Simple exclusion process: from randomness to determinism

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Abstract In this work I introduce a classical example of an Interacting Particle System: the Simple Exclusion Process. I present the notion of hydrodynamic limit, which is a Law of Large Numbers for the empirical measure and an heuristic argument to derive from the microscopic dynamics between particles a partial differential equation describing the evolution of the density profile. For the Simple Exclusion Process, in the Symmetric case ($p = 1/2$) we will get to the heat equation while in the Asymmetric case ($p \neq 1/2$) to the inviscid Burgers equation. Finally, I introduce the Central Limit Theorem for the empirical measure and the limiting process turns out to be a solution of a stochastic partial differential equation.

1 Introduction

In this work I am presenting some well known results and some of the latest developments on a classical interacting particle system: the simple exclusion process (SEP). Interacting particle systems were introduced by Spitzer in the late 70's and since then, their study has attracted the attention of researchers of several fields of Mathematics. The problems that initially appeared, have arisen from the physicists and the goal was to give precise answers to conjectures and experiments done by the physics community. Now I describe the idea behind the problems that we usually deal with. Suppose that one is interested in analyzing the evolution of some physical system, constituted by a large number of components, for example, a fluid or a gas. Due to the large number of molecules it becomes hard to analyze the microscopic evolution of the system, and as a consequence it is more relevant to ana-

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lyze the macroscopic evolution of the structure of that system. Following the approach proposed by Boltzmann from the Statistical Mechanics, first one finds the equilibrium states of this physical system and characterizes them through macroscopic quantities, called thermodynamical quantities that one is interested in analyzing such as the pressure, temperature, density... The natural question that follows is to analyze the behavior of that physical system out of equilibrium. The characterization and study of phenomena out of equilibrium is one of the biggest challenge of the Statistical Physics and despite its long history, nowadays, it still has not been found a satisfactory answer to this kind of problems. From this approach some differential equations arise that provide some information about the macroscopic evolution of the thermodynamical quantities of the system. Usually, and at least heuristically, these equations can be deduced from the scaling limit of a system and this deduction gives validity to this equation. When approaching these problems, due to the huge complexity of its analysis, some simplifications need to be introduced. With that purpose, usually one assumes that the underlying microscopic dynamics, i.e. the dynamics between molecules, is stochastic, in such a way that a probabilistic analysis of the system can be done. Assuming that the particles (or molecules) behave as interacting random walks subjected to random local restrictions, arise the so called **Interacting Particle Systems** [6]. Nowadays, there exists a well developed theory to deal with this kind of problems, that consists on the microscopic analysis of a particle system - a continuous time Markov process, whose macroscopic evolution of the density profile is governed by one (or system) partial differential equation, denominated by *Hydrodynamic Limit* [5]. This research field, deals and answers to the discretization of several partial differential equations, which have solutions with different qualitative behavior and whose microscopic dynamics has originated the study of different particle systems with hydrodynamic behavior.

Usually, for all the studied systems, the behavior of the associated partial differential equation, gives information about the behavior of the particle system. Nevertheless, there are several hard phenomena, that are very difficult to analyze in the analytical point of view of the solutions of the partial differential equation which can be analyzed through the study of the underlying microscopic system, as for example the partial differential equations that exhibit shocks, as the Burgers equation [2] (see [3] and references therein). The development of this theory has also provided some answers to questions related to the behavior of physical systems out of equilibrium, see [10].

Here is an outline of these notes. On the second section I introduce the simple exclusion process, generator and the invariant measures. On the third section I give the notion of hydrodynamic limit and an heuristic argument to get to the hydrodynamic equation for two cases, symmetric and asymmetric jumps. Then I give the notion of equilibrium and non-equilibrium fluctuations. Finally at section five by superposing both dynamics one obtains the WASEP and the results above are also stated for this process.

2 The Simple Exclusion Process

In this section I introduce the one-dimensional Exclusion Process. In this process, particles evolve on $\mathbb{Z}$ according to interacting random walks with an exclusion rule which prevents more than one particle per site. The dynamics can be informally described as follows. Fix a probability $p(\cdot)$ on $\mathbb{Z}$. Each particle, independently from the others, waits a mean one exponential time, at the end of which being at the site x it jumps to $x+y$ at rate $p(y)$. If the site is occupied the jump is suppressed to respect the exclusion rule. In both cases, the particle waits a new exponential time. The space state of the Markov process η_t is $\{0,1\}^{\mathbb{Z}}$ and we denote the configurations by the Greek letter η, so that $\eta(x)=0$ if the site x is vacant and $\eta(x)=1$ otherwise. The case in which $p(y)=0\ \forall |y|>1$ is referred as the Simple Exclusion process (SEP) and for the Asymmetric Simple Exclusion process (ASEP) the probability $p(\cdot)$ is such that $p(1)=p$, $p(-1)=1-p$ with $p\neq 1/2$ while in the Symmetric Simple Exclusion process (SSEP) $p=1/2$. The case in which $p=1$ is denoted by TASEP and means totally asymmetric simple exclusion process, since particles can perform jumps only to the right.

The dynamics of the SEP can be translated by means of a generator given on local functions by

$$\mathcal{L}f(\eta)=\sum_{x\in\mathbb{Z}}\sum_{y=x\pm 1}c(x,y,\eta)[f(\eta^{x,y})-f(\eta)],$$

where $c(x,y,\eta)=p(x,y)\eta(x)(1-\eta(y))$ and

$$\eta^{x,y}(z)=\begin{cases}\eta(z), \text{ if } z\neq x,y\\ \eta(y), \text{ if } z=x\\ \eta(x), \text{ if } z=y\end{cases}.$$

To keep notation simple we denote by $\mathcal{L}_S$ ($\mathcal{L}_A$) the generator of the SSEP (ASEP).

Before proceeding we give the definition of an equilibrium state of the system. Let $\eta_{\cdot}$ denote a Markov Process with generator Ω and semigroup $(S(t))_{t\geq 0}$. Let $\mathcal{P}$ denote the set of probability measures on $\{0,1\}^{\mathbb{Z}}$. A probability measure $\mu\in\mathcal{P}$ is said to be an invariant measure for the Markov process if $\mu S(t)=\mu$ for all $t\geq 0$, which is the same as saying that the distribution of $(\eta_t)_t$ does not depend on the time t. There is a nice criterium to find equilibrium states for a Markov Process and we recall it from [6]:

Proposition 1. *Let $\mathcal{I}$ denote the set of probability measures in $\{0,1\}^{\mathbb{Z}}$ and $\eta_{\cdot}$ be a Markov Process with generator Ω. Then*

$$\mathcal{I}=\{\mu:\int\mathcal{L}f(\eta)\mu(d\eta)=0,\forall f \text{ local}\}. \tag{1}$$

For $0 \leq \alpha \leq 1$, denote by ν_α the Bernoulli product measure on $\{0,1\}^{\mathbb{Z}}$ with density α. This means that the random variables $(\eta(x))_{x\in\mathbb{Z}}$ are independent with Bernoulli distribution:

$$\nu_\alpha(\eta(x)=1)=\alpha \tag{2}$$

It is known that ν_α is an invariant measure for the SEP and in fact, that all invariant and translation invariant measures are convex combinations of ν_α if $p(.)$ is such that $p_t(x,y)+p_t(y,x)>0$, $\forall x,y\in\mathbb{Z}^d$ and $\sum_x p(x,y)=1$, $\forall y\in\mathbb{Z}^d$.

3 Hydrodynamic Limit

3.1 From microscopic to macroscopic

In order to investigate the hydrodynamic limit, we need to settle some notation. We are going to consider the physical system evolving in a continuum space - the **macroscopic space**. The idea is to discretize this set by relating it to another one, but this last being a discrete set - the **microscopic space**. In the discrete space we define a particle system and since we want to study the temporal evolution of the density profile we have two different scales for time as well: a **macroscopic time** denoted by t and a **microscopic time** denoted by $t\theta(N)$. This function $\theta(N)$ depends on the subjacent microscopic dynamics and as we will see, for the SSEP we need $\theta(N)=N^2$ while for the ASEP $\theta(N)=N$ is enough. In order to simplify the exposition we suppose that we take the macroscopic space to be the one-dimensional torus $\mathbb{T}$. Then, we fix an integer N and split it in small interval of size $\frac{1}{N}$. The relation between this two sets is that if $u\in\mathbb{T}$ it corresponds to $[uN]$ in the microscopic space while if $x\in\mathbb{T}_N$ it corresponds to x/N in the macroscopic space $\mathbb{T}$.

Suppose now that the simple exclusion process is evolving on $\mathbb{T}_N$. For a given configuration η we define the the **empirical measure** π^N as the positive measure on $\mathbb{T}$ which gives to each particle a mass $1/N$, namely

$$\pi^N(\eta,du)=\frac{1}{N}\sum_{x\in\mathbb{T}_N}\eta(x)\delta_{\frac{x}{N}}(du), \tag{3}$$

where δ_u denotes the Dirac measure at u. Then we consider the time evolution of this measure defined by

$$\pi^N_t(du)=\frac{1}{N}\sum_{x\in\mathbb{T}_N}\eta_t(x)\delta_{\frac{x}{N}}(du) \tag{4}$$

where as usual η_t is the process at time t which is generated by $\mathcal{L}$ when the configuration at time zero is η.

Fix now, an initial profile $\rho_0 : \mathbb{T} \to [0,1]$ and denote by $(\mu_N)_{N\geq 1}$ a sequence of probability measures on $\{0,1\}^{\mathbb{T}_N}$. Depending on the model itself the initial profile ρ_0 needs to satisfy certain conditions that we shall impose later.

Assume that a time 0, the system starts from a initial measure μ_N that is associated to the initial profile ρ_0, ie the empirical measure at time 0 satisfies a law of large numbers:

Definition 1. A sequence $(\mu^N)_{N\geq 1}$ **is associated to** ρ_0, if for every continuous function $H : \mathbb{T} \to \mathbb{R}$ and for every $\delta > 0$

$$\lim_{N\to+\infty} \mu_N \Big[\eta : \Big|\frac{1}{N}\sum_{x\in\mathbb{T}_N} H\Big(\frac{x}{N}\Big)\eta(x) - \int_{\mathbb{T}^d} H(u)\rho_0(u)du\Big| > \delta\Big] = 0. \tag{5}$$

Note that the first term corresponds to the integral of H with respect to π^N, thus the above definition corresponds to asking that the sequence $\pi^N(\eta, du)$ converges in μ_N-probability to $\rho_0(u)du$.

The goal in hydrodynamic limit consists in showing that, if at time $t = 0$ the empirical measures are associated to some initial profile ρ_0, at the macroscopic time t (i.e. the microscopic time $t\theta(N)$) they are associated to a profile ρ_t which is the solution of the some partial differential equation. In other words the aim is to prove that the random measures $\pi^N_{t\theta(N)}$ converge in probability to the deterministic measure $\rho(t,u)du$, which is absolutely continuous with respect to the Lebesgue measure whose density evolves according to some partial differential equation - called hydrodynamic equation.

For the SSEP it was shown that starting from a sequence of measures $(\mu_N)_N$ associated to a profile $\rho_0(\cdot)$, under the **parabolic time scale** tN^2,

$$\pi^N_{tN^2} \xrightarrow[N\to+\infty]{} \rho(t,u)du \tag{6}$$

in $\mu^N S^S_N(t)$-probability, where $\rho(t,u)$ is a weak solution of the parabolic equation

$$\partial_t \rho(t,u) = \frac{1}{2}\Delta\rho(t,u) \tag{7}$$

and S^S_N is the semigroup associated to the generator $\mathcal{L}_S$.

For a proof of last result one can see for example chapter 4 of [5] where the entropy method is applied.

On the other hand, for the ASEP starting from a sequence of measures $(\mu_N)_N$ associated to a profile $\rho_0(.)$ and some additional hypotheses (see [8]) under the **hyperbolic time scale** tN

$$\pi^N_{tN} \xrightarrow[N\to+\infty]{} \rho(t,u)du, \tag{8}$$

in $\mu^N S_N^A(t)$-probability, where $\rho(t,u)$ is the entropy solution of the hyperbolic equation

$$\partial_t \rho(t,u) + (p-q)(1-2\rho(t,u))\nabla\rho(t,u) = 0 \tag{9}$$

known as the inviscid **Burgers equation** and $S_N^A(t)$ is the semigroup associated to the generator $\mathcal{L}_A$.

For a proof of last result we refer the interested reader to [8].

3.2 Hydrodynamic equation

As we have seen above, for the simple exclusion process we obtain the heat equation when considering symmetric jump rates while in the asymmetric jumps one gets to the inviscid Burgers equation. So one can ask, why defining similar microscopic jump rates can we get to completely different macroscopic behaviors. Here I present an heuristic argument relying on the microscopic dynamics to get the hydrodynamic equation for the two different processes, see [5].

In a general setting let η_t denote a Markov process whose generator is denoted by Ω and suppose it is evolving on the microscopic time scale $t\theta(N)$. It is known from the classical theory of Markov processes that, for a test function $H : \mathbb{T} \to \mathbb{R}$

$$M_t^{N,H} =< \pi_t^N, H > - < \pi_0^N, H > - \int_0^t \Omega < \pi_s^N, H > ds \tag{10}$$

is a martingale with respect to the natural filtration $\mathcal{F}_t = \sigma(\eta_s, s \leq t)$, whose quadratic variation is given by

$$\int_0^t \Omega(< \pi_s^N, H >)^2 - 2 < \pi_s^N, H >)\Omega < \pi_s^N, H > ds. \tag{11}$$

Here $< \pi_t^N, H >$ denotes the integral of H with respect to π_t^N. Using the explicit definition of the empirical measure, the integral part of the martingale is written as

$$\int_0^t \frac{1}{N} \sum_{x\in\mathbb{T}_N} H\Big(\frac{x}{N}\Big)\Omega\eta_s(x)ds. \tag{12}$$

Consider now the SEP with generator $\mathcal{L}$. It is easy to see that

$$\mathcal{L}(\eta(x)) = W_{x-1,x}(\eta) - W_{x,x+1}(\eta), \tag{13}$$

where for a site x and a configuration η, $W_{x,x+1}(\eta)$ is the instantaneous current between the sites x and $x+1$, namely

$$W_{x,x+1}(\eta) = p(x,x+1)\eta(x)(1-\eta(x+1)) - p(x+1,x)\eta(x+1)(1-\eta(x)). \tag{14}$$

Since the generator applied to $\eta_s(x)$ is written as gradient, this allows us to perform a summation by parts in the integral part of the martingale:

$$M_t^{N,H} =< \pi_t^N, H > - < \pi_0^N, H > - \int_0^t \frac{1}{N^2} \sum_{x \in \mathbb{T}_N} \nabla^N H\Big(\frac{x}{N}\Big) W_{x,x+1}(\eta_s) ds, \tag{15}$$

where $\nabla^N H$ denotes the discrete derivative of H.

Now we restrict ourselves to the SSEP. In this case the instantaneous current between the sites x and $x+1$, denoted by $W^S_{x,x+1}$ is given by a gradient:

$$W^S_{x,x+1}(\eta) = \frac{1}{2}(\eta(x) - \eta(x+1)).$$

This allows us to perform another summation by parts and write the martingale as

$$M_t^{N,H} =< \pi_t^N, H > - < \pi_0^N, H > - \int_0^t \frac{1}{2N^3} \sum_{x \in \mathbb{T}_N} \Delta_N H\Big(\frac{x}{N}\Big) \eta_s(x) ds, \tag{16}$$

where $\Delta_N H$ denotes the discrete laplacian of H.

Since we want to close the integral part of the martingale in terms of the empirical measure we have to rescale time by tN^2. This together with a change of variables gives us that

$$M_{tN^2}^{N,H} = \frac{1}{N} \sum_{x \in \mathbb{T}_N} H\Big(\frac{x}{N}\Big)\Big(\eta_{tN^2}(x) - \eta_0(x)\Big) - \int_0^t \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_{sN^2}(x) \frac{1}{2} \Delta^N H\Big(\frac{x}{N}\Big) ds. \tag{17}$$

Since this martingale vanishes at time 0 its expectation is equal to zero uniformly in time.

Now we recall the notion of *conservation of local equilibrium* which means, loosely speaking, that for a macroscopic time t, the expectation of η_{tN^2} with respect to the distribution of the system at the microscopic time $t\theta(N)$ is close to the expectation of $\eta(0)$ with respect to $\nu_{\rho(t,x/N)}$.

Then applying expectation to the equality above, we obtain:

$$\frac{1}{N} \sum_{x \in \mathbb{T}_N} H\Big(\frac{x}{N}\Big)\Big(\rho(t, x/N) - \rho(0, x/N)\Big) = \int_0^t \frac{1}{N} \sum_{x \in \mathbb{T}_N} \rho(s, x/N) \frac{1}{2} \Delta^N H\Big(\frac{x}{N}\Big) ds \tag{18}$$

Taking the limit as $N \to +\infty$ if follows that $\rho(t,u)$ is a weak solution of the heat equation:

$$\begin{cases} \partial_t \rho(t,u) = \frac{1}{2} \Delta \rho(t,u) \\ \rho(0,\cdot) = \rho_0(\cdot) \end{cases} . \tag{19}$$

On the other hand for the ASEP, the instantaneous current between x and $x+1$, here denoted by $W^A_{x,x+1}(\eta)$ is given by

$$W^A_{x,x+1}(\eta) = p\eta(x)(1-\eta(x+1)) - q\eta(x+1)(1-\eta(x)). \tag{20}$$

This allows *just one* summation by parts which together with the re-scaling of time by tN and the convergence to local equilibrium gives us

$$\frac{1}{N}\sum_{x\in\mathbb{T}_N} H\Big(\frac{x}{N}\Big)\Big(\rho(t,x/N)-\rho(0,x/N)\Big)+\int_0^t \frac{1}{N}\sum_{x\in\mathbb{T}_N} F(\rho(s,x/N))\nabla^N H\Big(\frac{x}{N}\Big)ds=0$$

where $F(\rho) = (p-q)\rho(1-\rho))$. Taking the limit as $N \to +\infty$ it follows that $\rho(t,u)$ is a weak solution of the inviscid Burgers equation:

$$\begin{cases}\partial_t\rho(t,u) + \nabla F(\rho(t,u)) = 0\\ \rho(0,\cdot) = \rho_0(\cdot)\end{cases}. \tag{21}$$

4 Central Limit Theorem for the empirical measure

4.1 Equilibrium case

Fix $\alpha \in (0,1)$ and take the SEP starting from the invariant state ν_α. Let $k \in \mathbb{N}$ and denote by $\mathcal{H}_k$ the Hilbert space induced by $S(\mathbb{R})$ and $< f,g >_k=< f,(x^2-\Delta)^k g >$, where $< \cdot,\cdot >$ denotes the inner product of $L^2(\mathbb{R})$ and by $\mathcal{H}_{-k}$ the dual of $\mathcal{H}_k$, relatively to this inner product. For a Markov process $\eta_.$ define the density fluctuation field acting on functions $H \in S(\mathbb{R})$ as

$$Y^N_t(H) = \frac{1}{\sqrt{N}}\sum_{x\in\mathbb{Z}} H\Big(\frac{x}{N}\Big)(\eta_t(x)-\alpha). \tag{22}$$

Consider the function below

$$D(\mathbb{R}_+,\{0,1\}^{\mathbb{Z}}) \longrightarrow D(\mathbb{R}_+,\mathcal{H}_{-k})$$

$$\eta_. \longrightarrow Y^N(\eta_.)$$

and let $\mathbb{P}^N_{\nu_\alpha}$ the probability measure on $D(\mathbb{R}_+,\{0,1\}^{\mathbb{Z}})$ induced by ν_α and by the Markov process $\eta_.$ speeded up by $t\theta(N)$; Q_N be the probability measure on $D(\mathbb{R}_+,\mathcal{H}_{-k})$ induced by the density fluctuation $Y^N_.$ and ν_α.

Theorem 1. *(Ravishankar [7]) Fix an integer $k > 3$. Let $\eta_.$ be the SSEP evolving on the parabolic time scale tN^2 starting from ν_α and let Q_N be the probability measure on $D(\mathbb{R}_+,\mathcal{H}_{-k})$ induced by the density fluctuation $Y^N_.$ and ν_α. Let Q be the probability measure on $C(\mathbb{R}^+,\mathcal{H}_{-k})$ corresponding to a stationary mean zero generalized Ornstein-Uhlenbeck process with characteristics $\mathfrak{A} = 1/2\Delta$ and $\mathfrak{B} = \sqrt{\chi(\alpha)}$. Then $(Q_N)_N$ converges weakly to Q.*

Theorem 2. *(G. [4]) Fix an integer $k > 2$. Let $\eta_\cdot$ be the ASEP evolving on the hyperbolic time scale tN starting from ν_α and let Q_N be the probability measure on $D(\mathbb{R}_+, \mathcal{H}_{-k})$ induced by the density fluctuation $Y^N_\cdot$ and ν_α. Let Q be the probability measure on $C(\mathbb{R}^+, \mathcal{H}_{-k})$ corresponding to a stationary Gaussian process with mean 0 and covariance given by*

$$E_Q[Y_t(H)Y_s(G)] = \chi(\alpha)\int_{\mathbb{R}} H(u + v(t-s))G(u)du \tag{23}$$

for every $0 \le s \le t$ and H, G in $\mathcal{H}_k$. Here $\chi(\alpha) = \boldsymbol{Var}(\nu_\alpha, \eta(0)) = \alpha(1-\alpha)$ and $v = (p-q)(1-2\alpha)$. Then, $(Q_N)_N$ converges weakly to Q.

In order to complete the exposition I just give a short presentation of the proof. The idea is to verify that $(Q_N)_N$ is tight and to characterize the limit field. The proof of tightness is technical and details can be found in [5]. So we proceed by characterizing the limit field.

We start by the symmetric case. Fix $H \in S(\mathbb{R})$ and note that

$$M_t^{N,H} = Y_t^N(H) - Y_0^N(H) - \int_0^t \frac{1}{2\sqrt{N}}\sum_{x\in\mathbb{Z}} \Delta_N H\Big(\frac{x}{N}\Big)\eta_s(x)ds \tag{24}$$

$$N_t^{N,H} = (M_t^{N,H})^2 - \int_0^t \frac{1}{N}\sum_{x\in\mathbb{Z}}\Big(\nabla^N H\Big(\frac{x}{N}\Big)\Big)^2\Big[c^S(x,x+1,\eta_s) + c^S(x+1,x,\eta_s)\Big]ds, \tag{25}$$

are martingales with respect to the filtration $\mathcal{F}_t = \sigma(\eta_s, s \le t)$.
Here $c^S(x, x+1, \eta)$ denotes the jump rate from x to $x+1$ in η. It is easy to show that

$$\lim_{N\to+\infty} \mathbb{E}_{\nu_\alpha}\Big[\int_0^t \frac{1}{N}\sum_{x\in\mathbb{Z}} H\Big(\frac{x}{N}\Big)\Big(c^S(x,x+1,\eta) - \frac{1}{2}\alpha(1-\alpha)\Big)\Big]^2 = 0. \tag{26}$$

Then, the limit of the martingale $N_t^{N,H}$ denoted by N_t^H equals to

$$(M_t^H)^2 - ||\mathfrak{B}H||_2^2 t, \tag{27}$$

where M_t^H denotes the limit of the martingale $M_t^{N,H}$ and $\mathfrak{B} = \sqrt{\chi(\alpha)}\nabla$, with $\chi(\alpha) = \alpha(1-\alpha)$. Note that

$$M_t^H = Y_t(H) - Y_0(H) - \int_0^t Y_s(\mathfrak{A}H)ds \tag{28}$$

where $\mathfrak{A} = \frac{1}{2}\Delta$. For each $H \in S(\mathbb{R})$, $B_t^H = ||\mathfrak{B}H||_2^{-1} M_t^H$ is a martingale whose quadratic variation is equal to t which implies that B_t^H is a Brownian motion. Then

$$Y_t(H) = Y_0(H) - \int_0^t Y_s(\mathfrak{A}H)ds + ||\mathfrak{B}H||_2 B_t^H, \tag{29}$$

which means that Y_t satisfies:

$$dY_t = \frac{1}{2}\Delta Y_t dt + \sqrt{\chi(\alpha)}\nabla dB_t \tag{30}$$

Then, one identifies Y_t as a generalized Ornstein-Uhlenbeck process with characteristics $\mathfrak{A} = \frac{1}{2}\Delta$ and $\mathfrak{B} = \sqrt{\chi(\alpha)}\nabla$.

For the asymmetric case fix as well a function $H \in S(\mathbb{R})$. Then

$$M_t^{N,H} = Y_t^N(H) - Y_0^N(H) - \int_0^t \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{Z}} \nabla^N H\Big(\frac{x}{N}\Big) W_{x,x+1}^A(\eta_s)ds \tag{31}$$

is a martingale with respect to $\tilde{\mathfrak{F}}_t = \sigma(\eta_s, s \le t)$, whose quadratic variation is given by

$$\int_0^t \frac{1}{N^2}\sum_{x\in\mathbb{Z}} \Big(\nabla H\Big(\frac{x}{N}\Big)\Big)^2 [p\eta(x)(1-\eta(x+1)) + q\eta(x+1)(1-\eta(x))]ds. \tag{32}$$

Since $\sum_{x\in\mathbb{Z}} \nabla^N H(\frac{x}{N}) = 0$, the integral part of the martingale can be written as:

$$\int_0^t \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{Z}} \nabla^N H\Big(\frac{x}{N}\Big)\Big[W_{x,x+1}^A(\eta_s) - E_{\nu_\alpha}(W_{x,x+1}^A(\eta_s))\Big]ds. \tag{33}$$

As we need to write the expression inside last integral in terms of the fluctuation field Y_s^N, we are able to replace $W_{x,x+1}^A(\eta_s) - E_{\nu_\alpha}(W_{x,x+1}^A(\eta_s))$ by $(p-q)\chi'(\alpha)[\eta_s(x)-\alpha]$, with the use of the:

Theorem 3. *(G. [4])(Boltzmann-Gibbs Principle) For every local function g, for every $H \in S(\mathbb{R})$ and every $t > 0$,*

$$\lim_{N\to\infty} \mathbb{E}_{\nu_\alpha}\Big[\int_0^t \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{Z}} H\Big(\frac{x}{N}\Big)\Big\{\tau_x g(\eta_s) - \tilde{g}(\alpha) - \tilde{g}'(\alpha)[\eta_s(x)-\alpha]\Big\}ds\Big]^2 = 0 \tag{34}$$

where $\tilde{g}(\alpha) = E_{\nu_\alpha}[g(\eta)]$.

Since $\lim_{N\to+\infty} \mathbb{E}_{\nu_\alpha}(M_t^{N,H})^2 = 0$ and by the Boltzmann-Gibbs Principle, the limit density field satisfies

$$Y_t(H) = Y_0(H) - \int_0^t Y_s(\mathfrak{C}H)ds, \tag{35}$$

where $\mathfrak{C} = v\nabla$ with $v = (p-q)(1-2\alpha)$, which in turn means that Y_t satisfies:

$$dY_t = v\nabla Y_t dt. \tag{36}$$

In this case we obtain a simple expression for Y_t given by $Y_t(H) = Y_0(T_t H)$ with $T_t H(u) = H(u + vt)$, which is the semigroup associated to $\mathfrak{C}$.

For $t \geq 0$, let $\mathcal{F}_t$ be the σ-algebra on $D([0,T], \mathcal{H}_{-k})$ generated by $Y_s(H)$ for $s \leq t$ and H in $S(\mathbb{R})$. Restricted to $\mathcal{F}_0$, Q is a Gaussian field with covariance given by

$$E_Q(Y_0(G)Y_0(H)) = \chi(\alpha) < G, H > . \tag{37}$$

I remark here that if one takes $\alpha = 1/2$ then $v = 0$ and as a consequence $Y_t(H) = Y_0(H)$, which means that there is no temporal evolution of the density fluctuation field, so in order to have some non trivial temporal evolution we have to speed up the process in a longer time scale. It is shown in [4] that until the time scale $tN^{4/3}$ the same behavior is observed. Nevertheless it is conjectured by Spohn in [10] that this same behavior is expected until the time scale $tN^{3/2}$.

4.2 Non-equilibrium case

Here I start by stating the Central limit theorem for the empirical measure starting from a Bernoulli product measure of varying parameter for the SSEP. Fix a profile $\rho_0 : \mathbb{R} \to [0,1]$ and denote by $\nu_{\rho_0(\cdot)}$ the product measure on $\{0,1\}^{\mathbb{Z}}$ such that for a site $x \in \mathbb{Z}$:

$$\nu_{\rho_0(\cdot)}(\eta(x) = 1) = \rho_0(x/N). \tag{38}$$

Let $k \in \mathbb{N}$ and define $\mathcal{H}_k$ as above. Let $\rho_t(x) = \mathbb{E}_{\nu_{\rho_0(\cdot)}}[\eta_t(x)]$.

Define the density fluctuation field acting on functions $H \in \mathcal{H}_k$ as

$$Y_t^N(H) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}} H\Big(\frac{x}{N}\Big)(\eta_t(x) - \rho_t(x)). \tag{39}$$

Let Q_N be the probability measure on $D(\mathbb{R}_+, \mathcal{H}_{-k})$ induced by the density fluctuation $Y_\cdot^N$ and $\nu_{\rho_0(\cdot)}$. It was shown by Galves, Kipnis and Spohn the following:

Theorem 4. *Fix $k \geq 4$. Let $\eta(\cdot)$ be the SSEP evolving on the time scale tN^2 and starting from $\nu_{\rho_0(\cdot)}$. Let Q be the probability measure concentrated on $C(\mathbb{R}^+, \mathcal{H}_{-k})$ corresponding to the Ornstein-Uhlenbeck process Y_t with mean zero and covariance given by*

$$E_Q\Big[Y_t(H)Y_s(G)\Big] = \int_{\mathbb{R}} (T_{t-s}H)G\chi_s du - \int_0^s \int_{\mathbb{R}} (T_{t-r}H)(T_{s-r}G)\{\partial_r \chi_r - \Delta \chi_r\} dudr \tag{40}$$

for $0 \leq s < t$ and G, H in$\mathcal{H}_k$. In this expression $(T_t)_t$ denotes the semigroup associated to the Laplacian and χ_s for the function $\chi(s,u) = \rho(s,u)(1 - \rho(s,u))$. Then, the sequence $(Q_N)_N$ converges to Q.

On the other hand the Central Limit Theorem for the empirical measure for the TASEP was shown by Rezakhanlou in [9]. The idea of the proof is to consider the TASEP as a growth model, ie the configuration space consists of functions h: $0 \leq h(i+1) - h(i) \leq 1$ for all $i \in \mathbb{Z}$. With rate one, each $h(i)$ increases by one unit provided that the resulting configuration does not leave the configuration space; otherwise the growth is suppressed. The Central Limit Theorem is established for $\rho_N(x,t) = \frac{1}{N} h([xN, tN])$. Assuming initially that the probability law of $\rho_N(x,0)$ is the same as $g(x) + \sqrt{1/N} B(x) + o(\sqrt{1/N})$ for a continuous function g (piecewise convex) and a continuous random process $B(\cdot)$, then at later times $\rho_N(x,t)$ can be stochastically represented as $\bar{\rho}(x,t) + \sqrt{1/N} Z(x,t) + o(\sqrt{1/N})$ where $\bar{\rho}$ is the unique solution of the corresponding Hamilton-Jacobi equation and $Z(x,t)$ is a random process that is given by a variational expression involving $B(\cdot)$. For more general initial conditions the problem is still open.

5 Superposition of both dynamics

In this section I consider a superposition of both dynamics defined above. This process is called Weakly Asymmetric Simple Exclusion (WASEP) and its generator, denoted by $\mathfrak{L}_W$, is given by:

$$\mathfrak{L}_W = \mathfrak{L}_S + \frac{1}{N} \mathfrak{L}_A, \tag{41}$$

with $\mathfrak{L}_S$ and $\mathfrak{L}_A$ defined as above.

Suppose that the asymmetric part of the generator is given with totally asymmetric jumps to the right. Starting this process from a sequence of measures $(\mu_N)_N$ associated to a profile $\rho_0(\cdot)$, under the **parabolic time scale** tN^2,

$$\pi^N_{tN^2} \xrightarrow[N \to +\infty]{} \rho(t,u) du \tag{42}$$

in $\mu^N S^W_N(t)$-probability, where $\rho(t,u)$ is a weak solution of the Burgers equation with viscosity

$$\partial_t \rho(t,u) + \nabla F(\rho(t,u)) = \frac{1}{2} \Delta \rho(t,u) \tag{43}$$

Here S^W_N is the semigroup associated to the generator $\mathfrak{L}_W$ and $F(\rho) = \rho(1-\rho)$.

On the other hand for the equilibrium Central Limit theorem for the empirical measure for the process speeded up by tN^2 it follows that the density fluctuation field defined as above, converges to a generalized Ornstein-Uhlenbeck process, ie the limit density fluctuation field is the solution of

$$dY_t = (1-2\alpha)\nabla Y_t dt + \frac{1}{2}\Delta Y_t dt + \sqrt{\alpha(1-\alpha)}\nabla dW_t, \tag{44}$$

where W_t is a Brownian motion.

For more general initial conditions I refer the interested reader to [1].

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