spin against mass squared

As the metric is also dependent on the worldsheet coordinates this must also be invariant. A 2d metric diffeomorphism is given as.

$$h_{\alpha\beta}(\sigma) \Rightarrow h'_{\alpha\beta}(\sigma') = \partial'_\alpha \sigma^\lambda \partial'_\beta \sigma^\epsilon h_{\lambda\epsilon}(\sigma) \tag{3.0.9}$$

The derivative is $\partial'_\alpha = \frac{\partial}{\partial \sigma'^\alpha}$. There is a third and final symmetry that only affects the worldsheet and it is a local rescaling of the metric, known as Weyl invariance.

$$h_{\alpha\beta} \Rightarrow e^{2\omega(\sigma)} h_{\alpha\beta} \tag{3.0.10}$$

For suitably small $\omega(\sigma)$ the transformation can be seen to be $\delta h_{\alpha\beta} = 2\omega(\sigma) h_{\alpha\beta}$. The combination of both local rescaling of the metric (Weyl invariance) and combined with the diffeomorphism invariance means that in 2 dimensions we can define the fiducial metric $\hat{h}$ which can locally take any form [26]. The choice can be made for the Polyakov string that the worldsheet metric is the Minkowski metric.

$$S = \frac{-T}{2} \int \mathrm{d}^2\sigma \; \partial_\alpha X^\mu \partial^\alpha X_\mu \tag{3.0.11}$$

Which is the same as the action for a free scalar field in 2 dimensions. Bearing in mind that after having made the choice for the worldsheet metric the choice still has to satisfy the the equations of motion for the worldsheet metric. Which is

$$\delta S = \frac{-T}{2} \int \mathrm{d}^2\sigma \; \delta h^{\alpha\beta} \big(\sqrt{-h} \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} \sqrt{-h} h_{\alpha\beta} h^{\lambda\epsilon} \partial_\lambda X^\mu \partial_\epsilon X_\mu \big) = 0 \tag{3.0.12}$$

This expression is obtained by using the identity for the variation of a metric determinant $\delta(h) = h h^{\alpha\beta} \delta h_{\alpha\beta}$, in this case the transformation needed is $\delta\sqrt{-h} = \frac{1}{2}\sqrt{-h} h^{\alpha\beta} \delta h_{\alpha\beta}$. The Stress tensor for the theory can be found by multiplying out the string tension and $\sqrt{-h}$ and via functional differentiation of the action with respect to the worldsheet metric.

$$T^{ab} = \frac{-2}{T} \frac{1}{\sqrt{-h}} \frac{\delta S}{\delta h_{\alpha\beta}} \tag{3.0.13}$$

$$= \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} h_{\alpha\beta} \big(\partial_\gamma X^\mu \partial^\gamma X_\mu \big) \tag{3.0.14}$$

3.1 Path Integral Quantisation in String theory

Above is the description of the classical relativistic string. However, String theory is used to produce Gauge theories and as such it is important to understand how the theory is quantised. The Path integral method is much the same as in quantum field theory in so much as the starting point is the integral over all possible fields.

$$\int \mathcal{D}X \mathcal{D}h \; \exp(-S) \tag{3.1.1}$$

However as a result of the gauge fixing discussed prior this integration results in an over counting. So after the fixing of the metric, usually to a flat metric, the integration over the metric can be removed as the coordinates are invariant and so is the local scaling under the Weyl invariance. A common way of writing the gauge fixing is as dividing over the volume of the gauge group.

$$\int \frac{\mathcal{D}X\mathcal{D}h}{\mathrm{Vol}_{\mathrm{Diff}\times\mathrm{Weyl}}}\ \exp(-S) \tag{3.1.2}$$

After this consideration the only measure that is integrated over is $\mathcal{D}X$ This is important to isolate the physical configurations not related under gauge transformations. As has been said in the context of Yang-Mills the method for producing a sensible integral for the generating functional is the Faddeev-Popov method. To do so consider the overall gauge transformation of both Diffeomorphism and Weyl.

$$h_{\alpha\beta}(\sigma) = h^{(g)}_{\alpha\beta}(\sigma') = e^{2\omega(\sigma)}\partial'_\alpha\sigma^\lambda\partial'_\beta\sigma^\epsilon h_{\lambda\epsilon}(\sigma) \tag{3.1.3}$$

Where the combined transformation is denoted by the super script (g) as in $(g) : h \to h^{(g)}$.The Faddeev-Popov determinant is given as.

$$1 = \Delta_{FP}(h)\int \mathcal{D}(g)\ \delta(h - \hat{h}^{(g)}) \tag{3.1.4}$$

This integral is done over a the measure $\mathcal{D}(g)$ which is the measure of the whole gauge group. Note the change to the fiducial metric $\hat{h}$ this will then change the the dependence of the generating functional to the choice of the fiducial metric. Inserting 3.1.4 into 3.1.2 results in and integrating over the metric to solve the delta function.

$$Z[\hat{h}] = \int \frac{\mathcal{D}(g)\mathcal{D}X}{\mathrm{Vol}_{\mathrm{Diff}\times\mathrm{Weyl}}}\ \Delta_{FP}[\hat{h}]\exp(-S) \tag{3.1.5}$$

As now the only gauge dependent part of the integral is the measure for the gauge group, integrating over this produces the volume of the gauge group which cancels with the denominator in the generating functional and produces.

$$Z[\hat{h}] = \int \mathcal{D}X\ \Delta_{FP}(\hat{h})\exp(-S) \tag{3.1.6}$$

Now all that remains to be done is to pin down what the Faddeev-Poppov determinant is. The way to do this is to look at the combined diffeomorphism and Weyl transformations as infinitesimal transformations. The Weyl transformation is parametrised by ω and in the diffeomorphism the infinitesimal variation of the coordinate is parametrised by some field $\delta\sigma^\alpha = c^\alpha(\sigma)$. The infinitesimal transformation is:

$$\delta\hat{h}_{\alpha\beta} = 2\omega\hat{h}_{\alpha\beta} + D_\alpha c_\beta + D_\beta c_\alpha \tag{3.1.7}$$

Where D_α is the covariant derivative required for the infinitesimal transformation $D_\alpha c_\beta = \partial_\alpha c_\beta - \Gamma^\gamma_{\alpha\beta} c_\gamma$. The tensor in the second term for the covariant derivative is the Levi-Civita connection to the worldsheet metric.

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} h^{\gamma\delta}(\partial_\alpha h_{\beta\delta} + \partial_\beta h_{\delta\alpha} - \partial_\delta h_{\alpha\beta})$$

The expression for the Faddeev-Popov determinant 3.1.4 can be rearranged to give an expression for the inverse determinant. Then inserting the infinitesimal transformation as the delta function and subsequently changing the measure of integration to be over the parametrisation of the transformation gives the inverse determinant as.

$$\Delta^{-1}_{FP}[\hat{h}] = \int \mathcal{D}\omega \mathcal{D}c \ \delta(2\omega \hat{h}_{\alpha\beta} + D_\alpha c_\beta + D_\beta c_\alpha) \tag{3.1.8}$$

Writing the delta functional in it's integral form gives the expression[25].

$$\delta(2\omega \hat{h}_{\alpha\beta} + D_\alpha c_\beta + D_\beta c_\alpha) = \int \mathrm{d}b \ \exp\left(2\pi i \int \mathrm{d}^2\sigma \ \sqrt{\hat{h}} b^{\alpha\beta}(2\omega \hat{h}_{\alpha\beta} + D_\alpha c_\beta + D_\beta c_\alpha)\right) \tag{3.1.9}$$

Inserting 3.1.9 into 3.1.8 and integrating over the Weyl parametrisation requires that $b^{\alpha\beta}\hat{h}_{\alpha\beta} = 0$ meaning $b^{\alpha\beta}$ is a symmetric traceless matrix. This leads to

$$\Delta^{-1}_{FP}[\hat{h}] = \int \mathcal{D}b\mathcal{D}c \ \exp\left[4\pi i \int \mathrm{d}^2\sigma \ \sqrt{\hat{h}} b^{\alpha\beta} D_\alpha c_\beta\right] \tag{3.1.10}$$

Taking the inverse of this the determinant looks like this.

$$\Delta_{FP}[\hat{h}] = \int \mathcal{D}b\mathcal{D}c \ \exp\left[\frac{i}{4\pi} \int \mathrm{d}^2\sigma \ \sqrt{\hat{h}} b_{\alpha\beta} D^\alpha c^\beta\right] \tag{3.1.11}$$

The factor of $\frac{1}{4\pi}$ can be changed to affect the normalisation by rescaling b and c. Now going back to the generating functional 3.1.6 and inserting the determinant 3.1.11.

$$Z[\hat{h}] = \int \mathcal{D}X\mathcal{D}b\mathcal{D}c \ \exp\Big(- S_p - S_{ghost}\Big) \tag{3.1.12}$$

S_p is the Polyakov action and S_{ghost} represents the ghost action and is the exponent of the Faddeev-Poppov determinant [22]. The subject of ghosts and their action will feature more in the discussion of String Theory as a conformal field theory.

3.2 String Theory as a Conformal Field Theory and the operator product expansion

Writing the Polyakov action out with the choice of a euclidean world sheet gives

$$S = \frac{T}{2} \int \mathrm{d}^2\sigma \ \partial_1 \ X^\mu \partial_1 X_\mu + \partial_2 X^\mu \partial_2 X_\mu. \tag{3.2.1}$$

This can be written more simply with the use of complex coordinates $z = \sigma^1 + i\sigma^2$ and $\bar{z} = \sigma^1 - i\sigma^2$. This requires the defining of integrals with respect to these coordinates which are

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_1 - i\partial_2), \tag{3.2.2}$$

$$\bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2). \tag{3.2.3}$$

The integration also then needs to change to represent the change of coordinates $\mathrm{d}z\,\mathrm{d}\bar{z}z = 2\,\mathrm{d}\sigma^1\,\mathrm{d}\sigma^2$. So the new action is

$$S = \frac{1}{2\pi\alpha'}\int \mathrm{d}z\,\mathrm{d}\bar{z}\;\partial X \cdot \bar{\partial} X, \tag{3.2.4}$$

and as one would expect the equations of motion are

$$\partial(\bar{\partial} X^\mu) = \bar{\partial}(\partial X^\mu) = 0. \tag{3.2.5}$$

From this it can be seen that ∂X^μ is holomorphic and $\bar{\partial} X^\mu$ is antiholomorphic. Converting this to example to a Minkowski world sheet metric is a simple matter of imposing the relation $\sigma^2 = i\sigma^0$ then holomorphic functions are only dependent of $\sigma^0 - \sigma^1$ and antiholomorphic is only dependent on $\sigma^0 + \sigma^1$ [8]. Another Useful part of the theory is to look at how the stress tensor is written in complex coordinates. It is derived in the same way as before and is traceless but becomes

$$T = -\frac{1}{\alpha'}\partial X \partial X \tag{3.2.6}$$

$$\bar{T} = -\frac{1}{\alpha'}\bar{\partial} X \bar{\partial} X \tag{3.2.7}$$

and satisfies $\bar{\partial} T_z = \partial T_{\bar{z}\bar{z}} = 0$. Which means $T \equiv T_{zz}$ is holomorphic and $\bar{T} \equiv T_{\bar{z}\bar{z}}$ is antiholomorphic. After producing the action for the conformal field theory the next thing to do is to quantise and the form of the generating functional stays the same as it has before. The difference in this case is that it is useful to look at what the expectation of a product of local operators is. Defining the expectation of some functional $A[X]$ as was given in 2.3.10

$$\langle A[X] \rangle = \int \mathcal{D}X\;\; \exp(-S) A[X] \tag{3.2.8}$$

elaborating on this and writing the functional $A[x]$ as a string of some local operators on the world sheet the expectation is:

$$\langle \mathcal{O}_1 \mathcal{O}_2 \ldots \mathcal{O}_n \rangle = \int \mathcal{D}X\;\; \exp(-S) \mathcal{O}_{a_1} \mathcal{O}_{a_2} \cdots \mathcal{O}_{a_n} \tag{3.2.9}$$

[8] It is from these definitions that holomorphic functions are often described as left moving and antiholomorphic as right moving

(This is the form that will be used when discussing vertex operators and scattering amplitudes).$\mathcal{O}_a$ describes a local set of operators related through

$$\mathcal{O}_a(z,\bar{z})\mathcal{O}_b(z',\bar{z}') = C^c_{ab}(z-z',\bar{z}-\bar{z}')\mathcal{O}_c(z',\bar{z}') \tag{3.2.10}$$

This is what is known as the operator product expansion [25], [27] which is the describes what happens when there are two operators within close proximity of each other. C^c_{ab} is a set of functions dependent on the separation of the two operators[9], these are known as the coefficient functions. It then stands to reason that the insertion of an expectation of a set of operators can be written in a similar way replacing $\mathcal{O}_c$ with $\langle\mathcal{O}_c\rangle$. Now that the bare bones of the theory has been described in complex coordinates how ghosts enter into a CFT can be looked at. By inspection of the ghost action given in 3.1.11 redefining the coordinate system to a complex coordinates gives the action

$$S_{ghost}\frac{1}{2\pi}\int \mathrm{d}z\,\mathrm{d}\bar{z}\,\left(b\bar{\partial}c + \bar{b}\partial\bar{c}\right) \tag{3.2.11}$$

Where b and c are holomorphic and $\bar{b}$ and $\bar{c}$ are antiholomorphic and are short hand ways of writing b_{zz}, c^z ,$b_{\bar{z}\bar{z}}$ and $c^{\bar{z}}$. The Stress tensor for the ghosts can then be found for the ghost action as

$$T = 2\partial cb + c\partial b \tag{3.2.12}$$

$$\bar{T} = 2\bar{\partial}\bar{c}\bar{b} + \bar{c}\bar{\partial}\bar{b} \tag{3.2.13}$$

Now that there is an action for the ghosts as a Conformal field theory it is important to define the operator product expansions of the two ghosts. The way this is done is by looking at a total a functional derivative of the expectation of one fields. As this results in a path integral of a total derivative the overall result must be zero so it leads to an easily manipulable expression. Looking at the case of

$$\frac{\delta}{\delta c(z)}\langle c(z')\rangle = \int \mathcal{D}b\mathcal{D}c\,\frac{\delta}{\delta c(z)}\big(\exp(-S_{ghost})c(z')\big) \tag{3.2.14}$$

This results in:

$$-\frac{1}{2\pi}\bar{\partial}b(z)c(z') + \delta(z-z') = 0 \tag{3.2.15}$$

Rearranging and using the formula:

$$\bar{\partial}\left(\frac{1}{z}\right) = 2\pi\delta(z,\bar{z}) \tag{3.2.16}$$

doing this again for the expectation of a b ghost results in a similar expression and helps produce the OPEs

$$b(z)c(z') = \frac{1}{z-z'} \tag{3.2.17}$$

$$c(z')b(z) = \frac{1}{z'-z} \tag{3.2.18}$$

[9] This is assuming that the operators are time ordered, or in some cases radially ordered

It is also at this point worth noting that $c(z)c(z') = 0$ and $b(z)b(z') = 0$ due to them being grassmann variables. The same method can be used to find the propagator for the scalar field X [25]in this theory which for the holomorphic part results from the OPE:

$$X(z)X(z') = -\frac{\alpha'}{2}\ln(z - z') \tag{3.2.19}$$

as X enters into the action only as a derivative term it is more useful to treat the derivative of X as an operator and the OPE can be obtained in the same way as before or mores imply by differentiating the OPE with respect to z and z' to obtain.

$$\partial X(z)\partial X(z') = -\frac{\alpha'}{2}\frac{1}{(z - z')^2} \tag{3.2.20}$$

As can be seen from these examples the result of the OPE changes depending on the ordering of the operators. Using this OPE, radial ordering[28] [10] and Wick's theorem, the OPE of the two stress tensors can be found.

$$R[T(z)T(z')] = \frac{D}{2(z - z')^4} + \frac{2T(z')}{(z - z')^2} + \frac{\partial T(z')}{(z - z')} \tag{3.2.22}$$

The above result has been expanded out by a Taylor expansion and then be used to produce the Virasoro algebra [22], this will not be covered in this thesis. The D term has arisen from the multiple metric terms that have been omitted from previous expressions of the stress tensor for the sake of tidiness but it is important to remember that they are there and if they take the form of the Minkowski metric then D has arisen from $\eta_{\mu\nu}\eta_{\lambda\rho}\eta^{\mu\lambda}\eta^{\nu\rho} = \delta^{\lambda}_{\nu}\delta^{\lambda}_{\nu}$.

3.3 BRST Formalism

The method of Quantisation that is of interest to this thesis is known as BRST quantisation which is based around having a theory with a symmetry that the constraints follow the form

$$\{G^a, G^b\} = f^{ab}_c G^c \tag{3.3.1}$$

Where f^{ab}_c are the structure constants of the closed Lie-Algebra and satisfies the Jacobi identity [22]. The process also requires a gauge fixing condition, this is usually done by setting a function or functional of the matter fields to be zero.

$$G^a(X^\mu) = 0 \tag{3.3.2}$$

[10]Radial ordering fixes the order of operators in an expression depending on the which sits at a larger complex radius eg.

$$R[\mathcal{O}(z)\mathcal{P}(z')] = \begin{cases} \mathcal{O}(z)\mathcal{P}(z') & \text{if } |z| > |z'| \\ (-1)\mathcal{P}(z')\mathcal{O}(z) & \text{if } |z'| > |z| \end{cases} \tag{3.2.21}$$

The (-1) only appears if the operators are fermionic.

As this technique being described is obviously requires some non-abelian gauge theory as has been previously discussed the standard procedure is the inclusion of ghost fields b and c and are defined ass conjugate to each other. From this there exists a Classical BRST charge, Q can be constructed from any gauge symmetry [22]. The BRST charge is the generator of the BRST transformations under which all fields in the theory transform. The classical BRST charge must be nilpotent i.e.

$$\{Q, Q\} = 0 \tag{3.3.3}$$

The BRST charge for a theory that can be described as above takes the form.

$$c_a G^a - \frac{1}{2} f^{ab} c_a c_b b^c \tag{3.3.4}$$

One other thing to keep in mind is the BRST charge has ghost number +1 and will always increase the ghost number by +1 when acting on a field [11] When looking at the Bosonic String in the complex coordinate space that was the topic of the previous subsection the the BRST charge is

$$Q = \oint \mathrm{d}z \quad cT_m + \frac{1}{2} cT_{gh} - \oint \mathrm{d}\bar{z} \ \bar{c}\bar{T}_m + \frac{1}{2}\bar{c}\bar{T}_{gh} \tag{3.3.5}$$

Or when written fully

$$Q = \oint \mathrm{d}z \, \mathrm{d}\bar{z} \left[-\frac{1}{\alpha'} c\partial X \partial X + \frac{1}{\alpha'} \bar{c}\bar{\partial} X \bar{\partial} X + bc\partial c - \bar{b}\bar{c}\bar{\partial}\bar{c} \right] \tag{3.3.6}$$

Normally there is a 2nd order differential of the c and $\bar{c}$ ghosts in the integrand but I have omitted it in this case. Acting this charge and using the OPE relations the BRST transformations are found. The charge more commonly considered is the purely holomorphic version[25]. One interesting result of the BRST charge is that it can be used to reproduce the conformal anomaly from the OPE of two stress tensors 3.2.22 but in this case the same can be done for the Ghost Stress tensor and can be compared.

$$R[T_{gh}(z)T_{gh}(z')] = -\frac{-13}{(z-z')^4} + \frac{2T_{gh}}{(z-z')^2} + \frac{\partial_{z'} T_g h(z')}{z - z'} \tag{3.3.7}$$

Equating this to the result 3.2.22 the value for D is 26 and is a requirement for the BRST charge to be nilpotent[22]. The reason for this is because the only terms that have a factor of $\frac{1}{(z-z')^4}$ are the first terms in 3.2.22 and 3.3.7 which must sum to zero for Q to be nilpotent. At this point it is worth defining the terms "BRST exact" and "BRST closed". Closed states are those which when the BRST charge acts on them then they reduce to zero, whereas, a BRST exact state is obtained

[11] Ghost number is a property of ghost fields: c has ghost number +1 and b has ghost number -1

through the action of the BRST charge. The BRST charge can be used to produce the Virasoro algebra of String theory, however, this will not be covered here for the sake of brevity. Overall the BRST formalsim provides an elegant way of encoding all of the symmetries of a theory into one term and how that can be inserted into the action of a theory and serve to gauge fix the theory in section 8.

3.4 Scattering Amplitudes

As is the case in QFT the computation of S-matrix elements are important as they represent the interactions of the theory. So far the only parts of string theory that has been shown is the free theory. The most common example considered is the 4-point S-matrix. The world sheet picture of this can be modelled as a sphere with 4 insertion points for the strings. at each of these insertion points are what are known as vertex operators V_{α_i}, these arise from conformal field theory and define a local operator that represents a state.In the case that will be considered here the topology of the worldsheet can be considered to be a sphere and the Vertex operators are local operators on the worldsheet. The choice of these vertex operators defines what states are present in the String scattering. However, to consider the full S-matrix all possible topologies of the world sheet must be considered[25]. When discussed above the shape mentioned was a sphere, however, it is possible for the shape to also be a torus or any other higher genus shape. So a way to include this is to make add a term to the action.

$$S = S_{poly} + \lambda \frac{1}{4\pi} \int \mathrm{d}^2\sigma \, \sqrt{h} R \tag{3.4.1}$$

λ is a coupling constant weighing the contribution of the integral that has been added. Which is the 2 dimensional Einstein-Hilbert action where R is the Ricci scalar of the world sheet metric[26]. In 2-dimensions this integral is very different to the 4 dimensional case, here it simply serves to count the genus of the worldsheet. Representing the integral as Ω then the value changes with respect to the genus as,

$$\Omega = 2(1 - g), \tag{3.4.2}$$

and g is the number of holes in the world sheet or in other words, its genus. Now that this has been outlined the matter of the coupling constant needs to be considered to understand how much higher order genus world sheets contribute to the S-matrix. As the coupling constant λ [12] will enter into the partition function as an exponential then it can be written as a new constant $g_s = e^{\lambda}$ This can now be made analogous to perturbation theory in QFT. The expectation for n Vertex operators on spherical

[12] This is actually represents the term known as the dilaton [25] which is a scalar field that plays an important part in string theory

world sheet is

$$\langle V_{\alpha_1} \cdots V_{\alpha_n} \rangle = \int \mathcal{D}X \ \exp\left[-\frac{1}{2\pi\alpha'} \int \mathrm{d}z\, \mathrm{d}\bar{z}\ \partial X \bar{\partial} X \right] \prod_{i=1}^{n} V_{\alpha_i}(p_i) \tag{3.4.3}$$

There is a subtlety hidden in this expression and it lies in the gauge fixing. As the geometry of the world sheet is now a sphere, one might think that the fixing of the metric to being euclidean no longer works. However due to the Weyl and diffeomorphisms any metric on a sphere can locally be mapped to a flat plane and therefore the gauge fixing holds. The general form for the scattering amplitude is

$$\mathcal{A}(\alpha_i, p_i) = \frac{1}{\mathrm{Vol}} \sum_{topologies} g_s^{-\Omega} \langle V_{\alpha_1} \cdots V_{\alpha_n} \rangle \tag{3.4.4}$$

The volume term that has appeared in this expression requires some explaining, even though the action has been gauge fixed to a choice of metric there is a remnant symmetry arising, with the structure $\mathbf{SL}(2;\mathbf{C})$[13] which is a result of the infinite dimensional Virasoro algebra [26]. This symmetry allows the movement of any three points on a sphere to three other points and once that is done the gauge is fixed. The simplest Amplitude to compute is the 4 Tachyon Tree level (genus zero world sheet) scattering amplitude. The vertex operators for the Tachyon[14] are [26]

$$V(p_i) = g_s \int \mathrm{d}z\, \mathrm{d}\bar{z}\ e^{ip_i X} \tag{3.4.5}$$

This then allows the expression for the amplitude to be written as

$$\mathcal{A}^{(4)} = \frac{g_s^2}{\mathrm{Vol}(\mathbf{SL}(2;\mathbf{C}))} \int \prod_{i=1}^{4} \mathrm{d}z_i\, \mathrm{d}\bar{z}_i \int \mathcal{D}X \ \exp\left[-\frac{1}{2\pi\alpha'} \int \mathrm{d}z\, \mathrm{d}\bar{z}\, \partial X \bar{\partial} X \right] \exp\left[i \sum_{i=1}^{4} p_i X \right] \tag{3.4.6}$$

Although this can seem daunting at first really this can be seen to be a Gaussian integral that occur in Quantum field theory with the free section of the theory plus a source term which can then be written with respect to the propagator of the scalar fields[15], which have already been derived 3.2.19, and the source terms which will arise from the vertex operator. So, in actuality the amplitude looks like this

$$\mathcal{A}^{(4)} = \frac{g_s^2}{\mathrm{Vol}(\mathbf{SL}(2;\mathbf{C}))} \int \prod_{i=1}^{4} \mathrm{d}z_i\, \mathrm{d}\bar{z}_i \ \exp\left[\frac{\pi\alpha'}{2} \int \mathrm{d}z\, \mathrm{d}\bar{z}\, \mathrm{d}z'\, \mathrm{d}\bar{z}'\ j(z,\bar{z})(P) j(z',\bar{z}') \right] \tag{3.4.7}$$

[13]The **SL** stands for "Special Linear" and it is a group of complex 2X2 matrices with unit determinant

[14]This is derived from the State operator mapping in conformal field theory and come from the mode expansions of the string to contour integrals

[15]It will look different here as the propagator being considered will be considering both coordinates z and $\bar{z}$ whereas before the only consideration was for the holomorphic case

where (P) is the propagator of the scalar fields X which is

$$(P) = \frac{1}{2\pi} \ln |z - z'|^2 \tag{3.4.8}$$

From the standard way of inserting the source term $(ij \cdot X)$ and comparing with the vertex operator 3.4.5 the source term can be found to be

$$j(z, \bar{z}) = \sum_{i=1}^{4} p_i \delta(z - z_i, \bar{z} - \bar{z}_i) \tag{3.4.9}$$

This makes the amplitude simpler to compute

$$\mathcal{A}^{(4)} = \frac{g_s^2}{\mathrm{Vol}(\mathbf{SL}(2;\mathbf{C}))} \delta^{26}\left(\sum_i p_i\right) \int \prod_{i=1}^{4} \mathrm{d}z_i \, \mathrm{d}\bar{z}_i \;\; \exp\left[\frac{\alpha'}{2} \sum_{j,k} p_j p_k \ln |z_j - z_k|\right] \tag{3.4.10}$$

The delta term arises to encode the momentum conservation and arises as the form of the source term insertion is the Fourier transform of the delta function. One last thing to note is that normal ordering will remove the terms where $j = k$. The final thing that needs to be done is simplifying the exponential and gauge fix by moving three of the insertion points to $z_1 = \infty$, $z_2 = 0$, $z_3 = z$, $z_4 = 1$ [25]. The choice of these points is for simplicity of calculation and any points could have been chosen, the result of doing this and only having one point which isn't fixed there is only one integral to do.

$$\mathcal{A}^{(4)} = g_s^2 \delta^{26}\left(\sum_i p_i\right) \int \; \mathrm{d}z \, \mathrm{d}\bar{z} \; |z|^{\alpha' p_2 p_3} |1 - z|^{\alpha' p_3 p_4} \tag{3.4.11}$$

After this there is the matter of assigning which are the incoming and outgoing states and setting the value of α'. The z integration can be turned into a Gaussian integral through subbing in:

$$|z|^{-a} = \frac{1}{\Gamma(\frac{a}{2})} \int \; \mathrm{d}x \; x^{\frac{a}{2}} e^{-x|z|^2} \tag{3.4.12}$$

and the same can be done for the $|1 - z|$.This can be solved using the Euler-beta function. Then defining the Mandelstam variables [16] as $s = -(p_1 + p_2)^2 = -(p_3 + p_4)^2$, $t = -(p_1 + p_3)^2 = -(p_4 + p_2)^2$ and $u = -(p_1 + p_4)^2 = -(p_3 - p_2)^2$ and setting the value of $\alpha' = 1/2$ then the result is.

$$\mathcal{A}^4 = g_s^2 \delta^{26}\left(\sum_i p_i\right) \frac{\Gamma(-1 - \frac{s}{8})\Gamma(-1 - \frac{t}{8})\Gamma(-1 - \frac{u}{8})}{\Gamma(2 + \frac{s}{8})\Gamma(2 + \frac{t}{8})\Gamma(2 + \frac{u}{8})} \tag{3.4.13}$$

[16] Mandelstam variables are a simple way to represent the ingoing and outgoing momenta in combined variables that sum to give the sum of masses squared, which for Tachyons is negative

where the Gamma functions are a shorthand way of writing the integral

$$\Gamma(a) = \int_0^\infty \mathrm{d}x \; x^{a-1} e^- x \tag{3.4.14}$$

This is a short introduction to the method of computing scattering amplitudes with one of the simplest examples that has a nice result. For more examples and a more rigorous treatments of scattering amplitudes see [25] and [22] .Scattering amplitudes will be revisited in the context of the Ambitwistor string in section 4.2.

4 Twistor Space and Twistor String Theory

Twistor string theory was first fully explored by Witten in 2003. Since then there have been many explorations into String theory in Twistor space as it appears to provide an alternate framework for producing gauge theories and scattering amplitudes [3] [29] [2] [30] [8] [31] and [32] are just a few examples. Twistor Space/ Twistor theory was produced by Roger Penrose as an attempt to provide a way to quantize gravity [33][34] [35]. The core concept of Twistor theory is that points in space and time are secondary constructs, formed from Twistors. That is to say that a single Twistor (light ray) represents a single point in Twitsor Space and that all Twisors that intersect a single spacetime point form the points on the surface of a Reimann Sphere $\mathbb{CP}^1$.

4.1 An Introduction to 4 Dimensional Twistor Space

The most well understood form of Twistor space is the 4 dimensional case, which I will explore here to give some context. However, it is important to note that in the following research sections we will be looking at theories in 10 dimensional Ambitwistor space which is still an area of ongoing research and the construction of Ambitwistor space is much more complicated when not in 4 Dimensions

This model of Twistor space is constructed by an from 4 dimensional Minkowski space-time. Then the Twistor Z^α where $\alpha = 0, 1, 2, 3$ where each component of can be related to the real space-time coordinates $t, x, y$and z through the incidence relation given by Penrose.

$$\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix} \tag{4.1.1}$$

The sign convention in the matrix above is a result of the choice of signature for the 4 dimensional Minkowski space. From this there exists a case where the product of a Twistor and a dual Twistor (For a definition of a dual Twistor see 4.1.11 for the representation in Spinor variables and compare to the expression for a Twsitor 4.1.6) is zero, in the case where they arise from a null geodesic.

A dual Twistor, in flat complex spacetime $\mathbb{C}^4$, can be defined as the complex conjugate of a Twistor with components such that $\bar{Z}_0 = \bar{Z}^2, \bar{Z}_1 = \bar{Z}^3, \bar{Z}_2 = \bar{Z}^0, \bar{Z}_3 = \bar{Z}^1$. Defining this also fixes the signature of the spacetime metric.The spacetime points incident with null Twistors can be seen to fall along null lines in spacetime or light rays. The space of these Null Twistors is what Penrose originally described as Twistor Space . According to Penrose [33] non-null twistors they can be considered to represent classical massless particles and as a result of this interpretation it defines constraints on the parameter of momentum, p^μ

$$p^\mu p_\mu = 0 \tag{4.1.2}$$

and are moving forward in time i.e.

$$p^0 > 0 \tag{4.1.3}$$

A useful way of representing Twistors is as 2 Spinor parts, λ_a and μ^a. From the starting point of 4 dimensional complex spacetime $\mathbb{C}^4$ with standard coordinates x^μ then through the introduction of λ_a as spinor degrees of freedom the space is elevated to 6 dimensional complex space $\mathbb{C}^6$. This can then be reduced by the introduction of the incidence relation:

$$\mu^a = x^{a\dot{a}}\lambda_a \tag{4.1.4}$$

$$x^{a\dot{a}} = x^\mu \sigma_\mu^{a\dot{a}} \tag{4.1.5}$$

The Spinors being defined here μ^a and λ_a that their values from 4.1.1 as

$$Z^0 = \mu^0,\ \ Z^1 = \mu^1,\ \ Z^2 = \lambda_{\dot{0}} \text{ and } Z^3 = \lambda_{\dot{1}}. \tag{4.1.6}$$

The indices a and $\dot{a}$ take the values 0 or 1 and $\dot{a}$ represent spinors of opposite chrality to those indexed by a where $\sigma_\mu^{a\dot{a}}$ are the 2x2 Pauli spin matrices that map spinors of opposite chirality to each other and are obtained from the Dirac gamma matrices can be used to write x^μ as a bi-spinor, $x^{a\dot{a}}$ .With these incidence relations imposed we can see that the space is reduced to a 3 dimensional complex projective space $\mathbb{CP}^3$ when.

$$t \in \mathbb{C}^4 \tag{4.1.7}$$

$$Z^\alpha \sim tZ^\alpha \tag{4.1.8}$$

if we define the Twistors as follows.

$$Z^\alpha = \begin{pmatrix} \mu^a \\ \lambda_{\dot{a}} \end{pmatrix} \tag{4.1.9}$$

A similar process for the translation of the x^μ to a bi-spinor can be done with the momentum vector P_μ.

$$P_{a\dot{a}} = \sigma^\mu_{a\dot{a}} P_\mu \quad \text{where} \quad P_{a\dot{a}} = \lambda_{\dot{a}} \tilde{\lambda}_a \tag{4.1.10}$$

$\tilde{\lambda}$ arises in the same way as λ has in 4.1.4 except they arise from the dual twistor 4.1.14 and the spinor parts are in the reverse order from before

$$W_0 = \tilde{\lambda}_0, \; W_1 = \tilde{\lambda}_1, \; W_2 = \tilde{\mu}^{\dot{0}} \text{ and } W_3 = \tilde{\mu}^{\dot{1}}. \tag{4.1.11}$$

The translation from these vectors to bi-spinors necessitates that

$$P^2 = P_\mu P^\mu = \det(P_{a\dot{a}}) \tag{4.1.12}$$

From the definition outlined prior for the space of non null Twistors describing massless particles where $P^2 = 0$ requires the determinant of the bi-spinor $P_{a\dot{a}}$ to be zero and from the incidence relations forces .

$$P_{a\dot{a}} = \lambda_{\dot{a}} \tilde{\lambda}_a \tag{4.1.13}$$

The notion of Ambitwistor space can be described easily as a space composed of both Twistors and dual Twistors. A dual Twistor, defined using the same notation and procedure as the Twistor written down earlier, is written as

$$W_\alpha = \begin{pmatrix} \tilde{\lambda}_a \\ \tilde{\mu}^{\dot{a}} \end{pmatrix} \tag{4.1.14}$$

Both the Twistor 4.1.9 and the dual Twistor 4.1.14 arise from a null line when they satisfy.

$$Z^\alpha W_\alpha = 0 \tag{4.1.15}$$

This occurs due to the incidence relations for a Twistor arising from null geodesic with momentum as shown in 4.1.13

$$Z^\alpha = \begin{pmatrix} i x^{a\dot{a}} \lambda_{\dot{a}} \\ \lambda_{\dot{a}} \end{pmatrix} \quad \text{and} \quad W_\alpha = \begin{pmatrix} \tilde{\lambda}_a \\ -i x^{a\dot{a}} \tilde{\lambda}_a \end{pmatrix} \tag{4.1.16}$$

Ambitwistor space is then given as a space defined by the Coordinates (Z, W) where both arise from a null line.

$$\mathbb{A} = \{Z, W \in \mathbb{CP}^3 | Z \cdot W = 0\} \tag{4.1.17}$$

4.2 Ambitwistor String Theory

The purpose of Investigating String theories in Twistor space is to provide a framework for constructing scattering amplitudes for field theories and has been done so succesfully by Witten [3] [29]. Following that, more expressions for scattering amplitudes were found and were expanded on by Cachazo, He and Yuan describing massless scattering amplitudes (CHY Amplitudes)[5][6][4][7]. Mason and Skinner then sought to provide String Theories existing in the space of Complex Null geodesics that can replicate the CHY amplitudes. The Bosonic action is obtained from the worldline of a massless particle [36].

$$S = \frac{1}{2\pi} \int_l P_\mu \partial_\tau X^\mu - \frac{e}{2} P^\mu P_\mu \tag{4.2.1}$$

This can be seen to be equivalent to 3.0.2 by solving the equations of motion for P_μ which results in $e^{-1}\partial_\tau X^\mu = P^\mu$. Subbing this back into 4.2.1 returns 3.0.2 when $m = 0$. Which has the following transformations.

$$\delta X^\mu = \alpha P^\mu, \qquad \delta P_\mu = 0, \qquad \delta e = \mathrm{d}\alpha \tag{4.2.2}$$

The action for the bosonic string is obtained by complexifying the worldline and the target space. This can be seen by replacing the differential $\partial_\tau X^\mu$ with $\bar{\partial} X^\mu$ using the notation introduced in section 3.2

$$S[X, P] = \frac{1}{2\pi} \int_\Sigma P_\mu \bar{\partial} X^\mu - \frac{e}{2} g^{\mu\nu} P_\mu P_\nu \tag{4.2.3}$$

where the holomorphic worldsheet coordinate σ has been suppressed. The modifications mean that the action describes the worldsheet as a complex Riemann surface Σ living in a space of null geodesics with the gauge fixing term that forces $P^2 = 0$ when solving the equation of motion for e. Where P^μ is a complex (1,0) form on the worldsheet.This means that e must represent a (0,1) form on the worldsheet which has values in the tangent space to the worldsheet and is therefore a Beltrami differential. X^μ are the coordinates on complex space-time. The transformations of the fields in this action are the same as those given in the particle theory in so much as the transformation of X^μ is given as a scaling along a null direction and can be given as equivalent. We know that P^μ must be null by construction and this allows for the natural consideration and this theory in a space of null geodesics. The symmetries imposed on this action

$$\delta X^\mu = \alpha P^\mu, \qquad \delta P_\mu = 0, \qquad \delta e = \bar{\partial}\alpha \tag{4.2.4}$$

describe a transformation on the worldsheet which can then be gauge fixed by setting $e = 0$. Doing this and fixing the worldsheet metric to be holomorphic such that these are the only degrees of freedom that enter this theory requires the introduction of 2 sets of ghosts.

$$S = \frac{1}{2\pi}\int_\Sigma P_\mu \bar{\partial} X^\mu + b\bar{\partial}c + \tilde{b}\bar{\partial}\tilde{c} \tag{4.2.5}$$

The bc ghosts enter in the usual way and the $\tilde{b}\tilde{c}$ ghosts are as a result of the gauge fixing of $P^2 = 0$. Now that the action has been gauge fixed and the conformal and gauge transformations are known the construction of the BRST operator is simple as it only requires the stress tensor and the gauge constraint [17].

$$Q = \oint cT + \frac{\tilde{c}}{2}P^2 \tag{4.2.6}$$

The formulation of scattering amplitudes requires the definition of vertex operators. However,the simplest vertex operators can be found by a local variation in the space-time metric with momentum eigenstates such that

$$\delta g^{\mu\nu}(X) = \epsilon^{\mu\nu} e^{ik\cdot X}$$

Where k is representing the momentum insertion involved in the scattering process. Requiring $\epsilon^{\mu\nu}$ to be symmetric and traceless. Leading to the vertex operator expression

$$c\tilde{c}V = c\tilde{c}P_\mu P_\nu \epsilon^{\mu\nu} e^{ik\cdot X} \tag{4.2.7}$$

The $P_\mu P_\nu$ term arises from the variation of the space-time metric in the action. This occurs as for the path integral the action sits in a partition function and the alteration of the metric in the action from $g^{\mu\nu} \to g^{\mu\nu} + \delta(g^{\mu\nu})$ partition function becomes.

$$Z[X,P] = \int \mathcal{D}X\mathcal{D}P \quad e^S e^{\int_\Sigma (\epsilon^{\mu\nu} e^{ik\cdot X} P^\mu P^\nu)} \tag{4.2.8}$$

If this change in the metric is small then a series expansion results in.

$$Z[X,P] = \int \mathcal{D}X\mathcal{D}P \quad e^S \left(1 + \int_\Sigma \epsilon^{\mu\nu} e^{ik\cdot X} P^\mu P^\nu\right) \tag{4.2.9}$$

The insertion of ghosts here is shorthand for the delta functions of the ghosts evaluated at that point on the worldsheet as they are grassmann. $\epsilon^{\mu\nu}$ is traceless as a consequence of forcing $P^2 = 0$ Writing the Vertex operator in it's usual integrated form over the worldsheet.

$$\int_\Sigma \mathcal{V} = \int_\Sigma \bar{\delta}(k\cdot P) P_\mu P_\nu \epsilon^{\mu\nu} e^{ik\cdot X} \tag{4.2.10}$$

This integral is interpreted as a local deformation in space-time as is the normal procedure and is analogous to the usual vertex operators of String theory/conformal

[17]This will not be the case when the BRST charge is constructed for the theory in Ambitwistor variables

field theory. The factor $\bar{\delta}(k \cdot P)$ does the same job as the usual delta function in so much as it forces $k \cdot P = 0$ but the $\bar{\delta}$ function is the form of the delta function that arises from the Cauchy integral formula.

$$\bar{\delta}(z - z') = \frac{1}{2\pi i} \oint \mathrm{d}z' \, \frac{1}{z - z'} \tag{4.2.11}$$

The application of this delta function can be seen to be applied in 4.2.16. However the $\bar{\delta}(k \cdot P)$ exists simply due to the construction of the theory in Ambitwistor space describing momentum eigenstates in flat spacetime and is fully described in section 2 of Mason and Skinner [2]. Once the Vertex operator expression is formalised then obtaining the tree level scattering amplitudes for an arbitrary number of insertions is obtained by the insertion of 3 fixed vertex operators $V = P_\mu P_\nu \epsilon^{\mu\nu} e^{ik \cdot X}$ and then an arbitrary amount of integrated vertex operators to compute the correlation function which gives the amplitude for n particle states.

$$\mathcal{M} = \left\langle c_1 \tilde{c}_1 V_1 c_2 \tilde{c}_2 V_2 c_3 \tilde{c}_3 V_3 \int \mathcal{V}_4 \ldots \int \mathcal{V}_n \right\rangle \tag{4.2.12}$$

As the Vertex operators are not polynomial in X the plane wave terms $e^{ik_i \cdot X}$can be inserted into the action as a summation for the number of vertex operators. so the action becomes

$$S = \frac{1}{2\pi} \int_\Sigma P_\mu \bar{\partial} X^\mu + i \sum_{i=1}^{n} k_i \cdot X \delta^2(\sigma - \sigma_i) \tag{4.2.13}$$

The action now includes all of the X dependence and included in the path integral. The delta function in the expression is used to fix the insertion of the vertex operation on the Reimann surface. Through integration the kinetic term $P_\mu \bar{\partial} X^\mu$ they decouple from the zero modes. And the normal result of a delta function for momentum conservation over all dimensions of space time, $\delta^{26}(\sum_i k_i)$ occurs from said integration just like in 3.4. Integrating out X for the non-zero modes results in the field equation.

$$\bar{\partial} P_\mu = 2\pi i \sum_i k_{i\mu} \delta^2(\sigma - \sigma_i) \tag{4.2.14}$$

On a worldsheet of genus zero i.e a sphere this has the unique solution for P_μ obtained from the definition that $P = P_\sigma(\sigma)\, \mathrm{d}\sigma$

$$P_\mu = \mathrm{d}\sigma \sum_{i=1}^{n} \frac{k_{i\mu}}{\sigma - \sigma_i} \tag{4.2.15}$$

This expression can be inserted into the expression of the vertex operators for the momentum term in the delta function $\bar{\delta}(k_i \cdot P(\sigma_i))$.

$$k \cdot P = \sum_{i \neq j}^{n} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0 \tag{4.2.16}$$

The requirement for $i \neq j$ is the result of the on shell condition $k_i^2 = 0$ and removes the redundant terms. These are the Scattering equations of Gross and Mende [37]. One thing that must be kept in mind as is the case for all theories of this type as it it still theory on a conformal map is the $\mathbf{SL}(2;\mathbf{C})$ symmetry and as such the final form of the amplitude is:

$$\mathcal{M} = \delta^{26}\left(\sum_i k_i\right) \int \frac{\mathrm{d}^2\sigma}{\text{Vol }(\mathbf{SL}(2;\mathbf{C}))} \prod_{i=1}^{n} \frac{\epsilon_i^{\mu\nu} k_{i\mu} k_{i\nu}}{(\sigma - \sigma_i)^2} \prod_j \bar{\delta}(k_j \cdot P(\sigma_j)) \tag{4.2.17}$$

5 Berkovits Superparticle in Supertwistor Variables

This section serves to provide the background for our research and is informed by the previous section as Berkovits constructed a model for a superparticle in twistor space. The reason for interest in this theory is not simply due to it's interesting structure but it has already been shown in more recent literature [2] that through the extension of superparticle theories in Ambitwistor to string theories can give the CHY scattering amplitudes for massless particles.

The action Berkovits starts with is the standard expression for an infinite tension limit string (particle), known as the Brink-Schwarz particle

$$S = \int d\tau P_\mu \partial_\tau X^\mu + \frac{g}{2} P_\mu P^\mu \tag{5.0.1}$$

The P^μ and X^μ are the bosonic vectors, g gives the world-line metric. Berkovits outlines that the, spacetime supersymmetric, superparticle model normally encounters a problem when $P^2 = 0$ as the κ symmetries reducing the degrees of freedom result in a increasing chain of ghosts in an attempt to obtain the usual degrees of freedom [1]. Starting From the above action, the equation of motion for e gives $P^2 = 0$. Berkovits introduced the 10D twistor variable such that the 10D momentum is

$$P^\mu = \lambda^a \Gamma^\mu_{ab} \lambda^b \tag{5.0.2}$$

and as has been outlined in 4.1.4 there exists incidence relations between the spinor components of a Twistor. In this case the incidence relation is.

$$\omega_a = X_\mu \Gamma^\mu_{ab} \lambda^b \tag{5.0.3}$$

This leads to the 5.0.1 being able to be rewritten as

$$S = \int \mathrm{d}\tau \bigg(- 2\omega_a \partial \lambda^a \bigg) \tag{5.0.4}$$

In this case I have deliberately omitted the terms that give rise to supersymmetry as the theory constructed later in this thesis is a Bosonic theory. so by constructing the

Twistor variables λ^a and ω_a . Note that these are no longer the 4D Twistor variables from before but are 10 dimensional Ambitwistors and λ is related to 10D momentum 5.0.2 .Γ^μ_{ab} is a real symmetric matrix that such that $\Gamma^\mu_{ab} = \Gamma^{ab}_\mu$

$$\gamma^\mu = \begin{pmatrix} 0 & \Gamma^{\mu ab} \\ \Gamma^\mu_{ab} & 0 \end{pmatrix} \tag{5.0.5}$$

and as such $\{\gamma^\mu, \gamma^\nu\} = \Gamma^\mu_{ab}\Gamma^{\nu bc} + \Gamma^\nu_{ab}\Gamma^{\mu bc} = 2\eta^{\mu\nu}\delta^c_a$. The variables λ^a, ω_a exist as the spinor components of Ambitwistors described in 4.1.9 or 4.1.14 satisfy the constraints for the gauge. The expression for momentum in 10D 5.0.2 is different to the expression in 4D 4.1.13 this is because in 10D the chiralities of the λ terms are the same. The Gamma matrix identity

$$\Gamma^\mu_{ab}\Gamma_{\mu cd} + \Gamma^\mu_{ca}\Gamma_{\mu bd} + \Gamma^\mu_{bc}\Gamma_{\mu ad} = 0 \tag{5.0.6}$$

forces the reduction of the degrees of freedom of λ^a and ω from $2d - 4$ to $d - 3$ as the extra degrees of freedom are constrained by the expression

$$G^a = (\lambda^b\Gamma^\mu_{bc}\lambda^c)\Gamma^\mu_{ad}\omega_d - 2\lambda^a(\lambda^b\omega_b) = 0 \tag{5.0.7}$$

This constraint generates the Bosonic transformation of Twistor variables.

$$\delta\lambda^a = \Big(- (\lambda^c\Gamma^\mu_{cd}\lambda^d)\Gamma^{ab}_\mu + 2\lambda^a\lambda^b\Big)\varepsilon_b \tag{5.0.8}$$

$$\delta\omega_a = \Big(2\Gamma^\mu_{ac}\lambda^c(\Gamma^{de}_\mu\omega_e) - 2\delta^d_a(\lambda^e\omega_e) - 2\lambda^d\omega_a\Big)\varepsilon_d \tag{5.0.9}$$

and the combination of 5.0.6 and 5.0.7 results in what are known as the reducibility constraints. These will be expanded upon later when the theory is looked at in the context of BRST quantisation.

$$\lambda^a\Gamma^\mu_{ab}G^b = 0 \tag{5.0.10}$$

$$\lambda^a\Gamma^\mu_{ab}(\lambda^c\Gamma_{\mu cd}\lambda^d) = 0 \tag{5.0.11}$$

The above constraints are the origin of the reducibility as 5.0.10 shows that $G^a = 0$ is not an independent constraint and 5.0.11 shows that $\lambda^a\Gamma^\mu_{ab} = 0$ isn't independent either. We can see that the incidence relation that defines 5.0.3 is invariant under the transformation $\delta X^\mu = \alpha\lambda^a\Gamma^\mu_{ab}\lambda^b$ and as a result ω_a and λ^a shows the transformation for $\delta X^\mu = \alpha P^\mu$. The transformations of these are then characterised by scaling along the world line or interactions with the gauge constraints. Now that the structure of the bosonic theory has been outlined it is simple enough to produce the worldline action by the inclusion of the constraint 5.0.7 into the action with the insertion of a Lagrange multiplier h_a

$$S = \int \mathrm{d}\tau \ - 2\omega_a\partial\lambda^a + h_aG^a \tag{5.0.12}$$

From this point we have enough to go on to produce a string theory from these variables.

6 Ambitwistor String Theory in the Spinor Formalism

The construction of a String theory in the same formalism as the berkovits superparticle is done by simply extending the worldline theory to a worldsheet theory. As such the theory being produced can be seen to be equivalent to the theory explored by Mason and Skinner of a Reimann sphere in the space of null geodesics. Taking the bosonic version of that theory as a starting point with worldsheet Σ parametrized by z

$$S = \int_\Sigma P_\mu \bar{\partial} X^\mu + \frac{e}{2} P^2 \tag{6.0.1}$$

The same holomorphic gauge transformations generated by the stress tensor $T = P_\mu \partial X^\mu$ and symmetries caused by $P^2 = 0$ hold in this case. Introducing (bosonic) worldsheet spinors $\lambda^a(z)$ and ω_a such that

$$P^\mu(z) = \lambda^a(z) \Gamma^\mu_{ab} \lambda^b(z) \tag{6.0.2}$$

and the incidence relation

$$\omega_a(z) = X_\mu(z) \Gamma^\mu_{ab} \lambda^b(z). \tag{6.0.3}$$

allows for the reformulation of the action to become.

$$S = \frac{1}{2} \int_\Sigma \omega_a \bar{\partial} \lambda^a - \lambda^a \bar{\partial} \omega_a \tag{6.0.4}$$

This removes the need for the Beltrami differential as the kinetic terms are formed from the spinor components of the twistors and $P^2 = 0$ is naturally encoded into this representation from the gamma matrix identities. To write this action in its manifestly Twistorial form define the Twistor

$$Z^I = \begin{pmatrix} \lambda^a \\ \omega_a \end{pmatrix} \tag{6.0.5}$$

The differential $\bar{\partial}$ can be expanded as $\bar{\partial} \to \bar{\partial} + \mu \partial$ this can be done by setting $\mu = 0$ and essentially reinserts the metric. Where μ is a Beltrami differential which take the form such that

$$\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z} \tag{6.0.6}$$

and arise from the expression for a change in the metric [25].

$$\mu^\gamma_{j\beta} = \frac{1}{2} \hat{h}^{\gamma\delta} \partial_j \hat{h}_{\beta\delta} \tag{6.0.7}$$

The kinetic term of the action can then be written as

$$S = \frac{1}{2} \int_\Sigma Z^I \bar{\partial} Z_I + \mu Z^I \partial Z_I \tag{6.0.8}$$

where the index is raised and lowered by the antisymmetric traceless metric

$$\Omega_{IJ} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{6.0.9}$$

as to satisfy the equation

$$Z^I \bar{\partial} Z_I = \Omega_{IJ} Z^I \bar{\partial} Z^J \tag{6.0.10}$$

The Stress tensor, follows the standard form of any field theory and is written as $T = \frac{\delta S}{\delta \mu}$

$$T(z) = \frac{1}{2} Z^I \partial Z_I = \frac{1}{2} \omega_a \partial \lambda^a - \lambda^a \partial \omega_a \tag{6.0.11}$$

This theory also is constrained by the same gauge symmetries as the Berkovits super particle so there exists such an expression that

$$G^a = (\lambda^b \Gamma^\mu_{bc} \lambda^c) \Gamma^\mu_{ad} \omega_d - 2\lambda^a (\lambda^b \omega_b) = 0 \tag{6.0.12}$$

In order to fix this constraint it is inserted into the action.

$$S = \int_\Sigma Z^I \bar{\partial} Z_I + \mu T + h_a G^a \tag{6.0.13}$$

where μ and h_a are gauge fields such that $\mu T = \mu^z_{\bar{z}} T_{zz}$, $h_a = (h_a)_{\bar{z}} d\bar{z}$ and act as Lagrange multipliers in the action. G^a gives a set of constraints that must be satisfied, but these are not independent and there are further constraints that arise from the gamma matrix identities and that $P^2 = 0$.

$$\Gamma^\mu_{ab} \lambda^b G^a = 0 \tag{6.0.14}$$

This constraint is also not independent and there exists another.

$$\lambda^c \Gamma_{\mu cd} \lambda^d \Gamma^\mu_{ab} \lambda^b = 0 \tag{6.0.15}$$

6.1 Conformal Symmetries, Gauge Symmetries and reducibility

The first thing that must be defined when considering constructing a conformal field theory and investigating conformal transformations are the Operator product expansions. In this case we have the simple OPEs of the matter fields being.

$$\lambda^a(z) \omega_b(z') = \frac{\delta^a_b}{z - z'} + \cdots \tag{6.1.1}$$

The stress tensor for the matter fields in this theory generates the conformal transformations.

$$\delta_T(v) \lambda^a(z) = \oint dz' \quad T(z') v(z') \lambda^a(z) = \frac{1}{2} \oint dz' \quad (\omega_a \partial \lambda^a - \lambda^a \partial \omega_a) v(z') \lambda^a(z) \tag{6.1.2}$$

$$\delta_T(v) \omega_a(z) = \oint dz' \quad T(\omega) v(\omega) \omega_a(z) = \frac{1}{2} \oint dz' \quad (\omega_a \partial \lambda^a - \lambda^a \partial \omega_a) v(z') \omega_a(z) \tag{6.1.3}$$

Around the contour $z = z'$. Using 6.1.1 and integrating over poles as delta functions i.e

$$\oint dz' \quad \frac{f(z')}{(z-z')^n} = \frac{\partial^{n-1}(f(z))}{(n-1)!} \tag{6.1.4}$$

these transformations become

$$\delta_T(v)\lambda^a = v\partial\lambda^a + \frac{1}{2}\lambda^a\partial v \tag{6.1.5}$$

$$\delta_T(v)\omega_a = -v\partial\omega_a - \frac{1}{2}\omega_a\partial v \tag{6.1.6}$$

More generally in the form of arbitrary holomorphic vector fields the algebra is

$$[T(v_1), T(v_2)] = -T(v_3) \tag{6.1.7}$$

Where the generators are defined in a similar way to the transformations derived above.

$$T(v_1) = \oint dz \quad v_1(z)T(z) \tag{6.1.8}$$

and in this case

$$v_3 := [v_1, v_2] = v_1\partial v_2 - v_2\partial v_1 \tag{6.1.9}$$

In addition, the constraint $G^a(z) = 0$, which defines some of the physics of the worldsheet theory to be that of the massless particle, results in the gauge symmetry where the Lagrange multiplier $h_a(z)$ transforms as

$$\delta h_a = -\bar{\partial}\varepsilon_a - 4\delta_a^{[b}\lambda^{c]}\varepsilon_c h_b, \tag{6.1.10}$$

and the 'matter' fields transform as

$$\delta\lambda^a = \Big(-(\lambda^c\Gamma^\mu_{cd}\lambda^d)\Gamma^{ab}_\mu + 2\lambda^a\lambda^b\Big)\varepsilon_b \tag{6.1.11}$$

$$\delta\omega_a = \Big(2\Gamma^\mu_{ac}\lambda^c(\Gamma^{de}_\mu\omega_e) - 2\delta^d_a(\lambda^e\omega_e) - 2\lambda^d\omega_a\Big)\varepsilon_d \tag{6.1.12}$$

where $\varepsilon_a(z)$ is a vector-valued worldsheet spinor such that.

$$G(\varepsilon) = \oint dz \quad \varepsilon_a(z)G^a(z) \tag{6.1.13}$$

From this we can now construct the full algebra for the gauge symmetry and the conformal transformations

$$[G(\varepsilon), G(\varepsilon')] = G(\breve{\varepsilon}), \tag{6.1.14}$$

where $\breve{\varepsilon}_c$ is given by.

$$\breve{\varepsilon}_c = -4\delta_c^{[a}\lambda^{b]}\varepsilon_a{\varepsilon'}_b \tag{6.1.15}$$

It is also possible to investigate the cross terms between the gauge symmetry and conformal symmetries generated by the stress tensor and the gauge constraint.

$$[T(v), G(\varepsilon)] = -G(\tilde{\varepsilon}), \tag{6.1.16}$$

$$\tilde{\varepsilon}_a = \frac{1}{2}\varepsilon_a \partial v - v\partial\varepsilon_a \tag{6.1.17}$$

This interaction between the conformal and gauge transformations hint at some interaction terms between the ghosts and we will see later that these appear in the BRST charge and can be included in the action through the method outlined in 8. This confirms that the field G^a has conformal weight 3/2 which stands to reason as each term in G^a is quadratic in λ^a and linear in ω_a which each have conformal weight 1/2. At this point we have enough to introduce the standard conformal ghost system b and c with conformal weights +2 and -1 respectively and Faddeev-Popov ghosts b^a and c_a which are fermionic with conformal weights 3/2 and -1/2. This results in the introduction of the ghost terms and gauge fixing terms into the action.

$$S = \int_\Sigma Z_I \bar{\partial} Z^I + b\bar{\partial}c + b^a \bar{\partial} c_a + \mu T + h_a G^a + \cdots \tag{6.1.18}$$

Where $\cdots$ represent the interaction terms for the ghosts that will be elaborated on in section 8. From this point we can begin to construct the BRST charge from the theory where we have terms for the ghost stress tensors written as T_{gh} and the structure function for the lie algebroid formed by the gauge symmetry.

$$[G^a, G^b] = f_c^{ab} G^c \tag{6.1.19}$$

Wheref_c^{ab} is the structure function for the gauge symmetry and can be seen from 6.1.14 and 6.1.15 meaning that.

$$f_c^{ab} = -4\lambda^{[a}\delta_c^{b]} \tag{6.1.20}$$

This leads to the BRST charge

$$Q = \oint \mathrm{d}z \; c\big(T + \frac{1}{2}T_{gh}\big) + c_a\big(G^a + \frac{1}{2} f_c^{ab} c_b b^c\big) \tag{6.1.21}$$

However, it is known from Berkovits that this system is reducible of level of level 2 [1]. The constraints $\Gamma^\mu_{ab}\lambda^b G^a = 0$ means that the constraint $G^a = 0$ isn't independent and through gauge fixing and the introduction of the Faddeev-Poppov ghosts b^a and c_a serves to over fix the degrees of freedom of the system. This will require the insertion of more ghosts to reintroduce degrees of freedom and as such must have the same statistics as the matter fields λ^a and ω_a which have Bose-Einstein statistics.However, it doesn't stop there as the identity $(\lambda^c \Gamma_{\mu cd}\lambda^d)\Gamma^\mu_{ab}\lambda^b = 0$ corresponding to the 2nd level constraints implies that the $\Gamma^\mu_{ab}\lambda^b G^a = 0$ is not independent either, so through the reinsertion of degrees of freedom once again there is an overcompensation and too many degrees of freedom have been reinserted and some of these must be fixed so another ghost system must be introduced and have the same statistics as b^a and c_a. The full process of this will be covered in the next section.

$$Z^\mu_a := \Gamma^\mu_{ab}\lambda^b \quad \text{and} \quad Z_\mu := P_\mu = \lambda^a \Gamma_{\mu ab}\lambda^b \tag{6.1.22}$$

6.1.22 are the expressions that will account for the reducibility. $Z^\mu_a G^a = 0$ is the level 1 constraint and $Z^\mu_a Z_\mu = 0$ is the level 2 constraint.

7 BRST Charge for a Reducible Gauge system

The procedure for constructing a BRST charge for a reducible system is covered in detail in [13]. This section will follow the procedure given by Henneaux in the context of this theory and extend it to a level 2 system. Starting by defining our bosonic spinor fields as canonically conjugate variables with ω_a being the conjugate of λ^a so that G^a is linear in the conjugate momentum

$$G^a(\lambda, \omega) = \xi^{ab}(\lambda)\omega_b \tag{7.0.1}$$

in this case we can define ξ^{ab} as

$$\xi^{ab} := P^\mu \Gamma^{ab}_\mu - 2\lambda^a \lambda^b \tag{7.0.2}$$

As we have already determined the structure functions for the gauge transformations are only dependent on λ we can rewrite the constraints $\{G^a, G^b\}$ as

$$\{G^a, G^b\} = 2\xi^{[a|d}\partial_d \xi^{|b]c} = f^{ab}_c G^c \tag{7.0.3}$$

$\{,\}$ here are the classical Poisson brackets and will be from from here onwards. Poisson brackets are:

$$\{M, N\} = \frac{\partial M}{\partial x}\frac{\partial N}{\partial p} - \frac{\partial M}{\partial p}\frac{\partial N}{\partial x} \tag{7.0.4}$$

where x is position and p is momentum. For the theory that will be explored in this case λ^a is the position variable and ω_a is the conjugate momenta. Using notation where ∂_c is shorthand for $\frac{\partial}{\partial \lambda^c}$ then the expression for the structure function can be verified to be

$$f^{ab}_c = -4\lambda^{[a}\delta^{b]}_c \tag{7.0.5}$$

For a reducible theory there exists some constraints that follow the form of:

$$Z^{\alpha_1}_{\alpha_0} \neq 0 \tag{7.0.6}$$

$$Z^{\alpha_1}_{\alpha_0} G^{\alpha_0} = 0 \tag{7.0.7}$$

These are the reducibility constraints Using notation where α_i is indexed for the level of reducibility so in this case $i = 0, 1, 2$. For the theory being considered these functions exist 6.1.22.

$$Z^\mu_a = \Gamma^\mu_{ab}\lambda^b \tag{7.0.8}$$

$$Z^\mu_a G^a = 0 \tag{7.0.9}$$

therefore $\alpha_0 = a$ and $\alpha_1 = \mu$ an important consequence of the level one constraints is that they give rise to the requirement for bosonic ghosts c_μ and b^μ of ghost number 2 and -2. As this theory is level 2 reducible there must exist more constraints

$$Z^{\alpha_1}_{\alpha_0} Z^{\alpha_2}_{\alpha_1} = 0 \tag{7.0.10}$$

which have previously been shown as

$$\Gamma^{\mu}_{ab}\lambda^{b}(\lambda^{c}\Gamma_{\mu cd}\lambda^{d}) = Z^{\mu}_{a}Z_{\mu} = 0 \tag{7.0.11}$$

This then implies that in the theory being described there is no index that corresponds to α_2 and therefore will be left blank. There still will be the need for Ghosts for this level of reducibility with ghost number 3 and -3. These will be denoted by $\mathbf{C}$ and $\mathbf{B}$. For this theory we have no need to consider any further constraints as everything is accounted for however if there were more levels of reducibility then it would follow the general form that there exist functions such that.

$$Z^{\alpha_i}_{\alpha_i-1}Z^{\alpha_i+1}_{\alpha_i} = 0 \tag{7.0.12}$$

where $i = 0, 1, \cdots, L$ and L gives the level of reducibility of the theory. Now that the reducibility of the theory is outlined then the BRST charge can be derived. From the starting point of the Jacobi identity for the gauge constraint.

$$\{\{G^{[a}, G^{b}\}, G^{c]}\} = 0 \tag{7.0.13}$$

From the algebra 7.0.3 this produces the expression.

$$\left(\partial_d \xi^{[a|d} f^{|bc]}_{e} - f^{[ab}_{d} f^{c]d}_{e}\right)G^{e} = 0 \tag{7.0.14}$$

And for a reducible theory this implies the existence of functions M^{abc}_{μ} that satisfy

$$\partial_d \xi^{[a|d} f^{|bc]}_{e} = f^{[ab}_{d} f^{c]d}_{e} - \frac{2}{3}M^{abc}_{\mu}Z^{\mu}_{e} \tag{7.0.15}$$

The existence of the functions Z^{μ}_{a} implies other functions of the form $D^{a\mu}_{b}$ exist to satisfy identities such as

$$\xi^{bd}\partial_d Z^{\mu}_{c} + Z^{\mu}_{d} f^{bd}_{c} = D^{b\mu}_{\nu}Z^{\nu}_{c} \tag{7.0.16}$$

This arises from the expression.

$$\left(\xi^{bd}\partial_d Z^{\mu}_{c} + Z^{\mu}_{d} f^{bd}_{c}\right)\xi^{ce} = 0 \tag{7.0.17}$$

if this is then multiplied by the first level reducibility constraint contracting with one of the indices and the fact that

$$\xi^{ab}Z^{\mu}_{b} = 0 \tag{7.0.18}$$

which can be verified by the definitions of these functions and the constraints outlined, The resulting expression is

$$Z^{[\mu}_{a}Z^{\nu]}_{b} f^{ab}_{c} = Z^{\mu}_{c}Z^{\lambda}_{d}D^{\nu d}_{\lambda} \tag{7.0.19}$$

if this is antisymmetric over μ and ν then the left hand side is exactly 0 and as such

$$D^{a[\mu}_{\lambda}Z^{\nu]}_{a}Z^{\lambda}_{c} = 0 \tag{7.0.20}$$

Considering the second level reducibility constraint that $Z_a^\mu Z_\mu = 0$ because of the antisymmetric case being satisfied above, the symmetric part suggests the existence of the following:

$$D_\lambda^{a\mu} Z_a^\nu + D_\lambda^{a\nu} Z_a^\mu = B^{\mu\nu} Z_\lambda \tag{7.0.21}$$

These functions will form the main part of the BRST generator. However there exist more functions of a similar form that exist to satisfy the level of reducibility required. The method for producing these functions is from the construction of the BRST generator and applying the fact that $\{Q, Q\} = 0$[18] The Ghosts introduced obey the algebra:

$$gh(c_{\alpha_i}) = -gh(b^{\alpha_i}) = i + 1 \tag{7.0.22}$$

The ghosts introduced alternate statistics for each level of reducibility. For even levels in this case $i = 0$ or $i = 2$ then the ghosts obey Fermi-Dirac statistics, they can be seen as constraining the degrees of freedom of the theory. The ghosts introduced for the odd levels of reducibility obey Bose-Einstein statistics and can be seen as reintroducing previously constrained degrees of freedom. The BRST charge for the gauge system is constructed as follows.

$$Q = c_a G^a + \sum_{i=0}^{L-1} c_{\alpha_i+1} Z_{\alpha_i}^{\alpha_i+1} b^{\alpha_i} + \cdots \tag{7.0.23}$$

The Summation term above takes care of the reducibility constraints and embeds them into the BRST charge. From this we can consider the BRST ccharge to be constructed from different parts each arising from the different levels of reducibility. In the case of the theory we are dealing with this can be broken down first of all into.

$$Q_{(0)} = c_a G^a + \dots \tag{7.0.24}$$
$$Q_{(1)} = c_\mu Z_a^\mu b^a + \dots \tag{7.0.25}$$
$$Q_{(2)} = \mathbf{C} Z_\mu b^\mu + \dots \tag{7.0.26}$$

As is usually the approach with the BRST charge it must also encode the symmetries of the theory as a result of $\{Q, Q\} = 0$ and starting from 7.0.24

$$\frac{1}{2}\{Q_0, Q_{(0)}\} = \frac{1}{2} c_a c_b \{G^a, G^b\} = \frac{1}{2} c_a c_b f_c^{ab} G^c \tag{7.0.27}$$

As Q is linear in conjugate momenta and the b^{α_i} ghosts are defined as conjugate to the c_{α_i} ghosts this leads to the equivalence of the last term above to

$$\frac{1}{2} c_a c_b f_c^{ab} G^c \Rightarrow \frac{1}{2} f_c^{ab} c_a c_b b^c \quad \text{as} \quad \{Q_0, b^c\} = G^c. \tag{7.0.28}$$

[18]At the moment the only symmetries being dealt with are the result of G^a and the Reducibility. The terms arising from the conformal symmetry will be introduced later.

For the zero level ghosts the numerical pre factors of terms can seen to be the inverse of the number of c_a terms ie: G^a is the coefficient of c_a term $\frac{1}{2}f_c^{ab}$ is that for $c_a c_b b^c$ and therefore for the term $M^{\alpha 1}_{\alpha_0 \beta_0 \gamma_0}$ should require $\frac{1}{3}M^{\mu}_{abc}$ is the coefficient for $c_a c_b c_c b^\mu$. The functions previously shown for level one reducibility have a general form for higher level reducibility as coefficients for the higher level ghosts.

$$D^{\alpha_0 \beta_i}_{\gamma_i} \quad \text{exists for terms} \quad c_a c_{\beta_i} b^{\gamma_i} \quad \text{for} \quad i = 1, \cdots, L \tag{7.0.29}$$

$$B^{\alpha_1 \beta_i}_{\gamma_{i+1}} \quad \text{exists for terms} \quad c_\mu c_{\beta_i} b^{\gamma_{i+1}} \quad \text{for} \quad i = 1, \cdots, L-1 \tag{7.0.30}$$

$$M^{\beta_0 \gamma_0 \delta_{i-1}}_{\alpha_i} \quad \text{exists for terms} \quad c_a c_b c_{\gamma_{i-1}} b^{\alpha_i} \quad \text{for} \quad i = 1, \cdots, L-1 \tag{7.0.31}$$

So for the theory being considered where $L = 2$ the suitable ansatz for the BRST charge can be written as

$$Q_G = \sum_{k=0}^{3} Q_{G(k)} \tag{7.0.32}$$

Where each of the terms are

$$Q_{G(0)} = c_a G^a \tag{7.0.33}$$

$$Q_{G(1)} = c_\mu Z^\mu_a b^a + \frac{1}{2} f^{ab}_c c_a c_b b^c \tag{7.0.34}$$

$$Q_{G(2)} = \mathbf{C} Z_\mu b^\mu - D^{a\mu}_\nu c_a c_\mu b^\mu + \frac{1}{3} M^{abc}_\mu c_a c_b c_c b^\mu \tag{7.0.35}$$

$$Q_{G(3)} = \frac{1}{2} B^{\mu\nu} c_\mu c_\nu \mathbf{B} - D^a c_a \mathbf{C}\mathbf{B} + M^{ab\mu} c_a c_b c_\mu \mathbf{B} \tag{7.0.36}$$

This is the ansatz for the section of the BRST charge that takes care of the gauge symmetry. As the theory being considered also has conformal symmetry this needs to be included the BRST charge. The way that this is achieved is to simply introduce terms for the stress tensors for matter fields and the ghosts

$$Q = c\big(T_m + \frac{1}{2} T^{(c)}_{gh} + T^{(0)}_{gh} + T^{(1)}_{gh} + T^{(2)}_{gh}\big) + Q_G \tag{7.0.37}$$

T_m is the stress tensor for the matter fields and $T^{(c)}_{gh}$ is for the conformal bc ghosts and $T^{(i)}_{gh}$ for $i = 0, 1, 2$ are the terms corresponding to the ghost arising due to the gauge symmetry and level 2 reducibility. T_m has already been derived previously but I will show it's calculation along side the other stress tensors.For any field with the action.

$$\int_\Sigma b^{\alpha_i} \bar{\partial} c_{\alpha_i} \tag{7.0.38}$$

The stress tensor can be found using the formula:

$$T = \partial(b^{\alpha_i}) c_{\alpha_i} - \Lambda \partial(b^{\alpha_i} c_{\alpha_i}) \tag{7.0.39}$$

Where Λ denotes the weight of the conjugate field b^{α_i}. As the weights of the fields dictate the OPE of the two fields and we have the general OPE of a field with it's conjugate to be $\frac{\delta^{\alpha_i}_{\beta_i}}{z-z'}$ then the product of a field and it's conjugate must satisfy $\{b^{\alpha_i}, c_{\alpha_i}\} = +1$ because of this these are the weights of the fields and their conjugates:

$$[\lambda^a] = +\frac{1}{2} \quad \text{and} \quad [\omega_a] = +\frac{1}{2} \tag{7.0.40}$$

$$[c] = -1 \quad \text{and} \quad [b] = +2 \tag{7.0.41}$$

$$[c_a] = -\frac{1}{2} \quad \text{and} \quad [b^a] = +\frac{3}{2} \tag{7.0.42}$$

$$[c_\mu] = -1 \quad \text{and} \quad [b^\mu] = +2 \tag{7.0.43}$$

$$[\mathbf{C}] = -2 \quad \text{and} \quad [\mathbf{B}] = +3 \tag{7.0.44}$$

The Stress tensors that are then derived from these and using 7.0.39 gives.

$$\begin{aligned} T_m &= (\partial\omega_a)\lambda^a - \frac{1}{2}\partial(\omega_a\lambda^a) \\ &= \frac{1}{2}\lambda^a\partial\omega_a - \frac{1}{2}\omega_a\partial\lambda^a \end{aligned} \tag{7.0.45}$$

$$\begin{aligned} T^{(c)}_{gh} &= (\partial b)c - 2\partial(bc) \\ &= -(\partial b)c - 2b\partial c \end{aligned} \tag{7.0.46}$$

$$\begin{aligned} T^{(0)}_{gh} &= (\partial b^a)c_a - \frac{3}{2}\partial(b^a c_a) \\ &= -\frac{1}{2}(\partial b^a)c_a - \frac{3}{2}b^a\partial c_a \end{aligned} \tag{7.0.47}$$

$$\begin{aligned} T^{(1)}_{gh} &= (\partial b^\mu)c_\mu - 2\partial(b^\mu c_\mu) \\ &= -(\partial b^\mu)c_\mu - 2b^\mu\partial c_\mu \end{aligned} \tag{7.0.48}$$

$$\begin{aligned} T^{(2)}_{gh} &= (\partial\mathbf{B})\mathbf{B} - 3\partial(\mathbf{BC}) \\ &= -2(\partial\mathbf{B})\mathbf{C} - 3\mathbf{B}\partial\mathbf{C} \end{aligned} \tag{7.0.49}$$

Now that all parts of the BRST Charge have been derived the full expression can be written.

$$\begin{aligned} Q = \oint \mathrm{d}z\ c\Big[&\big(\frac{1}{2}\lambda^a\partial\omega_a - \frac{1}{2}\omega_a\partial\lambda^a\big) + \frac{1}{2}\big(-(\partial b)c - 2b\partial c\big) + \big(-\frac{1}{2}(\partial b^a)c_a - \frac{3}{2}b^a\partial c_a\big) \\ &+ \big(-(\partial b^\mu)c_\mu - 2b^\mu\partial c_\mu\big) + \big(-2(\partial\mathbf{B})\mathbf{C} - 3\mathbf{B}\partial\mathbf{C}\big)\Big] + c_a G^a + c_\mu Z^\mu_a b^a + \frac{1}{2}f^{ab}_c c_a c_b b^c \\ &+ \mathbf{C}Z_\mu b^\mu - D^{a\mu}_\nu c_a c_\mu b^\nu + \frac{1}{3}M^{abc}_\mu c_a c_b c_c b^\nu + \frac{1}{2}B^{\mu\nu}c_\mu c_\nu\mathbf{B} - D^a c_a\mathbf{CB} + M^{ab\mu}c_a c_b c_\mu\mathbf{B} \end{aligned} \tag{7.0.50}$$

It is important to keep in mind that the way that this BRST charge has been constructed has been done so to satisfy $\{Q, Q\} = 0$ when deriving the functions that arise from the reducibility. Keep in mind that the argument presented here is an entirely classical procedure so the prospect of there being some quantum anomaly has not been explored. The full form of the BRST charge is known the next step is to define constraints of the structure functions that appear in it and the most straight forward way of doing that is to have the charge act on a field twice and see what resulting constraints need to be imposed for the expression to be zero as by definition the BRST charge action on a field twice should be zero. The Structure functions in 7.0.50 can be derived using the process above through the insertion of the variables and expressions relevant to the theory being investigated. They are derived to be.

$$f_c^{[ab]} = -4\lambda^{[a}\delta_c^{b]} \tag{7.0.51}$$

$$D_\nu^{a\mu} = 2\Gamma^{\mu ac}\Gamma_{\nu cb}\lambda^b \tag{7.0.52}$$

$$B^{\mu\nu} = 4g^{\mu\nu} \tag{7.0.53}$$

$$D^a = 2\lambda^a \tag{7.0.54}$$

$$M_\mu^{abc} = 0 \tag{7.0.55}$$

$$M^{ab\mu} = 0 \tag{7.0.56}$$

With the above structure functions inserted into the BRST charge the full expression is.

$$\begin{aligned} Q = \oint \mathrm{d}z\ c\Big[&\big(\frac{1}{2}\lambda^a\partial\omega_a - \frac{1}{2}\omega_a\partial\lambda^a\big) + \frac{1}{2}\big(-(\partial b)c - 2b\partial c\big) + \big(-\frac{1}{2}(\partial b^a)c_a - \frac{3}{2}b^a\partial c_a\big) \\ &+ \big(-(\partial b^\mu)c_\mu - 2b^\mu\partial c_\mu\big) + \big(-2(\partial\mathbf{B})\mathbf{C} - 3\mathbf{B}\partial\mathbf{C}\big)\Big] + c_aG^a + c_\mu Z_a^\mu b^a \\ &- 2\lambda^{[a}\delta_c^{b]}c_ac_bc^c + \mathbf{C}Z_\mu b^\mu + 2\Gamma^{\mu ac}\Gamma_{\nu cb}\lambda^b c_a c_\mu b^\nu + 2g^{\mu\nu}c_\mu c_\nu\mathbf{B} - 2\lambda^a c_a\mathbf{CB} \end{aligned} \tag{7.0.57}$$

However for the sake of generality and to formally derive the constraints for the structure functions, the nilpotency calculations will be done using the general form of the BRST charge given by 7.0.50. However in the cases where applicable the fact that 7.0.55 and 7.0.56 are zero will be inserted to simplify the expressions.

7.1 Nilpotency of BRST Charge on Matter fields and c ghosts

Due to the nature of the BRST charge and the fact that G^a is quadratic in λ^a the more straight forward calculations to carry out are the BRST transformations for λ and the c ghosts. This should suffice in producing the constraints for the structure functions. As the matter fields and ghosts are all defined as conjugate pairs the operator product expansions for them are simply.

$$\lambda^a(z)\omega_a(z') = \frac{\delta^a_b}{(z-z')} \tag{7.1.1}$$

$$c(z)b(z') = \frac{1}{(z-z')} \tag{7.1.2}$$

$$c_a(z)b^b(z') = \frac{\delta^b_a}{(z-z')} \tag{7.1.3}$$

$$c_\mu(z)b^\nu(z') = \frac{\delta^\nu_\mu}{(z-z')} \tag{7.1.4}$$

$$\mathbf{C}(z)\mathbf{B}(z') = \frac{1}{(z-z')} \tag{7.1.5}$$

Rather than doing the BRST transformations twice on every field it is best to obtain the expressions for the transformations happening once so they can be inserted into the expressions later on. Starting with the transformation on λ^a, Using the notation of $s(\phi) = \{Q, \phi\} = \delta_Q\phi$. to represent an infinitesimal BRST transformation. This can be written as an integration over the BRST current J and the field which is undergoing the transformation.

$$s(\lambda^a(z)) = \oint \mathrm{d}\sigma \ J(\sigma)\lambda^a(z) \tag{7.1.6}$$

where $J(\sigma)$is the Integrand of Q Using the OPEs 7.1.1, 7.1.2,7.1.3, 7.1.4, 7.1.5 and the expression for 7.0.50. For the sake of brevity I will only include the terms of Q that will affect the expression.

$$s(\lambda^a) = \oint \mathrm{d}\sigma \ \left[c\big(\frac{1}{2}\lambda^b\partial\omega_b - \frac{1}{2}\omega_b\partial\lambda^b\big) + c_bG^b\right]\lambda^a \tag{7.1.7}$$

The stress tensor terms return a similar term to 6.1.5 with a c ghost, likewise the G^b term produces a transformation analagous to 6.1.11. So the resultant transformation is.

$$s(\lambda^a) = c\partial\lambda^a + \frac{1}{2}\lambda^a\partial c - \xi^{ab}c_b \tag{7.1.8}$$

It is important to remember that for Grassmann variables such as the ghosts that when one ghost moves past another the term picks up a sign change due to the change of order. The matter fields λ and ω obey Bose-Einstein statistics and are therefore not Grassmannian. The Conformal ghosts (the bc system) the Gauge ghosts (the b_ac_a system) and the Ghosts arising from the 2nd level reducibility (the **BC** system) obey Fermi-Dirac statistics and are Grassmanian. The Ghosts from the 1st level reducibility obey Bose-Einstein statistics as they compensate an over fixing of the degrees of freedom by the Gauge fixing. The next transformation is for the conformal c ghosts.

$$s(c) = \oint \mathrm{d}\sigma \; Q(\sigma)c(z) \tag{7.1.9}$$

$$= \oint \mathrm{d}\sigma \; \left[\frac{1}{2}c\big(-(\partial b)c - 2b\partial c\big)\right]c \tag{7.1.10}$$

$$= c\partial c \tag{7.1.11}$$

This is the standard result for any free conformal field theory see [25]. The next logical transformation to do is for the 1st gauge ghost

$$s(c_a) = \oint \mathrm{d}\sigma \; Q(\sigma)c_a(z) \tag{7.1.12}$$

$$= \oint \mathrm{d}\sigma \; \left[c\big(-\frac{1}{2}(\partial b^b)c_b - \frac{3}{2}b^b\partial c_b\big) + c_\mu Z^\mu_b b^b + \frac{1}{2}f^{bc}_d c_b c_c b^d\right]c^a \tag{7.1.13}$$

$$= c_\mu Z^\mu_a - \frac{1}{2}\partial c c_a + c\partial c_a + \frac{1}{2}f^{bc}_a c_b c_c \tag{7.1.14}$$

This already starts to bring in the reducibility constraints into the expressions that wouldn't be present in the transformations for a non reducible gauge system. Looking at the ghost from the 1st level reducibility.

$$s(c_\mu) = \oint \mathrm{d}\sigma \; Q(\sigma)c_\mu(z) \tag{7.1.15}$$

$$= \oint \mathrm{d}\sigma \; \left[c\big(-(\partial b^\nu)c_\nu - 2b^\nu\partial c_\nu\big) + \mathbf{C}Z_\nu b^\nu - D^{a\nu}_\lambda c_a c_\nu b^\lambda + \frac{1}{3}M^{abc}_\lambda c_a c_b c_c b^\lambda\right]c_\mu \tag{7.1.16}$$

$$= -\mathbf{C}P_\mu - \partial c c_\mu + c\partial c_\mu + D^{a\nu}_\mu c_a c_\nu + \frac{1}{3}M^{abc}_\mu c_a c_b c_c \tag{7.1.17}$$

Finally obtaining the transformations for the ghosts from the level 2 reducibility.

$$s(\mathbf{C}) = \oint \mathrm{d}\sigma \; Q(\sigma)\mathbf{C} \tag{7.1.18}$$

$$= \oint \mathrm{d}\sigma \; \left[c\big(-2(\partial\mathbf{B})\mathbf{C} - 3\mathbf{B}\partial\mathbf{C}\big) + \frac{1}{2}B^{\mu\nu}c_\mu c_\nu\mathbf{B} - D^a c_a\mathbf{C}\mathbf{B} + \frac{1}{2}M^{ab\mu}c_a c_b c_\mu\mathbf{B}\right]\mathbf{C} \tag{7.1.19}$$

$$= -2\partial c\mathbf{C} + c\partial\mathbf{C} + \frac{1}{2}B^{\mu\nu}c_\mu c_\nu - D^a c_a\mathbf{C} + \frac{1}{2}M^{ab\mu}c_a c_b c_\mu \tag{7.1.20}$$

As the single transformations for the matter field, λ^a and the c ghosts are known the other functions or terms that it is useful to have expressions for their transformations can be identified. ξ^{ab}, P_μ, Z^μ_a, f^{bc}_a, $D^{a\nu}_\mu$, M^{abc}_μ, $B^{\mu\nu}$, D^a and $M^{ab\mu}$. The transformations of these functions can be obtained by finding their dependence on λ. This can be done by looking at the definition of the function for ξ^{ab}, P_μ and Z^μ_a or by looking at the conformal weights of the ghosts associated with the functions as each term in the BRST charge must have conformal weight 1.

$$[f_a^{bc}] = \frac{1}{2} \quad \Rightarrow \quad \text{homogenous in } \lambda^a \text{ of order 1} \tag{7.1.21}$$

$$[D_\mu^{a\nu}] = \frac{1}{2} \quad \Rightarrow \quad \text{homogenous in } \lambda^a \text{ of order 1} \tag{7.1.22}$$

$$[M_\mu^{abc}] = \frac{1}{2} \quad \Rightarrow \quad \text{homogenous in } \lambda^a \text{ of order 1} \tag{7.1.23}$$

$$[B^{\mu\nu}] = 0 \quad \Rightarrow \quad \text{independent of } \lambda^a \tag{7.1.24}$$

$$[D^a] = \frac{1}{2} \quad \Rightarrow \quad \text{homogenous in } \lambda^a \text{ of order 1} \tag{7.1.25}$$

$$[M^{ab\mu}] = 0 \quad \Rightarrow \quad \text{independent of } \lambda^a \tag{7.1.26}$$

From these results the single transformations for the functions are

$$s(\xi^{ab}) = \partial_c \xi^{ab}(c\partial\lambda^c + \frac{1}{2}\lambda^c\partial c - \xi^{cd}c_d) \tag{7.1.27}$$

$$s(P_\mu) = (\partial_a P_\mu)(s(\lambda^a)) = \partial c P_\mu + c\partial P_\mu \tag{7.1.28}$$

$$s(Z_a^\mu) = \Gamma_{ab}^\mu(c\partial\lambda^b + \frac{1}{2}\lambda^b\partial c - \xi^{bc}c_c) \tag{7.1.29}$$

$$s(f_a^{bc}) = \partial_d f_a^{bc}(c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c - \xi^{de}c_e) \tag{7.1.30}$$

$$s(D_\mu^{a\nu}) = \partial_d D_\mu^{a\nu}(c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c - \xi^{de}c_e) \tag{7.1.31}$$

$$s(M_\mu^{abc}) = \partial_d M_\mu^{abc}(c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c - \xi^{de}c_e) \tag{7.1.32}$$

$$s(B^{\mu\nu}) = 0 \tag{7.1.33}$$

$$s(D^a) = (\partial_d D^a)(c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c - \xi^{de}c_e) \tag{7.1.34}$$

$$s(M^{ab\mu}) = 0 \tag{7.1.35}$$

After working out all of the transformations on all functions and fields for the single operations of the BRST charge all that needs to be done now is to apply those transformations to the fields and functions that appear after one operation. Starting with λ^a the double operation is obtained by the formula below

$$s^2(\lambda^a(z)) = \oint \mathrm{d}\sigma \; Q(\sigma)s(\lambda^a) \tag{7.1.36}$$

$$s^2(\lambda^a) = (s(c))\partial\lambda^a - c(\partial(s(\lambda^a))) + \frac{1}{2}(\partial(s(c)))\lambda^a - \frac{1}{2}\partial c(s(\lambda^a)) - (s(\xi^{ab}))c_b - \xi^{ab}(s(c_b)) \tag{7.1.37}$$

After inserting the expressions for the transformations into the expression above:

$$s^2(\lambda^a) = c\partial c\partial\lambda^a - c\partial\big(c\partial\lambda^a + \frac{1}{2}\lambda^a\partial c - \xi^{ab}c_b\big) + \frac{1}{2}\partial\big(c\partial c\big)\lambda^a$$
$$- \frac{1}{2}\partial c\big(c\partial\lambda^a + \frac{1}{2}\lambda^a\partial c - \xi^{ab}c_b\big) - \big(\partial_c\xi^{ab}(c\partial\lambda^c + \frac{1}{2}\lambda^c\partial c - \xi^{cd}c_d)\big)c_b$$
$$- \xi^{ab}\big(c_\mu Z^\mu_b - \frac{1}{2}\partial c c_b + c\partial c_b + \frac{1}{2}f^{de}_b c_d c_e\big) \tag{7.1.38}$$

Expanding all of the terms above

$$s^2(\lambda^a) = c\partial c\partial\lambda^a - c\partial c\partial\lambda^a - cc\partial^2\lambda^a - \frac{1}{2}c\partial\lambda^a\partial c - \frac{1}{2}c\lambda^a\partial^2 c + c\partial(\xi^{ab})c_b$$
$$+ c\xi^{ab}\partial c_b + \frac{1}{2}\partial c\partial c\lambda^a + \frac{1}{2}c\partial^2 c\lambda^a - \frac{1}{2}\partial cc\partial\lambda^a - \frac{1}{4}\partial c\lambda^a\partial c + \frac{1}{2}\partial c\xi^{ab}c_b$$
$$- (\partial_c\xi^{ab})c\partial\lambda^c c_b - \frac{1}{2}(\partial_c\xi^{ab})\lambda^c\partial c c_b + (\partial_c\xi^{ab})\xi^{cd}c_d c_b - \xi^{ab}c_\mu Z^\mu_b + \frac{1}{2}\xi^{ab}\partial c c_b$$
$$- \xi^{ab}c\partial c_b - \frac{1}{2}\xi^{ab}f^{de}_b c_d c_e \tag{7.1.39}$$

Straight away there are terms that can be eliminated. specifically $cc\partial^2\lambda^a = 0$, $\frac{1}{2}\partial c\partial c\lambda^a = 0$ and $\frac{1}{4}\partial c\lambda^a\partial c = 0$ this is because of the Grassmanality of c that any terms of c^2 are automatically zero. The next terms that can be removed are the ones that can be summed together, in some cases to zero, $c\partial c\partial\lambda^a - c\partial c\partial\lambda^a = 0$, $-\frac{1}{2}c\partial\lambda^a\partial c - \frac{1}{2}\partial cc\partial\lambda^a = 0$ by reordering of c to match the terms and $-\frac{1}{2}c\lambda^a\partial^2 c + \frac{1}{2}c\partial^2 c\lambda^a = 0$. Also as a results of the constraints of the reducibility 7.0.18 means $\xi^{ab}Z^\mu_b c_\mu = 0$. This reduces the expression to the terms that arise from the gauge symmetry and the cross terms between conformal and gauge symmetries.

$$s^2(\lambda^a) = c\partial(\xi^{ab})c_b + c\xi^{ab}\partial c_b + \frac{1}{2}\partial c\xi^{ab}c_b - (\partial_c\xi^{ab})c\partial\lambda^c c_b - \frac{1}{2}(\partial_c\xi^{ab})\lambda^c\partial c c_b$$
$$+ (\partial_c\xi^{ab})\xi^{cd}c_d c_b + \frac{1}{2}\xi^{ab}\partial c c_b - \xi^{ab}c\partial c_b - \frac{1}{2}\xi^{ab}f^{de}_b c_d c_e \tag{7.1.40}$$

By first looking at the derivatives of ξ^{ab}

$$c\partial(\xi^{ab}c_b) \Rightarrow c\partial_c\xi^{ab}\partial\lambda^c c_b + c\xi^{ab}\partial c_b \tag{7.1.41}$$
$$\therefore$$
$$c\partial_c\xi^{ab}\partial\lambda^c c_b - \partial_c\xi^{ab}(c\partial\lambda^c + \frac{1}{2}\lambda^c\partial c - \xi^{cd}c_d)c_b$$
$$= \partial_c\xi^{ab}(c\partial\lambda^c - c\partial\lambda^c - \frac{1}{2}\lambda^c\partial c - \xi^{cd}c_d)c_b \tag{7.1.42}$$
$$\tag{7.1.43}$$

after cancellation the only terms that are derivatives of ξ^{ab}.

$$-\frac{1}{2}\lambda^c\partial_c\xi^{ab}\partial c c_b + \xi^{cd}\partial_c\xi^{ab}c_d c_b \tag{7.1.44}$$

Now considering terms linear in ξ^{ab}:

$$\xi^{ab}(c\partial c_b + \frac{1}{2}\partial c c_b + \frac{1}{2}\partial c c_b - c\partial c_b - \frac{1}{2}f^{cd}_b c_c c_d) \tag{7.1.45}$$

The first and fourth term in this expression cancel through addition and summing the second and third terms leaves.

$$\xi^{ab}\partial c c_b - \frac{1}{2}\xi^{ac}f^{bd}_c c_b c_d \tag{7.1.46}$$

Combining both linear terms and derivative terms we have

$$s^2(\lambda^a) = (-\frac{1}{2}\lambda^c\partial_c\xi^{ab} + \xi^{ab})\partial c c_b + \frac{1}{2}(\xi^{bc}\partial_c\xi^{ad} - \xi^{dc}\partial_c\xi^{ab} - \xi^{ac}f^{bd}_c)c_b c_d \tag{7.1.47}$$

as ξ^{ab} is homogeneous in λ of order $+2$. we know that $\lambda^c\partial_c\xi^{ab} \equiv 2\xi^{ab}$. The $\partial c c_b$ terms cancel through addition. For Q to be Nilpotent on λ^a then the remaining terms allow the derivation of f^{bc}_a as

$$\xi^{bc}\partial_c\xi^{ad} - \xi^{dc}\partial_c\xi^{ab} - \xi^{ac}f^{bd}_c = 0$$

Resulting in

$$f^{bc}_a = -4\lambda^{[b}\delta^{c]}_a$$

This is the same result for the structure function for the lie algebroid formed by the gauge constraint and confirms that it is linear in λ.

7.2 Nilpotency of Q on conformal ghosts and 1st order gauge ghosts

From here-on out the results of the BRST operations will be summarised and full details of the calculations are given in the appendix A. The procedure for the nilpotency of of Q acting on c is the same for most free conformal field theories.

$$s^2(c) = 2c\partial c\partial c + cc\partial c \tag{7.2.1}$$
$$\therefore s^2(c) = 0 \tag{7.2.2}$$

Both terms cancel exactly to zero, due to the grassmanality of c, as is expected. Moving on to the gauge ghost transformations will begin to produce some of the constraints for the structure functions that arise due to the reducibility.

$$s^2(c_a) = (-\mathbf{C}P_\mu - \partial c c_\mu + c\partial c_\mu + D_\mu^{a\nu} c_a c_\nu + \frac{1}{3} M_\mu^{abc} c_a c_b c_c) Z_a^\mu \tag{7.2.3}$$

$$+ c_\mu \Gamma^\mu_{ab}(c\partial\lambda^b + \frac{1}{2}\lambda^b \partial c - \xi^{bc} c_c) - \frac{1}{2}\partial(c\partial c) c_a \tag{7.2.4}$$

$$+ \frac{1}{2}\partial c(c_\mu Z_a^\mu - \frac{1}{2}\partial c c_a + c\partial c_a + \frac{1}{2} f_a^{bc} c_b c_c) + c\partial c \partial c_a \tag{7.2.5}$$

$$- c\partial(c_\mu Z_a^\mu - \frac{1}{2}\partial c c_a + c\partial c_a + \frac{1}{2} f_a^{bc} c_b c_c) \tag{7.2.6}$$

$$+ \frac{1}{2}\partial_d f_a^{bc}(c\partial\lambda^d + \frac{1}{2}\lambda^d \partial c - \xi^{de} c_e) c_b c_c \tag{7.2.7}$$

$$+ f_a^{bc}(c_\mu Z_b^\mu - \frac{1}{2}\partial c c_b + c\partial c_b + \frac{1}{2} f_b^{de} c_d c_e) c_c \tag{7.2.8}$$

Terms in this expression can be identified as zero are $-\mathbf{C}P_\mu Z_a^\mu = 0$ as it is the constraint that leads to the second level reducibility and $-\frac{1}{4}\partial d\partial c c_a = 0$ as $c^2 = 0$. Once expanded out and reordered into coefficients of the ghosts more terms can be seen to be zero, they include: c_a, ∂c_a, and $c_b \partial c_c$ through summation of the terms and c^2 terms, c_μ terms also vanish as $c\partial Z_a^\mu = \Gamma^\mu_{ab} c\partial\lambda^b$.The coefficient of $c_b c_c$ is zero due to the homogeneity of $f_a^{bc} = -4\lambda^{[b}\delta_a^{c]}$. The remaining terms do not cancel algebraically and the double transformation becomes.

$$s^2(c_a) = c_\mu c_b (D_\nu^{b\mu} Z_a^\nu - \Gamma^\mu_{ab}\xi^{bc} + f_a^{bc} Z_c^\mu) \tag{7.2.9}$$

$$+ c_b c_c c_d (-\frac{1}{2}\partial_d f_a^{bc} \xi^{de} + \frac{1}{3} M_\mu^{bcd} Z_a^\mu + \frac{1}{2} f_a^{ed} f_e^{bc}) \tag{7.2.10}$$

From this the following constraints are obtained.

$$D_\nu^{b\mu} Z_a^\nu - (\partial_c Z_a^\mu)\xi^{bc} + f_a^{bc} Z_c^\mu = 0 \tag{7.2.11}$$

And

$$-\frac{1}{2}(\partial_e f_a^{[cd|})\xi^{e|b]} + \frac{1}{2} f_a^{ed} f_e^{bc} + \frac{1}{3} M_\mu^{bcd} Z_a^\mu = 0 \tag{7.2.12}$$

Using the derived values for the M_μ^{abc} functions 7.0.55 the constraint 7.2.12 can be given as.

$$(\partial_e f_a^{[cd|})\xi^{e|b]} = f_a^{ed} f_e^{bc} \tag{7.2.13}$$

These constraints are required to be true for the BRST charge to be nilpotent on the field c_a

7.3 Nilpotency of Q on 2nd order gauge ghosts

Acting the BRST charge twice on the 2nd order ghosts results in the expression

$$
\begin{aligned}
s^2(c_\mu) &= -(-2\partial c\mathbf{C} + c\partial\mathbf{C} + \frac{1}{2}B^{\mu\nu}c_\mu c_\nu - D^a c_a\mathbf{C} + \frac{1}{2}M^{ab\mu}c_a c_b c_\mu)P_\mu \\
&+ \mathbf{C}(\partial c P_\mu + c\partial P_\mu) - c\partial(-\mathbf{C}P_\mu - \partial c c_\mu + c\partial c_\mu + D^{a\nu}_\mu c_a c_\nu + \frac{1}{3}M^{abc}_\mu c_a c_b c_c) \\
&- \partial(c\partial c)c_\mu + \partial c(-\mathbf{C}P_\mu - \partial c c_\mu + c\partial c_\mu + D^{a\nu}_\mu c_a c_\nu + \frac{1}{3}M^{abc}_\mu c_a c_b c_c) + (c\partial c)\partial c_\mu \\
&+ \partial_d D^{a\nu}_\mu (c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c - \xi^{de}c_e)c_a c_\nu \\
&+ D^{a\nu}_\mu (c_\mu Z^\mu_a - \frac{1}{2}\partial c c_a + c\partial c_a + \frac{1}{2}f^{bc}_a c_b c_c)c_\nu \\
&- D^{a\nu}_\mu c_a(-\mathbf{C}P_\nu - \partial c c_\nu + c\partial c_\nu + D^{a\lambda}_\nu c_a c_\nu + \frac{1}{3}M^{abc}_\nu c_a c_b c_c) \\
&+ \frac{1}{3}\partial_d M^{abc}_\mu (c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c - \xi^{de}c_e)c_a c_b c_c \\
&+ M^{abc}_\mu (c_\nu Z^\nu_a - \frac{1}{2}\partial c c_a + c\partial c_a + \frac{1}{2}f^{de}_a c_d c_e)c_b c_c
\end{aligned}
\tag{7.3.1}
$$

Once again by ordering this into the coefficients of the different ghosts and there derivatives there are terms that are zero. These are the coefficients of cc_μ, $c\partial\mathbf{C}$, $\partial c\mathbf{C}$, $c\partial c_\mu$, $c_a\partial c_\nu$, $c\mathbf{C}$, $\partial c_a c_\nu$ and $c_a\mathbf{C}$. For the Nilpotency of the BRST charge to be satisfied then these constraints must be applied.

$$D^a P_\mu = D^{a\nu}_\mu P_\nu \tag{7.3.2}$$

$$B^{\nu\lambda}P_\mu = 2D^{a[\nu}_\mu Z^{\lambda]}_a \tag{7.3.3}$$

$$(\partial_e M^{[abc|}_\mu)\xi^{e|d]} + \frac{3}{2}M^{e[cd}_\mu f^{ab]}_e - D^{[a|\nu}_\mu M^{|bcd]}_\nu = 0 \tag{7.3.4}$$

$$(\partial_d D^{[a|\nu}_\mu)\xi^{d|b]} + \frac{1}{2}D^{c\nu}_\mu f^{ab}_c - D^{[a|\nu}_\lambda D^{|b]\lambda}_\mu + M^{cab}_\mu Z^\nu_c - \frac{1}{2}M^{ab\nu}P_\mu = 0 \tag{7.3.5}$$

by inspection the above constraint 7.3.4 can be reduced to zero due to the derived expression of 7.0.55. By the same token 7.3.5 can be simplified by using 7.0.55 and 7.0.56 to give.

$$(\partial_d D^{[a|\nu}_\mu)\xi^{d|b]} + \frac{1}{2}D^{c\nu}_\mu f^{ab}_c - D^{[a|\nu}_\lambda D^{|b]\lambda}_\mu = 0 \tag{7.3.6}$$

The application of these constraints reduces the expression to zero as $(\partial_d D^{a\nu}_\mu)\lambda^d = D^{a\nu}_\mu$ and $(\partial_d M^{abc}_\mu)\lambda^d = M^{abc}_\mu$. Therefore, by imposing the above constraints Q is nilpotent on c_μ.

7.4 Nilpotency of Q on 3rd order gauge ghosts

The expression obtained by acting the BRST charge on the field $\mathbf{C}$ is

$$\begin{aligned}
s^2(\mathbf{C}) =& -2(c\partial c)\mathbf{C} + 2\partial c(-2\partial c\mathbf{C} + c\partial\mathbf{C} + \frac{1}{2}B^{\mu\nu}c_\mu c_\nu - D^a c_a\mathbf{C} + \frac{1}{2}M^{ab\mu}c_a c_b c_\mu) \\
&+ c\partial c\partial\mathbf{C} - c\partial(-2\partial c\mathbf{C} + c\partial\mathbf{C} + \frac{1}{2}B^{\mu\nu}c_\mu c_\nu - D^a c_a\mathbf{C} + \frac{1}{2}M^{ab\mu}c_a c_b c_\mu)
\end{aligned} \tag{7.4.1}$$

$$\begin{aligned}
&+ \frac{1}{2}s(B^{\mu\nu})c_\mu c_\nu + B^{\mu\nu}(-\mathbf{C}P_\mu - \partial c c_\mu + c\partial c_\mu + D^{a\nu}_\mu c_a c_\nu + \frac{1}{3}M^{abc}_\mu c_a c_b c_c)c_\nu \\
&- (\partial_d D^a)(c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c - \xi^{de}c_e)c_a\mathbf{C} - D^a(c_\mu Z^\mu_a - \frac{1}{2}\partial c c_a + c\partial c_a + \frac{1}{2}f^{bc}_a c_b c_c)\mathbf{C} \\
&+ D^a c_a(-2\partial c\mathbf{C} + c\partial\mathbf{C} + \frac{1}{2}B^{\mu\nu}c_\mu c_\nu - D^b c_b\mathbf{C} + \frac{1}{2}M^{bc\mu}c_b c_c c_\mu) \\
&+ \frac{1}{2}(\partial_d M^{ab\mu})(c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c - \xi^{de}c_e)c_a c_b c_\mu
\end{aligned} \tag{7.4.2}$$

$$\begin{aligned}
&+ M^{ab\mu}(c_\nu Z^\nu_a - \frac{1}{2}\partial c c_a + c\partial c_a + \frac{1}{2}f^{cd}_a c_c c_d)c_b c_\mu \\
&- \frac{1}{2}M^{ab\mu}c_a c_b(-\mathbf{C}P_\mu - \partial c c_\mu + c\partial c_\mu + D^{c\nu}_\mu c_c c_\nu + \frac{1}{3}M^{cde}_\mu c_c c_d c_e)
\end{aligned} \tag{7.4.3}$$

The transformations for $B^{\mu\nu}$ and $M^{ab\mu}$ have already been shown to be trivial 7.1.33 and 7.1.35. Reorganising the terms into coefficients of the ghosts as has been done before yields an expression that allows for the easy identification of trivial terms. Once that is done the coefficients that are removed are$c\mathbf{C}$, $c\partial\mathbf{C}$, $\partial c_\mu c_\nu$, $\partial c c_a\mathbf{C}$, $c\partial c_\mu c_\nu$, $\partial c c_a c_b c_\mu$, $c c_a\partial\mathbf{C}$, $c\partial c_\mu c_\nu$, $c c_a\partial\mathbf{C}$, $c\partial c_a c_b c_\mu$ and $c c_a c_b\partial c_\mu$ which leaves the expression from which the constraints are obtained.

$$\begin{aligned}
s^2(\mathbf{C}) =& \,\partial c c_a\mathbf{C}(D^a - \frac{1}{2}(\partial_d D^a)\lambda^d + \frac{1}{2}D^a + 2D^a) \\
&+ \mathbf{C}c_\nu(-B^{\mu\nu} - D^a Z^\nu_a) + c_a c_\lambda c_\nu(B^{\mu\nu}D^{a\lambda}_\mu + \frac{1}{2}D^a B^{\nu\lambda} + M^{ab\lambda}Z^\nu_b) \\
&+ c_a c_b c_c c_\nu(\frac{1}{3}B^{\mu\nu}M^{abc}_\mu + \frac{1}{2}D^a M^{bc\nu} + \frac{1}{2}M^{ad\nu}f^{bc}_d - \frac{1}{2}M^{ab\mu}D^{a\nu}_\mu) \\
&+ c_a c_b\mathbf{C}(-(\partial_d D^a)\xi^{db} - \frac{1}{2}D^c f^{ab}_c - D^a D^b + \frac{1}{2}M^{ab\mu}P_\mu) \\
&- c_a c_b c_c c_d c_e(\frac{1}{6}M^{ab\mu}M^{cde}_\mu)
\end{aligned} \tag{7.4.4}$$

For the Nilpotency of Q to hold, the constraints that must be applied are

$$M^{[ab|\mu}M_{\mu}^{|cde]} = 0 \tag{7.4.5}$$

$$B^{\mu\nu}P_{\mu} = -D^{a}Z_{a}^{\nu} \tag{7.4.6}$$

$$B^{\mu[\nu|}D_{\mu}^{a|\lambda]}+\frac{1}{2}D^{a}B^{\nu\lambda} + M^{ab[\lambda|}Z_{a}^{|\nu]} = 0 \tag{7.4.7}$$

$$(\partial_{d}D^{[a|})\xi^{d|b]}+\frac{1}{2}D^{c}f_{c}^{ab} + D^{[a}D^{b]} - \frac{1}{2}M^{ab\mu}P_{\mu} = 0 \tag{7.4.8}$$

$$\frac{1}{3}B^{\mu\nu}M_{\mu}^{abc} + \frac{1}{2}D^{[a|}M^{|bc]\nu}-\frac{1}{2}(\partial_{d}M^{[ab|\nu})\xi^{d|c]} + \frac{1}{2}M^{[a|d\nu}f_{d}^{|bc]} - \frac{1}{2}M^{[ab|\mu}D_{\mu}^{|c]\nu} = 0 \tag{7.4.9}$$

Once again applying the expressions 7.0.55 and 7.0.56 then 7.4.9 becomes zero by definition and 7.4.7 and 7.4.8 become.

$$B^{\mu[\nu|}D_{\mu}^{a|\lambda]} = -\frac{1}{2}D^{a}B_{a}^{\nu\lambda} \tag{7.4.10}$$

$$(\partial_{d}D^{[a|})\xi^{d|b]}+\frac{1}{2}D^{c}f_{c}^{ab} + D^{[a}D^{b]} = 0 \tag{7.4.11}$$

The fact that the BRST charge is classically nilpotent on this many fields under these constraint is good evidence to show that the expression for Q is correct. There remains the question as to what the critical dimension of the theory by computing Q^2 The issue with computing this is that it is a very long calculation not simply just because of the length of the expression as the single transformations of the b ghosts are not difficult to obtain. The real difficulty lies in deriving the transformation of ω_a as there are many functions that are linear or quadratic in λ for the conjugate field contract with.

8 Gauge fixing procedure and extended action

Now that a full expression for Q has been obtained we need to have an action that serves to gauge fix the extra ghosts that have arisen because of the reducibility and also accounts for the cross terms between the gauge symmetry, conformal symmetry and reducibility. Thankfully a recent paper by Ohmori [11] a way of producing a Gauge fixed, BRST exact action can be done by inserting terms in a method that is analogous to Witten's paper that sheds light on the concept of the picture changing operator [38]. The theory explored by Ohmori is the previously discussed Ambitwistor String theory 4.2 produced by Mason and Skinner [2] Which we hope to be physically equivalent to the theory outlined in this thesis. Ohmori's treatment begins by defining a 2d conformal field theory with T^m being the holomorphic stress tensor and $\tilde{T}^m$ is the gauge term that appears as $-\frac{1}{2}P^2$ in this theory and as before to impose the same gauge constraint there is the insertion of the Beltrami differential

e to act as the Lagrange multiplier. This leads to the inclusion of the term $e\tilde{T}^m$ in the action. When considering the theory the ghost systems must be taken into account. In the Mason-Skinner theory these systems are bc and $\tilde{b}\tilde{c}$ with stress tensors. $T_{bc} = -\partial bc - 2b\partial c$ and $T_{\tilde{b}\tilde{c}} = -\partial\tilde{b}\tilde{c} - 2\tilde{b}\partial\tilde{c}$. These are used to construct the BRST charge.

$$Q = \oint \mathrm{d}z \left[cT^m + cT_{\tilde{b}\tilde{c}} + \frac{1}{2}cT_{bc} + \tilde{c}\tilde{T}^m \right] \tag{8.0.1}$$

I have omitted the total derivative term that usually appears at the end of the current as it only serves to make sure the integrand is a current and will play no further part in the following calculations. From the BRST charge when it interacts with the b ghosts it returns a term for the combined stress tensor for matter and ghost fields

$$\begin{aligned} Q \cdot b &= T^m + T_{\tilde{b}\tilde{c}} + T_{bc} &(8.0.2)\\ &= T &(8.0.3) \end{aligned}$$

This can be inserted into the action coupled with a Lagrange multiplier μ. The Transformation for the $\tilde{b}$ ghost produces

$$Q \cdot \tilde{b} = \tilde{T}^m - (\partial\tilde{b})c - 2\tilde{b}\partial c = \tilde{T} \tag{8.0.4}$$

The term $\tilde{T}^m$remains but there are the more terms which arise from cross terms between the conformal ghosts and gauge ghosts these are then grouped into one term $\tilde{T}$. This term is then inserted with a Beltrami differential, e rather than simply inserting a term for the gauge constraints. With the BRST transformation on the Beltrami differential defined as $\{Q, e\} = \delta e$.The action can then be made BRST exact if it is extended with the addition of a $(\delta e\ \tilde{b})$ term. Likewise for the stress tensor term with a Beltrami differential μ so that $\{Q, \mu\} = \delta\mu$ the addition of $(\delta\mu\ T)$ will make the action BRST exact.

However, for this method to be used in the context of the theory being explored in this thesis it must be altered due to the reducibility of the gauge theory. The procedure stays much the same only there are more ghost systems and therefore more terms inserted into the action for gauge fixing. A suitable starting point is to compute the results of the BRST charge acting on the b ghosts.

$$\begin{aligned} s(b) = \oint \mathrm{d}\sigma \Big[& c\big(\frac{1}{2}\lambda^a\partial\omega_a - \frac{1}{2}\omega_a\partial\lambda^a\big) + \frac{c}{2}\big(- (\partial b)c - 2b\partial c\big) \\ & + c\big(- \frac{1}{2}(\partial b^a)c_a - \frac{3}{2}b^a\partial c_a\big) + c\big(- \partial b^\mu)c_\mu - 2b^\mu\partial c_\mu\big) \\ & + c\big(- 2(\partial\mathbf{B})\mathbf{C} - 3\mathbf{B}\partial\mathbf{C}\big)\Big] b(z) \end{aligned} \tag{8.0.5}$$

The only term in this that requires much thought is the second term but it can be shown to simply reduce to the original stress tensor term from which it is derived

$$
\begin{aligned}
\oint \mathrm{d}\sigma\ \frac{c}{2}\big(-(\partial b)c - 2b\partial c\big)b(z) &= \oint \mathrm{d}\sigma\ \big(-cb\partial c\big)b(z) \\
&= -b\partial c + \partial(cb) \\
&= -2b\partial c - c\partial b \\
&= T^{(c)}_{gh}
\end{aligned} \tag{8.0.6}
$$

Once this has been done all of the other terms are simple to work out and leads to a summation of all stress tensor terms 7.0.45, 7.0.46, 7.0.47, 7.0.48 and 7.0.49 therefore.

$$
s(b) = T_m + T^{(c)}_{gh} + T^{(0)}_{gh} + T^{(1)}_{gh} + T^{(2)}_{gh} \tag{8.0.7}
$$

The transformation for b^a is given by:

$$
\begin{aligned}
s(b^a) = \oint \mathrm{d}\sigma\ \Big[&c_b G^b - \frac{1}{2}c\partial b^b c_b - \frac{3}{2}cb^b\partial c_b + \frac{1}{2}f^{bc}_{d}c_bc_cb^d - D^{b\nu}_{\mu}c_bc_\nu b^\mu \\
&+ \frac{1}{3}M^{bcd}_{e}c_bc_cc_db^e - D^b c_b\mathbf{C}\mathbf{B} + \frac{1}{2}M^{bc\mu}c_bc_cc_\mu\mathbf{B}\Big]b^a(z)
\end{aligned} \tag{8.0.8}
$$

This leads to the transformation being

$$
s(b^a) = G^a - 2c\partial b^a - \frac{3}{2}\partial cb^a - f^{ab}_{c}c_bb^c - D^{a\nu}_{\mu}c_\nu b^\mu + M^{abc}c_bc_cb^\mu - D^a\mathbf{C}\mathbf{B} + M^{ab\mu}c_bc_\mu\mathbf{B} \tag{8.0.9}
$$

Breaking down the transformation there one can see clearly the terms that are from the gauge constraint, the cross terms and the functions that exist as a consequence of the reducibility. The other ghosts transform as

$$
\begin{aligned}
s(b^\mu) = \oint \mathrm{d}\sigma\ \Big[&c_\nu Z^\nu_a b^a - c\partial b^\mu c_\mu - 2cb^\mu\partial c_\mu - D^{a\lambda}_{\nu}c_ac_\lambda b^\nu \\
&+ \frac{1}{2}B^{\nu\lambda}c_\nu c_\lambda\mathbf{B} + \frac{1}{2}M^{ab\nu}c_ac_bc_\nu\mathbf{B}\Big]b^\mu(z)
\end{aligned} \tag{8.0.10}
$$

This becomes

$$
s(b^\mu) = Z^\mu_a b^a - 3c\partial b^\mu - 2\partial cb^\mu + D^{a\mu}_{\nu}c_ab^\nu + \frac{1}{2}B^{\mu\nu}c_\nu\mathbf{B} + \frac{1}{2}M^{ab\mu}c_ac_b\mathbf{B} \tag{8.0.11}
$$

Finally the last transformation is

$$
s(\mathbf{B}) = \oint \mathrm{d}\sigma\ \Big[\mathbf{C}P_\mu b^\mu - 2c\partial\mathbf{B}\mathbf{C} - 3c\mathbf{B}\partial\mathbf{C} - D^a c_a\mathbf{C}\mathbf{B}\Big]\mathbf{B}(z) \tag{8.0.12}
$$

$$
= P_\mu b^\mu - 5c\partial\mathbf{B} - 3\partial c\mathbf{B} + D^a c_a\mathbf{B} \tag{8.0.13}
$$

These can be inserted into the action, along with the standard gauge fixing terms for ghosts as follows.

$$S = \int_\Sigma Z^I \bar{\partial} Z_I + b\bar{\partial}c + b^a\bar{\partial}c_a + b^\mu\bar{\partial}c_\mu + \mathbf{B}\bar{\partial}\mathbf{C} + \mu(s(b)) + h_a(s(b^a)) + h_\mu(s(b^\mu)) + \mathbf{h}(s(\mathbf{B})) \tag{8.0.14}$$

Then by imposing the BRST transformations of the Lagrange multipliers as follows

$$\begin{aligned} s(\mu) &= \delta\mu \\ s(h_a) &= \delta h_a \\ s(h_\mu) &= \delta h_\mu \\ s(\mathbf{h}) &= \delta\mathbf{h} \end{aligned}$$

The new action can be modified so that it is BRST exact by the addition of the extended action

$$S_{ext} = \int_\Sigma \delta\mu b + \delta h_a b^a + \delta h_\mu b^\mu + \delta\mathbf{h}\mathbf{B} \tag{8.0.15}$$

So we now have a new action, $\hat{S} = S + S_{ext}$ and $s(S_{ext}) = -\delta\mu(s(b)) - \delta h_a(s(b^a)) - \delta h_\mu(s(b^\mu)) - \delta\mathbf{h}(s(\mathbf{B}))$ assuming the lagrange multipliers are BRST exact. This can be further simplified by grouping all of these new terms into one expression.

$$\mathcal{W} = \mu b + h_a b^a + h_\mu b^\mu + \mathbf{h}\mathbf{B} \tag{8.0.16}$$

so that the action can be written as:

$$\hat{S} = \int_\Sigma Z^I \bar{\partial} Z_I + b\bar{\partial}c + b^a\bar{\partial}c_a + b^\mu\bar{\partial}c_\mu + \mathbf{B}\bar{\partial}\mathbf{C} + \{Q, \mathcal{W}\} \tag{8.0.17}$$

Where $\mathcal{W}$ is the expression given in 8.0.16 and now the action is fairly simple representing a free topological theory.

9 Conclusion

The research conducted in this thesis has outlined a string theory in the form of Twistor variables which as it has been derived through transformation of the same action as was discussed in [2] the assumption can be made that they represent the same theory in a different context. The gauge transformations and conformal transformations were laid out to describe the theory and the reducibility of the gauge theory. Then the full form of the BRST charge for a reducible gauge theory was derived along with the structure functions that arise due to said reducibility. The BRST charge has been shown to be nilpotent on the λ matter fields and all c ghosts under certain constraints which strongly suggests that the BRST charge constructed is correct. Once the BRST charge was obtained an alternative method for gauge fixing was explored that had previously been applied to the Ambitwistor string. Following that method the extended action for the theory which incorporates all of the new ghost systems and the there stress tensors is written along with a BRST exact term that encodes the BRST symmetry and the reducibility in one term and keeps the action BRST invariant.

9.1 Future questions

Unfortunately due to time constraints I have not been able to explore this theory more and as such there are still a lot of questions that would be interesting to find the answer and I hope to outline some here.

Critical dimension and supersymmetry.

There still remains the question of what the critical dimension of this theory is. Naively, one would expect it to be 26 as is the case with Bosonic string theories. However, that may not be the case, and the full quantum calculations must be done. There is also the question of how supersymmetry would affect the BRST charge. Also the question of what type of supersymmetry would be most appropriate. The supersymmetry used in [1] by Berkovits is space-time supersymmetry, but now that the theory has been extended to a worldsheet theory there is the possibility for the application of RNS supersymmetry.

Vertex operators and Scattering amplitudes

Another interesting thing to do with this theory would be to produce scattering amplitudes as was done in [2] and investigate them. But before that the vertex operators for the theory need to be worked out. One would assume that it would not be as simple as inserting the incidence relations into the expressions for vertex operators in section 4.2 due to the requirement of the extra ghost systems and the requirement for ghosts to be inserted with the vertex operators. One alternative method for deriving the vertex operator insertions can be found in [38] and perhaps this method could shed some light on the form of vertex operators in this theory.

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A Full Nilpotency Calculations

A.1 Nilpotency on conformal ghost

The nilpotency for the conformal c ghost is trivial and the same example can be found in many conformal field theory textbooks.

$$s^2(c) = (s(c))\partial c - c\partial(s(c)) \tag{A.1.1}$$
$$s^2(c) = (c\partial c)\partial c - c\partial(c\partial c) \tag{A.1.2}$$
$$s^2(c) = 2c\partial c\partial c + cc\partial c \tag{A.1.3}$$
$$\therefore s^2(c) = 0 \tag{A.1.4}$$

A.2 Nilpotency on first order gauge ghost

$$s^2(c_a) = (-\mathbf{C}P_\mu - \partial c c_\mu + c\partial c_\mu + D^{a\nu}_\mu c_a c_\nu + \frac{1}{3}M^{abc}_\mu c_a c_b c_c)Z^\mu_a \tag{A.2.1}$$
$$+ c_\mu \Gamma^\mu_{ab}(c\partial\lambda^b + \frac{1}{2}\lambda^b\partial c - \xi^{bc}c_c) - \frac{1}{2}\partial(c\partial c)c_a \tag{A.2.2}$$
$$+ \frac{1}{2}\partial c(c_\mu Z^\mu_a - \frac{1}{2}\partial c c_a + c\partial c_a + \frac{1}{2}f^{bc}_a c_b c_c) + c\partial c\partial c_a \tag{A.2.3}$$
$$- c\partial(c_\mu Z^\mu_a - \frac{1}{2}\partial c c_a + c\partial c_a + \frac{1}{2}f^{bc}_a c_b c_c) \tag{A.2.4}$$
$$+ \frac{1}{2}\partial_d f^{bc}_a(c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c - \xi^{de}c_e)c_b c_c \tag{A.2.5}$$
$$+ f^{bc}_a(c_\mu Z^\mu_b - \frac{1}{2}\partial c c_b + c\partial c_b + \frac{1}{2}f^{de}_b c_d c_e)c_c \tag{A.2.6}$$

The terms $-\mathbf{C}P_\mu Z^\mu_a = 0$ as it is the constraint that leads to the second level reducibility and $-\frac{1}{4}\partial d\partial c c_a = 0$ as $c^2 = 0$. By reorganising the expression into coefficients of the ghosts and there derivatives.

$$s^2(c_a) = c_\mu(-\partial c Z^\mu_a + \frac{1}{2}\partial c Z^\mu_a + \Gamma^\mu_{ab}c\partial\lambda^b + \frac{1}{2}\Gamma^\mu_{ab}\lambda^b\partial c - c\partial Z^\mu_a) \tag{A.2.7}$$
$$+ \partial c_\mu(-cZ^\mu_a + cZ^\mu_a) + c_a(-\frac{1}{2}\partial(c\partial c) + \frac{1}{2}c\partial^2 c) + c_\mu c_b(D^{b\mu}_\nu - \Gamma^\mu_{ac}\xi^{bc} + f^{ba}_c Z^\mu_b) \tag{A.2.8}$$
$$+ \partial c_a(\frac{1}{2}\partial c c + c\partial c + \frac{1}{2}c\partial c - c\partial c) + c_b c_c(\frac{1}{4}\partial c f^{bc}_a - \frac{1}{2}c\partial\lambda^d(\partial_d f^{bc}_a) \tag{A.2.9}$$
$$+ \frac{1}{2}(\partial_d f^{bc}_a)(c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c) - \frac{1}{2}f^{bc}_a\partial c) + c_b\partial c_c(cf^{bc}_a - cf^{bc}_a) \tag{A.2.10}$$
$$+ c_b c_c c_d(-\frac{1}{2}(\partial_e f^{cd}_a)\xi^{eb} + \frac{1}{2}f^{ed}_a f^{bc}_e + \frac{1}{3}M^{bcd}_\mu Z^\mu_a) \tag{A.2.11}$$

By inspection the coefficients of c_a, ∂c_a, and $c_b\partial c_c$ are zero through summation of the terms and c^2 terms. The c_μ terms also vanish as $c\partial Z^\mu_a = \Gamma^\mu_{ab}c\partial\lambda^b$. Two terms in the coefficient of $c_b c_c$ cancel $-\frac{1}{2}c\partial\lambda^d(\partial_d f^{bc}_a) = \frac{1}{2}(\partial_d f^{bc}_a)c\partial\lambda^d = 0$ simplifying the expression further to

$$s^2(c_a) = c_\mu c_b(D^{b\mu}_\nu - \Gamma^\mu_{ac}\xi^{bc} + f^{ba}_c Z^\mu_b) \tag{A.2.12}$$
$$+ c_b c_c(\frac{1}{4}\partial c f^{bc}_a + \frac{1}{2}(\partial_d f^{bc}_a)(\frac{1}{2}\lambda^d\partial c) - \frac{1}{2}f^{bc}_a\partial c) \tag{A.2.13}$$
$$+ c_b c_c c_d(-\frac{1}{2}(\partial_e f^{cd}_a)\xi^{eb} + \frac{1}{2}f^{ed}_a f^{bc}_e + \frac{1}{3}M^{bcd}_\mu Z^\mu_a) \tag{A.2.14}$$

The coefficient of $c_b c_c$ is zero due to the homogeneity of $f^{bc}_a = -4\lambda^{[b}\delta^{c]}_a$

$$(\partial_d f^{bc}_a) = -\frac{1}{2}\partial_d(4\lambda^b\delta^c_a - 4\lambda^c\delta^b_a) = -2(\delta^b_d\delta^c_a - \delta^c_d\delta^b_a) \equiv -4\delta^{bc}_{da} \tag{A.2.15}$$
$$\therefore \frac{1}{4}(\partial_d f^{bc}_a)\lambda^d\partial c = -\frac{1}{4}\lambda^d 4\delta^{bc}_{da}\partial c = -4\lambda^{[b}\delta^{c]}_a\partial c = \frac{1}{4}f^{bc}_a\partial c \tag{A.2.16}$$

The remaining terms do not cancel algebraically and the double transformation becomes.

$$s^2(c_a) = c_\mu c_b(D^{b\mu}_\nu Z^\nu_a - \Gamma^\mu_{ab}\xi^{bc} + f^{bc}_a Z^\mu_c) \tag{A.2.17}$$

$$+ c_b c_c c_d(-\frac{1}{2}\partial_d f^{bc}_a \xi^{de} + \frac{1}{3}M^{bcd}_\mu Z^\mu_a + \frac{1}{2}f^{ed}_a f^{bc}_e) \tag{A.2.18}$$

This then implies the following constraints when writing $\Gamma^\mu_{ab}\xi^{bc} = (\partial_c Z^\mu_a)\xi^{ab}$:

$$D^{b\mu}_\nu Z^\nu_a - (\partial_c Z^\mu_a)\xi^{bc} + f^{bc}_a Z^\mu_c = 0 \tag{A.2.19}$$

And

$$-\frac{1}{2}(\partial_e f^{[cd|}_a)\xi^{e|b]} + \frac{1}{2}f^{ed}_a f^{bc}_e + \frac{1}{3}M^{bcd}_\mu Z^\mu_a = 0 \tag{A.2.20}$$

These constraints make it so that the BRST charge is nilpotent on c_a.

A.3 Nilpotency for 2nd order ghost terms

Moving on to the next level of reducibility.

$$s^2(c_\mu) = -(s(\mathbf{C})P_\mu + \mathbf{C}(s(P_\mu)) - \partial(s(c))c_\mu + \partial c(s(c_\mu)) + (s(c))\partial c_\mu \tag{A.3.1}$$

$$- c\partial(s(c_\mu)) + (s(D^{a\lambda}_\mu)c_a c_\lambda + D^{a\lambda}_\mu(s(c_a))c_\lambda - D^{a\lambda}_\mu c_a(s(c_\lambda)) \tag{A.3.2}$$

$$+ \frac{1}{3}(s(M^{abc}_\mu))c_a c_b c_c + M^{abc}_\mu(s(c_a))c_b c_c \tag{A.3.3}$$

inserting the transformations into the above expression leads to the following

$$\begin{aligned}
s^2(c_\mu) = &-(-2\partial c\mathbf{C} + c\partial\mathbf{C} + \frac{1}{2}B^{\mu\nu}c_\mu c_\nu - D^a c_a\mathbf{C} + \frac{1}{2}M^{ab\mu}c_a c_b c_\mu)P_\mu \\
&+ \mathbf{C}(\partial cP_\mu + c\partial P_\mu) - c\partial(-\mathbf{C}P_\mu - \partial cc_\mu + c\partial c_\mu + D^{a\nu}_\mu c_a c_\nu + \frac{1}{3}M^{abc}_\mu c_a c_b c_c) \\
&- \partial(c\partial c)c_\mu + \partial c(-\mathbf{C}P_\mu - \partial cc_\mu + c\partial c_\mu + D^{a\nu}_\mu c_a c_\nu + \frac{1}{3}M^{abc}_\mu c_a c_b c_c) + (c\partial c)\partial c_\mu \\
&+ \partial_d D^{a\nu}_\mu(c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c - \xi^{de}c_e)c_a c_\nu \\
&+ D^{a\nu}_\mu(c_\mu Z^\mu_a - \frac{1}{2}\partial cc_a + c\partial c_a + \frac{1}{2}f^{bc}_a c_b c_c)c_\nu \\
&- D^{a\nu}_\mu c_a(-\mathbf{C}P_\nu - \partial cc_\nu + c\partial c_\nu + D^{a\lambda}_\nu c_a c_\nu + \frac{1}{3}M^{abc}_\nu c_a c_b c_c) \\
&+ \frac{1}{3}\partial_d M^{abc}_\mu(c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c - \xi^{de}c_e)c_a c_b c_c \\
&+ M^{abc}_\mu(c_\nu Z^\nu_a - \frac{1}{2}\partial cc_a + c\partial c_a + \frac{1}{2}f^{de}_a c_d c_e)c_b c_c
\end{aligned} \tag{A.3.4}$$

Once again collecting coefficients of the ghosts and there derivatives makes it easier to see which terms are zero through addition or because of identities.

$$
\begin{aligned}
s^2(c_\mu) &= cc_\mu\big(\partial^2 c - \partial^2 c\big) + c\partial\mathbf{C}\big(-P_\mu + P_\mu\big) + \partial c\mathbf{C}\big(2P_\mu - P_\mu - P_\mu\big)\\
&+ c\partial c_\mu\big(\partial c - \partial c - \partial c + \partial c\big) + c_a\partial c_\nu\big(cD^{a\nu}_\mu - cD^{a\nu}_\mu\big) + c\mathbf{C}\big(P_\mu - P_\mu\big)\\
&+ \partial c_a c_\nu\big(cD^{a\nu}_\mu - cD^{a\nu}_\mu\big) + c_a\mathbf{C}(D^a P_\mu - D^{a\nu}_\mu P_\nu) + (-\frac{1}{2}B^{\nu\lambda}P_\mu + D^{a\nu}_\mu Z^\lambda_a)c_\nu c_\lambda\\
&+ \big[(\partial_d D^{a\nu}_\mu)(c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c) - \frac{1}{2}D^{a\nu}_\mu\partial c - D^{a\nu}_\mu\partial c + \partial c D^{a\nu}_\mu - c(\partial_d D^{a\nu}_\mu)\partial\lambda^d\big]c_a c_\nu\\
&+ \big[-\frac{1}{2}M^{ab\nu}P_\mu + (\partial_d D^{a\nu}_\mu)\xi^{db} + \frac{1}{2}D^{c\nu}_\mu f^{ab}_c - D^{a\lambda}_\mu D^{b\nu}_\lambda + M^{abc}_\mu Z^\nu_c\big]c_a c_b c_\nu\\
&+ \big[\frac{1}{3}\partial c M^{abc}_\mu - \frac{1}{3}c(\partial_d M^{abc}_\mu)\partial\lambda^d + \frac{1}{3}(\partial_d M^{abc}_\mu)(c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c) - \frac{1}{2}M^{abc}_\mu\partial c\big]c_a c_b c_c\\
&+ \big[-\frac{1}{3}D^{a\nu}_\mu M^{bcd}_\nu + \frac{1}{3}(\partial_e M^{abc}_\mu)\xi^{de} + \frac{1}{2}M^{ecd}_\mu f^{ab}_e\big]c_a c_b c_c c_d
\end{aligned}
\tag{A.3.5}
$$

I have grouped together the terms that sum to zero at the start of the expression so the coefficients of cc_μ, $c\partial\mathbf{C}$, $\partial c\mathbf{C}$, $c\partial c_\mu$, $c_a\partial c_\nu$, $c\mathbf{C}$, $\partial c_a c_\nu$ and $c_a\mathbf{C}$ can be removed from the expression. The remaining terms in the expression must each be zero for Q to be nilpotent on c_μ so from this expression constraints must be applied.

$$
\begin{aligned}
s^2(c_\mu) &= (D^a P_\mu - D^{a\nu}_\mu P_\nu)c_a\mathbf{C} + (-\frac{1}{2}B^{\nu\lambda}P_\mu + D^{a\nu}_\mu Z^\lambda_a)c_\nu c_\lambda\\
&+ \big[(\partial_d D^{a\nu}_\mu)(c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c) - \frac{1}{2}D^{a\nu}_\mu\partial c - D^{a\nu}_\mu\partial c + \partial c D^{a\nu}_\mu - c(\partial_d D^{a\nu}_\mu)\partial\lambda^d\big]c_a c_\nu\\
&+ \big[-\frac{1}{2}M^{ab\nu}P_\mu + (\partial_d D^{a\nu}_\mu)\xi^{db} + \frac{1}{2}D^{c\nu}_\mu f^{ab}_c - D^{a\lambda}_\mu D^{b\nu}_\lambda + M^{abc}_\mu Z^\nu_c\big]c_a c_b c_\nu\\
&+ \big[\frac{1}{3}\partial c M^{abc}_\mu - \frac{1}{3}c(\partial_d M^{abc}_\mu)\partial\lambda^d + \frac{1}{3}(\partial_d M^{abc}_\mu)(c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c) - \frac{1}{2}M^{abc}_\mu\partial c\big]c_a c_b c_c\\
&+ \big[-\frac{1}{3}D^{a\nu}_\mu M^{bcd}_\nu + \frac{1}{3}(\partial_e M^{abc}_\mu)\xi^{de} + \frac{1}{2}M^{ecd}_\mu f^{ab}_e\big]c_a c_b c_c c_d
\end{aligned}
\tag{A.3.6}
$$

The simplest constraints that can be seen are from the coefficients of $c_a\mathbf{C}$ and $c_\nu c_\lambda$ as for each of these terms to be zero then the constraints must be satisfied

$$D^a P_\mu = D^{a\nu}_\mu P_\nu \tag{A.3.7}$$
$$B^{\nu\lambda}P_\mu = 2D^{a[\nu}_\mu Z^{\lambda]}_a \tag{A.3.8}$$

] Naively they can be considered to be suitable as the terms on the right hand side match the the conformal weights of the terms of the left hand side. The next constraint is from the coefficient of $c_a c_b c_c c_d$ through anti-symmetrization over a, b, c and d

$$(\partial_e M^{[abc|}_\mu)\xi^{e|d]} + \frac{3}{2}M^{e[cd}_\mu f^{ab]}_e - D^{[a|\nu}_\mu M^{|bcd]}_\nu = 0 \tag{A.3.9}$$

The next constraint that must be imposed is.

$$(\partial_d D^{[a|\nu}_{\mu})\xi^{d|b]} + \frac{1}{2}D^{c\nu}_{\mu} f^{ab}_{c} - D^{[a|\nu}_{\lambda} D^{|b]\lambda}_{\mu} + M^{cab}_{\mu} Z^{\nu}_{c} - \frac{1}{2}M^{ab\nu} P_{\mu} = 0 \qquad \text{(A.3.10)}$$

Once these constraints are applied the only terms left are

$$s^2(c_\mu) = \big[(\partial_d D^{a\nu}_{\mu})(c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c) - \frac{1}{2}D^{a\nu}_{\mu}\partial c - D^{a\nu}_{\mu}\partial c + \partial c D^{a\nu}_{\mu} - c(\partial_d D^{a\nu}_{\mu})\partial\lambda^d\big] c_a c_\nu$$
$$+ \big[\frac{1}{3}\partial c M^{abc}_{\mu} - \frac{1}{3}c(\partial_d M^{abc}_{\mu})\partial\lambda^d + \frac{1}{3}(\partial_d M^{abc}_{\mu})(c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c) - \frac{1}{2}M^{abc}_{\mu}\partial c\big] c_a c_b c_c \qquad \text{(A.3.11)}$$

These remaining terms are zero as $(\partial_d D^{a\nu}_{\mu})\lambda^d = D^{a\nu}_{\mu}$ because $D^{a\nu}_{\mu}$ is linear in λ and as such the coefficient of $c_a c_\nu$ sums to zero and $(\partial_d M^{abc}_{\mu})\lambda^d = M^{abc}_{\mu}$ so the same process occurs for the coefficient of $c_a c_b c_c$. Therefore, by imposing the above constraints Q is nilpotent on c_μ

A.4 Nilpotency for 3rd order gauge ghosts

$$s^2(\mathbf{C}) = -2(\partial(s(c)))\mathbf{C} + 2\partial c(s(\mathbf{C})) + (s(c))\partial\mathbf{C} - c(\partial(s(\mathbf{C}))) + \frac{1}{2}(s(B^{\mu\nu}))c_\mu c_\nu$$
$$+ B^{\mu\nu}(s(c_\mu))c_\nu - s(D^a)c_a\mathbf{C} - D^a(s(c_a))\mathbf{C} + D^a c_a(s(\mathbf{C})) + \frac{1}{2}s(M^{ab\mu})c_a c_b c_\mu$$
$$+ M^{ab\mu}(s(c_a))c_b c_\mu + \frac{1}{2}M^{ab\mu}c_a c_b(s(c_\mu)) \qquad \text{(A.4.1)}$$

By inserting the expressions for the transformations this becomes.

$$s^2(\mathbf{C}) = -2(c\partial c)\mathbf{C} + 2\partial c(-2\partial c\mathbf{C} + c\partial\mathbf{C} + \frac{1}{2}B^{\mu\nu}c_\mu c_\nu - D^a c_a\mathbf{C} + \frac{1}{2}M^{ab\mu}c_a c_b c_\mu)$$
$$+ c\partial c\partial\mathbf{C} - c\partial(-2\partial c\mathbf{C} + c\partial\mathbf{C} + \frac{1}{2}B^{\mu\nu}c_\mu c_\nu - D^a c_a\mathbf{C} + \frac{1}{2}M^{ab\mu}c_a c_b c_\mu) \qquad \text{(A.4.2)}$$

$$+ \frac{1}{2}s(B^{\mu\nu})c_\mu c_\nu + B^{\mu\nu}(-\mathbf{C}P_\mu - \partial c c_\mu + c\partial c_\mu + D^{a\nu}_{\mu}c_a c_\nu + \frac{1}{3}M^{abc}_{\mu}c_a c_b c_c)c_\nu$$
$$- (\partial_d D^a)(c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c - \xi^{de}c_e)c_a\mathbf{C} - D^a(c_\mu Z^\mu_a - \frac{1}{2}\partial c c_a + c\partial c_a + \frac{1}{2}f^{bc}_{a}c_b c_c)\mathbf{C}$$
$$+ D^a c_a(-2\partial c\mathbf{C} + c\partial\mathbf{C} + \frac{1}{2}B^{\mu\nu}c_\mu c_\nu - D^b c_b\mathbf{C} + \frac{1}{2}M^{bc\mu}c_b c_c c_\mu)$$
$$+ \frac{1}{2}(\partial_d M^{ab\mu})(c\partial\lambda^d + \frac{1}{2}\lambda^d\partial c - \xi^{de}c_e)c_a c_b c_\mu \qquad \text{(A.4.3)}$$

$$+ M^{ab\mu}(c_\nu Z^\nu_a - \frac{1}{2}\partial c c_a + c\partial c_a + \frac{1}{2}f^{cd}_{a}c_c c_d)c_b c_\mu$$
$$- \frac{1}{2}M^{ab\mu}c_a c_b(-\mathbf{C}P_\mu - \partial c c_\mu + c\partial c_\mu + D^{c\nu}_{\mu}c_c c_\nu + \frac{1}{3}M^{cde}_{\mu}c_c c_d c_e) \qquad \text{(A.4.4)}$$

As as been defined earlier in 7.1.33 and 7.1.35 the transformations of these functions must be zero as the conformal weights of the functions imply that these functions are independent of λ and any other field therefore the transformations of $B^{\mu\nu}$ and $M^{ab\mu}$ is trivial. As done previously the expression is split into coefficients of the ghosts and the derivatives.

$$\begin{aligned}
s^2(\mathbf{C}) &= c\mathbf{C}(-2\partial^2 c + 2\partial^2 c) + c\partial\mathbf{C}(2\partial c + \partial c - \partial c - 2\partial c) + \partial c_\mu c_\nu(B^{\mu\nu} - B^{\mu\nu}) \\
&+ \partial c c_a \mathbf{C}(-2D^a - \frac{1}{2}(\partial_d D^a)\lambda^d + \frac{1}{2}D^a + 2D^a) + c\partial c_\mu c_\nu(-B^{\mu\nu} + B^{\mu\nu}) \\
&+ \partial c c_a c_b c_\mu(M^{ab\mu} - \frac{1}{2}M^{ab\mu} - \frac{1}{2}M^{ab\mu}) + c c_a \partial\mathbf{C}((\partial_d D^a)\partial\lambda^d - (\partial_d D^a)\partial\lambda^d) \\
&+ c\partial c_\mu c_\nu(D^a - D^a) + c c_a \partial\mathbf{C}(D^a - D^a) + c\partial c_a c_b c_\mu(-M^{ab\mu} + M^{ab\mu}) \\
&+ c c_a c_b \partial c_\mu(-\frac{1}{2}M^{ab\mu} + \frac{1}{2}M^{ab\mu}) + \mathbf{C}c_\nu(-B^{\mu\nu}P_\mu - D^a Z_a^\nu) \\
&+ c_a c_\lambda c_\nu(B^{\mu\nu}D_\mu^{a\lambda} + \frac{1}{2}D^a B^{\lambda\nu} + M^{ab\lambda}Z_b^\nu) + c_a c_b c_c c_d c_e(-\frac{1}{6}M^{ab\mu}M_\mu^{cde}) \\
&+ c_a c_b c_c c_\nu(\frac{1}{3}B^{\mu\nu}M_\mu^{abc} + \frac{1}{2}D^a M^{bc\nu} + \frac{1}{2}M^{ad\nu}f_d^{bc} - \frac{1}{2}M^{ab\mu}D_\mu^{c\nu}) \\
&+ c_a c_b \mathbf{C}(-(\partial_d D^a)\xi^{db} - \frac{1}{2}D^c f_c^{ab} - D^a D^b + \frac{1}{2}M^{ab\mu}P_\mu)
\end{aligned} \tag{A.4.5}$$

After the omission of the trivial transformations it can be seen that the terms in the coefficients of $c\mathbf{C}$, $c\partial\mathbf{C}$, $\partial c_\mu c_\nu$, $\partial c c_a\mathbf{C}$, $c\partial c_\mu c_\nu$, $\partial c c_a c_b c_\mu$, $c c_a \partial\mathbf{C}$, $c\partial c_\mu c_\nu$, $c c_a\partial\mathbf{C}$, $c\partial c_a c_b c_\mu$ and $c c_a c_b \partial c_\mu$ all sum to zero leaving only

$$\begin{aligned}
s^2(\mathbf{C}) &= \partial c c_a \mathbf{C}(D^a - \frac{1}{2}(\partial_d D^a)\lambda^d + \frac{1}{2}D^a + 2D^a) \\
&+ \mathbf{C}c_\nu(-B^{\mu\nu} - D^a Z_a^\nu) + c_a c_\lambda c_\nu(B^{\mu\nu}D_\mu^{a\lambda} + \frac{1}{2}D^a B^{\nu\lambda} + M^{ab\lambda}Z_b^\nu) \\
&+ c_a c_b c_c c_\nu(\frac{1}{3}B^{\mu\nu}M_\mu^{abc} + \frac{1}{2}D^a M^{bc\nu} + \frac{1}{2}M^{ad\nu}f_d^{bc} - \frac{1}{2}M^{ab\mu}D_\mu^{a\nu}) \\
&+ c_a c_b \mathbf{C}(-(\partial_d D^a)\xi^{db} - \frac{1}{2}D^c f_c^{ab} - D^a D^b + \frac{1}{2}M^{ab\mu}P_\mu) \\
&- c_a c_b c_c c_d c_e(\frac{1}{6}M^{ab\mu}M_\mu^{cde})
\end{aligned} \tag{A.4.6}$$

The constraints that must be satisfied for the Q to be nilpotent are as follows.

$$M^{[ab|\mu}M_\mu^{|cde]} = 0 \tag{A.4.7}$$

$$B^{\mu\nu}P_\mu = -D^a Z_a^\nu \tag{A.4.8}$$

$$B^{\mu[\nu|}D_\mu^{a|\lambda]} + \frac{1}{2}D^a B^{[\nu\lambda]} + M^{ab[\lambda|}Z_a^{|\nu]} = 0 \tag{A.4.9}$$

$$\partial_d D^{[a|}\xi^{d|b]} + \frac{1}{2}D^c f_c^{ab} + D^{[a}D^{b]} - \frac{1}{2}M^{ab\mu}P_\mu = 0 \tag{A.4.10}$$

$$\frac{1}{3}B^{\mu\nu}M_\mu^{abc} + \frac{1}{2}D^{[a|}M^{|bc]\nu} - \frac{1}{2}(\partial_d M^{[ab|\nu})\xi^{d|c]} + \frac{1}{2}M^{[a|d\nu}f_d^{|bc]} - \frac{1}{2}M^{[ab|\mu}D_\mu^{|c]\nu} = 0 \tag{A.4.11}$$