THE UNIVERSITY OF HULL

Convergence in Incomplete Market Models

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in memoriam
August Jacobsen
(1913–1992)

Abstract

The problem of pricing and hedging of contingent claims in incomplete markets has lead to the development of various valuation methodologies. This thesis examines the mean-variance and variance-optimal approaches to risk-minimisation and shows that these are robust under the convergence from discrete- to continuous-time market models. This property yields new convergence results for option prices, trading strategies and value processes in incomplete market models.

Techniques from nonstandard analysis are used to develop new results for the lifting property of the minimal martingale density and risk-minimising strategies. These are applied to a number of incomplete market models:

The restriction of hedging dates in a general class of discrete- and continuous-time models is studied and it is shown that the convergence of the underlying models implies the convergence of strategies and value processes.

Similar results are obtained for multinomial models and approximations of the Black-Scholes model by direct observation of the price process. The concept of D^2-convergence is extended to these classes of models, including the construction of discretisation schemes. This yields new convergence results for these models as well as for option prices in a jump-diffusion model.

The computational aspects of these approximations are examined and numerical results are provided in the case of European and Asian options.

For ease of reference a summary of the main results from nonstandard analysis in the context of mathematical finance is given as well as a brief introduction to mean-variance hedging and variance-optimal pricing.

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Introduction

Let us consider a simplified model of a financial market in which some asset S (a company share, government bond, foreign currency etc.) is traded; the price of S follows a random process. We also assume the existence of a risk-free bank account which allows us to transfer money over time. Let us suppose that the price process of S is *arbitrage-free*: this means that it is not possible to invest in S in such a way that, with some positive probability, the return on this investment exceeds the return on the bank account.

A *contingent claim* H is a contract with a prescribed maturity date T which results in a certain cash value for its holder at time T, the amount depending on events which took place during the "life" of the claim (usually this value depends on the behaviour of the risky asset up to time T). These claims are often *options* which means that the contract gives its holder the right to this cash value (which may be negative) but allows him to default.

Suppose we were able to trade in S in such a way that, starting from some initial balance c of the bank account and not adding or withdrawing extra funds, we could match the value of the claim H in all possible events. Then this *self-financing trading strategy* and H would be equivalent and the initial investment c would be an arbitrage-free price for this claim. Note that in this case the risk in issuing the claim is zero as the above hedging strategy insures the issuer against all eventualities. If every claim H in this model can be replicated in this way the model is called *complete*.

In probabilistic terms it can be shown that (after discounting all entities at the risk-free rate) the above situation is equivalent to the existence of a unique martingale measure for the stochastic process S and that c has to be the expectation of H under this measure.

The idea of perfect replication is fundamental for the methodology of arbitrage-free pricing in complete markets. However, in most cases — in particular in practical situations — markets are incomplete: this means that there exist claims whose intrinsic risk cannot be reduced to zero.

When faced with an incomplete market several questions arise: How should the risk associated with a claim be measured? How should an "optimal" strategy be chosen in order to minimise this risk? What replaces the arbitrage-free price c for the claim in this situation?

The question of optimal trading is of particular practical importance for financial institutions: not only has a claim to be priced in the market but its issuer also has to manage the risk associated with this instrument. The choice of a suitable pricing and hedging methodology is still a topic of most active discussion and research.

At the same time the question of discrete- vs. continuous-time modelling still poses

many open problems, in particular in connection with convergence in these models: If a sequence of discrete-time market models converges to a continuous-time model, under which conditions does this imply that the prices of contingent claims and optimal trading strategies in these models converge as well?
One basic requirement for a pricing methodology for incomplete markets should be precisely this *stability under convergence* for option values and trading strategies[1]. In this thesis we consider two related approaches (mean-variance hedging and variance-optimal pricing) for which we will show that they have this stability property. As a suitable mode of convergence from discrete- to continuous-time stochastic models we are using the notion of D^2-convergence which originates from a natural lifting condition in nonstandard analysis.

Outline of this Thesis

Chapter 1 presents a summary of the mean-variance hedging and variance-optimal pricing methodologies, both in discrete and continuous time. We emphasise the connections between these approaches, in particular in the transition from discrete- to continuous-time models with its resulting technical difficulties. Most of the results presented here are taken from recent research articles and this material has not previously been presented in this unified form.
In Chapter 2 we review briefly the main results from nonstandard measure theory and the nonstandard approach to stochastic integration in the context of their application to mathematical finance. We define the notion of D^2-convergence and summarise the main convergence results in the context of the complete Cox-Ross-Rubinstein and Black-Scholes models.
As a first application of nonstandard methods in incomplete markets we consider in Chapter 3 the problem of restricted hedging in discrete- and continuous-time models for a finite number of possible hedging dates. The results in this chapter extend previous work by Mercurio and Vorst [MV96] to a larger class of models. Moreover, we obtain convergence results for trading strategies and value processes in these models.
New nonstandard results which are of particular importance in the study of incomplete market models are obtained Chapter 4. We first develop a criterion which allows us to deduce the lifting property of the density for the minimal martingale measure from a decomposition of the price process (or, alternatively, the return process) of the risky asset[2]. This is then applied to two incomplete models which can be regarded as internal versions of the models considered by Runggaldier, Schweizer and

[1] This requirement is in general not satisfied: e.g. it has been shown in [Pri95] that the upper and lower option prices $c_{\max}$ and $c_{\min}$ as defined in [HK79] are not stable under convergence. Here $c_{\max} = \sup E[H]$ and $c_{\min} = \inf E[H]$ with the sup / inf taken over all martingale measures for S.

[2] A corresponding standard criterion was obtained independently by Prigent [Pri97].

others [RS95, MV96]. Finally, we are able to extend the results of Cutland, Kopp and Willinger [CKW91, CKW93b] on lifting properties for trading strategies and value processes to the case of incomplete discrete-time models which have complete continuous-time models as their standard parts.

These results are then applied in Chapter 5 to two incomplete discrete-time approximations of the Black-Scholes model, using multinomial trees and direct observations of the continuous-time price process, respectively. We can extend the notion of D^2-convergence to these models and obtain new convergence results for trading strategies and value processes, again extending previous results in [CKW93a, RS95, MV96].

In the final Chapter 6 we examine the practical aspects of these models in terms of their utility for numerical approximations. Explicit examples and numerical results are provided for European and Asian options.

We conclude with a brief discussion of the results and suggestions for further research.

Chapter 1

Pricing and Hedging in Incomplete Markets

Under the assumption of no-arbitrage any option price has to be the expectation of the discounted value of the option at maturity under a martingale measure for discounted asset prices[1]. However, this martingale measure is only unique for complete market models, so that in general we are left with an infinite number of choices (the collection of martingale measures is a convex set). Different optimality criteria for the pricing and hedging of claims will lead to different choices for this "pricing measure".

This chapter presents two possible optimality criteria: *mean-variance hedging* and *variance-optimal pricing*, as developed in a series of papers by Föllmer, Schweizer and Sondermann (the main references are [FS86, FS91, Sch91, Sch93b, Sch94b]). We will see in the following chapters that these pricing concepts have a very appealing stability property with respect to convergence of option values and trading strategies when continuous time models are approximated by discrete time ones. There are of course numerous other approaches to option pricing under incompleteness, e.g. [Dav97, EQ95, KLSX91, Cvi97].

A note on our terminology: some authors prefer the use of "option value" instead of "option price" in incomplete market models to indicate that these values are not obtained by arbitrage considerations (cf. [MV96] or [RS95]). However we will continue to use the word "option price", keeping in mind that this price usually depends on individual preferences of the market agent.

In order to examine the convergence from discrete to continuous time models we present the mean-variance hedging approach for both types of models in Section 1.1 and 1.2. The final Section 1.3 then summarises the connections between mean-variance hedging and variance-optimal pricing. These first sections also introduce

[1] In continuous time the notion of no-arbitrage is made precise by the concept of "no free lunch with vanishing risk" (see [DS94]).

the notation and terminology which will be used in the following chapters.
We will assume throughout that all entities in our models are already discounted, i.e. the bank account process is constant equal to one. This will simplify the notation and does not cause any loss of generality since all other price and value processes can be viewed relative to the (possibly stochastic) instantaneous short rate (cf. [HK79]).

1.1 Mean-Variance Hedging in Discrete Time

Let $T \in \mathbb{N}$ and define the discrete time line

$$\mathbb{T} := \{0, 1, \dots, T\}.$$

$\mathbb{T}$ will represent the set of possible trading dates in this discrete time economy, i.e. changes in portfolios are only possible at times $t \in \mathbb{T}$.
The price of the risky asset is given as a stochastic process S on some complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$. We assume that $\mathcal{F}_0$ is trivial (i.e. it contains only sets of measure zero and one, so that every $\mathcal{F}_0$-measurable random variable is constant) and that $\mathcal{F} = \mathcal{F}_T$. For S to be a *price process* we require that $S = (S_t)_{t\in\mathbb{T}}$ is $\mathbb{F}$-adapted and square integrable, i.e. $S_t \in L^2(P)$ for all $t \in \mathbb{T}$.

Notation 1.1.1. For any process $X = (X_t)_{t\in\mathbb{T}}$ define $\Delta X_t := X_{t+1} - X_t$, $t \in \mathbb{T}$.

Note. This definition of "forward" rather than "backward" increments for discrete time processes is different from the notation used in [Sch88]. However, it matches exactly with the definition used in the nonstandard theory of stochastic integration (cf. Section 2.2) and therefore we will use it throughout.

A *portfolio* is a pair $(\theta_t, \psi_t) \in \mathbb{R}^2$ where θ_t represents the number of units in the risky asset held between time t and $t+1$ (more precisely, θ_t is held over the interval $(t, t+1]$) and ψ_t the number of units of currency held in the risk-free bank account. However, θ_t, ψ_t have to be chosen *after* the prices at time t are announced to prevent the use of "insider knowledge" about the price S_t. After the terminal time T we sell the risky asset and allow one final adjustment of the bank account[2]. These assumptions are reflected in the following definition:

Definition 1.1.2. A *trading strategy* ϕ is a pair of processes $\phi = (\theta, \psi) = (\theta_t, \psi_t)_{t\in\mathbb{T}}$ satisfying the following conditions:

(1) θ is $\mathbb{F}$-adapted, i.e. θ_t is $\mathcal{F}_t$-measurable for $t \in \mathbb{T}$, and $\theta_T = 0$;

[2] This extra freedom is crucial for the replication of options in incomplete markets as it allows the addition or withdrawal of cash to "balance the account". For complete market models this final adjustment is not needed.

(2) $\sum_{u=0}^{t-1} \theta_u \Delta S_u \in L^2(P)$ for $t \in \mathbb{T}$;

(3) ψ is $\mathbb{F}$-adapted;

(4) $\theta_t S_t + \psi_t \in L^2(P)$ for $t \in \mathbb{T}$.

Then the following processes can be associated with a trading strategy ϕ:

$$\begin{array}{llcl} \text{the value process:} & V_t(\phi) & := & \theta_t S_t + \psi_t, \\ \text{the (accumulated) gains process:} & G_t(\phi) & := & \sum_{u=0}^{t-1} \theta_u \Delta S_u, \\ \text{the (cumulative) cost process:} & C_t(\phi) & := & V_t(\phi) - G_t(\phi), \end{array}$$

for $t \in \mathbb{T}$. The gains process represents the accumulated change in the portfolio market value due to price changes up to (and including) time t. The value process gives the value of the portfolio after it has been adjusted at time t.
We see that a *self-financing* strategy (as defined e.g. in [HK79, HP81]) has a constant cost process $C \equiv C_0$. This is the motivation for the following definition:

Definition 1.1.3. A trading strategy ϕ is called *mean-self-financing* if its cost process $C(\phi)$ is a martingale.

Note that any mean-self-financing strategy is completely determined by its first component θ together with its terminal value V_T.
A *contingent claim* — or European *option* — is a random variable $H \in L^2(P)$ representing a random payoff at time T.

Definition 1.1.4. Let $H \in L^2(P)$. A trading strategy ϕ is called *H-admissible* if $V_T(\phi) = H$, P-a.s.. We then say that ϕ *generates H*.

Given a claim H it is our aim to find an "optimal" strategy which generates H, where we use the following optimality criterion:

Local Risk-Minimisation

Let $H \in L^2(P)$ be a contingent claim. Define the *risk process* $R(\phi)$ of a trading strategy ϕ by

$$R_t(\phi) := E\left[\left(C_T(\phi) - C_t(\phi)\right)^2 \middle| \mathcal{F}_t \right], \qquad t \in \mathbb{T}. \tag{1.1}$$

We could then look for an H-admissible *risk-minimising* strategy ϕ in the sense that $R_t(\phi) \le R_t(\tilde{\phi})$ for any $t \in \mathbb{T}$, where ϕ and $\tilde{\phi}$ agree up to time t, and both ϕ and $\tilde{\phi}$ generate H.

However, it turns out that ϕ is risk-minimising if and only if it is mean-self-financing and $R_t(\phi) \le R_t(\tilde{\phi})$, $t \in \mathbb{T}$, for *all* H-admissible strategies $\tilde{\phi}$, not only for those sharing a "common past" with ϕ (Proposition I.4 in [Sch88]). Precisely this very strict characterisation prevents a general solution of this optimisation problem; an example of a situation in which there does not exist any risk-minimising strategy is given in [Sch88, pp.16–17].

This leads to a weaker notion of risk-minimisation in which the *local risk* defined by

$$\begin{aligned} r_t(\phi) &:= E\left[\left(C_{t+1}(\phi) - C_t(\phi)\right)^2 \middle| \mathcal{F}_t \right] \\ &= E\left[\left(V_{t+1}(\phi) - \theta_t \Delta S_t - V_t(\phi)\right)^2 \middle| \mathcal{F}_t \right] \end{aligned} \tag{1.2}$$

for $t \in \mathbb{T} \setminus \{T\}$ is to be minimised by an appropriate choice of θ_t and ψ_t.

As in the risk-minimising case above it can be shown that any locally risk-minimising strategy is mean-self-financing (Lemma I.7 in [Sch88]). A candidate for such a strategy can be found by a backward sequential regression procedure, starting from the requirement $V_T = H =: V_T^H$ and setting[3]

$$\theta_t^H := \frac{\operatorname{Cov}\left[V_{t+1}^H, \Delta S_t \middle| \mathcal{F}_t \right]}{\operatorname{Var}\left[\Delta S_t \middle| \mathcal{F}_t \right]} \tag{1.3}$$

(where $\theta_t^H := 0$ if $\operatorname{Var}[\Delta S_t | \mathcal{F}_t] = 0$)

$$V_t^H := E\left[V_{t+1}^H - \theta_t^H \Delta S_t \middle| \mathcal{F}_t \right], \tag{1.4}$$

$$\psi_t^H := V_t^H - \theta_t^H S_t \tag{1.5}$$

for $t \in \mathbb{T} \setminus \{T\}$. Finally, set $\theta_T^H := 0$ and $\psi_T^H := H$. In (1.3) θ_t^H can be viewed as the best linear estimate for V_{t+1}^H based on the information at time t (cf. (1.2) and see [FS89] for a more expository account). (1.4) and (1.5) simply ensure that the strategy $\phi^H = (\theta^H, \psi^H)$ remains mean-self-financing. In particular, $V_t(\phi^H) = V_t^H$ for all $t \in \mathbb{T}$. If (1.3)–(1.5) define a trading strategy then it is the unique locally risk-minimising strategy for the claim H (Proposition I.8 in [Sch88]).

However, (1.3) does not guarantee that the integrability conditions (2) and (4) in Definition 1.1.2 are satisfied. This can be achieved by imposing a *nondegeneracy condition* on the price process S:

$$\frac{(E[\Delta S_t | \mathcal{F}_t])^2}{\operatorname{Var}[\Delta S_t | \mathcal{F}_t]} \le L, \quad P\text{-a.s., for } t \in \mathbb{T} \setminus \{0\} \tag{1.6}$$

for some constant $L \in \mathbb{R}$.

[3] The conditional variance and covariance is defined in the usual sense: Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and $X, Y \in L^2(P)$. Then $\operatorname{Var}[X|\mathcal{G}] := E[X^2|\mathcal{G}] - E[X|\mathcal{G}]^2$ and $\operatorname{Cov}[X,Y|\mathcal{G}] := E[XY|\mathcal{G}] - E[X|\mathcal{G}]E[Y|\mathcal{G}]$.

Remark 1.1.5. In terms of the behaviour of S as a risky asset condition (1.6) means that the expected gains from holding S — given by $E[\Delta S_t|\mathcal{F}_t]$ — are balanced by a sufficiently strong random behaviour of S — expressed by the conditional variance. In order to realise a profit from holding S an investor therefore has to take a certain risk. This corresponds to the usual assumption of no-arbitrage which means that the possibility of a riskless profit from an investment is excluded.

If (1.6) is satisfied then

$$E\left[(\theta_t \Delta S_t)^2\right] \leq E\left[V_{t+1}^2\right] < \infty \tag{1.7}$$

(see [Sch88], Lemma A.3.3), so that (1.3)–(1.5) indeed define a trading strategy ϕ^H for H, which is then the unique locally risk-minimising strategy generating H. The initial investment V_0^H for this strategy can be considered a *fair hedging price* for the claim H. The process V^H is usually called the *intrinsic value process* of the claim H.

Remark 1.1.6. In the special case where the market model is complete we know that there exists a self-financing H-admissible strategy ϕ^H. For this strategy the risk process $R(\phi^H) \equiv 0$ and hence ϕ^H is the unique risk-minimising strategy generating H. In this case V_{t+1}^H and ΔS_t are linearly dependent, so that the estimate in (1.3) provides a perfect fit. (cf. [FS89]).

A Decomposition of the Contingent Claim

From now on we assume that the nondegeneracy condition (1.6) is satisfied.

To gain a better understanding of the structure of the locally risk-minimising strategy we define the discrete *Doob-Meyer decomposition* of the price process S by

$$\begin{aligned} S &= S_0 + M + A \\ \text{where} \quad \Delta A_t &:= E\left[\Delta S_t \middle| \mathcal{F}_t\right], \\ \Delta M_t &:= \Delta S_t - \Delta A_t \\ \text{and} \quad A_0 := 0, & \qquad M_0 := 0. \end{aligned}$$

Then M is a martingale and A a predictable process; both M and A are square-integrable. Equation (1.4) can then be written as

$$V_t^H = E\left[V_{t+1}^H \middle| \mathcal{F}_t\right] - \theta_t^H \Delta A_t. \tag{1.8}$$

Following [Sch94a] we define a square-integrable martingale L by

$$\begin{aligned} \Delta L_t^H &:= V_{t+1}^H - E\left[V_{t+1}^H \middle| \mathcal{F}_t\right] - \theta_t^H \Delta M_t, \quad t \in \mathbb{T} \setminus \{T\}, \\ L_0^H &:= 0. \end{aligned} \tag{1.9}$$

Then the martingales M and L^H are orthogonal, i.e. their product is a martingale, or equivalently

$$E\left[\Delta M_t \, \Delta L_t^H \middle| \mathcal{F}_t\right] = 0 \quad P\text{-a.s.}. \tag{1.10}$$

Note that $\Delta L_t^H = \Delta C_t(\phi^H)$, so that L^H represents the extra cost $C(\phi^H) - C_0(\phi^H)$ required by the strategy ϕ^H. This additional cost process is orthogonal to the martingale part of the price process S.
Using (1.8) we have

$$\begin{aligned} \Delta V_t^H &= \theta_t^H \Delta S_t + \Delta L_t^H \quad \text{for } t \in \mathbb{T} \setminus \{T\} \\ \text{and} \quad V_0^H &= E\Big[H - \sum_{s=0}^{T} \theta_s^H \Delta A_s\Big], \end{aligned} \tag{1.11}$$

so we have obtained a decomposition

$$H = V_0^H + \sum_{u=0}^{T-1} \theta_u^H \Delta S_u + L_T^H, \tag{1.12}$$

where L^H is a square-integrable martingale orthogonal to the martingale part of S. (1.12) is the discrete time version of the so-called *Föllmer-Schweizer decomposition* (see Section 1.2). Note that in discrete time this decomposition exists under the sole assumption of the nondegeneracy condition (1.6).

Remark 1.1.7. In the case where the price process S is a martingale the decomposition (1.12) simplifies to the (discrete time) Kunita-Watanabe decomposition of H (see e.g. [Mét82] for the general existence and uniqueness of this decomposition) and the option price is given as $V_0^H = E[H]$. It turns out that in this case the locally risk-minimising strategy defined in (1.3)–(1.5) is in fact risk-minimising (see [Sch88, pp.23–24]).

The Minimal Martingale Measure

As mentioned at the beginning of this chapter any option price in an arbitrage-free market has to be given as the expectation under some martingale measure for the underlying price process. We will now describe the martingale measure associated to the value V_0^H in (1.11).
Define

$$\alpha_t := \frac{\Delta A_t}{\operatorname{Var}\left[\Delta S_t | \mathcal{F}_t\right]} = \frac{E\left[\Delta S_t | \mathcal{F}_t\right]}{E\left[(\Delta M_t)^2 | \mathcal{F}_t\right]}, \quad t \in \mathbb{T} \setminus \{T\} \tag{1.13}$$

and set

$$\hat{Z}_t := \prod_{s=0}^{t-1} \left(1 - \alpha_s \Delta M_s\right). \tag{1.14}$$

The process $\hat{Z}$ is a square-integrable martingale (Proposition 2.3 in [Sch93b]). The random variable $\hat{Z}_T$ will become the density for our martingale measure. However, $\hat{Z}_T$ may take negative values. We therefore introduce the notion of a signed martingale measure (cf. [Sch93b]):

Definition 1.1.8. A signed measure Q on $(\Omega, \mathcal{F})$ is called a *signed martingale measure* for S if $Q(\Omega) = 1$, $Q \ll P$ on $\mathcal{F}$ with $dP/dQ \in L^2(P)$ and

$$E\left[\frac{dP}{dQ}\Delta S_t \middle| \mathcal{F}_t\right] = 0 \qquad P\text{-a.s. for } t \in \mathbb{T}.$$

We then have the following result (see [Sch93b], in particular Lemma 2.7 for details).

Proposition 1.1.9. *If S satisfies the nondegeneracy condition* (1.6) *then*

$$d\hat{P} := \hat{Z}_T \, dP \tag{1.15}$$

defines a signed martingale measure for S and it follows from (1.10) *and* (1.12) *that*

$$V_0^H = \hat{E}[H] := E[\hat{Z}_T H].$$

Furthermore

$$V_t^H = \hat{E}[H|\mathcal{F}_t] := E[\hat{Z}_T H|\mathcal{F}_t].$$

The option price corresponding to the locally risk-minimising strategy is therefore indeed given as the expectation under a (signed) martingale measure for price of the underlying asset.
The measure $\hat{P}$ is called the *(signed) minimal martingale measure*[4] for S and will play a crucial rôle in the problem of risk-minimisation in continuous time as well as in the study of convergence from discrete to continuous time models.
We note that the decomposition of the claim H can also be computed under the minimal martingale measure, with equations (1.3) and (1.9) simplifying to

$$\theta_t^H = \frac{\hat{E}\left[V_{t+1}^H \Delta S_t \middle| \mathcal{F}_t\right]}{\hat{E}\left[(\Delta S_t)^2 \middle| \mathcal{F}_t\right]} \quad \text{and} \quad \Delta L_t^H = \Delta V_t^H - \theta_t^H \Delta S_t. \tag{1.16}$$

(where the conditional expectations are defined as in Definition 1.1.8 are assumed to exist). In particular, L remains a martingale under $\hat{P}$ which is orthogonal to S.

1.2 Mean-Variance Hedging in Continuous Time

In continuous time the set of trading dates is $\mathbb{T} := [0, T] \subset \mathbb{R}$ for $T > 0$. We assume that $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{T}}$ is a filtration on some complete probability space $(\Omega, \mathcal{F}, P)$ which

[4]The reason for this name will become clear in the following section (see Remark 1.2.7).

satisfies the usual conditions (i.e. $\mathbb{F}$ is right-continuous and complete), with $\mathcal{F}_0$ trivial and $\mathcal{F}_T = \mathcal{F}$. The price of the risky asset is now given by a semimartingale $S \in \mathcal{S}^2(P)$, i.e. S admits a Doob-Meyer decomposition

$$S = S_0 + M + A$$

where M is a square-integrable martingale and A is a predictable process with square-integrable variation. Then the predictable quadratic variation $\langle M \rangle$ exists (see [Mét82] or [Ell82] for definitions) and we make the additional assumption that

$$A \text{ is absolutely continous with respect to } \langle M \rangle. \tag{1.17}$$

This means that there exists a predictable process α such that the Doob-Meyer decomposition of S is given by

$$S = S_0 + \int \alpha \, d\langle M \rangle + M \tag{1.18}$$

Assumption (1.17) is a nondegeneracy condition similar to (1.6) in the discrete case, again corresponding to the assumption of no-arbitrage in the model.
We can now define trading strategies and their associated value, gains and cost processes analogously to the previous section, replacing the finite sums by stochastic integrals:

Definition 1.2.1. A *trading strategy* ϕ is a pair of processes $(\theta, \psi) = (\theta_t, \psi_t)_{t \in \mathbb{T}}$ with θ predictable and ψ adapted such that

(1) $\theta \in L^2(S)$, i.e.

$$E\left[\int_0^T \theta_s^2 d\langle M \rangle_s + \left(\int_0^T |\theta_s| \, d\,|A|_s\right)^2\right] < \infty.$$

(2) the value process $V_t(\phi) := \theta_t S_t + \psi_t$ is square-integrable.

Let ϕ be a trading strategy. Because of (1) the stochastic integral $\int \theta dS$ exists and is square integrable. Therefore the gains process $G_t(\phi) := \int_0^t \theta_u dS_u$ and the cost process $C_t(\phi) := V_t(\phi) - G_t(\phi)$ are square integrable.

Local Risk Minimisation

The risk process R can now be defined exactly as in (1.1) in discrete time and one could again consider the question of finding a risk-minimising strategy ϕ^H for a given claim H in the same sense as above. It can again be shown that any risk-minimising strategy is mean-self-financing, i.e. $C(\phi^H)$ is a martingale.

In the case where S is a square-integrable martingale under P (i.e. $A \equiv 0$) this problem was solved in [FS86] by using the Kunita-Watanabe decomposition of H (cf. Remark 1.1.7 above):

$$H = E[H] + \int_0^T \theta_u^H dS_u + L_T \tag{1.19}$$

where $\theta \in L^2(S)$ and L is a square-integrable martingale strongly orthogonal to S.

Theorem 1.2.2 ([FS86, Theorem 2]). *Suppose S is a square-integrable martingale. Then there exists a unique H-admissible risk-minimising strategy $\phi^H = (\theta^H, \psi^H)$ given by θ^H in (1.19) and $\psi_t^H := E[H|\mathcal{F}_t] - \theta_t^H S_t$.*

But we already know from Section 1.1 that there is in general no risk-minimising strategy when P is no longer a martingale measure for S.
It is therefore natural to define the notion of local risk-minimisation as in discrete time, replacing the definition of $r(\phi)$ in (1.2) with an infinitesimal version. This involves considerable technicalities using the concept of "small perturbations" of a trading strategy which leads to the study of differentiation of semimartingales; details can be found in [Sch91].
However, under the following further assumptions on S it is possible to obtain an alternative characterisation of locally risk-minimising strategies which translates our problem into a question about orthogonality of martingales.
We assume from now on:

(A1) A is continuous

(A2) the density α in (1.18) satisfies $E_M[|\alpha| \cdot \log^+ |\alpha|] < \infty$ (here $E_M[\,\cdot\,]$ denotes the expectation with respect to the Doléans measure induced by M).

(A3) S is continuous at T, P-a.s..

Under these assumptions the main result in [Sch90] is used to obtain:

Theorem 1.2.3 ([Sch91, Theorem 2.3]). *Let ϕ be an H-admissible trading strategy. Then the following are equivalent:*

(a) ϕ is locally risk-minimising

(b) ϕ is mean-self financing and the cost process $C(\phi)$ is orthogonal to M.

Remark 1.2.4. Many authors use the characterisation (b) in the above theorem as a *definition* of locally risk-minimising strategies. This eliminates the need for the assumptions (A1)–(A3) at this stage. However, the derivation of this definition from the discrete time concept of local risk-minimisation is then somewhat obscured. We will therefore continue with the above assumptions.

In view of Theorem 1.2.3 and the discrete time decomposition (1.12) it is natural to look for a so-called *Föllmer-Schweizer (FS) decomposition* of the claim H in the form

$$H = V_0^H + \int_0^T \theta_u^H \, dS_u + L_T^H \tag{1.20}$$

where $V_0^H \in \mathbb{R}$, $\theta^H \in L^2(S)$ and L^H is a square-integrable martingale orthogonal to M.

The mean-self-financing H-admissible strategy ϕ^H defined by θ^H in (1.20) has cost process $C(\phi^H) = V_0^H + L^H$ and it is therefore easy to see that the existence of a locally risk-minimising strategy for H is indeed equivalent to the existence of a FS decomposition (1.20) of H. We saw that in discrete time such a decomposition exists for any square-integrable claim H, provided the nondegeneracy condition (1.6) is satisfied. However, in continuous time the problem of existence of a FS decomposition for a given claim H becomes more subtle.

We saw above that in the case where P is a martingale measure for S, (1.20) is given by the Kunita-Watanabe decomposition of H with respect to the stable subspace generated by S. However, such a projection theorem is not available in our case where S is only a semimartingale. One possible solution is the following: Find a martingale measure $\hat{P}$ for S which also preserves orthogonality relations, then use the Kunita-Watanabe decomposition of H under this new measure to obtain (1.20).

The Minimal Martingale Measure

Definition 1.2.5. A martingale measure $\hat{P} \approx P$ will be called *minimal* if $\hat{P} = P$ on $\mathcal{F}_0$ and if any square-integrable martingale which is orthogonal to M under P remains a martingale under $\hat{P}$.

An existence and uniqueness result for the minimal martingale measure was first obtained in [FS91] for a continuous price process S; the following extension can be found in [MR97, Chapter 26]:

Theorem 1.2.6. *Suppose $\alpha_t \Delta M_t < 1$ for all $t \in \mathbb{T}$ (here $\Delta M_t := M_t - M_{t-}$).*

(i) The minimal martingale measure $\hat{P}$ exists if and only if the process

$$\hat{Z}_t := \mathcal{E}\Big(-\int \alpha \, dM\Big)_t \tag{1.21}$$

is square-integrable under P (here $\mathcal{E}(\cdot)$ denotes the stochastic exponential; see e.g. [Pro90, p.78] for a definition). In that case, $\hat{P}$ is given by

$$\frac{d\hat{P}}{dP} := \hat{Z}_T. \tag{1.22}$$

(ii) $\hat{P}$ is uniquely determined.

We introduce the *mean-variance tradeoff process* $\tilde{K}$ of S defined as

$$\tilde{K}_t := \int_0^t \alpha_s^2 d\langle M\rangle_s;$$

this process will also play an important rôle in the context of variance-optimal pricing as discussed in the following Section 1.3.

Remark 1.2.7. Even if S does not satisfy the assumptions of Theorem 1.2.6 it is always possible to define a signed local martingale measure $\hat{P}$ for S by (1.22) provided the process $\hat{Z}$ in (1.21) is a martingale. $\hat{P}$ is then called the *minimal signed local martingale measure* for S. This measure is minimal in the following sense: Under the assumption that the process $\tilde{K}$ is deterministic, $\hat{P}$ is the unique measure that minimises the distance $\left\|\frac{dQ}{dP} - 1\right\|_{L^2(P)}$ over all signed local martingale measures Q for S with density $\frac{dQ}{dP} \in L^2(P)$ (Theorem 7 in [Sch95]).

We now have the following analogue to Proposition 1.1.9:

Proposition 1.2.8 ([Sch94b, Lemma 17]). *Suppose the mean-variance tradeoff process $\tilde{K}$ is bounded and $H \in L^2(P)$ admits a FS decomposition* (1.20). *Then*

$$V_0^H = \hat{E}[H] := E\left[\hat{Z}_T H\right].$$

If $\hat{Z}$ is strictly positive then we also have

$$V_t^H := H_0 + \int_0^t \theta_u^H dS_u + L_t^H = \hat{E}[H|\mathcal{F}_t] := E\left[\hat{Z}_T H\middle|\mathcal{F}_t\right].$$

On the problem of existence and uniqueness of a FS decomposition (1.20) we have the following results: If the price process S is continuous and the density $\hat{Z}$ is square-integrable then every $H \in L^2(P)$ has a unique Kunita-Watanabe decomposition under the minimal martingale measure $\hat{P}$ which does indeed yield the P-orthogonality of L^H and M (Theorem 3.14 in [FS91]). The necessary integrability conditions on θ^H and L^H can be deduced from additional assumptions on $\tilde{K}$ and H:

Lemma 1.2.9 ([Sch95, Corollary 10]). *Suppose S is continuous and the mean-variance tradeoff process $\tilde{K}$ is bounded. If $H \in L^r(P)$ for some $r > 2$ then $\theta^H \in L^2(S)$ and L^H is a square-integrable P-martingale.*

Unfortunately, if S is not continuous the decomposition (1.20) can no longer be obtained by using the Kunita-Watanabe decomposition of H under $\hat{P}$. This is due to the fact that the $\hat{P}$-martingale $\hat{L}$ from the decomposition under $\hat{P}$ will typically not be P-orthogonal to M if M has jumps (see [Sch93a, p.106] and [Sch88, II.5.3]).

In the special case of a jump-diffusion model an explicit construction of the FS decomposition can be obtained from a martingale representation theorem for jump-diffusion processes (see [Sch93a, Section II.8] for details).
The question of existence and uniqueness of the FS decomposition has be settled recently by Monat and Stricker [MS95]; without using the minimal martingale measure the following result has been obtained:

Theorem 1.2.10 ([MS95, Theorem 3.4]). *If the mean-variance tradeoff process $\tilde{K}$ is bounded then every $H \in L^2(P)$ admits a FS decomposition. This decomposition is unique in the following sense: If*

$$H = V_0 + \int_0^T \theta_u dS_u + L_T = \bar{V}_0 + \int_0^T \bar{\theta}_u dS_u + \bar{L}_T$$

are two FS decompositions of H then $V_0 = \bar{V}_0$, $\theta = \bar{\theta}$ in $L^2(M)$, and $L_T = \bar{L}_T$ P-a.s..

A counterexample in the same paper shows that there exists a claim $H \in L^2(P)$ without FS decomposition if $\tilde{K}$ is not bounded.

1.3 Variance-Optimal Pricing

Complete market models allow the perfect replication of any claim by a self-financing strategy. Under incompleteness this property no longer holds. We saw above that extending the set of admissible strategies to mean-self-financing ones (i.e. allowing strategies which are "on average" self-financing) leads to an optimality criterion with respect to the minimisation of the expected additional cost.
Another possible choice for the set of strategies is one where we still insist on the self-financing property but give up the requirement of perfect replication. Using the notation of the previous section the value process of a self-financing strategy ϕ is completely determined by its initial investment $c = V_0(\phi)$ together with its gains process $G_t(\phi)$. Note that only the first component ξ of a strategy[5] $\phi = (\xi, \psi)$ enters the definition of the gains process; we will therefore use the notation $G(\xi)$ for the gains process in this section.
The problem is now the following: for a given claim H we are looking for a self-financing strategy which replicates this claim as closely a possible. In discrete time this means we want to

$$\text{choose } c \in \mathbb{R},\ \xi \in \Xi \text{ such that } E\left[(c + G_T(\xi) - H)^2\right] \text{ is minimal,} \tag{1.23}$$

[5] Here we denote the number of units of the risky asset by ξ to distinguish it from the strategies in Sections 1.1 and 1.2.

where Ξ is the set of all adapted processes ξ such that $\xi_t \Delta S_t \in L^2(P)$ for $t \in \mathbb{T} \setminus \{T\}$ (cf. Definition 1.1.2). We will denote the solution to this problem (if it exists) by (c^H, ξ^H).

In this section we will outline the solution of (1.23) in discrete and continuous time and explain the connections to the mean-variance hedging approach of Sections 1.1 and 1.2. In the continuous case we make some additional assumptions on the price process S which allow us to obtain a solution of (1.23) in an explicit form. Finally, we mention more general assumptions under which a solution to (1.23) exists in continuous time.

Discrete Time

We first consider a simpler version of (1.23): for given $c_0 \in \mathbb{R}$ and $H \in L^2(P)$

$$\text{choose } \xi \in \Xi \text{ such that } E\left[(c_0 + G_T(\xi) - H)^2\right] \text{ is minimal.} \tag{1.24}$$

Under the nondegeneracy condition (1.6) a solution to (1.24) exists in discrete time. An elegant proof of this result was given in [Sch93b] where it was shown that in this case the space of stochastic integrals (identified with their terminal values) $G(\Xi) := \{G_T(\xi) : \xi \in \Xi\}$ is closed in the Hilbert space $L^2(P)$. The optimal strategy ξ^{H,c_0} can then be found by projecting $H - c_0$ onto $G(\Xi)$.

A counterexample in [Sch93b] shows that there is in general no solution to (1.24) if the nondegeneracy condition (1.6) is not satisfied.

However, this existence result does not reveal any information about the structure of the optimal strategy and the choice of the initial portfolio value c^H in (1.23). In the following we will see how these can be obtained from the decomposition

$$H = V_0^H + \sum_{i=0}^{T-1} \theta_i^H \Delta S_i + L_T^H \tag{1.25}$$

(cf. (1.12)) of H.

In discrete time the mean-variance tradeoff process $\tilde{K}$ of S is defined as

$$\tilde{K}_t := \sum_{i=0}^{t-1} \frac{\left(E\left[\Delta S_t \middle| \mathcal{F}_t\right]\right)^2}{\operatorname{Var}\left[\Delta S_t \middle| \mathcal{F}_t\right]} = \sum_{i=0}^{t-1} \alpha_i \, \Delta A_i \quad t \in \mathbb{T} \tag{1.26}$$

(where we have used the notation introduced in equation (1.13)). Note that the nondegeneracy condition (1.6) is equivalent to the boundedness of $\tilde{K}$. We now assume that the mean-variance tradeoff process is deterministic[6], so that the nondegeneracy condition (1.6) is satisfied if $\operatorname{Var}[\Delta S_t | \mathcal{F}_t] > 0$ with positive probability. Under these assumptions we have the following:

[6]Under this assumption the problem (1.24) was first solved in [Sch94a]. While the general solution for bounded $\tilde{K}$ can also be calculated explicitly (see [Sch93b], in particular Theorem 2.4 and 2.8) the results for deterministic $\tilde{K}$ will be sufficient for our purposes.

Theorem 1.3.1 ([Sch93b, Corollary 3.2 and Proposition 4.3]). *If the mean-variance tradeoff process $\tilde{K}$ is deterministic then the solution (c^H, ξ^H) of (1.23) exists and is given by*

$$c^H = V_0^H = \hat{E}[H]$$

(where $\hat{E}[\,\cdot\,]$ denotes the expectation with respect to the minimal martingale measure of Section 1.1) and

$$\xi_t^H = \theta_t^H + \frac{\Delta A_t}{E\left[(\Delta S_t)^2 \,|\, \mathcal{F}_t\right]}\left(V_t^H - c^H - G_t(\xi^H)\right), \quad t \in \mathbb{T}, \tag{1.27}$$

where

$$V_t^H = V_0^H + \sum_{i=0}^{t-1} \theta_i^H \Delta S_i + L_t^H, \quad t \in \mathbb{T},$$

is the intrinsic value process of the claim H.

Remark 1.3.2. If S is a P-martingale then the nondegeneracy condition (1.6) is trivially satisfied and the mean-variance tradeoff process $\tilde{K}$ is constant zero. In this case $c^H = E[H]$ and $\xi^H = \theta^H$, so that the mean-variance and variance-optimal pricing methodologies yield the same option prices and optimal trading strategies.

Finally, if the claim H is attainable (i.e. $L^H \equiv 0$ in (1.25)) then the solution to (1.23) is given by $(c^H, \xi^H) = (V_0^H, \theta^H)$ which shows that the variance-optimal pricing approach is consistent with the usual arbitrage-pricing methodology in complete markets (see also Remark 1.1.6).

Continuous Time

Let us again assume that S has a Doob-Meyer decomposition of the form

$$S = S_0 + A + M = S_0 + \int \alpha \, d\langle M \rangle + M$$

(cf. assumption (1.17) and equation (1.18) on page 11) and recall that the mean-variance tradeoff process[7] $\tilde{K}$ of S is defined as

$$\tilde{K}_t := \int_0^t \alpha_s^2 \, d\langle M \rangle_s = \int_0^t \alpha_s \, dA_s. \tag{1.28}$$

Recall that a random variable $H \in L^2(P)$ admits an FS decomposition if

$$H = V_0^H + \int_0^T \theta_u^H \, dS_u + L_T^H \tag{1.29}$$

[7] In continuous time there are in fact two processes corresponding to the discrete time mean-variance tradeoff process (see [Sch94b] for details). However, these two processes are indistinguishable if A is continuous.

(cf. equation (1.20)) where $V_0^H \in \mathbb{R}$, $\theta^H \in L^2(S)$ and L^H is a square-integrable martingale orthogonal to M.

Assuming that A is continuous we now have the following analogue to Theorem 1.3.1:

Theorem 1.3.3 ([Sch94b, Theorem 2.3 and Corollary 4.10]). *If the mean-variance tradeoff process $\tilde{K}$ is deterministic and H admits an FS decomposition* (1.29) *then the problem* (1.23) *has a solution (c^H, ξ^H), given by*

$$c^H = V_0^H$$

and

$$\xi_t^H = \theta_t^H + \alpha_t \big(V_{t-}^H - c^H - G_{t-}(\xi^H)\big) \tag{1.30}$$

with

$$V_t^H = H_0 + \int_0^t \theta_u^H \, dS_u + L_t^H.$$

Note. Under the assumption that the finite variation process A is continuous we see that α_t corresponds to the term $\Delta A_t / E[(\Delta S_t)^2 | \mathcal{F}_t]$ in discrete time, so that (1.30) is indeed the continuous time analogue of (1.27).

Remark 1.3.4. If S is a P-martingale the assumptions of Theorem 1.3.3 are satisfied immediately (the decomposition (1.29) given by the Kunita-Watanabe decomposition of H), and the solution $(c^H, \xi^H) = (V_0^H, \theta^H)$ again.

A general result on the existence and uniqueness of an FS decomposition has already been stated in Theorem 1.2.10. In the same paper [MS95] it has been shown that the space $G_T(\Xi)$ of stochastic integrals with respect to S is indeed closed under the sole assumption that the mean-variance tradeoff process $\tilde{K}$ is bounded[8]. The same projection argument as in the discrete case can then be applied to show the existence of a solution to (1.24). However, an explicit formula for the optimal strategy as in (1.30) or even for the optimal choice of the initial portfolio is not available in this general case.

[8] A counterexample in [MS95] shows that this no longer holds if $\tilde{K}$ is not bounded.

Chapter 2

Nonstandard Methods in Complete Markets

In this chapter we review various concepts and results from nonstandard analysis in the context of their application to complete market models in mathematical finance. We assume familiarity with the basic notions of nonstandard analysis, including the extensions of the real numbers to a set of hyperreals ${}^*\mathbb{R}$, the nonstandard approach to calculus as well as the construction and properties of the superstructure $V({}^*\mathbb{R})$, including the notions of internal sets and functions, the transfer principle and the standard part map. Introductions to these concepts can be found, e.g. in [Cut88, CC95, ACH97].

The first two sections of this chapter deal with the theory of Loeb measure and integration and the nonstandard approach to stochastic integration. These sections contain most of the results used later in this thesis; proofs can be found in [CC95, AFHL86] for Section 2.1 and [AFHL86, HP83, Lin80] for Section 2.2; see [Ell82, Mét82, Pro90] for the corresponding standard theory of stochastic integration. In the last two sections we introduce the Cox-Ross-Rubinstein and Black-Scholes pricing models and review the main results of Cutland, Kopp and Willinger [CKW91, CKW93a, CKW95] regarding the nonstandard approach to option pricing and convergence results in these models. These methods will be extended to incomplete market models in Chapter 4 and we will obtain new convergence results in Chapter 5.

2.1 Loeb Measure and Integration Theory

An *internal measure space* is a triple $(\Omega, \mathcal{F}, P)$, where Ω is an internal set, $\mathcal{F}$ an internal algebra of subsets of Ω and $P : \mathcal{F} \to {}^*\mathbb{R}$ is an internal finitely additive set function; we assume that $P(\Omega)$ is finite. Important examples of this setup are:

(1) The *hyperfinite time line* $\mathbb{T} := \{0, \Delta t, 2\Delta t, \dots (N-1)\Delta t, T\}$, where $T \in \mathbb{R}^+$ and

$\Delta t := T/N$ for some infinite integer $N \in {}^*\mathbb{N}\backslash\mathbb{N}$. In this case $\mathcal{F}$ is the algebra of all internal subsets of $\mathbb{T}$ and a set function $\Lambda : \mathcal{F} \to {}^*[0,T]$ is given by $\Lambda(\{t\}) = \Delta t$.

(2) *Internal probability spaces*, where $P : \mathcal{F} \to {}^*[0,1]$. Often (but not always) the set Ω is hyperfinite, $\mathcal{F}$ the algebra of all internal subsets and P is the normalised counting measure on Ω, i.e. $P(\{\omega\}) = 1/\mathrm{card}(\Omega)$.

An important result by Loeb [Loe75] shows that internal measure spaces can be turned into standard measure spaces:

Theorem 2.1.1 (Loeb construction). *There is a unique σ-additive extension of ${}^\circ P$ to the σ-algebra $\sigma(\mathcal{F})$ generated by $\mathcal{F}$. The completion of this measure is the* Loeb measure $L(P)$ *and the completion of $\sigma(\mathcal{F})$ is the* Loeb σ-algebra $L(\mathcal{F})$. *Furthermore, for any $B \in L(\mathcal{F})$ there is $A \in \mathcal{F}$ such that $L(P)(A \triangle B) = 0$.*

Definition 2.1.2. The measure space $(\Omega, L(\mathcal{F}), L(P))$ (also written $(\Omega, \mathcal{F}_L, P_L)$) is called the *Loeb space* associated with the internal measure space $(\Omega, \mathcal{F}, P)$.

The Loeb space associated with the hyperfinite time line $\mathbb{T}$ can be identified with the interval $[0,T] \subset \mathbb{R}$ via the standard part map $\mathrm{st} : \mathbb{T} \to [0,T]$. Then $L(\Lambda) \circ (\mathrm{st}_{\mathbb{T}}^{-1})$ is just Lebesgue measure (here $\mathrm{st}_{\mathbb{T}}^{-1}(G) = \mathrm{st}^{-1}(G) \cap \mathbb{T}$ for $G \subset [0,T]$.) In the case of an internal probability space $(\Omega, \mathcal{F}, P)$ we obtain a standard probability space $(\Omega, \mathcal{F}_L, P_L)$. One application of Loeb theory is Anderson's [And76] construction of Brownian motion as a hyperfinite random walk: Let $\Omega := \{-1,+1\}^{\mathbb{T}\backslash\{T\}}$, $\mathcal{F}$ the algebra of all internal subsets of Ω and P the normalised counting measure. Define $W : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ by

$$W(\omega, t) := \sum_{s<t} \omega(s)\sqrt{\Delta t}$$

(here we have introduced the convention that the hyperfinite sum is taken over elements of $\mathbb{T}$; this will be used throughout). It can then be shown (see Theorem 3.3.5 in [AFHL86]) that $b : \Omega \times [0,T] \to \mathbb{R}$ defined by $b(\omega, u) := {}^\circ B(\omega, \bar{u})$, where $\bar{u}$ denotes the point in $\mathbb{T}$ immediately to the right of u, is a standard Brownian motion on the Loeb space $(\Omega, \mathcal{F}_L, P_L)$. One ingredient of the proof is the following nonstandard version of the central limit theorem:

Proposition 2.1.3 (Central Limit Theorem). *Let $(X_n)_{n \in {}^*\mathbb{N}}$ be an internal sequence of *independent random variables on some internal probability space $(\Omega, \mathcal{F}, P)$ with a common standard distribution function and with mean* 0 *and variance* 1. *Then for any $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ and any $a \in {}^*\mathbb{R}$*

$$P\left(\left\{\omega \in \Omega : \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i(\omega) \leq a\right\}\right) \approx {}^*\Phi(a)$$

where

$$\Phi(a) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} \exp\left(-\frac{x^2}{2}\right) dx$$

is the standard normal distribution function.

We now want to relate internal functions on an internal measure space $(\Omega, \mathcal{F}, P)$ to standard functions on the Loeb space $(\Omega, \mathcal{F}_L, P_L)$: A function $f : \Omega \to \mathbb{R}$ is called *Loeb measurable* if $f^{-1}(B) \in \mathcal{F}_L$ for all open sets $B \subset \mathbb{R}$. An internal function $F : \Omega \to {}^*\mathbb{R}$ is $\mathcal{F}$-measurable if $F^{-1}(A) \in \mathcal{F}$ for any *open set $A \subset {}^*\mathbb{R}$.

Definition 2.1.4. An internal $\mathcal{F}$-measurable function $F : \Omega \to {}^*\mathbb{R}$ is a lifting of the function $f : \Omega \to \mathbb{R}$ if $f(\omega) = {}^\circ F(\omega)$ for P_L-a.a. ω.

Theorem 2.1.5. *The function $f : \Omega \to \mathbb{R}$ is Loeb measurable if and only if it has a lifting F.*

For a $\mathcal{F}$-measurable and *integrable function $F : \Omega \to {}^*\mathbb{R}$ we have an internal integral $\int_\Omega F\,dP$ (or $E_P[F]$ in the case of an internal probability space). We may also consider the integral $\int_\Omega {}^\circ F\,dP_L$ (if it exists) on the Loeb space. To relate these two integrals the notion of $\mathcal{S}$-integrability is needed:

Definition 2.1.6. Let $F : \Omega \to {}^*\mathbb{R}$ be a $\mathcal{F}$-measurable internal function. Then F is *$\mathcal{S}$-integrable* if

(i) $\displaystyle\int_\Omega |F|\,dP$ is finite,

(ii) if $A \in \mathcal{F}$ and $P(A) \approx 0$, then $\displaystyle\int_A |F|\,dP \approx 0$

For $r > 0$ we write $\mathcal{S}L^r(P)$ for the set of internal functions F such that $|F|^r$ is $\mathcal{S}$-integrable.

Theorem 2.1.7. *(i) For any internal $\mathcal{F}$-measurable function $F : \Omega \to {}^*\mathbb{R}$ we have* $\displaystyle\int_\Omega {}^\circ|F|\,dP_L \leq {}^\circ\Big(\int_\Omega F\,dP\Big)$.

(ii) F is $\mathcal{S}$-integrable if and only if $\displaystyle\int_\Omega {}^\circ|F|\,dP_L = {}^\circ\Big(\int_\Omega |F|\,dP\Big) < \infty$.

(iii) $f : \Omega \to \mathbb{R}$ is P_L-integrable if and only if it has an $\mathcal{S}$-integrable lifting F, and in this case $\displaystyle\int_\Omega f\,dP_L = {}^\circ\Big(\int_\Omega F\,dP\Big)$.

We have the following useful test for $\mathcal{S}$-integrability:

Lemma 2.1.8 (Lindstrøm's Lemma [Lin80]). *Let $F : \Omega \to {}^*\mathbb{R}$ be an internal, $\mathcal{F}$-measurable function. If $\int_\Omega |F|^k dP < \infty$ for some $k > 1$, then F is $\mathcal{S}$-integrable.*

The next lemma relates an internal change of measure to the change of the associated Loeb measures. This is of particular importance in the discussion of equivalent martingale measures in mathematical finance.

Lemma 2.1.9 (cf. [CKW91, Lemma 2.1]). *If the internal measure Q is *absolutely continuous with respect to P and its density Z is $\mathcal{S}$-integrable, then Q_L is absolutely continuous with respect to P_L with density $^\circ Z$.*

2.2 Martingales and Stochastic Integration

We start with two regularity concepts for internal functions on the hyperfinite time line $\mathbb{T}$. Our terminology follows [HP83].

Definition 2.2.1. Let $F : \mathbb{T} \to {}^*\mathbb{R}$ be an internal function such that $F(t)$ is finite for all $t \in \mathbb{T}$.

(a) F is of class *SD* if for each $u \in [0,T]$ there are points $s, t \in \mathbb{T}$, $s \approx t \approx u$, such that if $\tilde{s}, \tilde{t} \in \mathbb{T}$ with $\tilde{s} \approx \tilde{t} \approx u$ and $\tilde{s} \leq s$, $t \leq \tilde{t}$, then $F(\tilde{s}) \approx F(s)$ and $F(t) \approx F(\tilde{t})$.

For an SD function we define the function $\mathrm{st}(F) : [0,T] \to \mathbb{R}$ by

$$\mathrm{st}(F)(u) := \lim_{\substack{^\circ t \downarrow u \\ t \in \mathbb{T}}} {}^\circ F(t), \quad u \in [0,T], \tag{2.1}$$

and call $\mathrm{st}(F)$ the *standard part* of F.

(b) F is of class *SDJ* if (a) holds with $s = t$ and also $F(t) \approx F(0)$ for all $t \approx 0$.

Recall that F is $\mathcal{S}$-continuous if $F(s) \approx F(t)$ whenever $s \approx t$, and that the standard part of an $\mathcal{S}$-continuous function is continuous, i.e. $\mathrm{st}(F) \in C[0,T]$. The standard part of an SD function is an element of the set $D[0,T]$ of right-continuous functions with left limits. The extra regularity of SDJ functions means that these functions have at most one non-infinitesimal jump in each monad and are $\mathcal{S}$-continuous at 0. It can be shown (Theorem 2.6 in [HP83]) that $\mathrm{st}|_{\mathrm{SDJ}}$, where st is the map defined in (2.1) is the standard part map for the Skorohod topology on $^*D[0,T]$ (see [Bil68] for a definition of this topology).

We now turn our attention to internal stochastic processes on the hyperfinite time line, i.e. to maps $X : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$, where $(\Omega, \mathcal{F}, P)$ is some internal probability space with associated Loeb space $(\Omega, \mathcal{F}_L, P_L)$. We say that a process is $\mathcal{S}$-continuous (or of class SD or SDJ, respectively) if the path $(X_t(\omega))_{t\in\mathbb{T}}$ has this property for P_L-almost all ω.

Definition 2.2.2. (a) An *internal filtration* $\mathbb{A}$ is an increasing internal sequence $(\mathcal{A}_t)_{t\in\mathbb{T}}$ of algebras on Ω.

(b) An internal process $X : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ is *nonanticipating* (with respect to $\mathbb{A}$) if the map $\omega \mapsto X_t(\omega)$ is $\mathcal{A}_t$-measurable for all $t \in \mathbb{T}$.

For an internal filtration $\mathbb{A}$ we can define a standard filtration $\mathbb{B} = (\mathcal{B}_u)_{u\in[0,T]}$ by setting

$$\mathcal{B}_u := \Big(\bigcap_{\substack{u<{}^\circ t\\ t\in\mathbb{T}}} L(\mathcal{A}_t)\Big) \cup \mathcal{N},$$

where $\mathcal{N}$ denotes the collection of P_L-null sets in $\mathcal{F}_L$. The filtration $\mathbb{B}$ satisfies the "usual conditions" (i.e. right-continuity and completeness), so that $(\Omega, \mathbb{B}, \mathcal{F}, P)$ is a standard filtered probability space.

Definition 2.2.3. (a) A nonanticipating process M is an *internal martingale* if M_t is *integrable (with respect to P) and

$$E\left[\Delta M_t \middle| \mathcal{A}_t\right] = 0 \quad \text{for } t \in \mathbb{T} \setminus \{T\},$$

where $\Delta M_t := M_{t+\Delta t} - M_t$ denotes the "forward increment" of M at t.

(b) M is a λ^2-*martingale* if it is an internal martingale and $E\left[M_t^2\right]$ is finite for all $t \in \mathbb{T}$.

(c) M is an SL^2-martingale if, in addition, $M_t \in SL^2(P)$ for all $t \in \mathbb{T}$.

Remark 2.2.4. Any λ^2-martingale M is of class SD and its standard part $\mathrm{st}(M)$ — which is defined pathwise according to (2.1) — is a $\mathbb{B}$-martingale (Theorem 5.2 in [HP83]).

Definition 2.2.5. The *internal quadratic variation* $[X] : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ of an internal process X is defined as

$$[X]_t := \sum_{s<t} (\Delta X_s)^2.$$

The following result shows that the quadratic variation is an important tool in the study of internal martingales. The second assertion in this theorem follows from [HP83, Theorems 6.4 and 7.18].

Theorem 2.2.6. *Let M be a λ^2-martingale.*

(i) M is S-continuous if and only if its quadratic variation $[M]$ is S-continuous.

(ii) If M is of class SDJ then $[M]$ is of class SDJ and $\mathrm{st}([M]) = [\mathrm{st}(M)]$, *$P_L$-a.s., where the process on the right denotes the standard (optional) quadratic variation of* $\mathrm{st}(M)$ *(see [Ell82, p.99] for a definition).*

We can now obtain stochastic integrals as hyperfinite Stieltjes sums: for an internal process $\Theta : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ we define the *internal stochastic integral* with respect to the internal martingale M as

$$\left(\sum \Theta \Delta M\right)_t := \sum_{s<t} \Theta_s \Delta M_s, \quad t \in \mathbb{T}. \tag{2.2}$$

However, for (2.2) to have a standard counterpart we require additional conditions on the integrand Θ.

Definition 2.2.7. Let $\mathcal{A}_{\Omega \times \mathbb{T}}$ be the internal algebra on $\Omega \times \mathbb{T}$ generated by the sets $\{A \times \{t\} : t \in \mathbb{T}, A \in \mathcal{A}_t\}$. For an internal martingale $M : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ we define the *internal Doléans measure* ν_M on $(\Omega \times \mathbb{T}, \mathcal{A}_{\Omega \times \mathbb{T}})$ by

$$\nu_M(A \times \{t\}) = E\left[\mathbf{1}_A (\Delta M_t)^2\right],$$

where $\mathbf{1}_A$ denotes the indicator function for $A \in \mathcal{A}_t$. The Loeb measure constructed from ν_M will be denoted by $\nu_{L,M}$.

Theorem 2.2.8. *(i) If M is an SL^2-martingale and $\Theta \in SL^2(\nu_M)$ (in particular, Θ is nonanticipating) then $\sum \Theta \Delta M$ is an SL^2-martingale.*

(ii) Furthermore, if M is of class SDJ, then so is $\sum \Theta \Delta M$.

(iii) If M is S-continuous then $\sum \Theta \Delta M$ is S-continuous.

Under the assumptions of Theorem 2.2.8 $\mathrm{st}(\sum \Theta \Delta M)$ is a $\mathbb{B}$-martingale according to Remark 2.2.4. We now want to establish the connection between this process and the standard stochastic integral with respect to $\mathrm{st}(M)$.

Lemma 2.2.9. *Let M be an SL^2-martingale of class SDJ and let $m := \mathrm{st}(M)$ be its standard part. Then the Doléans measure ν_m of m is the restriction of $\nu_{L,M} \circ \mathrm{st}_{\mathbb{T}}^{-1}$ to the predictable sets, where $\mathrm{st}_{\mathbb{T}} : \Omega \times \mathbb{T} \to \Omega \times [0,T]$ is defined by $\mathrm{st}_{\mathbb{T}}(\omega, t) := (\omega, {}^\circ t)$.*

To obtain a nonstandard representation of standard stochastic integrals we need to extend the notion of a lifting. We assume that M is an SL^2-martingale of class SDJ and let $m := \mathrm{st}(M)$.

Definition 2.2.10. Let $\theta : \Omega \times [0,1] \to \mathbb{R}$ be a predictable process in $L^2(\nu_m)$. A *2-lifting* of θ (with respect to M) is a process $\Theta : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ in $SL^2(\nu_M)$ such that ${}^\circ\Theta(\omega, t) = \theta(\omega, {}^\circ t)$ for $\nu_{L,M}$-a.a. (ω, t).

The following theorem now summarises the main results of the nonstandard theory of stochastic integration and its connection to standard stochastic integration.

Theorem 2.2.11. *Let M be an SL^2-martingale of class SDJ. Let $m := \mathrm{st}(M)$ be its standard part and assume that $\theta \in L^2(\nu_m)$. Then θ has a 2-lifting Θ such that*

$$\int \theta dm = \mathrm{st}\left(\sum \Theta \Delta M\right), \qquad P_L\text{-a.s..}$$

Furthermore, $\int \theta dm = \mathrm{st}\left(\sum \bar{\Theta} \Delta M\right)$, *$P_L$-a.s., for any other 2-lifting $\bar{\Theta}$ of θ.*

2.3 Complete Pricing Models

The Black-Scholes model is the most widely used option pricing model in the financial literature as well as in the financial industry. Most generalisations and extensions of pricing methods use this model as a reference point and benchmark. The underlying stochastic evolution of the price s of a risky asset in the Black-Scholes model follows a geometric Brownian motion[1]

$$\frac{ds_t}{s_t} = \mu t + \sigma w_t, \tag{2.3}$$

where w is a standard Brownian motion on some probability space and μ, σ are either constants or time-dependent (non-stochastic) parameters. Equation 2.3 can be used to model equity prices, stock indices and exchange rates as well as prices of bonds and futures (Black's model [Bla76]; see [Hul97] for a detailed exposition of these models). One essential property of the model (2.3) is its completeness, i.e. any claim in this model can be replicated by a self-financing strategy which allows the pricing of this claim by arbitrage considerations alone. This feature was used in [BS73] in the derivation of the celebrated Black-Scholes formula.

However, while the Black-Scholes formula provides a closed-form solution for the pricing problem in the case of European call options and their variants (i.e. options whose values only depend on the price of the risky asset at maturity) the valuation of American or exotic (i.e. path-dependent) options usually requires a combination of sophisticated analytical and numerical methods.

On the other hand, the use of binomial tree models, as first introduced[2] to option pricing problems by Cox, Ross and Rubinstein [CRR79], allows the pricing of claims and the calculation of trading strategies by simple algebra: backward induction methods provide robust numerical procedures for European and American style options which are flexible and easy to implement.

The intuitive idea behind the use of binomial trees in discrete time is that these models approximate — for small time steps — the continuous time model. However,

[1]Geometric Brownian motion with a positive drift was first used in [Sam65] to describe the movement of stock prices.

[2]A binomial option pricing formula was first developed by Sharpe (see [Sha78]). It was subsequently shown in [CRR79] that this yields the Black-Scholes formula in the limit.

this requires a theory of convergence which is not straightforward since it has to deal with convergence of stochastic processes, in particular the convergence of stochastic integrals and integrands.
Most standard convergence results in the literature are concerned with option prices [He90, RS95, MV96, Pri97] or the (weak) convergence of gains processes [DP92]. For a recent survey of convergence methods and results see [BK98, pp.211–221]. We will see below how a nonstandard approach yields a mode of convergence which is suitable for prices, strategies and gains and value processes.

In the following we summarise these models and outline the nonstandard approach together with its convergence results.

The Cox-Ross-Rubinstein Model

For each $n \in \mathbb{N}$ we define the *n-th Cox-Ross-Rubinstein (CRR) model* as follows: Let $T \in \mathbb{R}^+$. Define the discrete time line

$$\mathbb{T}_n := \{0, \Delta_n t, 2\Delta_n t, \ldots, (n-1)\Delta_n t, T\}, \quad \text{with} \quad \Delta_n t := T/n,$$

and let $(\Omega_n, \mathcal{F}_n, P_n)$ be the underlying probability space, where $\Omega_n := \{-1, 1\}^{\mathbb{T}_n \setminus \{T\}}$, $\mathcal{F}_n := \mathcal{P}(\Omega_n)$ and P_n is the normalised counting measure on Ω. A filtration $\mathbb{A}_n = (\mathcal{A}_{n,t})_{t \in \mathbb{T}_n}$ on $(\Omega_n, \mathcal{F}_n)$ is generated by the sets $[\omega]_t := \{\hat{\omega} \in \Omega_n : \omega_s = \hat{\omega}_s,\, s < t\}$. We denote the counting measure on $\mathbb{T}_n$ by Λ_n, i.e. $\Lambda_n(\{t\}) = \Delta_n t$.
A random walk $W_n : \Omega_n \times \mathbb{T}_n \to \mathbb{R}$ is defined by

$$\Delta W_{n,t} := \omega_t \sqrt{\Delta_n t}, \quad W_{n,0} := 0$$

for $\omega = (\omega_0, \ldots, \omega_{T-\Delta_n t}) \in \Omega_n$. Note that the filtration $\mathbb{A}_n$ is then also the natural filtration generated by the process W_n. The price process in the n-th CRR model is now defined as

$$S_{n,t} := s_0 \prod_{s<t} (1 + \mu \Delta_n t + \sigma \Delta W_{n,s}), \quad t \in \mathbb{T}_n,$$

where $\mu, \sigma, s_0 \in \mathbb{R}$ with $\sigma, s_0 > 0$, so that

$$\begin{aligned} S_{n,t+\Delta_n t} &= S_{n,t} \cdot \begin{cases} u := 1 + \mu\Delta_n t + \sigma\sqrt{\Delta_n t} & \text{with prob. } \frac{1}{2} \\ d := 1 + \mu\Delta_n t - \sigma\sqrt{\Delta_n t} & \text{with prob. } \frac{1}{2} \end{cases} \\ S_{n,0} &= s_0. \end{aligned} \tag{2.4}$$

Due to its binomial structure this model is complete (see [TW87] for a proof and a more detailed analysis of finite market models) and any claim H can be replicated by a unique self-financing strategy $\Phi^H = (\Theta^H, \Psi^H)$ (see [CKW91] for an explicit formula for Φ^H). Then

$$H = V_0^H + \sum_{t<T} \Theta_t^H \Delta S_{n,t}$$

for $V_0^H \in \mathbb{R}$. It is well-known that the completeness of the model is equivalent to the uniqueness of the equivalent martingale measure for S_n (see [HK79]). For the n-th CRR model this measure Q_n is given by the density

$$\frac{dQ_n}{dP_n} = \prod_{t<T} \Big(1 - \frac{\mu}{\sigma} \Delta W_{n,t}\Big).$$

The Black-Scholes Model

As mentioned above the price process in the Black-Scholes (BS) model is given by a geometric Brownian motion

$$s_t = s_0 \exp\Big((\mu - \frac{1}{2}\sigma^2)t + \sigma w_t\Big)$$

on the time line $\mathbb{T} := [0, T]$, where w is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ (the parameters T, μ, σ, s_0 are the same as in the CRR model). We assume without loss of generality that $\Omega = \mathcal{C} := \{x \in C[0, T] : x(0) = 0\}$ and that P is Wiener measure on $\mathcal{C}$. A filtration $\mathbb{A} = (\mathcal{A}_t)_{t \in \mathbb{T}}$ on $(\Omega, \mathcal{F})$ is generated by the process w, i.e. $\mathcal{A}_t = \sigma\{s_u : u \leq t\}$.
It follows from the representation theorem for Brownian motion (see e.g. [KS88, Theorem 3.4.2]) that any claim $h \in L^2(P)$ can be replicated by a unique self-financing strategy $\phi^h = (\theta^h, \psi^h)$, i.e.

$$h = v_0^h + \int_0^T \theta_u^h \, ds_u$$

for some $v_0^h \in \mathbb{R}$. Hence, the BS model is complete. The unique equivalent martingale measure Q for s is given by

$$\frac{dQ}{dP} = \exp\Big(-\frac{1}{2}(\frac{\mu}{\sigma})^2 T - \frac{\mu}{\sigma} w_T\Big).$$

The Hyperfinite CRR Model

For any fixed infinite $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ we have an internal CRR model on the hyperfinite filtered probability space $(\Omega_N, \mathcal{F}_N, \mathbb{A}_N, P_N)$ with a price process $S_N : \Omega_N \times \mathbb{T}_N \to {}^*\mathbb{R}$. When considering this model on the Loeb space $(\Omega_N, L(\mathcal{F}_N), L(P_N))$ we see that S_N is $\mathcal{S}$-continuous and

$$\mathrm{st}(S_N) = s, \quad L(P_N)\text{-a.s.}$$

(Lemma 3.1 in [CKW91]). Furthermore,

$$Q = L(Q_N),$$

so that we have indeed constructed a BS model on the Loeb space.

The main result of [CKW91] states that the valuation of claims and calculation of self-financing trading strategies in the hyperfinite CRR model is equivalent to the corresponding operations in the BS model:

Theorem 2.3.1. *Let $H : \Omega_N \to {}^*\mathbb{R}$ be an internal claim and $h \in L^2(Q)$. Then the following are equivalent:*

(i) H is an $SL^2(Q_N)$-lifting of h.

(ii) Θ^H is an $SL^2(\nu_{S_N})$-lifting of θ^h, and Ψ^H is an $SL^2(Q_N \times \Lambda_N)$-lifting of ψ^h.

(iii) $\Theta^H S_N$ and Ψ^H are $SL^2(Q_N \times \Lambda_N)$-liftings of $\theta^h s$ and ψ^h, respectively.

(iv) $V(\Phi^H)$ is $\mathcal{S}$-continuous and $\mathrm{st}\big(V(\Phi^H)\big) = V(\phi^h)$.

(v) $G(\Phi^H)$ is $\mathcal{S}$-continuous and $\mathrm{st}\big(V_0^H + G(\Phi^H)\big) = v_0^h + G(\phi^h)$.

2.4 Convergence Results

When discussing the convergence of random variables or processes on the sequence of CRR models to those on the BS model we have to consider that the underlying probability spaces are all different. This suggest the use of weak convergence, i.e. convergence of the distribution of values. However, the driving process in the CRR model is the binomial random walk W, and the Brownian motion w in the BS model. Contingent claims, strategies and value processes can therefore be regarded as functionals of W and w, respectively. It should then be possible to include the information about this functional relationship into a suitable mode of convergence.
Recall that $\Omega = \mathcal{C}$ is the path space for the BS model. Similarly, we can define the path space $\mathcal{C}_n := \{W_{n,\cdot}(\omega) : \omega \in \Omega_n\}$ for the n-th CRR model. By connecting the points of a path in $\mathcal{C}_n$ linearly the space $\mathcal{C}_n$ can be considered as a subset of the normed space $\mathcal{C}$ (with the uniform norm). The following general convergence result then relates the nonstandard notion of a lifting of a function to the standard concept of "weak convergence along the graph". Note that this concept is genuinely stronger than weak convergence (see [CKW93a] for an example).

Theorem 2.4.1 ([CKW95, Theorem 3.2]). *Let $\mathcal{Y}$ be a separable metric space with a Borel probability μ, and suppose that (μ_n) is a family of probabilities on Borel sets $\mathcal{Y}_n \subset \mathcal{Y}$ converging weakly to μ.*
Equivalently, $\mu = L(\mu_N) \circ \mathrm{st}^{-1}$ for any infinite $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, where $\mathrm{st} : {}^*\mathcal{Y} \to \mathcal{Y}$ is the standard part mapping (see [AR78]).

If $F_n : \mathcal{Y}_n \to \mathbb{R}$ is a family of measurable functions and $f : \mathcal{Y} \to \mathbb{R}$ is measurable then the following are equivalent:

(i) F_N is a lifting of f for all infinite N, i.e. for $L(\mu_N)$-a.a. $y \in \mathcal{Y}_N$, $F_N(y) \approx f({}^\circ y)$;

(ii) $\big(F_n(y), y\big) \to \big(f(y), y\big)$ weakly as $n \to \infty$. This means that the distribution of $\big(F_n(y), y\big) \in \mathbb{R} \times \mathcal{Y}_n$ with y distributed according to μ_n converges to the distribution of $\big(f(y), y\big) \in \mathbb{R} \times \mathcal{Y}$ with y distributed under μ.

Moreover, if (i) or (ii) holds, then for all $r > 0$, $F_N \in SL^r(\mu_N)$ for all infinite N if and only if $E_{\mu_n}[|F_n|^r] \to E_\mu[|f|^r]$ as $n \to \infty$.

Discretisation Schemes

An alternative characterisation of the above lifting property in the context of the paths spaces for the CRR and BS models can be obtained by introducing the idea of a discretisation scheme for mapping paths in $\mathcal{C}$ back into $\mathcal{C}_n$:

Definition 2.4.2. A family $(d_n)_{n \in \mathbb{N}}$ of mappings $d_n : \mathcal{C} \to \mathcal{C}_n$ is an *adapted Q-discretisation scheme* if

(i) d_n is $\mathbb{A}_n$-adapted; i.e. for each $t \in \mathbb{T}_n$ $(d_n(\cdot))(t)$ is $\mathcal{A}_{n,t}$-measurable.

(ii) d_n is Q-measure-preserving; i.e. $Q(d_n^{-1}(A)) = Q_n(A)$ for all $A \in \mathcal{F}_n$.

(iii) $d_n(\omega) \to \omega$ in Q-probability; i.e. for all $\varepsilon > 0$,

$$Q\big(\|d_n(\omega) - \omega\| < \varepsilon\big) \to 1 \quad \text{as } n \to \infty.$$

(Here $\|\cdot\|$ denotes the supremum norm on $\mathcal{C}$.)

Theorem 2.4.3 ([CKW93a, Theorem A.1]). *There is an adapted Q-discretisation scheme for the binomial CRR model.*

In [CKW93a] a Q-adapted discretisation scheme is constructed by using a modification of the random walk approximation to Brownian motion via a "Knight scheme" (see [IM65, p.39]). However, the convergence results in this section are independent of the particular construction of this discretisation scheme.
In this context we have the following extension of Theorem 2.4.1:

Theorem 2.4.4 ([CKW93a, Theorem 4.1]). *Let $(H_n)_{n \in \mathbb{N}}$ with $H_n : \Omega_n \to \mathbb{R}$ be a sequence of claims and let $h \in L^2(Q)$. Then the following are equivalent:*

(i) $H_n\big(d_n(\cdot)\big) \to h$ in $L^2(Q)$.

(ii) H_N is an $SL^2(Q_N)$-lifting of h for all infinite N.

(iii) $\big(H_n(\omega), \omega\big) \to \big(h(\omega), \omega\big)$ weakly and $E_{Q_n}\left[H_n^2\right] \to E_Q\left[h^2\right]$.

Definition 2.4.5. Let $(H_n)_{n\in\mathbb{N}}$ and h be claims as in Theorem 2.4.4. We say that H_n *D^2-converges* to h if any of the equivalent conditions (i)–(iii) in Theorem 2.4.4 hold. We then write $H_n \stackrel{D^2}{\to} h$.

Using an adapted Q-discretisation scheme we can construct a sequence of D^2-convergent claims in the following way:

Theorem 2.4.6 ([CKW93a, Theorem 4.4]). *Let $h \in L^2(Q)$. Define a sequence $H_n : \Omega_n \to \mathbb{R}$ by*

$$H_n(\omega) := E_Q\left[h(\tilde{\omega})|\, d_n(\tilde{\omega}) = \omega\right].$$

Then $H_n \stackrel{D^2}{\to} h$.

We can now extend the notion of D^2-convergence to trading strategies and value and gains processes. This requires a discretisation scheme for the time line $\mathbb{T}$ and the extension of Theorem 2.4.1 to functions taking values in the normed space $\mathcal{C}$, both of which are straightforward (see [CKW93a] for details). We can then reformulate Theorem 2.3.1 in terms of D^2-convergence:

Theorem 2.4.7. *Let $H_n : \Omega_n \to \mathbb{R}$ be a sequence of claims in the CRR models and $h \in L^2(Q)$ a claim in the BS model. Then the following are equivalent:*

(i) $H_n \stackrel{D^2}{\to} h$.

(ii) $\Phi^{H_n} \stackrel{D^2}{\to} \phi^h$.

(iii) $V(\Phi^{H_n}) \stackrel{D^2}{\to} V(\phi^h)$.

(iv) $G(\Phi^{H_n}) \stackrel{D^2}{\to} G(\phi^h)$ *and* $V_0^{H_n} \to V_0^h$.

Remark 2.4.8. This result shows that the concept of D^2-convergence is preserved by the operations of stochastic integration and differentiation (i.e. the calculation of replicating strategies). It therefore seems to be suitable for the analysis of convergence from discrete- to continuous-time market models; the use of D^2-convergence will be extended to incomplete discrete-time models in Chapter 5.
Results similar to Theorem 2.4.7 do not exist for weak convergence. Furthermore, it was shown in [CKWW97] that the above methods can be used to obtain D^2-convergence results for the optimal exercise times and the Snell envelopes in the pricing of American put options in the context of the CRR and BS models.

Chapter 3

Hedging at Fixed Trading Dates

As a first application of nonstandard methods (in particular, Loeb measure theory) in incomplete markets we consider the problem of introducing hedging restrictions into the complete Cox-Ross-Rubinstein and Black-Scholes models, allowing portfolio adjustments only at fixed trading dates. This problem has been studied in [MV96] where it was shown that option prices with respect to variance-optimal hedging[1] in these models converge.

We will give an alternative and more transparent proof of this result. Furthermore we will show that the optimal trading strategies and value processes converge as well for a more general class of models.

The completeness of the CRR and BS models as defined in Section 2.3 relies crucially on the assumption that hedging portfolios can be adjusted at the same times at which asset prices change. Besides the practical impossibility of continuous hedging in a Black-Scholes model the presence of transaction costs in real financial markets leads to the need of changing portfolios as little as possible.

We assume that a set of possible hedging dates $0 = h_0 < h_1 < \ldots < h_{\tilde{T}} = T$ $(\tilde{T} \in \mathbb{N})$ is exogenously given. Since asset prices in the CRR and BS models follow Markov processes the information available to an agent is completely given by the value of these processes at the hedging dates $h_0, \ldots, h_{\tilde{T}}$. This observation leads to modified pricing models: for example, in the BS model the price process $\tilde{S}$ observed at dates $h_0, \ldots, h_{\tilde{T}}$ satisfies

$$\begin{aligned} \tilde{S}_{h_{i+1}} &= \tilde{S}_{h_i} \exp\big((\mu - \frac{1}{2}\sigma^2)(h_{i+1} - h_i) + \sigma \Delta W_{h_i}\big), \quad i = 0, \ldots, \tilde{T} - 1 \\ \tilde{S}_0 &= S_0, \end{aligned}$$

where ΔW_{h_i} is a normal random variable with mean zero and variance $(h_{i+1} - h_i)$. A similar recursive equation applies to the CRR model. We will now introduce a

[1] In fact, in the models under consideration these prices agree with those implied by mean-variance hedging.

framework in which we can study the modified BS and CRR model and examine their convergence properties.

3.1 The Modified Black-Scholes Model

Let $\tilde{\mathbb{T}} := \{0, \dots, \tilde{T}\}$ for some $\tilde{T} \in \mathbb{N}$, $\Omega := \mathbb{R}^{\tilde{T}}$, $\mathcal{F} := \mathcal{B}(\Omega)$, set $\Delta h_i := h_{i+1} - h_i$ $(i = 0, \dots, \tilde{T} - 1)$ and let the probability P be defined by

$$\omega_i \sim \mathcal{N}(0, \Delta h_i),$$

where $\omega = (\omega_0, \dots, \omega_{\tilde{T}-1}) \in \Omega$ and $\omega_0, \dots, \omega_{\tilde{T}-1}$ are independent. This means that ω is a $\tilde{T}$-variate normal random variable, i.e.

$$P\big(\omega_i \leq a_i;\, i = 0, \dots, \tilde{T} - 1\big) = \int_{x \leq a} \prod_{i=0}^{\tilde{T}-1} \frac{1}{\sqrt{2\pi \Delta h_i}} \exp\left(-\frac{(x_i - x_{i-1})^2}{2\Delta h_i}\right) dx$$

where $a = (a_0, \dots, a_{\tilde{T}-1})$, $x = (x_0, \dots x_{\tilde{T}-1}) \in \mathbb{R}^{\tilde{T}}$ and $x_{-1} = 0$.
We can then define a process $W : \Omega \times \tilde{\mathbb{T}} \to \mathbb{R}$ by

$$\begin{aligned} \Delta W_i(\omega) = W_{i+1}(\omega) - W_i(\omega) &:= \omega_i, \quad i < \tilde{T} \\ W_0(\omega) &:= 0. \end{aligned}$$

A filtration $\mathbb{F} = (\mathcal{F}_i)_{i \in \tilde{\mathbb{T}}}$ on Ω is defined by the sequence of σ-algebras generated by W, so that $\mathcal{F}_i := \sigma(W_i)$, $i \in \tilde{\mathbb{T}}$. Equivalently, $\mathcal{F}_i$ is generated by the sets $[\omega]_i := \{\hat{\omega} : \omega_k = \hat{\omega}_k,\, k < i\}$. Observe that $\mathcal{F} = \mathcal{F}_{\tilde{T}}$, so that $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a filtered probability space satisfying the assumptions of Section 1.1. As mentioned above the price process $\tilde{S} : \Omega \times \tilde{\mathbb{T}} \to \mathbb{R}$ in the modified BS model is now given by

$$\begin{aligned} \tilde{S}_{i+1} &= \tilde{S}_i \exp\big((\mu - \tfrac{1}{2}\sigma^2)\Delta h_i + \sigma \Delta W_i\big), \quad i < \tilde{T} \\ \tilde{S}_0 &= S_0 \end{aligned}$$

(cf. equation (14) in [MV96]). Then $\tilde{S}$ is $\mathbb{F}$-adapted and $\tilde{S}_i \in L^2(P)$ for $i \in \tilde{\mathbb{T}}$. We have

$$\Delta \tilde{S}_i = \tilde{S}_i \Big(\exp\big((\mu - \tfrac{1}{2}\sigma^2)\Delta h_i + \sigma \Delta W_i\big) - 1\Big)$$

and can therefore calculate (for $i < \tilde{T}$)

$$\begin{aligned} E[\Delta \tilde{S}_i | \mathcal{F}_i] &= \tilde{S}_i \Big(\exp\big((\mu - \tfrac{1}{2}\sigma^2)\Delta h_i\big) E\left[\exp(\sigma \Delta W_i) | \mathcal{F}_i\right] - 1\Big) \\ &= \tilde{S}_i \big(\exp(\mu \Delta h_i) - 1\big) \end{aligned}$$

and

$$\begin{aligned}
E[(\Delta\tilde{S}_i)^2|\mathcal{F}_i] &= \tilde{S}_i^2 E\left[\left(\exp\left((\mu - \frac{1}{2}\sigma^2)\Delta h_i + \sigma\Delta W_i\right) - 1\right)^2 \middle| \mathcal{F}_i\right] \\
&= \tilde{S}_i^2\Big(\exp\left(2(\mu - \frac{1}{2}\sigma^2)\Delta h_i\right)E\left[\exp(2\sigma\Delta W_i)|\,\mathcal{F}_i\right] \\
&\quad - 2\exp(\mu\Delta h_i) + 1\Big) \\
&= \tilde{S}_i^2\left(\exp\left((2\mu + \sigma^2)\Delta h_i\right) - 2\exp(\mu\Delta h_i) + 1\right),
\end{aligned}$$

hence

$$\begin{aligned}
\operatorname{Var}[\Delta\tilde{S}_i|\mathcal{F}_i] &= \tilde{S}_i^2\Big(\exp\left((2\mu + \sigma^2)\Delta h_i\right) - 2\exp(\mu\Delta h_i) + 1 \\
&\quad - \exp(2\mu\Delta h_i) + 2\exp(\mu\Delta h_i) - 1\Big) \\
&= \tilde{S}_i^2\exp(2\mu\Delta h_i)\left(\exp(\sigma^2\Delta h_i) - 1\right),
\end{aligned}$$

where we have used the fact that $E[\exp(cX)] = \exp(\frac{1}{2}c^2 v)$ for $X \sim \mathcal{N}(0, v)$ and $c \in \mathbb{R}$. In particular, $\operatorname{Var}[\Delta\tilde{S}_i|\mathcal{F}_i] > 0$, P-a.s.. We see that the ratio

$$\frac{(E[\Delta\tilde{S}_j|\mathcal{F}_j])^2}{\operatorname{Var}[\Delta\tilde{S}_j|\mathcal{F}_j]} = \frac{\left(\exp(\mu\Delta h_j) - 1\right)^2}{\exp(2\mu\Delta h_j)\left(\exp(\sigma^2\Delta h_j) - 1\right)}$$

is deterministic. This implies that the nondegeneracy condition (1.6) is satisfied. We can therefore use the methodology of Section 1.1 to calculate the mean-variance optimal strategy ϕ^H with associated value process V^H for any $H \in L^2(P)$ (cf. equations (1.3)–(1.5)). Furthermore, also the variance-optimal strategy ξ^H exists and can be calculated in the recursive form (1.27). Finally, the variance-optimal price c^H agrees with the mean-variance price V_0^H (cf. Theorem 1.3.1).

3.2 The Modified Cox-Ross-Rubinstein Model

The n-th CRR model was defined in Section 2.3. We recall that the price process in this model follows the evolution equation

$$\begin{aligned}
S_{n,t+\Delta_n t} &= S_{n,t} \cdot \begin{cases} u := 1 + \mu\Delta_n t + \sigma\sqrt{\Delta_n t} & \text{with prob. } \frac{1}{2} \\ d := 1 + \mu\Delta_n t - \sigma\sqrt{\Delta_n t} & \text{with prob. } \frac{1}{2} \end{cases} \\
S_{n,0} &= S_0,
\end{aligned} \tag{3.1}$$

where $t \in \mathbb{T}_n = \{0, \Delta_n t, \dots, T\}$ with $\Delta_n t := 1/n$. When considering this model under restricted hedging we have to make sure that the hedging dates $h_0, \dots, h_{\tilde{T}}$ are elements of the set $\mathbb{T}_n$ of possible trading dates. We therefore assume from now on that $h_1, \dots, h_{\tilde{T}} \in \mathbb{Q}$ and define

$$\tilde{\mathbb{N}} := \left\{n \in \mathbb{N} : h_1 = \frac{n_1}{n}, \dots, h_{\tilde{T}} = \frac{n_{\tilde{T}}}{n} \text{ for some } n_1, \dots, n_{\tilde{T}} \in \mathbb{N}\right\}. \tag{3.2}$$

Due to our assumption on $h_1, \dots, h_{\tilde{T}}$ the set $\tilde{\mathbb{N}}$ is infinite. For $n \in \tilde{\mathbb{N}}$ we can now define the n-th modified CRR model:
The observed stock price in this model follows a multinomial evolution:

$$\begin{aligned} \tilde{S}_{n,i+1} &= \tilde{S}_{n,i} \cdot \begin{cases} u^{\beta_i} & \text{with prob. } 2^{-\beta_i} \\ \vdots & \vdots \\ u^j d^{\beta_i - j} & \text{with prob. } \binom{\beta_i}{j} 2^{-\beta_i} \text{ for } j = 1, \dots, \beta_i - 1 \\ \vdots & \vdots \\ d^{\beta_i} & \text{with prob. } 2^{-\beta_i} \end{cases} \\ \tilde{S}_{n,0} &= S_0, \end{aligned}$$

(cf. equation (15) in [MV96][2]) with $i \in \tilde{\mathbb{T}}$ and where $\beta_i := n_{i+1} - n_i$ with $n_1, \dots, n_{\tilde{T}}$ defined as in (3.2) is the number of price changes in the n-th CRR model between the hedging dates h_i and h_{i+1}.
We now set

$$\Omega_n := R_0 \times \cdots \times R_{\tilde{T}-1} \quad \text{with} \quad R_i := \{(-\beta_i + 2j)\sqrt{\Delta_n t} : j = 0, \dots, \beta_i\}.$$

Let $\mathcal{F}_n := \mathcal{P}(\Omega_n)$ and the measure P_n on $(\Omega_n, \mathcal{F}_n)$ is given by

$$P_n\big(\omega_i = (-\beta_i + 2j)\sqrt{\Delta_n t}\big) = \binom{\beta_i}{j} 2^{-\beta_i}, \quad j = 0, \dots, \beta_i,$$

where $\omega = (\omega_0, \dots, \omega_{\tilde{T}-1}) \in \Omega_n$ and $\omega_0, \dots, \omega_{\tilde{T}-1}$ are independent, so each ω_i is a $(\beta_i + 1)$-nomial random variable with mean zero and variance $\beta_i \Delta_n t = \Delta h_i$. We denote the expectation with respect to P_n by $E_n[\,\cdot\,]$.
The process $W_n : \Omega_n \times \tilde{\mathbb{T}}$ is now defined as in Section 3.1:

$$\begin{aligned} \Delta W_{n,i}(\omega) &= W_{n,i+1}(\omega) - W_{n,i}(\omega) := \omega_i, \quad i < \tilde{T} \\ W_{n,0}(\omega) &:= 0. \end{aligned}$$

Again, the filtration $\mathbb{F}_n = (\mathcal{F}_{n,i})_{i \in \tilde{\mathbb{T}}}$ on Ω_n is generated by W_n, i.e. $\mathcal{F}_{n,i}$ is the algebra generated by the sets $[\omega]_i := \{\hat{\omega} : \omega_k = \hat{\omega}_k,\ k < i\}$. Hence $\mathcal{F}_n = \mathcal{F}_{n,\tilde{T}}$. The dynamics of the price process observed at the hedging dates $\tilde{\mathbb{T}}$ can now be written as

$$\begin{aligned} \tilde{S}_{n,i+1} &= \tilde{S}_{n,i} u^{J_i} d^{\beta_i - J_i}, \quad i < \tilde{T} \\ \tilde{S}_{n,0} &= S_0, \end{aligned} \tag{3.3}$$

where $J_i := \frac{1}{2}\big(\frac{\Delta W_{n,i}}{\sqrt{\Delta_n t}} + \beta_i\big)$ is the number of "up"-movements of the underlying CRR price process between the hedging dates h_i and h_{i+1}. We see that $\tilde{S}_n$ is $\mathbb{F}_n$-adapted and $\tilde{S}_{n,i} \in L^2(P_n)$ for $i \in \tilde{\mathbb{T}}$ with

$$\Delta \tilde{S}_{n,i} = \tilde{S}_{n,i}\big(u^{J_i} d^{\beta_i - J_i} - 1\big),$$

[2]Note that our definition of the CRR model in Section 2.3 differs from the one used in [MV96]. However, this does not affect the convergence properties as we will see in the next section.

so that

$$E_n[\Delta\tilde{S}_{n,i}|\mathcal{F}_{n,i}] = \tilde{S}_{n,i}\Big(E\left[u^{J_i}d^{\beta_i-J_i}\Big|\,\mathcal{F}_{n,i}\right]-1\Big)$$
$$= \tilde{S}_{n,i}\big(2^{-\beta_i}(u+d)^{\beta_i}-1\big)$$
$$= \tilde{S}_{n,i}\big((1+\mu\Delta_n t)^{\beta_i}-1\big)$$

and

$$E_n[(\Delta\tilde{S}_{n,i})^2|\mathcal{F}_{n,i}] = \tilde{S}^2_{n,i}\Big(E\left[(u^2)^{J_i}(d^2)^{\beta_i-J_i}\Big|\,\mathcal{F}_{n,i}\right]-2(1+\mu\Delta_n t)^{\beta_i}+1\Big)$$
$$= \tilde{S}^2_{n,i}\Big(2^{-\beta_i}\big(u^2+d^2\big)^{\beta_i}-2(1+\mu\Delta_n t)^{\beta_i}+1\Big),$$

hence

$$\mathrm{Var}_n[\Delta\tilde{S}_{n,i}|\mathcal{F}_{n,i}] = \tilde{S}^2_{n,i}\Big(2^{-\beta_i}\big(u^2+d^2\big)^{\beta_i}-(1+\mu\Delta_n t)^{2\beta_i}\Big)$$
$$= \tilde{S}^2_{n,i}\Big(\big(1+(\sigma^2+2\mu)\Delta_n t+\mu^2(\Delta_n t)^2\big)^{\beta_i}$$
$$-\big(1+2\mu\Delta_n t+\mu^2(\Delta_n t)^2\big)^{\beta_i}\Big)$$

for $i<\tilde{T}$. Again the ratio $(E_n[\Delta S_{n,t}|\mathcal{F}_{n,t}])^2/\mathrm{Var}_n[\Delta S_{n,t}|\mathcal{F}_{n,t}]$ is deterministic so that for any $H_n \in L^2(P_n)$ the mean-variance strategy ϕ^{H_n} with value process V^{H_n} and the variance-optimal strategy ξ^{H_n} with optimal price $c^{H_n} = V_0^{H_n}$ can be calculated as in Section 3.1.

3.3 The Nonstandard Approach

We have introduced the two modified models

$$\boldsymbol{\Omega}_n := (\Omega_n, \mathbb{F}_n, \mathcal{F}_n, P_n, \tilde{S}_n) \qquad \text{and} \qquad \boldsymbol{\Omega} := (\Omega, \mathbb{F}, \mathcal{F}, P, \tilde{S})$$

and have noted that for contingent claims

$$H_n \in L^2(P_n) \qquad \text{and} \qquad H \in L^2(P)$$

the solutions

$$V^{H_n}, \phi^{H_n}, \psi^{H_n}, \xi^{H_n} \qquad \text{and} \qquad V^H, \phi^H, \psi^H, \xi^H$$

to the local risk-minimisation and variance optimisation problems exist and can be calculated using the results in Sections 1.1 and 1.3. We will now examine the convergence properties of these models, i.e. we want to show that the prices and optimal strategies in $\boldsymbol{\Omega}_n$ converge to those in $\boldsymbol{\Omega}$ if the contingent claims H_n converge to H in a suitable way.

The set $\tilde{\mathbb{N}}$ of possible parameters n for the modified CRR model (3.3) was defined in (3.2). It has a nonstandard extension ${}^*\tilde{\mathbb{N}}$ which by the Transfer Principle contains arbitrarily large infinite elements. We now fix one such $N \in {}^*\tilde{\mathbb{N}} \setminus \tilde{\mathbb{N}}$. So we have an internal hyperfinite probability space $(\Omega_N, \mathcal{F}_N, P_N)$ and its associated Loeb space $(\Omega_N, L(\mathcal{F}_N), L(P_N))$.

Observing that each ω_i for $\omega = (\omega_0, \dots, \omega_{\tilde{T}-1}) \in \Omega_N$ is the sum of β_i binomial random variables with mean zero and variance $\Delta_N t$ it follows from the nonstandard version of the Central Limit Theorem 2.1.3 that[3]

$$P = L(P_N) \circ \mathrm{st}_N^{-1}, \tag{3.4}$$

where $\mathrm{st}_N^{-1} : \Omega \to \Omega_N$ denotes the restriction of the inverse standard part map to Ω_N, and by the definition of W_N and W we have

$$^\circ W_{N,i}(\omega) = {}^\circ\Big(\sum_{j=1}^{i} \omega_j\Big) = \sum_{j=1}^{i} {}^\circ\omega_j = W_i({}^\circ\omega) \qquad L(P_N)\text{-a.s.},$$

for $i \in \tilde{\mathbb{T}}$. Furthermore, Lemma 3.1 in [CKW91] shows that

$$u^{J_i} d^{\beta_i - J_i} \approx \exp\big((\mu - \frac{1}{2}\sigma^2)\Delta h_i + \sigma \Delta W_{N,i}\big) \qquad L(P_N)\text{-a.s.},$$

for $i \in \tilde{\mathbb{T}}$, with both sides in $SL^r(P_N)$ for $r \in [1, \infty)$. By the definition of $\tilde{S}_N$ and $\tilde{S}$ this implies that

$$^\circ\tilde{S}_{N,i}(\omega) = \tilde{S}_i({}^\circ\omega) \quad \text{for } i \in \tilde{\mathbb{T}}, \quad L(P_N)\text{-a.s.}, \tag{3.5}$$

so that, for $i \in \tilde{\mathbb{T}}$, $\tilde{S}_{N,i}$ is a $SL^r(P_N)$-lifting of $\tilde{S}_i$ ($r \in [1, \infty)$).

In the following we will see that the properties (3.4) and (3.5) are sufficient for the convergence of the optimal trading strategies and option prices in the sequence of models $(\mathbf{\Omega}_n)_{n \in \mathbb{N}}$ to those in $\mathbf{\Omega}$. We therefore consider a general sequence of finite models $\mathbf{\Omega}_n = (\Omega_n, \mathbb{F}_n, \mathcal{F}_n, P_n, S_n)$ ($n \in \tilde{\mathbb{N}}$ where $\tilde{\mathbb{N}} \subset \mathbb{N}$ is infinite) together with a model $\mathbf{\Omega} = (\Omega, \mathbb{F}, \mathcal{F}, P, S)$ on the set $\mathbb{T} = \{0, \dots, T\}$ satisfying the following assumptions:

(M1) $\Omega_n = R_0 \times \cdots \times R_{T-1}$ for finite sets $R_i \subset \mathbb{R}$, $\mathcal{F}_n = \mathcal{P}(\Omega_n)$ and $\Omega = \mathbb{R}^T$, $\mathcal{F} = \mathcal{B}(\Omega)$.

(M2) For $\omega = (\omega_0, \dots, \omega_{T-1}) \in \Omega_n$ (Ω, resp.) the random variables $(\omega_i)_{i \in \mathbb{T} \setminus \{T\}}$ are independent under the measure P_n (P, resp.). The filtrations $\mathbb{F}_n$ and $\mathbb{F}$ are generated by the sets $[\omega]_i = \{\hat{\omega} : \omega_j = \hat{\omega}_j, \, j < i\}$. $\mathcal{F}_n = \mathcal{F}_{n,T}$ and $\mathcal{F} = \mathcal{F}_T$.

[3] see the proof of Theorem 3.3.5 in [AFHL86] or the proof of Lemma 3.3.7 below for a similar argument.

(M3) The price processes S_n (S, resp.) are square-integrable, the ratio

$$\frac{\left(E_n[\Delta S_{n,i}|\mathcal{F}_{n,i}]\right)^2}{\mathrm{Var}_n[\Delta S_{n,i}|\mathcal{F}_{n,i}]} \quad \text{is deterministic for } i < T$$

and $\mathrm{Var}_n[\Delta S_{n,i}|\mathcal{F}_{n,i}] > 0$ for $i < T$, with the analogous conditions satisfied for S.

(M4) For all $N \in {}^*\tilde{\mathbb{N}} \setminus \tilde{\mathbb{N}}$ (3.4) and (3.5) are satisfied, i.e.

$$P = L(P_N) \circ \mathrm{st}_N^{-1} \qquad \text{and} \qquad {}^\circ S_{N,i}(\omega) = S_i({}^\circ\omega) \quad L(P_N)\text{-a.s., } i \in \mathbb{T}.$$

Furthermore, we assume that $E_N[|S_{N,i}|^k] < \infty$ for $i \in \mathbb{T}$ and some $k > 2$.

We have already verified that (M1)–(M4) are satisfied for the sequence of modified CRR models of Section 3.2 and the modified BS model of Section 3.1.
The next lemma shows that in this framework the lifting property (3.5) extends to the internal processes involved in finding the locally risk-minimal and variance-optimal strategies.

Lemma 3.3.1. *Let $F : \Omega_N \to {}^*\mathbb{R}$ be a lifting of $f : \Omega \to \mathbb{R}$ (i.e. ${}^\circ(F(\omega)) = f({}^\circ\omega)$ for $L(P_N)$-a.a. ω) which is S-integrable. Then*

$${}^\circ(E_N[F|\mathcal{F}_{N,i}](\omega)) = E[f|\mathcal{F}_i]({}^\circ\omega) \quad \text{for } i \in \mathbb{T},\ L(P_N)\text{-a.s..}$$

Proof. Note that $E_N[F|\mathcal{F}_{N,T}] = E_N[F|\mathcal{F}_N] = F$, $L(P_N)$-a.s. by our assumption on $\mathbb{F}_N$, and similarly for f. Hence the assertion is true for $i = T$.
For $i < T$ we define a subspace $\Omega^{(i)}$ of Ω and $\Omega_N^{(i)}$ of Ω_N by

$$\begin{aligned} \Omega^{(i)} &:= \mathbb{R}^{T-i} \\ \Omega_N^{(i)} &:= R_i \times \cdots \times R_{T-1} \end{aligned}$$

with probability measures $P_N^{(i)}$ and $P^{(i)}$ defined by

$$\begin{aligned} P^{(i)}(A) &:= P(\mathbb{R}^i \times A) \\ P_N^{(i)}(A_N) &:= P_N(R_0 \times \cdots R_{i-1} \times A_N) \end{aligned}$$

for $A \in \Omega^{(i)}$ and $A_N \in \Omega_N^{(i)}$.
Since F (and f, respectively) is of the form $F(\omega) = F(\omega_0, \dots, \omega_i, \dots, \omega_{T-1})$ with $\omega_j \in R_j$ ($\omega_j \in \mathbb{R}$, resp.) we have

$$\begin{aligned}
{}^\circ(E_N[F|\mathcal{F}_{N,i}](\omega)) &= {}^\circ\Big(\int_{\Omega_N^{(i)}} F(\omega_1,\dots,\omega_i,\hat{\omega})dP_N^{(i)}(\hat{\omega})\Big) \\
&= \int_{\Omega_N^{(i)}} {}^\circ F(\omega_1,\dots,\omega_i,\hat{\omega})dL(P_N^{(i)})(\hat{\omega}) \qquad (3.6)\\
&= \int_{\mathrm{st}_{N,i}^{-1}(\Omega^{(i)})} {}^\circ F(\omega_1,\dots,\omega_i,\hat{\omega})dL(P_N^{(i)})(\hat{\omega}) \qquad (3.7)\\
&= \int_{\mathrm{st}_{N,i}^{-1}(\Omega^{(i)})} f({}^\circ\omega_1,\dots,{}^\circ\omega_i,{}^\circ\hat{\omega})dL(P_N^{(i)})(\hat{\omega})\\
&= \int_{\Omega^{(i)}} f({}^\circ\omega_1,\dots,{}^\circ\omega_i,\hat{\omega})dP^{(i)}(\hat{\omega}) \qquad (3.8)\\
&= E[f|\mathcal{F}_i]({}^\circ\omega)
\end{aligned}$$

(here $\mathrm{st}_{N,i}^{-1} : \Omega^{(i)} \to \Omega_N^{(i)}$ denotes the restriction of the standard part inverse map to $\Omega_N^{(i)}$). Equality (3.6) follows from the $\mathcal{S}$-integrability of F and Keisler's Fubini Theorem [AFHL86, Theorem 3.2.14], (3.7) from the fact that elements of $\Omega_N^{(i)}$ are $L(P_N)$-a.s. finite and (3.8) from the measurability of the standard part map together with (3.4) and the "measurable transformation theorem" [Hal50, Theorem 39.C]. □

Since the optimal trading strategies and value processes are defined in terms of conditional expectations (cf. (1.3)–(1.4)) we can use Lemma 3.3.1 to show that all the processes on $\Omega_N \times \mathbb{T}$ involved in the solution of the risk-minimisation problem are liftings of their respective counterparts on $\Omega \times \mathbb{T}$. We only have to make sure that the condition of $\mathcal{S}$-integrability is satisfied at all stages.

Option Prices and Trading Strategies

Let $H \in L^2(P)$ be a claim on Ω and $H_N \in SL^2(P_N)$ be a lifting of H. We will see later how we can obtain such a lifting in a natural way.
By assumption (M4) and Lindstrøm's Lemma 2.1.8, $\Delta S_{N,i}$ is an SL^2-lifting of ΔS_i for $i \in \mathbb{T}$. Lemma 3.3.1 then implies that $\mathrm{Var}[\Delta S_{N,i}|\mathcal{F}_{N,i}]$ is a lifting of $\mathrm{Var}[\Delta S_i|\mathcal{F}_i]$ for $i \in \mathbb{T}\setminus\{T\}$. In particular, by assumption (M3),

$$\mathrm{Var}_N\,[\Delta S_{N,i}|\,\mathcal{F}_{N,i}] \not\approx 0 \quad \text{for } i \in \mathbb{T}\setminus\{T\}.$$

Furthermore, since $E_N[H_N^2] < \infty$ and $E_N[|\Delta S_{N,i}|^k] < \infty$ for some $k > 2$ we have, setting

$$r := \frac{2k}{k+2}, \quad p := \frac{2}{r} = \frac{k+2}{k}, \quad q := \frac{k}{r} = \frac{k+2}{2},$$

and using Hölder's inequality (note that $1/p + 1/q = 1$),

$$\begin{aligned}
E_N\left[|H_N\,\Delta S_{N,i}|^r\right] &\le \left(E_N\left[|H_N|^{rp}\right]\right)^{1/p}\left(E_N\left[|\Delta S_{N,i}|^{rq}\right]\right)^{1/q}\\
&= \left(E_N\left[H_N^2\right]\right)^{1/p}\left(E_N\left[|\Delta S_{N,i}|^k\right]\right)^{1/q} < \infty
\end{aligned}$$

As $r = 2k/(k+2) > 1$, Lindstrøm's Lemma implies that $H_N \Delta S_{N,T-1}$ is $\mathcal{S}$-integrable. It follows from Lemma 3.3.1 that $\mathrm{Cov}_N\left[H_N, \Delta S_{N,T-1} | \mathcal{F}_{N,T}\right]$ is an $\mathcal{S}$-integrable lifting of $\mathrm{Cov}[H, \Delta S_{T-1} | \mathcal{F}_T]$, and hence

$$\theta_{T-1}^{H_N} := \frac{\mathrm{Cov}_N\left[H_N, \Delta S_{N,T-1} | \, \mathcal{F}_{N,T-1}\right]}{\mathrm{Var}_N\left[\Delta S_{N,T-1} | \, \mathcal{F}_{N,T-1}\right]}$$

(cf. equation (1.3)) is a lifting of its counterpart θ_{T-1}^{H} on Ω.
Since $E_N\left[(\theta_{T-1}^{H_N} \Delta S_{N,T-1})^2\right] < \infty$ (see inequality (1.7)) also

$$V_{T-1}^{H_N} := E_N\left[H_N - \theta_{T-1}^{H_N} \Delta S_{N,T-1} \middle| \, \mathcal{F}_{N,T-1}\right]$$

is a lifting of its counterpart V_{T-1}^{H}. Moreover, $E_N\left[(V_{T-1}^{H_N})^2\right] < \infty$. We can then repeat the above argument backwards for $i = T-2, \ldots, 0$ to obtain

$$\left(V_i^{H_N}, \theta_i^{H_N}\right)_{i \in \mathbb{T}} \quad \text{is a lifting of} \quad (V_i^{H}, \theta_i^{H})_{i \in \mathbb{T}}.$$

In particular,

$$V_0^{H_N} \approx V_0^{H}.$$

The corresponding lifting property of the second component of the strategy $\phi^{H_N} = (\theta^{H_N}, \psi^{H_N})$ then follows immediately from its definition $\psi^{H_N} := V^{H_N} - \theta^{H_N} S_N$ (see equation (1.8)).
Assumption (M3) implies that the internal mean-variance tradeoff process $\tilde{K}_N$ as defined in (1.26) is deterministic[4], so that the variance-optimal strategy ξ^{H_N} can be calculated in the recursive form of equation (1.27):

$$\xi_i^{H_N} = \theta_i^{H_N} + \frac{E_N\left[\Delta S_{N,i} | \, \mathcal{F}_{N,i}\right]}{E_N\left[(\Delta S_{N,i})^2 | \, \mathcal{F}_{N,i}\right]} \left(V_i^{H_N} - V_0^{H_N} - \sum_{j=0}^{i-1} \xi_j^{H_N} \Delta S_{N,j}\right)$$

for $i \in \mathbb{T}$. An induction argument then shows that the term on the right hand side of the above equation is a lifting of its counterpart on Ω (note that $\xi_0^{H_N} = \theta_0^{H_N}$), so that

$$\xi^{H_N} \quad \text{is a lifting of} \quad \xi^{H}.$$

We summarise our results in the following

Theorem 3.3.2. *Suppose the models Ω_N and Ω satisfy the assumptions (M1)-(M4). If $H_N \in SL^2(P_N)$ is a lifting of $H \in L^2(P)$ then*

$$\left(V^{H_N}, \theta^{H_N}, \psi^{H_N}, \xi^{H_N}\right) \quad \text{is a lifting of} \quad (V^{H}, \theta^{H}, \psi^{H}, \xi^{H}).$$

[4] In fact, our arguments also show that $\tilde{K}_N$ is a lifting of the mean-variance tradeoff process $\tilde{K}$ on $\Omega \times \mathbb{T}$.

The Minimal Martingale Measure

Since $S_{N,i}$ is an SL^2-lifting of S_i Lemma 3.3.1 shows that the processes A_N, M_N defined by

$$\Delta A_{N,i} := E_N\left[\Delta S_{N,i} | \mathcal{F}_{N,i}\right], \quad \Delta M_{N,i} := \Delta S_{N,i} - \Delta A_{N,i}, \quad A_{N,0} := M_{N,0} := 0$$

are SL^2-liftings of their respective counterparts A, M on $\Omega \times \mathbb{T}$.
We then define α_N and α according to (1.13), so that

$$\alpha_{N,i} = \frac{\Delta A_{N,i}}{\mathrm{Var}_N\left[\Delta S_{N,i} | \mathcal{F}_{N,i}\right]} = \frac{\Delta A_{N,i}}{E_N\left[(\Delta M_{N,i})^2 | \mathcal{F}_{N,i}\right]}$$

is a lifting of α_i. This shows that

$$\hat{Z}_N := \prod_{s=0}^{T-1}\left(1 - \alpha_{N,i}\Delta M_{N,i}\right) \quad \text{is a lifting of} \quad \hat{Z} := \prod_{s=0}^{T-1}\left(1 - \alpha_i \Delta M_i\right). \tag{3.9}$$

We know from Section 1.1 that $E_N[\hat{Z}_N^2] < \infty$, hence $\hat{Z}_N$ is $\mathcal{S}$-integrable.
We can now define the minimal martingale measures $\hat{P}_N$ on Ω_N and $\hat{P}$ on Ω by setting

$$\frac{d\hat{P}_N}{dP_N} = \hat{Z}_N \quad \text{and} \quad \frac{d\hat{P}}{dP} = \hat{Z}.$$

Remark 3.3.3. It was mentioned before that the minimal martingale measure is in general a signed measure. In fact, in the modified BS and CRR models the densities $\hat{Z}$ and $\hat{Z}_N$, respectively, will take negative values with positive probability: for example, in the modified BS model the calculations in Section 3.1 show that

$$\alpha_i \Delta M_i = \frac{1}{\tilde{S}_i}\Big(\frac{\exp(c_1 + \sigma \Delta W_i) - 1}{c_2} - c_3\Big)$$

for constants $c_1, c_2, c_3 \in \mathbb{R}$ with $c_2 > 0$. Hence

$$1 - \alpha_i \Delta M_i < 0 \quad \text{if} \quad \exp(c_1 + \sigma\Delta W_i) > 1 + c_2(\tilde{S}_i + c_3)$$

which is true with positive probability since $P(\Delta W_i > k) > 0$ for any $k \in \mathbb{R}$.
To deal with this we can define the positive and negative parts of $\hat{Z}_N$ and $\hat{Z}$ by

$$\hat{Z}_N^{\pm} := \frac{1}{2}\big(|\hat{Z}_N| \pm \hat{Z}_N\big) \quad \text{and} \quad \hat{Z}^{\pm} := \frac{1}{2}\big(|\hat{Z}| \pm \hat{Z}\big).$$

Again, $\hat{Z}_N^{\pm}$ is $\mathcal{S}$-integrable and $\hat{Z}_N^{\pm}$ is a lifting of $\hat{Z}^{\pm}$, respectively. The positive and negative variations of the measures $\hat{P}_N$ and $\hat{P}$ (see [Rud87] for a definition) are then given by

$$d\hat{P}_N^{\pm} = \hat{Z}_N^{\pm} dP_N \quad \text{and} \quad d\hat{P}^{\pm} = \hat{Z}^{\pm} dP,$$

i.e. $\hat{P}_N = \hat{P}_N^{+} - \hat{P}_N^{-}$ and similarly for $\hat{P}$. Lemma 2.1.9 then implies that

$$\frac{dL(\hat{P}_N^{\pm})}{dL(P_N)}(\omega) = {}^{\circ}\Big(\frac{d\hat{P}_N^{\pm}}{dP_N}(\omega)\Big) = {}^{\circ}\big(\hat{Z}_N^{\pm}(\omega)\big) = \hat{Z}^{\pm}({}^{\circ}\omega).$$

Proposition 3.3.4. *For $N \in {}^*\mathbb{N} \setminus \mathbb{N}$,*

$$\hat{P}^{\pm} = L(\hat{P}_N^{\pm}) \circ \mathrm{st}_N^{-1}.$$

Hence

$$\hat{P} = L(\hat{P}_N) \circ \mathrm{st}_N^{-1}.$$

Proof. Given $A \in \mathcal{F}$,

$$\begin{aligned}
\left(L(\hat{P}_N^{\pm}) \circ \mathrm{st}_N^{-1}\right)(A) &= \int_{\mathrm{st}_N^{-1}(A)} dL(\hat{P}_N^{\pm}) \\
&= \int_{\mathrm{st}_N^{-1}(A)} {}^{\circ}\hat{Z}_N^{\pm}\, dL(P_N) \\
&= \int_A \hat{Z}^{\pm}\, dP = \int_A d\hat{P}^{\pm} = \hat{P}^{\pm}(A).
\end{aligned}$$

□

We will close this section with two other examples of models satisfying assumptions (M1)–(M4).

Further Examples

Example 3.3.5 (Approximation of the Cox-Ross Jump Model). The Poisson process model of Cox and Ross [CR76] was obtained in [CKW93b] as the limit of a sequence of binomial models. More precisely, the process

$$S_t = S_0 \exp(-\mu t + bN_t),$$

(with constants $\mu, b \in \mathbb{R}$, $b > 0$), where N is a Poisson process with intensity parameter $\lambda > 0$, is approximated by the sequence of processes (S_n) defined by

$$S_{n,t+\Delta_n t} := S_{n,t} \cdot \begin{cases} u := 1 - \mu\Delta_n t + (e^b - 1) & \text{with prob. } p := \lambda\Delta_n t \\ d := 1 - \mu\Delta_n t & \text{with prob. } 1 - p \end{cases}$$

for $t \in \mathbb{T}_n = \{0, \Delta_n t, \dots, T\}$ with $\Delta_n t := 1/n$ (cf. (3.1) for the CRR price process). We may now consider these models under the same hedging restrictions as before: using the notation of Sections 3.1 and 3.2 we set

$$\Omega_n := R_0 \times \cdots \times R_{\tilde{T}-1} \quad \text{with} \quad R_i := \{j : j = 0, \dots, \beta_i\}$$

with a probability P_n given by

$$P_n(\omega_i = j) = \binom{\beta_i}{j} p^j (1-p)^{\beta_i - j},$$

and $\Omega = \mathbb{N}^{\tilde{T}}$ with $P(\omega_i = j) = \frac{(\lambda\Delta h_i)^j}{j!}\exp(-\lambda\Delta h_i)$. The modified processes $\tilde{S}$ and $\tilde{S}_n$ on $\tilde{\mathbb{T}}$ then satisfy

$$\begin{aligned} \tilde{S}_{i+1} &= \tilde{S}_i \exp(-\mu\Delta h_i + b\,\omega_i) \\ \text{and} \quad \tilde{S}_{n,i+1} &= \tilde{S}_{n,i}\, u^{\omega_i} d^{\beta_i - \omega_i}. \end{aligned}$$

Using the results in [CKW93b] it is now straightforward to check that the assumptions (M1)–(M4) are satisfied[5], so that Theorem 3.3.2 and Proposition 3.3.4 can be applied in this context.

Example 3.3.6 (Alternative Definition of the CRR Model). As mentioned earlier the definition of the CRR model in [MV96] is different from the one used in Section 3.2 in that the evolution equation (3.1) takes the form

$$\begin{aligned} S_{n,t+\Delta_n t} &= S_{n,t} \cdot \begin{cases} u := \exp(\sigma\sqrt{\Delta_n t}) & \text{with prob. } p := \frac{e^{\mu\Delta_n t} - d}{u-d} \\ d := \frac{1}{u} = \exp(-\sigma\sqrt{\Delta_n t}) & \text{with prob. } 1-p \end{cases} \\ S_{n,0} &= 0. \end{aligned} \tag{3.10}$$

The set $\tilde{\mathbb{N}}$ is again defined as in (3.2) and for $n \in \tilde{\mathbb{N}}$ we set $\Omega_n := R_0 \times \cdots \times R_{\tilde{T}-1}$, with $R_i := \{(-\beta_i + 2j)\sigma\sqrt{\Delta_n t} : j = 0, \dots, \beta_i\}$, $\mathcal{F}_n = \mathcal{P}(\Omega_n)$ and

$$P_n\big(\omega_i = (-\beta_i + 2j)\sigma\sqrt{\Delta_n t}\big) = \binom{\beta_i}{j} p^j (1-p)^{\beta_i - j}$$

for $\omega = (\omega_0, \dots, \omega_{\tilde{T}-1}) \in \Omega_n$. Each ω_i is therefore a sum of β_i binomial random variables taking values $\pm\sigma\sqrt{\Delta_n t}$ with probabilities p and $1-p$, respectively, hence

$$E_n[\omega_i] = (2p-1)\sigma\sqrt{\Delta h_i} \quad \text{and} \quad E_n[\omega_i^2] = \sigma^2 \Delta h_i.$$

The filtration $\mathbb{F}_n$ is again generated in the natural way.
We also set $\Omega = \mathbb{R}^{\tilde{\mathbb{T}}}$, $\mathcal{F} = \mathcal{B}(\Omega)$ and define the probability P by

$$\omega_i \sim \mathcal{N}\big((\mu - \frac{1}{2}\sigma^2)\Delta h_i, \sigma^2 \Delta h_i\big)$$

for $\omega = (\omega_0, \dots, \omega_{\tilde{T}-1}) \in \Omega$. The filtration $\mathbb{F}$ is defined as before.
The following lemma shows that assumption (M4) is satisfied for this sequence of models:

Lemma 3.3.7. *With Δh_i and β_i defined as above, let $(X_j)_{j=1,\dots,\beta_i}$ be a sequence of binomial random variables, taking values $\pm\sigma\sqrt{\Delta_N t}$ with probability p and $1-p$, respectively, where $\Delta_N t = \Delta h_i/\beta_i$ and p is defined as in* (3.10). *Then*

$$P_N\Big(\sum_{j=1}^{\beta_i} X_j \le a\Big) \approx \Phi\left(\frac{{}^\circ a - (\mu - \frac{1}{2}\sigma^2)\Delta h_i}{\sigma\sqrt{\Delta h_i}}\right), \tag{3.11}$$

[5] It should be noted that also in this model $\tilde{S}_{N,i} \in \mathcal{S}L^r(P_N)$ for each $r \in [1, \infty)$.

where Φ is the standard normal distribution function. Furthermore,

$$E_N\left[\exp\Big(\sum_{j=1}^{\beta_i} X_j\Big)^r\right] < \infty$$

for any $r \in [1, \infty)$.

Proof. First note that each X_j has mean $m := (2p-1)\sigma\sqrt{\Delta_N t}$ and variance $v^2 := \sigma^2\Delta_N t - m^2$. Then

$$\begin{aligned}
P_N\Big(\sum_{j=1}^{\beta_i} X_j \le a\Big) &= P_N\Big(\sum_{j=1}^{\beta_i} \frac{X_j - m}{v} \le \frac{a - \beta_i m}{v}\Big) \\
&= P_N\Big(\frac{1}{\sqrt{\beta_i}}\sum_{j=1}^{\beta_i} \frac{X_j - m}{v} \le \frac{a - \beta_i m}{\sqrt{\beta_i} v}\Big) \\
&\approx \Phi\Big({}^\circ\Big(\frac{a - \beta_i m}{\sqrt{\beta_i} v}\Big)\Big) \qquad (3.12)
\end{aligned}$$

by the nonstandard Central Limit Theorem 2.1.3. A straightforward calculation using the Taylor expansion of the exponential function shows that

$$\frac{(2p-1)\sigma}{\sqrt{\Delta_N t}} \approx \mu - \frac{1}{2}\sigma^2, \quad \text{hence} \quad \beta_i m = \frac{\Delta h_i}{\Delta_N t}(2p-1)\sigma\sqrt{\Delta_N t} \approx (\mu - \frac{1}{2}\sigma^2)\Delta h_i.$$

Therefore

$$\sqrt{\beta_i} v = \sqrt{\beta_i \sigma^2 \Delta_N t - \beta_i m^2} \approx \sqrt{\sigma^2 \Delta h_i}.$$

Taking standard parts an substituting into (3.12) proves (3.11).
For the second assertion note that due to the independence of $(X_j)_{j=1,\dots,\beta_i}$

$$E_N\left[\exp\Big(\sum_{j=1}^{\beta_i} X_j\Big)^r\right] = \prod_{j=1}^{\beta_i} E_N\left[\exp(rX_j)\right].$$

Now

$$E_N\left[\exp(rX_j)\right] = \exp(r\sigma\sqrt{\Delta_N t})p + \exp(-r\sigma\sqrt{\Delta_N t})(1-p)$$

and another calculation using the Taylor expansion of the exponential function as in the proof of Lemma 3.1(b) in [CKW91] together with the above calculation of $2p-1$ yields

$$\prod_{j=1}^{\beta_i} E_N\left[\exp(rX_j)\right] \approx \exp\Big(\big(\mu - \frac{1}{2}\sigma^2(1-r)\big)r\Delta h_i\Big) < \infty.$$

□

With the above definitions the price process $\tilde{S}_n$ in this modified CRR model is given by

$$\tilde{S}_{n,i+1} = \tilde{S}_{n,i} \exp(\omega_i), \quad \tilde{S}_{n,0} = 0$$

and similar calculations to those in Section 3.2 show that assumption (M3) is satisfied (see also the proof of Lemma 3.4 in [MV96]). In the modified BS model the price process satisfies

$$\tilde{S}_{i+1} = \tilde{S}_i \exp(\omega_i), \quad \tilde{S}_0 = 0.$$

By Lemma 3.3.7, $P = L(P_N) \circ \mathrm{st}_N^{-1}$ for any infinite $N \in {}^*\tilde{\mathbb{N}} \setminus \tilde{\mathbb{N}}$, and hence

$${}^\circ\tilde{S}_{N,i}(\omega) = \tilde{S}_i({}^\circ\omega) \quad L(P_N)\text{-a.s. for } i \in \tilde{\mathbb{T}}$$

with $\tilde{S}_{N,i} \in SL^r(P_N)$ for any $r > 0$.
This shows that this sequence of models satisfies assumptions (M1)–(M4), hence Theorem 3.3.2 and Proposition 3.3.4 apply.

3.4 Convergence Results

We saw in the previous section how the lifting property of a claim on the internal model $\mathbf{\Omega}_N$ implies the lifting property of the optimal strategies and value processes (Theorem 3.3.2). Using the general convergence result 2.4.1 for functions on separable metric spaces we can now rewrite Theorem 3.3.2 as follows, using the equivalent (standard) formulation of assumption (M4):

(M4') $P_n \to P$ weakly and $\big(S_{n,i}(\omega), \omega\big) \to \big(S_i(\omega), \omega)\big)$ weakly as $n \to \infty$. Furthermore, for some $k > 2$, $E_n\left[|S_{n,i}|^k\right] < \infty$ for all $n \in \tilde{\mathbb{N}}$.

Theorem 3.4.1. *Suppose the sequence of models $(\mathbf{\Omega}_n)_{n\in\tilde{\mathbb{N}}}$ and the model $\mathbf{\Omega}$ satisfy assumptions (M1)–(M3) and (M4'). Let $H_n : \Omega_n \to \mathbb{R}$ ($n \in \tilde{\mathbb{N}}$) and $H : \Omega \to \mathbb{R}$ be measurable functions such that $\big(H_n(\omega), \omega\big) \to \big(H(\omega), \omega\big)$ weakly as $n \to \infty$, and $E_n\left[H_n^2\right] \to E\left[H^2\right]$ as $n \to \infty$. Then, for $i \in \mathbb{T}$,*

$$\Big(H_{n,i}(\omega), \theta_i^{H_n}(\omega), \psi_i^{H_n}(\omega), \xi_i^{H_n}(\omega), \omega\Big) \to \big(H_i(\omega), \theta_i^H(\omega), \psi_i^H(\omega), \xi_i^H(\omega), \omega\big)$$

weakly as $n \to \infty$. In particular,

$$H_{n,0} \to H_0 \quad \text{as } n \to \infty.$$

In the previous section we gave examples of models satisfying the assumptions of Theorem 3.4.1. All that remains is to show how we can obtain a sequence of claims

$(H_n)_{n\in\tilde{\mathbb{N}}}$ converging to a given claim H in the sense that H_N is an $\mathcal{S}L^2$-lifting of H for all $N \in {}^*\tilde{\mathbb{N}} \setminus \mathbb{N}$.

Any claim H on Ω is necessarily of the form $H(\omega) = H(\omega_0, \dots, \omega_{T-1})$ for $\omega \in \mathbb{R}^T$. In the case of the modified BS model (and the modified Cox-Ross jump model of Example 3.3.5) there is a one-to-one correspondence between ω and $\tilde{S}(\omega) := (\tilde{S}_1(\omega), \dots, \tilde{S}_{\tilde{T}}(\omega))$, so that any claim in these models is necessarily of the form $H = H(\tilde{S}_1, \dots, \tilde{S}_{\tilde{T}})$ for some function $H : \mathbb{R}^{\tilde{T}} \to \mathbb{R}$. This means that the value of any claim in these models can only depend on the values of the risky asset at the trading dates.

We now assume that the function H has only countably many points of discontinuity and satisfies a polynomial growth condition

$$|H(x_1, \dots, x_{\tilde{T}})| \leq \sum_{i=1}^{m} c_i \prod_{j=1}^{\tilde{T}} |x_j|^{k_{i,j}} \tag{3.13}$$

for suitable $m, k_{i,j} \in \mathbb{N}$ (cf. equation (19) in [MV96]). Any "reasonable" claim in these models can be represented by such a function H; this class of claims includes e.g. Asian, lookback, barrier and binary options.

For the approximating model Ω_n we can define the function $H_n : \Omega_n \to \mathbb{R}$ where $H_n(x) = H(x)$. Then H_N is a lifting of H for infinite N since

$$H_N(x) = {}^*H(x) \approx H({}^\circ x) \quad \text{for } L(P_N)\text{-a.a. } x \in \Omega_N,$$

and $H_N \in \mathcal{S}L^r(P_N)$ for any $r \in [1, \infty)$ due to condition (3.13) and the fact that $\tilde{S}_{N,i} \in \mathcal{S}L^r(P_N)$ for any $r \in [1, \infty)$.

Having established the existence of a sequence of claims satisfying the conditions of Theorem 3.4.1 we can now use the sequence (Ω_n) of modified CRR models to calculate approximations to the strategies and value processes in the modified BS model. This has important practical implications: although option prices in the modified BS model can in principle be calculated by integrating over a multivariate normal distribution (see Proposition 3.1 in [MV96] for an explicit formula) the numerical evaluation of these formulae is often problematic; we will see examples of this in Chapter 6. On the other hand, the calculation of prices and strategies in the multinomial tree of the modified CRR model can be done by straightforward algebra. Theorem 3.3.2 now ensures that these approximations converge to the corresponding values and strategies in the modified BS model.

Chapter 4

Nonstandard Methods in Incomplete Markets

We saw in Chapter 3 — in particular in (3.9) and Proposition 3.3.4 — that the lifting property of the price process implies the lifting property of the minimal martingale density in a discrete time setting. In continuous time such a result is not available and we have to examine the structure of the minimal martingale density and the Doob-Meyer decomposition of the price process in more detail. The first section of this chapter presents a general criterion under which the lifting property of the minimal density can be obtained. This general result is then applied to two internal incomplete models in Section 4.2 and Section 4.3, respectively.
In Section 4.4 we show that optimal strategies and value processes in internal incomplete models are liftings of their respective counterparts in a complete standard model if and only if they are associated to a claim which is a lifting of the corresponding claim in the standard model.
The final section contains the proofs of some technical results which are used in Sections 4.1 and 4.4.

Throughout this chapter $(\Omega, \mathbb{A}, \mathcal{F}, P)$ denotes an internal filtered probability space with associated Loeb space $(\Omega, \mathcal{F}_L, P_L)$ as defined in Section 2.2. Specific examples of this setup will be introduced in Sections 4.2 and 4.3. The hyperfinite time line $\mathbb{T}$ is defined as $\mathbb{T} := \{0, \Delta t, \dots, T\}$ for some $T \in \mathbb{R}^+$ and $\Delta t := T/N$ for some $N \in {}^*\mathbb{N}\backslash\mathbb{N}$.

4.1 The Minimal Martingale Measure

We saw in Section 1.2 that for a semimartingale s with Doob-Meyer decomposition

$$s = s_0 + \int \alpha \, d\langle m\rangle + m \tag{4.1}$$

the density for the minimal martingale measure is defined via the stochastic exponential

$$\hat{z} = \mathcal{E}\Big(-\int \alpha\, dm\Big) \tag{4.2}$$

In discrete time the density is given by the process

$$\hat{Z}_t = \prod_{s<t}(1 - \alpha_s \Delta M_s) \tag{4.3}$$

with

$$\alpha_t = \frac{E\left[\Delta S_t \mid \mathcal{A}_t\right]}{\mathrm{Var}\left[\Delta S_t \mid \mathcal{A}_t\right]} \quad \text{and} \quad \Delta M_t = \Delta S_t - E\left[\Delta S_t \mid \mathcal{A}_t\right] \tag{4.4}$$

(cf. equation (1.14)). Proposition 4.1.2 below will help to establish the connection between (4.2) and (4.3) on the hyperfinite time line. The following lemma is needed in the proof. It shows how the "pure jump" part of a cadlag function can be approximated by an internal function; the proof is given in Section 4.5. Elements of $\mathbb{T}$ are usually denoted by s, t while u, v are used for elements of $[0, T]$.

Lemma 4.1.1. *Let $F : \mathbb{T} \to {}^*\mathbb{R}$ be an SDJ function such that*

$$\sum_{s<T}(\Delta F_s)^2 \quad \text{is finite.} \tag{4.5}$$

Let $f := \mathrm{st}(F)$ be the standard part of F. Then there exists an internal subset $\tilde{\mathbb{T}}$ of $\mathbb{T}$ such that the functions $J : \mathbb{T} \to {}^*\mathbb{R}$, $j : [0,T] \to \mathbb{R}$ defined as

$$J(t) := \sum_{\substack{s<t \\ s\in\tilde{\mathbb{T}}}} \Delta F_s + \frac{1}{2}(\Delta F_s)^2 \qquad \text{and} \qquad j(v) := \sum_{u\leq v} \Delta f_u + \frac{1}{2}(\Delta f_u)^2$$

satisfy $j = \mathrm{st}(J)$. Furthermore, the functions $K : \mathbb{T} \to {}^*\mathbb{R}$, $k : [0,T] \to \mathbb{R}$ defined as

$$K(t) := \prod_{\substack{s<t \\ s\in\tilde{\mathbb{T}}}} (1 + \Delta F_s) \qquad \text{and} \qquad k(v) := \prod_{u\leq v} (1 + \Delta f_u)$$

satisfy $k = \mathrm{st}(K)$.

Proposition 4.1.2. *Let $X : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ be an internal martingale of class SDJ. Let $x : \Omega \times [0,T] \to \mathbb{R}$ be its standard part, $x := \mathrm{st}(X)$. Define an internal martingale $Z : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ by*

$$Z_t := \prod_{s<t}(1 + \Delta X_s).$$

Then Z is of class SDJ and $z := \mathrm{st}(Z) = \mathcal{E}(x)$, P_L-a.s., where

$$\mathcal{E}(x)_v = \exp\Big(x_v - \frac{1}{2}[x]_v\Big) \prod_{u\leq v} (1 + \Delta x_u) \exp\Big(-\Delta x_u + \frac{1}{2}(\Delta x_u)^2\Big)$$

denotes the stochastic exponential of x.

Proof. Let $\tilde{\Omega}$ be the subset of Ω such that $X(\omega)$ and $[X](\omega)$ are SDJ and $\mathrm{st}([X](\omega)) = [x](\omega)$ for all $\omega \in \tilde{\Omega}$. We know from Theorem 2.2.6(ii) that $P_L(\tilde{\Omega}) = 1$. Now fix $\omega \in \tilde{\Omega}$. Let $\tilde{\mathbb{T}}$ be the internal subset of $\mathbb{T}$ obtained by applying Lemma 4.1.1 to the path $X(\omega)$ (i.e. $\tilde{\mathbb{T}}$ contains all $s \in \mathbb{T}$ such that $^\circ|\Delta X_s(\omega)| > 0$). Set $\mathbb{T}'' := \mathbb{T} \setminus \tilde{\mathbb{T}}$ and write $Z(\omega) = Z' \cdot Z''$ (we will now suppress the explicit dependence on ω) with

$$\begin{aligned} Z'_t &:= \prod_{\substack{s<t \\ s\in\tilde{\mathbb{T}}}} (1+\Delta X_s) \\ &= \exp\Big(\sum_{\substack{s<t \\ s\in\tilde{\mathbb{T}}}} \Delta X_s - \frac{1}{2}(\Delta X_s)^2 - \Delta X_s + \frac{1}{2}(\Delta X_s)^2\Big) \prod_{\substack{s<t \\ s\in\tilde{\mathbb{T}}}} (1+\Delta X_s), \qquad (4.6) \\ Z''_t &:= \prod_{\substack{s<t \\ s\in\mathbb{T}''}} (1+\Delta X_s). \qquad (4.7) \end{aligned}$$

Since $|\log(1+y) - y + \frac{1}{2}y^2| \le |y|^3$ for $|y| \le \frac{1}{2}$ and $^\circ|\Delta X_s| = 0$ for all $s \in \mathbb{T}''$ we have

$$\log(Z''_t) = \sum_{\substack{s<t \\ s\in\mathbb{T}''}} \log(1+\Delta X_s) = \sum_{\substack{s<t \\ s\in\mathbb{T}''}} \Delta X_s - \frac{1}{2}(\Delta X_s)^2 + \varepsilon_s \quad \text{with} \quad |\varepsilon_s| \le |\Delta X_s|^3.$$

Let $\bar{\varepsilon} := \max\{|\Delta X_s| : s \in \mathbb{T}''\}$. Then $\bar{\varepsilon} \approx 0$ and

$$\Big|\sum_{\substack{s<t \\ s\in\mathbb{T}''}} \varepsilon_s\Big| \le \bar{\varepsilon} \sum_{s<t} (\Delta X_s)^2 = \bar{\varepsilon} \cdot [X]_t \approx 0$$

as $[X]$ is SDJ; hence, by the $\mathcal{S}$-continuity of the exponential function,

$$Z''_t \approx \exp\Big(\sum_{\substack{s<t \\ s\in\mathbb{T}''}} \Delta X_s - \frac{1}{2}(\Delta X_s)^2\Big).$$

Therefore, by (4.6) and (4.7),

$$\begin{aligned} Z_t &\approx \exp\Big(\sum_{s<t} \Delta X_s - \frac{1}{2}(\Delta X_s)^2\Big) \exp\Big(\sum_{\substack{s<t \\ s\in\tilde{\mathbb{T}}}} -\Delta X_s + \frac{1}{2}(\Delta X_s)^2\Big) \prod_{\substack{s<t \\ s\in\tilde{\mathbb{T}}}} (1+\Delta X_s) \\ &= \exp\Big(X_t - \frac{1}{2}[X]_t\Big) \prod_{\substack{s<t \\ s\in\tilde{\mathbb{T}}}} (1+\Delta X_s) \sum_{\substack{s<t \\ s\in\tilde{\mathbb{T}}}} \exp\Big(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2\Big). \end{aligned}$$

Observing that $\mathrm{st}(X) = x$ and $\mathrm{st}([X]) = [x]$ it then follows from the definition of $\tilde{\mathbb{T}}$ in Lemma 4.1.1 and the $\mathcal{S}$-continuity of the exponential function that

$$\mathrm{st}(Z)_v = \exp\Big(x_v - \frac{1}{2}[x]_v\Big) \prod_{u\le v} (1+\Delta x_u) \exp\Big(-\Delta x_u + \frac{1}{2}(\Delta x_u)^2\Big).$$

□

Corollary 4.1.3. *Let $M : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ be an SL^2-martingale of class SDJ and $m := \mathrm{st}(M)$ its standard part. Let a be a predictable process in $L^2(\nu_m)$ and A be an $SL^2(\nu_M)$-lifting of a. Then*

$$\mathrm{st}\Big(\prod(1 + A\Delta M)\Big) = \mathcal{E}\Big(-\int a\,dm\Big).$$

Proof. It follows from Theorems 2.2.8 and 2.2.11 that $\sum A\Delta M$ is a martingale of class SDJ and $\mathrm{st}(\sum A\Delta M) = \int a\,dm$. The result then follows from Proposition 4.1.2. □

Remark 4.1.4. It will be shown in Chapter 5 how Corollary 4.1.3 implies convergence results for option prices in a sequence of discrete-time models. Similar results are obtained in [Pri97]; however the above formulation was derived independently and makes use of different techniques. In [Pri97] the notion of *uniform tightness* of a sequence of martingales is used together with the results in [JMP89] to obtain the weak convergence of the sequence of associated stochastic integrals (see also [DP92]).

Corollary 4.1.3 provides a method to obtain the lifting property of the minimal density by calculating the processes α, M for a given internal price process S on $\Omega \times \mathbb{T}$ and comparing them with the processes α, m for a standard price process s on $\Omega \times [0, 1]$. However, most models are defined via a *return process*: in discrete time the price process S is then given by

$$S_{t+\Delta t} = S_t(1 + \Delta R_t), \quad S_0 = 0$$

for some (internal) process R with $R_0 = 0$. This process has itself a decomposition

$$\Delta R_t = \alpha_t^R E\left[(\Delta M_t^R)^2 \middle| \mathcal{A}_t\right] + \Delta M_t^R \tag{4.8}$$

with

$$\alpha_t^R := \frac{E\left[\Delta R_t \middle| \mathcal{A}_t\right]}{\mathrm{Var}\left[\Delta R_t \middle| \mathcal{A}_t\right]} \quad \text{and} \quad \Delta M_t^R := \Delta R_t - E\left[\Delta R_t \middle| \mathcal{A}_t\right].$$

Calculating α, M for S according to (4.4) and observing that $\Delta S_t = S_t \Delta R_t$ yields

$$\begin{aligned} \alpha_t &= \frac{S_t\, E\left[\Delta R_t \middle| \mathcal{A}_t\right]}{S^2\, \mathrm{Var}\left[\Delta R_t \middle| \mathcal{A}_t\right]} = \frac{1}{S_t}\alpha_t^R \\ \Delta M_t &= S_t\Big(\Delta R_t - E\left[\Delta R_t \middle| \mathcal{A}_t\right]\Big) = S_t \Delta M_t^R \end{aligned}$$

so that

$$\alpha_t \Delta M_t = \alpha_t^R \Delta M_t^R.$$

Similarly, in continuous time we assume that the price process s is defined as the solution to the stochastic differential equation

$$ds_t = s_{t-} dr_t$$

for a semimartingale r with Doob-Meyer decomposition

$$r_t = \int \alpha^r d\langle m^r \rangle + m^r \tag{4.9}$$

with a square-integrable martingale m^r and a predictable process α^r. The processes α, m in the Doob-Meyer decomposition (4.1) of s are then given by

$$\alpha_t = \frac{1}{s_{t-}} \alpha_t^r \quad \text{and} \quad dm_t = s_{t-} dm_t^r,$$

so that

$$\alpha_t dm_t = \alpha_t^r dm_t^r.$$

We therefore obtain the following criterion for the lifting property of the minimal density:

Corollary 4.1.5. *Let $R : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ and $r : \Omega \times \mathbb{T} \to \mathbb{R}$ be return processes with decompositions* (4.8) *and* (4.9)*, respectively. If M^R is an SL^2-lifting of m^r of class SDJ and α^R is an $SL^2(\nu_{M^R})$-lifting of α^r then the minimal densities $\hat{Z} : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ and $\hat{z} : \Omega \times [0, T] \to \mathbb{R}$ are given by*

$$\hat{Z} = \prod (1 + \alpha^R \Delta M^R) \quad \text{and} \quad \hat{z} = \mathcal{E}\Big(- \int \alpha^r dm^r \Big)$$

and satisfy $\hat{z} = \mathrm{st}(\hat{Z})$.

4.2 Example: A Multinomial Model

We will now apply the results of the previous section to a multinomial model for the evolution of the price process on the hyperfinite time line $\mathbb{T}$: let $\beta \in \mathbb{N}$ be a natural number; given the price at time $t \in \mathbb{T}$ there are now $\beta + 1$ possible prices at time $t + \Delta t$. However, if $\beta > 1$ the resulting model is in general not complete and we will use the methodology of mean-variance hedging to determine the value of contingent claims in such a model.

More specifically, the price process S is given by the evolution equation

$$\begin{aligned} S_{t+\Delta t} &= S_t \cdot \begin{cases} u^\beta & \text{with prob. } 2^{-\beta} \\ \vdots & \vdots \\ u^j d^{\beta - j} & \text{with prob. } \binom{\beta}{j} 2^{-\beta} \text{ for } j = 1, \ldots, \beta - 1 \\ \vdots & \vdots \\ d^\beta & \text{with prob. } 2^{-\beta} \end{cases} \qquad (4.10) \\ S_0 &= s_0 \end{aligned}$$

with

$$u := 1 + \mu\frac{\Delta t}{\beta} + \sigma\sqrt{\Delta t/\beta} \quad \text{and} \quad d := 1 + \mu\frac{\Delta t}{\beta} - \sigma\sqrt{\Delta t/\beta} \tag{4.11}$$

for $s_0, \mu, \sigma \in \mathbb{R}$, $\sigma, s_0 > 0$. Note that this model is a hyperfinite version of the modified CRR model of Section 3.2 corresponding to a hyperfinite number of hedging dates and a fixed number $(\beta - 1)$ of "skipped" price changes between hedging dates.
To define S in terms of an internal return process we set $\Omega := \{0, 1, \dots, \beta\}^{\mathbb{T}\setminus\{T\}}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and define a measure P on $(\Omega, \mathcal{F})$ with

$$P(\omega_t = j) = 2^{-\beta}\binom{\beta}{j}$$

for $\omega = (\omega_0, \dots, \omega_{T-\Delta t}) \in \Omega$ and $(\omega_t)_{t\in\mathbb{T}\setminus\{T\}}$ independent. The filtration $\mathbb{A}$ on $(\Omega, \mathcal{F})$ is generated in the usual way, i.e. $\mathcal{A}_t$ is the internal algebra generated by the sets $[\omega]_t := \{\hat{\omega} : \omega_s = \hat{\omega}_s,\ s < t\}$.
We now have a multinomial version of Anderson's random walk (see Section 2.1).

Proposition 4.2.1. *The internal process $W : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ defined by*

$$W_t = \sum_{s<t}(-\beta + 2\omega_s)\sqrt{\Delta t/\beta}$$

is S-continuous and $w := \mathrm{st}(W)$ is a standard Brownian motion on $(\Omega, \mathcal{F}_L, P_L)$.

Proof. This can be proved by arguments analogous to those used in the proof of Theorem 3.3.5 in [AFHL86]. Alternatively, W can be viewed as the process obtained from a binomial random walk on the finer time line $\mathbb{T}_\beta := \{0, \Delta t/\beta, 2\Delta t/\beta, \dots, T\}$ by evaluation at the points of the set $\mathbb{T} \subset \mathbb{T}_\beta$. □

We then define the return process $R : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ as

$$\Delta R_t := u^{\omega_t} d^{\beta-\omega_t} - 1, \quad R_0 := 0,$$

with u, d as in (4.11). The price process $S : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ given by $\Delta S_t := S_t \Delta R_t$, $S_0 := s_0$ satisfies the above evolution equation (4.10). It follows from the proof of Lemma 3.1 in [CKW91] that

$$S_t \approx s_0 \exp\Big((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\Big) \quad P_L\text{-a.s.},$$

and $S_t \in SL^r(P)$ for $r \in [1, \infty)$. Hence $s := \mathrm{st}(S)$ is a geometric Brownian motion with drift μ and volatility σ, i.e. s is a Black-Scholes price process on $(\Omega, \mathcal{F}_L, P_L)$. According to Section 2.3 there is a unique martingale measure for s, given by the density (relative to P_L)

$$\hat{z} := \exp\Big(-\frac{1}{2}(\frac{\mu}{\sigma})^2 T - \frac{\mu}{\sigma} w_T\Big). \tag{4.12}$$

We will now calculate α^R and ΔM^R in the decomposition (4.8):

$$\begin{aligned} E\left[\Delta R_t \middle| \mathcal{A}_t\right] &= E\left[u^{\omega_t} d^{\beta-\omega_t} \middle| \mathcal{A}_t\right] - 1 \\ &= \Big(\sum_{j=0}^{\beta} 2^{-\beta}\binom{\beta}{j} u^j d^{\beta-j}\Big) - 1 = 2^{-\beta}(u+d)^\beta - 1, \end{aligned}$$

so

$$\Delta M_t^R = u^{\omega_t} d^{\beta-\omega_t} - 2^{-\beta}(u+d)^\beta$$

and

$$\begin{aligned} E\left[(\Delta M_t^R)^2 \middle| \mathcal{A}_t\right] &= E\left[u^{2\omega_t} d^{2(\beta-\omega_t)} \middle| \mathcal{A}_t\right] - 2^{-2\beta}(u+d)^{2\beta} \\ &= 2^{-\beta}(u^2+d^2)^\beta - 2^{-2\beta}(u+d)^{2\beta}, \end{aligned}$$

hence

$$\alpha_t^R = \frac{(u+d)^\beta - 2^\beta}{(u^2+d^2)^\beta - 2^{-\beta}(u+d)^{2\beta}}$$

(in particular, α_t^R is a constant). Since $u+d = 2 + 2\mu\Delta t/\beta$ we see that

$$(u+d)^\beta = 2^\beta + \beta\, 2^\beta \mu \frac{\Delta t}{\beta} + 2^\beta \sum_{j=2}^{\beta} \binom{\beta}{j} (\mu \frac{\Delta t}{\beta})^j,$$

hence (by ignoring terms containing $(\Delta t)^k$ for $k > 1$)

$$\frac{(u+d)^\beta - 2^\beta}{\Delta t} \approx 2^\beta \mu. \tag{4.13}$$

A similar (though lengthier) calculation for $(u^2+d^2)^\beta - 2^{-\beta}(u+d)^\beta$ shows that

$$\frac{(u^2+d^2)^\beta - 2^{-\beta}(u+d)^{2\beta}}{\Delta t} \approx 2^\beta \sigma^2. \tag{4.14}$$

Hence

$$\alpha_t^R \approx \frac{\mu}{\sigma^2},$$

which is finite. We now consider the process $\sum \Delta M^R$: Using the Taylor expansion of the exponential function we see that

$$\begin{aligned} \frac{\beta}{\Delta t}\left| u - \exp\Big((\mu - \frac{1}{2}\sigma^2)\frac{\Delta t}{\beta} + \sigma\sqrt{\Delta t/\beta}\Big)\right| &\approx 0, \\ \frac{\beta}{\Delta t}\left| d - \exp\Big((\mu - \frac{1}{2}\sigma^2)\frac{\Delta t}{\beta} - \sigma\sqrt{\Delta t/\beta}\Big)\right| &\approx 0 \\ \text{and} \quad \frac{\beta}{\Delta t}\left| \frac{1}{2}(u+d) - \exp\Big(\mu\frac{\Delta t}{\beta}\Big)\right| &\approx 0, \end{aligned}$$

hence

$$\begin{aligned} 0 &\approx \frac{1}{\Delta t}\left| u^{\omega_t} d^{\beta-\omega_t} - \exp\Big((\mu - \frac{1}{2}\sigma^2)\Delta t + (-\beta + 2\omega_t)\sigma\sqrt{\Delta t/\beta}\Big)\right| \\ &= \frac{1}{\Delta t}\left| u^{\omega_t} d^{\beta-\omega_t} - \exp\big((\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma \Delta W_t\big)\right| \end{aligned}$$

and

$$0 \approx \frac{1}{\Delta t}\left| 2^{-\beta}(u+d)^{\beta} - \exp\big(\mu \Delta t\big)\right|.$$

This implies that

$$\begin{aligned} \sum_{s<t} \Delta M_s^R &= \sum_{s<t}\Big(u^{\omega_s} d^{\beta-\omega_s} - 2^{-\beta}(u+d)^{\beta}\Big) \\ &\approx \sum_{s<t}\Big(\exp\big((\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\Delta W_s\big) - \exp\big(\mu\Delta t\big)\Big) \\ &= \exp\big((\mu - \frac{1}{2}\sigma^2)\Delta t\big)\sum_{s<t}\Big(\exp(\sigma\Delta W_s) - \exp\big(\frac{1}{2}\sigma^2\Delta t\big)\Big) \\ &\approx \sum_{s<t}\Big(\exp(\sigma\Delta W_s) - \exp\big(\frac{1}{2}\sigma^2\Delta t\big)\Big) \\ &\approx \sum_{s<t}\Big(1 + \sigma\Delta W_s + \frac{1}{2}\sigma^2(\Delta W_s)^2 - 1 - \frac{1}{2}\sigma^2\Delta t\Big) \qquad (4.15) \\ &= \sum_{s<t}\sigma\Delta W_s + \frac{1}{2}\sigma^2\big((\Delta W_s)^2 - \Delta t\big) \\ &= \sigma W_t + \frac{1}{2}\sigma^2\big([W]_t - t\big) \end{aligned}$$

(where (4.15) again uses the Taylor expansion of the exponential function up to order 2). By Proposition 4.2.1 together with Theorem 2.2.6, $[W]$ is $\mathcal{S}$-continuous and $\mathrm{st}([W])_t = t$, P_L-a.s., hence $\sum \Delta M^R$ is $\mathcal{S}$-continuous and

$$\mathrm{st}\Big(\sum \Delta M^R\Big) = \sigma w, \quad P_L\text{-a.s..}$$

We have therefore shown that

$$\mathrm{st}\Big(\sum \alpha^R \Delta M^R\Big) = \frac{\mu}{\sigma} w, \quad P_L\text{-a.s..}$$

Proposition 4.2.2. *In the multinomial CRR model* (4.10) *the minimal density* $\hat{Z}$ *satisfies*

$$\mathrm{st}(\hat{Z})_u = \mathcal{E}\Big(-\int \frac{\mu}{\sigma} dw\Big)_u = \exp\Big(-\frac{1}{2}\big(\frac{\mu}{\sigma}\big)^2 u - \frac{\mu}{\sigma} w_u\Big) \quad \text{for all } u \in [0,T],\ P_L\text{-a.s..}$$

Furthermore, for any $r \in [1, \infty)$, $\hat{Z}_t \in \mathcal{S}L^r(P)$ *for all* $t \in \mathbb{T}$.

Proof. The first assertion follows from Corollary 4.1.5. It only remains to show the $\mathcal{S}$-integrability of $|\hat{Z}_t|^m$ for $m \in \mathbb{N}$: due to the independence of $(\omega_t)_{t\in\mathbb{T}\setminus\{T\}}$ we have

$$E\left[|\hat{Z}_t|^m\right] = E\Big[\prod_{s<t}|1-\alpha_s^R\Delta M_s^R|^m\Big] = \prod_{s<t} E\Big[|1-\alpha_s^R\Delta M_s^R|^m\Big].$$

Note that $\alpha := \alpha_s^R$ is a constant and the random variables $(\Delta M_s^R)_{s\in\mathbb{T}\setminus\{T\}}$ are identically distributed; in the following we will therefore just write ΔM instead of ΔM_s^R.

$$\begin{aligned} E\Big[|1-\alpha\Delta M|^m\Big] &\leq \sum_{i=0}^m \binom{m}{i} 1^{m-i}|\alpha|^i E\Big[|\Delta M|^i\Big] \\ &= 1+\sum_{i=1}^m \binom{m}{i} |\alpha|^i E\Big[|\Delta M|^i\Big] \end{aligned}$$

Calculating $|\Delta M|^i$, taking expectations and neglecting terms which contain $(\Delta t)^k$ for $k > 1$ — similarly to the calculations leading to equations (4.13) and (4.14) — shows that

$$E\Big[|\Delta M|^i\Big] = K_i \Delta t$$

for some finite constant $K_i \in {}^*\mathbb{R}$. Hence

$$\begin{aligned} E\left[|\hat{Z}_t|^m\right] &\leq \prod_{s<t}\Big(1+\sum_{i=1}^m \binom{m}{i}|\alpha|^i K_i\Delta t\Big) \\ &= \prod_{s<t}(1+\tilde{K}\Delta t) \\ &\approx \exp\big(\tilde{K}t\big) < \infty, \end{aligned}$$

where $\tilde{K}$ is a finite constant. □

We can now define the (internal) minimal martingale measure $\hat{P}$ on $(\Omega, \mathcal{F})$ by

$$\frac{d\hat{P}}{dP} := \hat{Z}_T.$$

Lemma 2.1.9 again shows that

$$\frac{d\hat{P}_L}{dP_L} = {}^\circ\Big(\frac{d\hat{P}}{dP}\Big) = {}^\circ\hat{Z}_T = \exp\Big(-\frac{1}{2}\big(\frac{\mu}{\sigma}\big)^2 T - \frac{\mu}{\sigma}w_T\Big),$$

so that the "minimal Loeb measure" $\hat{P}_L$ coincides with the unique martingale measure for s (cf. equation (4.12)). Therefore, if $h \in L^2(\hat{P}_L)$ is a claim in the BS model and $H \in \mathcal{S}L^2(P)$ is a lifting of h then $H\hat{Z}_T \in \mathcal{S}L(P)$ by Proposition 4.2.2 and Hölder's inequality, hence

$$E_{\hat{P}}[H] = E\left[\hat{Z}_T H\right] \approx E_{P_L}\left[{}^\circ(\hat{Z}_T H)\right] = E_{\hat{P}_L}[h],$$

i.e. option prices with respect to the internal minimal martingale measure are infinitesimally close to the corresponding prices in the BS model.

This property of the multinomial CRR model (4.10), namely that it is "complete in the limit", will be examined further in the next chapter.

4.3 Example: A Jump-Diffusion Model

As a second applications of the results of Section 4.1 we consider a continuous-time jump-diffusion model with a discretised version that is obtained by "direct discretisation" of the continuous-time price process. We therefore consider the model introduced in [RS95] on the hyperfinite time line. Note that this model is incomplete in both discrete and continuous time. It was shown in [RS95] that the minimal densities (and hence option prices) for the discrete time models converge to those in the continuous model. Using the results of this section we will obtain an alternative proof of this result in Chapter 5.

Let $(\Omega, \mathcal{F}, P)$ be an internal probability space carrying an internal Brownian motion $(\hat{W}_t)_{t\in {}^*[0,T]}$ and an internal Poisson process $(\hat{N}_t)_{t\in {}^*[0,T]}$ with intensity λ for $\lambda \in \mathbb{R}, \lambda > 0$. Then $\hat{W}$ and $\hat{N}$ are necessarily independent (see [Sch93a, p.92]).
Let $\mu, \sigma, \varphi, s_0 \in \mathbb{R}$ with $s_0 > 0$, $\varphi > -1$ and $\sigma^2 + \varphi^2\lambda > 0$. We define an internal price process $\hat{S} : \Omega \times {}^*[0,T] \to {}^*\mathbb{R}$ by

$$\hat{S}_t := s_0 \exp\Big((\mu - \frac{1}{2}\sigma^2)t + \sigma\hat{W}_t + \log(1+\varphi)\hat{N}_t\Big), \tag{4.16}$$

i.e. $\hat{S}$ is the solution to the internal stochastic differential equation

$$\frac{d\hat{S}_t}{\hat{S}_{t-}} = \mu dt + \sigma d\hat{W}_t + \varphi d\hat{N}_t, \quad \hat{S}_0 = s_0.$$

Note that, due to our assumption $\varphi > -1$, $\hat{S}$ is strictly positive. Furthermore, by transfer of [Sch93a, equation (II.8.17)],

$$E\Big[\sup_{t\in {}^*[0,T]} |\hat{S}_t|^r\Big] < \infty \quad \text{for } r \in [1,\infty),$$

so that $\hat{S}_t \in SL^r(P)$ for all $t \in {}^*[0,1]$, $r \in [1,\infty)$.
We now define discrete parameter internal processes $W, N, S : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ by evaluating $\hat{W}, \hat{N}, \hat{S}$ at points in $\mathbb{T}$:

$$W_t(\omega) := \hat{W}_t(\omega), \quad N_t(\omega) := \hat{N}_t(\omega), \quad S_t := \hat{S}_t(\omega) \quad \text{for } t \in \mathbb{T}, \omega \in \Omega.$$

This means that the increments $(\Delta W_t)_{t\in\mathbb{T}\setminus\{T\}}$ are internal i.i.d. $\mathcal{N}(0,\Delta t)$ random variables. Since, for $j \in {}^*\mathbb{N}$ and $s, t \in {}^*[0,T]$, $t + s \leq T$,

$$P(\tilde{N}_{t+s} - \tilde{N}_t = j) = \frac{(\lambda s)^j}{j!} \exp(-\lambda s) \tag{4.17}$$

the increments $(\Delta N_t)_{t\in\mathbb{T}\setminus\{T\}}$ are i.i.d. with a distribution given by

$$P(\Delta N_t = j) = \frac{(\lambda \Delta t)^j}{j!} \exp(-\lambda \Delta t) \quad \text{for } j \in {}^*\mathbb{N}.$$

An internal filtration $\mathbb{A} = (\mathcal{A}_t)_{t\in\mathbb{T}}$ on $(\Omega, \mathcal{F})$ is generated by the processes W and N, i.e. $\mathcal{A}_t := \sigma\{W_s, N_s : s \le t\}$.
The price process S is also $\mathcal{S}L^r$-integrable for any $r \in [1,\infty)$ and satisfies

$$\Delta S_t = S_t \Delta R_t, \quad S_0 = s_0, \tag{4.18}$$

where $R : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ is the discrete return process given by

$$\Delta R_t := \exp\Big((\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\Delta W_t + \log(1+\varphi)\Delta N_t\Big) - 1, \quad R_0 := 0.$$

Proposition 4.3.1. *(i) The process W is $\mathcal{S}$-continuous and $w := \mathrm{st}(W)$ is a standard Brownian motion.*

(ii) The process N is of class SDJ and $n := \mathrm{st}(N)$ is a Poisson process with intensity λ.

Proof. (i) This was proved in [Cut87, Theorem 2.2]. The proof is a modification of the construction of Brownian motion as a binomial random walk (see Section 2.1).
(ii) This is an adaptation of the proofs of [Cut83, Theorem 6.2] and [CKW93b, Proposition 2.1]: First define a process $\tilde{n} : \Omega \times [0,T] \to \mathbb{N} \cup \{\infty\}$ by

$$\tilde{n}(u,\omega) := \max_{t\approx u} {}^\circ\tilde{N}(t,\omega).$$

Then $\tilde{n}$ is an increasing process and, by (4.17), $P(\tilde{N}_T = k) = \frac{(\lambda T)^k}{k!}\exp(-\lambda T)$, so that (summing over $k \in \mathbb{N}$) $\tilde{N}_T$ is finite, P_L-a.s.. It also follows from (4.17) that, for fixed $t \in {}^*[0,T]$ and any $s \approx 0$, $P(\tilde{N}_{t+s} - \tilde{N}_t = 0) = \exp(\lambda s) \approx 1$, so that $\tilde{N}_{t+s} = \tilde{N}_t$ P_L-a.s.. Hence

$$P_L(\tilde{n}_{u+v} - \tilde{n}_u = j) = \frac{(\lambda v)j}{j!}\exp(-\lambda v)$$

for $j \in \mathbb{N}$ and $u, v \in [0,T]$, $u + v \le T$. Furthermore the increments of $\tilde{n}$ are independent by the corresponding property of $\tilde{N}$. To show that P_L-a.a. paths of $\tilde{n}$ are right-continuous fix $\omega \in \Omega$ such that $\tilde{n}_v(\omega) < \infty$ for all $v \in [0,T]$ and consider $u \in [0,T]$ with $\tilde{n}_u = j \in \mathbb{N}$. By definition of $\tilde{n}$ there exists $t \in \mathbb{T}, t \approx u$ with $\tilde{N}_t(\omega) = j$ and $\tilde{N}_{t+s}(\omega) = j$ for all $s \ge 0, s \approx 0$. By overflow there exists a real $\delta > 0$ such that $\tilde{N}_{t+s}(\omega) = j$ for $s \in {}^*[0,\delta)$, hence $\tilde{n}_{u+v}(\omega) = j$ for all $v \in [0,\delta)$.
We have shown that $\tilde{N}$ is of class SD and that $\tilde{n} = \mathrm{st}(\tilde{N}) = \mathrm{st}(N) = n$ is a standard Poisson process with intensity λ. In particular, the jumps of $\tilde{n}$ are of size one, P_L-a.s.. Hence N (as well as $\tilde{N}$) has at most one jump of size one in each monad, i.e. N is of class SDJ. ☐

It follows from Proposition 4.3.1 that the internal return and price processes R and S are of class SDJ and

$$\begin{aligned} r &:= \mathrm{st}(R) = \mu u + \sigma w_u + \varphi n_u, \\ s &:= \mathrm{st}(S) = s_0 \mathcal{E}(r)_u \quad \text{for } u \in [0,T], \quad P_L\text{-a.s.}, \end{aligned}$$

so that s is the stock price process in the jump-diffusion model in [RS95, equation (1.1)]. The Doob-Meyer decomposition of the return process r can be calculated as

$$dr_u = \alpha_u^r d\langle m^r \rangle_u + dm_u^r$$

with

$$\begin{aligned} \alpha_u^r &= \frac{\mu + \varphi\lambda}{\sigma^2 + \varphi^2 \lambda} \quad \text{and} \\ dm_u^r &= \sigma dw_u + \varphi(dn_u - \lambda u) \end{aligned}$$

(see [RS95]). On the other hand, calculating the internal decomposition of R and observing that $E\left[c^{\Delta N_t}\right] = \exp((c-1)\lambda \Delta t)$ for $c \in {}^*\mathbb{R}$ (see [Pro90, p.15]) yields

$$E\left[\Delta R_t | \mathcal{A}_t\right] = \exp\big((\mu + \varphi\lambda)\Delta t\big) - 1,$$

so

$$\begin{aligned} \Delta M_t^R &= \Delta R_t - E\left[\Delta R_t | \mathcal{A}_t\right] \\ &= \exp\big((\mu + \varphi\lambda)\Delta t\big) \\ &\quad \times \Big(\exp\big((-\frac{1}{2}\sigma^2 - \varphi\lambda)\Delta t + \sigma\Delta W_t + \log(1+\varphi)\Delta N_t\big) - 1\Big) \end{aligned} \tag{4.19}$$

and

$$E\left[(\Delta M^R)_t^2 | \mathcal{A}_t\right] = \exp\big(2(\mu + \varphi\lambda)\Delta t\big)\Big(\exp\big((\sigma^2 + \varphi^2\lambda)\Delta t\big) + 2 \cdot 1 - 1\Big),$$

hence

$$\alpha_t^R = \frac{\exp\big((\mu + \varphi\lambda)\Delta t\big) - 1}{\exp\big((\sigma^2 + \varphi^2\lambda)\Delta t\big) - 1} \exp\big(-2(\mu + \varphi\lambda)\Delta t\big).$$

We see that α_t^R is a constant and, using the Taylor expansion of the exponential function once again,

$${}^\circ\alpha_t^R = \frac{\mu + \varphi\lambda}{\sigma^2 + \varphi^2\lambda} = \alpha_t^r.$$

In order to show that $\mathrm{st}(M^R) = m^r$ it remains to prove the following lemma:

Lemma 4.3.2. *The process* $M : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ *defined by*

$$M_t := \sum_{s<t} \Big(\exp\big((-\frac{1}{2}\sigma^2 - \varphi\lambda)\Delta t + \sigma \Delta W_t + \log(1+\varphi)\Delta N_t \big) - 1 \Big)$$

satisfies

$$\mathrm{st}(M)_u = -\varphi\lambda u + \sigma w_u + \varphi n_u \quad \text{for } u \in [0,T],\ P_L\text{-a.s.}.$$

Proof. Note first that $e^{x+y} - 1 = (e^x - 1) + (e^y - 1) + (e^x - 1)(e^y - 1)$ for $x, y \in {}^*\mathbb{R}$. We will use this relation with

$$x_t := (-\varphi\lambda - \frac{1}{2}\sigma^2)\Delta t + \sigma \Delta W_t,$$
$$y_t := \log(1+\varphi)\Delta N_t,$$

so

$$M_t = \sum_{s<t} (e^{x_s} - 1) + (e^{y_s} - 1) + (e^{x_s} - 1)(e^{y_s} - 1). \tag{4.20}$$

By Theorem 2.2.6(ii) there is a set $\tilde{\Omega} \subset \Omega$ with $P_L(\tilde{\Omega}) = 1$ such that $(W, N)(\omega)$ is SDJ and $[w](\omega) = \mathrm{st}([W])(\omega)$ for $\omega \in \tilde{\Omega}$. Now fix $\omega \in \tilde{\Omega}$. Using the Taylor expansion of the exponential function shows that

$$\sum_{s<t} e^{x_s(\omega)} - 1 \approx (-\varphi\lambda - \frac{1}{2}\sigma^2)t + \frac{1}{2}\sigma^2 [W]_t(\omega) + \sigma W_t(\omega)$$

hence

$$\mathrm{st}\Big(\sum e^{x_\cdot(\omega)} - 1\Big)_u = -\varphi\lambda u + \sigma w_u(\omega) \quad \text{for } u \in [0,T].$$

Since $N(\omega)$ is SDJ we have $\Delta N_t(\omega) \in \{0,1\}$, so that

$$e^{y_s(\omega)} - 1 = (1+\varphi)^{\Delta N_s(\omega)} - 1 = \varphi \Delta N_s(\omega) \quad \text{for } s \in \mathbb{T},$$

hence

$$\mathrm{st}\Big(\sum e^{y_\cdot(\omega)} - 1\Big)_u = \varphi n_u(\omega) \quad \text{for } u \in [0,T].$$

Finally, let $\varepsilon := \max\{|e^{x_s(\omega)} - 1| : s \in \mathbb{T}\}$, so $\varepsilon \approx 0$. For the remaining term in (4.20) we then have

$$\Big|\sum_{s<t} (e^{x_s(\omega)} - 1)(e^{y_s(\omega)} - 1)\Big| = \Big|\sum_{s<t} (e^{x_s(\omega)} - 1)\varphi \Delta N_s(\omega)\Big| \leq \varepsilon\varphi N_t(\omega),$$

which is infinitesimal since $N_t(\omega)$ is finite. $\square$

Proposition 4.3.3. *In the discretised jump-diffusion model* (4.18) *the minimal density* $\hat{Z} = \prod(1 - \alpha^R \Delta M^R)$ *is of class SDJ and satisfies*

$$\mathrm{st}(\hat{Z})_u = \mathcal{E}\Big(-\int \alpha^r dm^r\Big)_u \quad \text{for all } u \in [0,T],\ P_L\text{-a.s..}$$

Furthermore, for any $r \in [1, \infty)$, $\hat{Z}_t \in \mathcal{S}L^r(P)$ *for all* $t \in \mathbb{T}$.

Proof. The first assertion follows again from Corollary 4.1.5. For the $\mathcal{S}$-integrability we can use the same argument as in the proof of Proposition 4.2.2, so it is enough to show that, for $i \in \mathbb{N}$, $E\left[|\Delta M_t^R|^i\right] = K_i \Delta t$ for some finite $K_i \in {}^*\mathbb{R}$. But this follows immediately from the expression (4.19) for ΔM_i^R together with the fact that

$$E\left[\exp(c\Delta W_t)\right] = \exp\big(\frac{1}{2}c^2\Delta t\big) \quad \text{and} \quad E\left[\exp(c\Delta N_t)\right] = \exp\big((\exp(c) - 1)\lambda\Delta t\big)$$

for any $c \in {}^*\mathbb{R}$. □

Remark 4.3.4. Note that the density $\mathcal{E}\big(-\int \alpha^r dm^r\big)$ in Proposition 4.3.3 may take negative values as m^r has jumps of size $\varphi\lambda$. We can ensure that the density remains strictly positive if we make the additional assumption on the coefficients $\lambda, \mu, \sigma, \varphi$ that $\varphi\alpha^r = \varphi\frac{\mu+\varphi\lambda}{\sigma^2+\varphi^2\lambda} < 1$ (cf. Theorem 1.2.6).

Proposition 4.3.3 is the exact analogue of Proposition 4.2.2 for the multinomial model in Section 4.2. Therefore the minimal Loeb measure $\hat{P}_L$ coincides with the minimal martingale measure for s and, for any $h \in L^2(P_L)$ with lifting $H \in \mathcal{S}L^2(P)$, we have

$$E_{\hat{P}}[H] \approx E_{\hat{P}_L}[h].$$

4.4 Trading Strategies and Value Processes

In this section we present a general result which we will use later to relate the optimal trading strategies for discrete incomplete market models to those in *complete* continuous time models. We will illustrate these results in the next chapter when we consider (incomplete) discrete time approximations of the Black-Scholes model.

We are back in the general setting of Section 4.1, i.e. $(\Omega, \mathbb{A}, \mathcal{A}, P)$ is an internal filtered probability space with associated Loeb space $(\Omega, \mathcal{F}_L, P_L)$. We will denote the internal expectation with respect to P by $E[\,\cdot\,]$ and the (standard) expectation with respect to P_L by $E_L[\,\cdot\,]$. The counting measure on the hyperfinite time line $\mathbb{T}$ is denoted by Λ. Suppose the process $S : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ is an internal $\mathcal{S}L^2(P)$-martingale of class SDJ. As usual we denote the internal Doléans measure of S by ν_S. Let $s := \mathrm{st}(S)$ be the standard part of S, so that $s : \Omega \times [0,T] \to \mathbb{R}$ is an $L^2(P_L)$-martingale.

We make the following assumptions (S1) and (S2) on the processes S and s. In Chapter 5 we will introduce two models for which these are satisfied. Furthermore, note that (S1) and (S2) are satisfied for the models considered in [CKW91, CKW93b].

(S1) For any nonanticipating process $\Theta : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ and any $\theta \in L^2(\nu_s)$ we have

$$\Theta \text{ is a } \mathcal{S}L^2(\nu_S)\text{-lifting of } \theta \quad \Leftrightarrow \quad \Theta S \text{ is a } \mathcal{S}L^2(P \times \Lambda)\text{-lifting of } \theta s.$$

(S2) Any $h \in L^2(P_L)$ can be represented as

$$h = v_0 + \int_0^T \theta_u ds_u \tag{4.21}$$

with a unique $\theta \in L^2(\nu_s)$ and $v_0 \in \mathbb{R}$.

Now let $h \in L^2(P_L)$ and v_0, θ be defined as in (4.21). Define processes g, v, ψ : $\Omega \times [0,T] \to \mathbb{R}$ by

$$g_t := \int_0^t \theta_u ds_u, \qquad v_t := v_0 + g_t, \qquad \psi_t := v_t - \theta_t s_t.$$

Let $H \in \mathcal{S}L^2(P)$ be a lifting of h. We can then define an internal decomposition of H as in Section 1.1: Let

$$V_t := E[H|\mathcal{A}_t], \qquad \Theta_t := \frac{E\left[V_{t+\Delta t}\Delta S_t|\, \mathcal{A}_t\right]}{E\left[(\Delta S_t)^2|\, \mathcal{A}_t\right]}.$$

(with $\Theta_t = 0$ if $E[(\Delta S_t)^2|\mathcal{A}_t] = 0$) and set $\Theta_T := 0$. So V is a $\mathcal{S}L^2$-martingale. By transfer of the inequality (1.7) we see that

$$E\left[(\Theta_t \Delta S_t)^2\right] < \infty.$$

The internal martingale L defined by

$$L_0 := 0, \qquad \Delta L_t := \Delta V_t - \Theta_t \Delta S_t$$

is internally orthogonal to S, i.e.

$$\begin{aligned} E\left[\Delta L_t \Delta S_t|\, \mathcal{A}_t\right] &= E\left[\Delta V_t \Delta S_t - \Theta_t(\Delta S_t)^2 | \mathcal{A}_t\right] \\ &= E\left[V_{t+\Delta t}\Delta S_t|\, \mathcal{A}_t\right] - V_t E\left[\Delta S_t|\, \mathcal{A}_t\right] - \Theta_t E\left[(\Delta S_t)^2\middle|\, \mathcal{A}_t\right] \\ &= 0 \quad \text{for } t \in \mathbb{T} \setminus \{T\} \end{aligned}$$

(where the last equality uses the martingale property of S and the definition of Θ_t). Then

$$V_t = V_0 + \sum_{u<t} \Theta_u \Delta S_u + L_t \quad \text{and} \quad V_T = H.$$

Finally, we define internal processes $G, \Psi : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ by

$$G_t := \sum_{u<t} \Theta_u \Delta S_u, \qquad \Psi_t := V_t - \Theta_t S_t.$$

We need the following two lemmata; the proofs are given in Section 4.5. The first characterises orthogonality of martingales in terms of orthogonality with respect to the space generated by stochastic integrals. The second gives a simple criterion for an internal martingale to be infinitesimal.

Lemma 4.4.1. *Let $X, Y : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ be internal martingales. Then X and Y are orthogonal, i.e.*

$$E\left[\Delta X_t \Delta Y_t | \mathcal{A}_t\right] = 0 \quad \text{for all } t \in \mathbb{T}, \tag{4.22}$$

if and only if for all nonanticipating $\Theta : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ the process Z defined by

$$Z_t = Y_t \sum_{s<t} \Theta_s \Delta X_s \tag{4.23}$$

is an internal martingale, i.e. Y and $\sum \Theta \Delta X$ are orthogonal.

Lemma 4.4.2. *Let $X : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ be an internal martingale with $X_0 = 0$. If $E[X_T^2] \approx 0$ then $X_t \approx 0$ for all $t \in \mathbb{T}$, P_L-a.s., i.e.* $\mathrm{st}(X) \equiv 0$.

We can now prove the main result of this section:

Theorem 4.4.3. *Under the assumptions (S1) and (S2) we have*

(i) Θ is an $SL^2(\nu_S)$-lifting of θ;

(ii) Ψ is an $SL^2(P \times \Lambda)$-lifting of ψ;

(iii) G and V are SL^2-martingales of class SDJ and satisfy, P_L-a.s.,

$$\begin{aligned} \mathrm{st}(G) &= g, \\ \mathrm{st}(V) &= v, \\ \text{hence} \quad \mathrm{st}(V) &= \mathrm{st}(V_0 + G). \end{aligned} \tag{4.24}$$

Proof. First note that $v_0 = E_L[h] \approx E[H] = V_0$. Since S is an SL^2-lifting of s and $\theta \in L^2(\nu_s)$ there exists a 2-lifting $\bar{\Theta}$ of θ such that

$$^\circ\Big(\sum_{t<T} \bar{\Theta}_t \Delta S_t\Big) = \int_0^T \theta_u ds_u, \quad P_L\text{-a.s.}$$

(Theorem 2.2.11). Then $\bar{H} := V_0 + \sum_{t<T} \bar{\Theta}_t \Delta S_t$ is also an SL^2-lifting of h, hence

$$
\begin{aligned}
0 \approx E\left[(H-\bar{H})^2\right] &= E\Big[\Big(\sum_{t<T} \Theta_t \Delta S_t + L_T - \sum_{t<T} \bar{\Theta}_t \Delta S_t\Big)^2\Big] \\
&= E\Big[\Big(\sum_{t<T} (\Theta_t - \bar{\Theta}_t)\Delta S_t\Big)^2\Big] + E\left[L_T^2\right] \\
&\qquad + 2E\Big[L_T \sum_{t<T} (\Theta_t - \bar{\Theta}_t)\Delta S_t\Big] \qquad (4.25)
\end{aligned}
$$

The internal process

$$M_t := L_t \sum_{u<t} (\Theta_u - \bar{\Theta}_u)\Delta S_u$$

is a martingale by Lemma 4.4.1 (recall that L and S are internally orthogonal). Since $M_0 = 0$ we have

$$E\Big[L_T \sum_{t<T} (\Theta_t - \bar{\Theta}_t)\Delta S_t\Big] = 0.$$

Therefore the final term in (4.25) disappears and

$$
\begin{aligned}
\int_{\Omega \times \mathbb{T}} (\Theta - \bar{\Theta})^2 \, d\nu_S = E\Big[\Big(\sum_{t<T} (\Theta_t - \bar{\Theta}_t)\Delta S_t\Big)^2\Big] &\approx 0 \\
\text{and} \qquad E\left[L_T^2\right] &\approx 0.
\end{aligned}
$$

Hence Θ is also a 2-lifting of θ with respect to ν_S and $\Theta \in SL^2(\nu_S)$. By assumption (S1) this is equivalent to ΘS being a $SL^2(P \times \Lambda)$-lifting of θs. Furthermore, G is an SL^2-martingale of class SDJ by Theorem 2.2.8, and Theorem 2.2.11 implies that

$$\mathrm{st}(G) = \mathrm{st}\Big(\sum \Theta \Delta S\Big) = \int \theta ds = g, \quad P_L\text{-a.s..}$$

As $E[L_T^2] \approx 0$ it follows from Lemma 4.4.2 that the paths of $\mathrm{st}(L)$ are constant zero, P_L-a.s., hence

$$\mathrm{st}(V) = \mathrm{st}(H_0 + G + L) = {}^\circ H_0 + \mathrm{st}(G) = h_0 + g_t = v \qquad P_L\text{-a.s.,}$$

Finally, this implies that $\Psi = V - \Theta S$ is an SL^2-lifting of $\psi = v - \theta s$. □

Remark 4.4.4. In the language of mathematical finance Theorem 4.4.3 shows that the lifting property of a claim H implies the lifting property of the associated locally-risk-minimising (or variance-optimal[1]) strategy and its value and gains process.

[1]Since S is a martingale the locally-risk-minimising and variance-optimal strategies coincide in this case (see Remark 1.3.2).

This result therefore includes Theorem 3.5 in [CKW91] and Theorem 4.1 in [CKW93b] as special cases. It should be noted however that the models in [CKW91, CKW93b] are *internally complete.* It is therefore possible to obtain an internally self-financing trading strategy generating the claim H, so that equation (4.24) is automatically satisfied: for self-financing strategies we have $V = V_0 + G$.
The crucial point in Theorem 4.4.3 is that the additional cost process L is infinitesimal, so that the internal strategies here are self-financing "in the limit". This last statement will be made precise in the next chapter.

We summarise the results of this section in the following theorem which is a generalisation of Theorem 2.3.1:

Theorem 4.4.5. *With the above notation and assumptions (S1) and (S2) the following are equivalent:*

(i) H *is an* $SL^2(P)$*-lifting of* h.

(ii) Θ *is an* $SL^2(\nu_S)$*-lifting of* θ, *and* Ψ *is an* $SL^2(P \times \Lambda)$*-lifting of* ψ.

(iii) ΘS *and* Ψ *are* $SL^2(P \times \Lambda)$*-liftings of* θs *and* ψ, *respectively.*

(iv) G *is an* SL^2*-martingale of class SDJ and* $\mathrm{st}(V_0 + G) = v_0 + g$, P_L*-a.s..*

(v) V *is an* SL^2*-martingale of class SDJ and* $\mathrm{st}(V) = v$, P_L*-a.s..*

4.5 Auxiliary Results

This section contains the proofs of some technical results which were used in the earlier parts of this chapter.

Proof of Lemma 4.1.1. Note first that F_s is finite for all $s \in \mathbb{T}$ and let $B := \max\{|\Delta F_s| + 1 : s \in \mathbb{T}\}$. Then B is finite and $|{}^\circ\Delta F_s| \leq B$ for all $s \in \mathbb{T}$.
There are at most countably many points $u \in [0,T]$ such that $|\Delta f_u| > 0$. Note also that since F is SDJ there exists a unique $s \in \mathbb{T}$ for every $u \in [0,T]$ with $|\Delta f_u| > 0$ such that $s \approx u$,

$$^\circ F(s) = f(u-) \quad \text{and} \quad {}^\circ F(s + \Delta t) = f(u), \quad \text{hence} \quad {}^\circ\Delta F_s = \Delta f_u. \tag{4.26}$$

Now let $\mathbb{J} := \{u \in [0,T] : |\Delta f_u| > 0\} = \{u_1, u_2, \dots\}$ be the (possibly) finite set of jump points of f.
Case (i): $\mathbb{J}$ is a finite — possibly empty — set. Let $\tilde{\mathbb{T}}$ be the set of corresponding jump points s of F as defined in (4.26). Then $\tilde{\mathbb{T}}$ is finite (or empty), hence internal

and, for $t \in \mathbb{T}$,

$$
\begin{aligned}
{}^\circ J(t) &= {}^\circ\Big(\sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}} \Delta F_s + \frac{1}{2}(\Delta F_s)^2\Big) = \sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}} {}^\circ\Delta F_s + \frac{1}{2}({}^\circ\Delta F_s)^2 &\quad (4.27)\\
{}^\circ K(t) &= {}^\circ\Big(\prod_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}(1+\Delta F_s)\Big) = \prod_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}(1+{}^\circ\Delta F_s). &\quad (4.28)
\end{aligned}
$$

Case (ii): $\mathbb{J}$ is countably infinite. Due to assumption (4.5) the set of $u \in [0,T]$ such that $|\Delta f_u| > c$ is finite for any fixed $c > 0$. We may therefore assume (by applying Case (i) to a subset of $\mathbb{J}$ if necessary) that $|\Delta f_u| < \frac{1}{2}$ for all $u \in \mathbb{J}$. Let $(u_i)_{i\in\mathbb{N}}$ be an enumeration of the points in $\mathbb{J}$ and $(s_i)_{i\in\mathbb{N}}$ the sequence of corresponding non-infinitesimal jump points in $\mathbb{T}$ as defined in (4.26). By Countable Comprehension (see e.g. Theorem 2.5.1 in [CC95]) this sequence can be extended to a sequence $(s_i)_{i\in{}^*\mathbb{N}}$ with $s_i \in \mathbb{T}$ for all $i \in {}^*\mathbb{N}$.

Now let $\bar{\varepsilon} := \max\{|F_s - {}^\circ F_s| : s \in \mathbb{T}\}$. Then $\bar{\varepsilon} \approx 0$ and

$$\max\{|\Delta F_s - {}^\circ\Delta F_s| : s \in \mathbb{T}\} \le 2\bar{\varepsilon}.$$

Choose $M \in {}^*\mathbb{N}\setminus\mathbb{N}$ such that $M\bar{\varepsilon} \approx 0$ and $M \le N$ (e.g. let $M = \min\{[1/\sqrt{\bar{\varepsilon}}], N\}$ where $[r]$ denotes the integer part of $r \in {}^*\mathbb{R}$; if $\bar{\varepsilon} = 0$ then let $M = N$).

Define $\tilde{\mathbb{T}} := \{s_i : i \le M\} \subset \mathbb{T}$. Then $\tilde{\mathbb{T}}$ is internal and

$$
{}^\circ J(t) = {}^\circ\Big(\sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}} \Delta F_s + \frac{1}{2}(\Delta F_s)^2\Big) = \sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}} {}^\circ\Delta F_s + \frac{1}{2}({}^\circ\Delta F_s)^2 \qquad (4.29)
$$

as

$$
\begin{aligned}
&\Big|\sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}} \Delta F_s - {}^\circ\Delta F_s + \frac{1}{2}\big((\Delta F_s)^2 - ({}^\circ\Delta F_s)^2\big)\Big|\\
\le\ & \sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}} |\Delta F_s - {}^\circ\Delta F_s| + \frac{1}{2}|\Delta F_s - {}^\circ\Delta F_s|\cdot|\Delta F_s + {}^\circ\Delta F_s|\\
\le\ & M(2\bar{\varepsilon} + 2B_{\bar{\varepsilon}}) \approx 0.
\end{aligned}
$$

Furthermore,

$$\Big|\log\big(\prod_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}(1+\Delta F_s)\big)-\log\big(\prod_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}(1+{}^\circ\Delta F_s)\big)\Big| \le \sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}\Big|\log(1+\Delta F_s)-\log(1+{}^\circ\Delta F_s)\Big|$$
$$=\sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}\Big|\log\Big(1+\frac{\Delta F_s-{}^\circ\Delta F_s}{1+{}^\circ\Delta F_s}\Big)\Big|$$

(writing $\log(1+x)=x+\varepsilon$)

$$=\sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}\Big|\frac{\Delta F_s-{}^\circ\Delta F_s}{1+{}^\circ\Delta F_s}+\varepsilon_s\Big|$$
$$\le\sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}\Big|\frac{\Delta F_s-{}^\circ\Delta F_s}{1+{}^\circ\Delta F_s}\Big|+|\varepsilon_s|.$$

Recall that $|\Delta f_u|<\frac{1}{2}$ for all $u\in\mathbb{J}$, so that, for $s\in\tilde{\mathbb{T}}$, $|{}^\circ\Delta F_s|\le\frac{1}{2}$, which implies that $1+{}^\circ\Delta F_s\ge\frac{1}{2}$. Since $|\log(1+x)-x|\le|x|^2$ for $|x|\le\frac{1}{2}$ we see that $|\varepsilon_s|\le 4|\Delta F_s-{}^\circ\Delta F_s|^2$ and hence the final sum is bounded by

$$\sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}2|\Delta F_s-{}^\circ\Delta F_s|+4|\Delta F_s-{}^\circ\Delta F_s|^2\le M(4\bar{\varepsilon}+16\bar{\varepsilon}^2)=4M\bar{\varepsilon}(1+4\bar{\varepsilon})\approx 0,$$

thus, by the $\mathcal{S}$-continuity of the exponential function

$$^\circ K(t)={}^\circ\Big(\prod_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}(1+\Delta F_s)\Big)=\prod_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}(1+{}^\circ\Delta F_s). \tag{4.30}$$

(Note that the use of the above inequality for the logarithm function also shows that

$$\log\Big(\prod_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}(1+\Delta F_s)\Big)=\sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}}\log(1+\Delta F_s)$$

is finite since $\sum_{s<t}(\Delta F_s)^2$ is finite by (4.5).)

We now show that $\mathrm{st}(J)=j$ and $\mathrm{st}(K)=k$ in either case: Let $v\in[0,T]$. Then there are points $v',v''\in\mathbb{J}$ such that $v'\le v<v''$ and $s\notin\mathbb{J}$ for all $v'<s<v''$ (i.e. v',v'' are the nearest jump points to the left and right of v — note that possibly $v'=v$). Let t',t'' be the corresponding points in $\tilde{\mathbb{T}}$ as defined in (4.26). Then $^\circ|v-t''|>0$. Hence, using (4.27) and (4.29) respectively, and observing that j only changes values at points $u \in \mathbb{J}$ while ${}^\circ J(\cdot)$ changes values only at the corresponding points $s \in \tilde{\mathbb{T}}$,

$$\begin{aligned}
\mathrm{st}(J)(v) = \lim_{{}^\circ t \downarrow v} {}^\circ J(t) &= \lim_{{}^\circ t \downarrow v} \sum_{\substack{s<t\\ s\in\tilde{\mathbb{T}}}} {}^\circ\Delta F_s + \frac{1}{2}({}^\circ\Delta F_s)^2 \\
&= \sum_{\substack{s<t'\\ s\in\tilde{\mathbb{T}}}} {}^\circ\Delta F_s + \frac{1}{2}({}^\circ\Delta F_s)^2 \\
&= \sum_{u\le v'} \Delta f_s + \frac{1}{2}(\Delta f_s)^2 = j(v).
\end{aligned}$$

An analogous argument (using (4.28) and (4.30)) then shows that $\mathrm{st}(K) = k$. □

Proof of Lemma 4.4.1. First assume that X and Y are orthogonal. Calculating ΔZ_t yields

$$\begin{aligned}
\Delta Z_t &= Y_{t+\Delta t} \sum_{s<t+\Delta t} \Theta_s \Delta X_s - Y_t \sum_{s<t} \Theta_s \Delta X_s \\
&= (Y_t + \Delta Y_t) \sum_{s<t+\Delta t} \Theta_s \Delta X_s - Y_t \sum_{s<t} \Theta_s \Delta X_s \\
&= Y_t \Theta_t \Delta X_t + \Delta Y_t \sum_{s<t+\Delta t} \Theta_s \Delta X_s \\
&= Y_t \Theta_t \Delta X_t + \Delta Y_t \sum_{s<t} \Theta_s \Delta X_s + \Delta Y_t \Theta_t \Delta X_t
\end{aligned}$$

Hence

$$\begin{aligned}
E[\Delta Z_t | \mathcal{A}_t] &= E[Y_t \Theta_t \Delta X_t | \mathcal{A}_t] + E\left[\Delta Y_t \sum_{s<t} \Theta_s \Delta X_s \middle| \mathcal{A}_t\right] + E[\Delta Y_t \Theta_t \Delta X_t | \mathcal{A}_t] \\
&= Y_t \Theta_t E[\Delta X_t | \mathcal{A}_t] + E[\Delta Y_t | \mathcal{A}_t] \sum_{s<t} \Theta_s \Delta X_s + \Theta_t E[\Delta Y_t \Delta X_t | \mathcal{A}_t] \\
&= 0
\end{aligned}$$

because of the martingale property and the orthogonality of X and Y (cf. equation 4.22).

For the converse note that the process

$$Z_t := X_t \sum_{s<t} \Delta Y_s = X_t Y_t$$

is a martingale (choose $\Theta \equiv 1$ in (4.23)). □

The following lemma is a nonstandard version of the standard result that two continuous processes which are versions of each other are already indistinguishable.

Lemma 4.5.1. *Let $X : \Omega \times \mathbb{T} \to {}^*\mathbb{R}$ be an $\mathcal{S}$-continuous process and $P_L(X_t \approx 0) = 1$ for each $t \in \mathbb{T}$. Then*

$$P_L(X_t \approx 0 \text{ for all } t \in \mathbb{T}) = 1.$$

Proof. Let $\tilde{\Omega} := \{\omega \in \Omega : X(\omega) \text{ is } \mathcal{S}\text{-continuous}\}$. Then $P_L(\tilde{\Omega}) = 1$. For $r \in [0,T]$ let $\bar{r} := \max\{t \in \mathbb{T} : t \leq r\}$, i.e. $\bar{r}$ is the element in $\mathbb{T}$ immediately to the left of r. Now let $\mathbb{S} := \{\bar{q} : q \in \mathbb{Q} \cap [0,T]\} \subset \mathbb{T}$. Since there is a one-to-one correspondence between the elements in $\mathbb{Q} \cap [0,T]$ and those in $\mathbb{S}$ we see that $\mathbb{S}$ is countable and $\mathcal{S}$-dense, i.e. $\mathrm{st}(\mathbb{S}) = [0,T]$.
For $s \in \mathbb{S}$ define

$$D_s := \{\omega \in \tilde{\Omega} : {}^\circ|X_s(\omega)| > 0\} = \{\omega \in \Omega : X_s(\omega) \not\approx 0\} \cap \tilde{\Omega}.$$

Then $P_L(D_s) = 0$ by assumption. Let

$$D := \bigcup_{s \in \mathbb{S}} D_s.$$

Then $P_L(D) = 0$ and $P_L(\tilde{\Omega} \setminus D) = 1$. For fixed $\omega \in \tilde{\Omega} \setminus D$ the path $X_.(\omega)$ is $\mathcal{S}$-continuous and $X_t(\omega) \approx 0$ for all $t \in \mathbb{S}$. Let $x := \mathrm{st}(X(\omega))$, so x is a continuous function which is zero on $\mathbb{Q} \cap [0,T]$. Hence $X_t(\omega) \approx 0$ for all $t \in \mathbb{T}$. □

Proof of Lemma 4.4.2. Let $[X]$ be the internal quadratic variation of X. We know that

$$E[X_t^2] = E[X_0^2] + E\big[[X]_t\big] = E\big[[X]_t\big] \quad \text{for } t \in \mathbb{T}.$$

It follows that $E\big[[X]_T\big] = E[X_T^2] \approx 0$. Since $[X]_T \geq 0$ this implies that $[X]_T(\omega) \approx 0$ for P_L-a.a. ω. But $[X]_0 = 0$ and $[X]$ is an increasing process, hence the path $[X]_.(\omega)$ is $\mathcal{S}$-continuous for P_L-a.a. ω. Therefore X is $\mathcal{S}$-continuous by Theorem 2.2.6.
Since $[X]$ is an increasing process we have

$$0 \leq E[X_t^2] = E\big[[X]_t\big] \leq E\big[[X]_T\big] \approx 0 \quad \text{for } t \in \mathbb{T}.$$

Hence, for fixed $t \in \mathbb{T}$, $X_t \approx 0$ P_L-a.s.. Lemma 4.5.1 then implies that $X_t \approx 0$ for all $t \in \mathbb{T}$, P_L-a.s.. □

Chapter 5

Approximations of the Black-Scholes Model

In this chapter we apply the results of Section 4.4 to two alternative approximations of the BS model. These can be viewed as the limits of the modified CRR and BS models, respectively, under restricted hedging (see Chapter 3) when the time between hedging dates decreases to zero. The results in this chapter therefore extend the results in [MV96] and [RS95] on the convergence of option prices.

5.1 The Multinomial Cox-Ross-Rubinstein Model

In Section 4.2 we introduced a hyperfinite multinomial version of the CRR model and showed that the minimal martingale measure on the Loeb space coincides with the unique martingale measure of the complete BS model; this implies that option prices in this multinomial model are infinitesimally close to the corresponding prices in the BS model. Using the results of Section 4.4 we now show that the same is true for the associated trading strategies and value processes. We then use the concept of D^2-convergence to obtain new convergence results for these processes. It has previously been shown in [MV96, Proposition 3.6] that option prices for a European call option in the sequence of multinomial models converge to the corresponding BS price.

Definition of the Model

Let $\beta \in \mathbb{N}$ be fixed and let $\mu, \sigma, s_0 \in \mathbb{R}$, $\sigma, s_0 > 0$. For $n \in \mathbb{N}$ we define the n-th $(\beta+1)$-*nomial CRR model* as follows: For $T \in \mathbb{R}^+$ let $\mathbb{T}_n := \{0, \Delta_n t, \ldots, T\}$ with $\Delta_n t := T/n$. Set $\Omega_n := \{0, 1, \ldots, \beta\}^{\mathbb{T}_n \setminus \{T\}}$ and $\mathcal{F}_n := \mathcal{P}(\Omega_n)$. The counting measure on $\mathbb{T}_n$ is denoted by Λ_n.

We want to specify a probability measure Q_n on Ω_n such that the price process

$$S_{n,t+\Delta} := S_{n,t} \cdot u^{\omega_t} d^{\beta-\omega_t}, \quad S_{n,0} := s_0 \tag{5.1}$$

with

$$u := 1 + \mu \frac{\Delta_n t}{\beta} + \sigma\sqrt{\Delta_n t/\beta} \quad \text{and} \quad d := 1 + \mu \frac{\Delta_n t}{\beta} - \sigma\sqrt{\Delta_n t/\beta}$$

is a martingale under Q_n. On possible choice would be the internal minimal martingale measure constructed in Section 4.2; however, we can use the extra information that our multinomial model is obtained by sampling the price process of a complete binomial CRR model (on the finer time set $\mathbb{T}_n^\beta := \{0, \Delta_n t/\beta, 2\Delta_n t/\beta, \dots, T\}$) at the points $t \in \mathbb{T}_n$. This "hidden" binomial model has a unique martingale measure given by the probabilities

$$q := \frac{1-d}{u-d} = \frac{1}{2}\left(1 - \frac{\mu}{\sigma}\sqrt{\Delta_n t/\beta}\right) \quad \text{and} \quad 1-q = \frac{u-1}{u-d} = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta_n t/\beta}\right)$$

for an "up-" or "down-movement" between times in $\mathbb{T}_n^\beta$. We therefore define the measure Q_n on $(\Omega_n, \mathcal{F}_n)$ by

$$Q_n(\omega_t = j) := \binom{\beta}{j} q^j (1-q)^{\beta-j}$$

for $\omega = (\omega_0, \dots, \omega_{T-\Delta_n t}) \in \Omega_n$ and $(\omega_t)_{t\in\mathbb{T}\setminus\{T\}}$ independent. We denote the expectation with respect to Q_n by $E_n[\,\cdot\,]$. A filtration $\mathbb{A}_n = (\mathcal{A}_{n,t})_{t\in\mathbb{T}_n}$ is again generated by the multinomial random walk $W_n : \Omega_n \times \mathbb{T}_n \to \mathbb{R}$ with

$$\Delta W_{n,t} := (-\beta + 2\omega_t)\sqrt{\Delta_n t/\beta}, \quad W_{n,0} := 0.$$

Since $\Delta W_{n,t}$ is the sum of β independent and identically distributed binomial trials, each with outcome $+\sqrt{\Delta_n t/\beta}$ (with probability q) or $-\sqrt{\Delta_n t/\beta}$ (with probability $1-q$), we see that

$$E_n\left[\Delta W_{n,t} \middle| \mathcal{A}_{n,t}\right] = \beta\sqrt{\Delta_n t/\beta}\,(2q-1) = -\frac{\mu}{\sigma}\Delta_n t,$$

i.e. W_n is not a martingale under Q_n. We therefore define an adjusted multinomial process $\tilde{W}_n : \Omega_n \times \mathbb{T}_n \to \mathbb{R}$ with

$$\Delta\tilde{W}_{n,t} := \Delta W_{n,t} + \frac{\mu}{\sigma}\Delta_n t, \quad \tilde{W}_{n,0} := 0,$$

so that

$$E_n\left[\Delta\tilde{W}_{n,t} \middle| \mathcal{A}_{n,t}\right] = 0 \tag{5.2}$$

$$\begin{aligned} \mathrm{Var}_n\left[\Delta\tilde{W}_{n,t} \middle| \mathcal{A}_{n,t}\right] &= \beta\Big(\Delta_n t/\beta - \big(\tfrac{\mu}{\sigma}\Delta_n t/\beta\big)^2\Big) \\ &= \Delta_n t\Big(1 - \big(\tfrac{\mu}{\sigma}\big)^2 \Delta_n t/\beta\Big). \end{aligned} \tag{5.3}$$

The n-th $(\beta+1)$-nomial CRR model satisfies the assumptions of Section 1.1, so that any claim $H:\Omega_n \to \mathbb{R}$ in this model can be replicated by a mean-self-financing strategy $\Phi^H = (\Theta^H, \Psi^H)$ which is risk-minimising since S_n is a martingale (cf. Remark 1.1.7 on page 9). Furthermore, the variance-optimal strategy ξ^H exists and coincides with Θ^H in this case (see Remark 1.3.2).

Remark 5.1.1. As mentioned above the multinomial CRR model can be regarded as the restriction of a binomial CRR model on $\mathbb{T}_n^\beta = \{0, \Delta_n t/\beta, \dots, T\}$ to the coarser time line $\mathbb{T}_n$. It will be useful later to have some notation for this binomial model: let $\Omega_n^\beta := \{-1,+1\}^{\mathbb{T}_n^\beta \setminus \{T\}}$ and Q_n^β be the measure on $(\Omega_n^\beta, \mathcal{P}(\Omega_n^\beta))$ given by the binomial probabilities q and $1-q$. Let W_n^β be a binomial random walk on $\mathbb{T}_n^\beta$ with step size $\pm\sqrt{\Delta_n t/\beta}$ and the price process S_n^β be given by

$$S_{n,t}^\beta(\omega) = s_0 \prod_{s<t} \Big(1 + \mu \frac{\Delta_n t}{\beta} + \sigma \Delta W_{n,s}^\beta\Big), \quad s,t \in \mathbb{T}_n^\beta, \tag{5.4}$$

where $\Delta W_{n,t}^\beta := W_{n,t+\Delta_n t/\beta}^\beta - W_{n,t}^\beta$.

The Hyperfinite Version

For any infinite $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ we have an internal $(\beta+1)$-nomial CRR model on the hyperfinite filtered probability space $(\Omega_N, \mathcal{F}_N, \mathbb{A}_N, Q_N)$ with associated Loeb space $(\Omega_N, L(\mathcal{F}_N), L(Q_N))$.

Lemma 5.1.2. *(i) The processes W_N and $\tilde{W}_N$ are $\mathcal{S}$-continuous.*

(ii) $\tilde{w} := \mathrm{st}(\tilde{W}_N)$ is a standard Brownian motion on $(\Omega_N, L(\mathcal{F}_N), L(Q_N))$.

(iii) The price process S_N is $SL^r(Q_N)$-integrable for each $r \in [1,\infty)$.

Proof. (i): By transfer of (5.2) and (5.3) $\tilde{W}_N$ is an internal λ^2-martingale, and calculating the quadratic variation of $\tilde{W}_N$ yields

$$\begin{aligned}[\tilde{W}_N]_t(\omega) &= \sum_{s<t} \Delta_N t\Big(\frac{1}{\beta}(-\beta+\omega_s)^2 + 2\frac{\mu}{\sigma}(-\beta+2\omega_s)\sqrt{\Delta_N t/\beta} + (\mu/\sigma)^2 \Delta_N t\Big) \\ &= \sum_{s<t} K_s(\omega)\Delta_N t,\end{aligned}$$

where $|K_s(\omega)| < \beta+1$ for all s and ω. Hence $[\tilde{W}_N](\omega)$ is $\mathcal{S}$-continuous which implies that $\tilde{W}_N$ is $\mathcal{S}$-continuous by Theorem 2.2.6(i). Since $W_{N,t} = \tilde{W}_{N,t} - \frac{\mu}{\sigma}t$ also W_N is $\mathcal{S}$-continuous.

(ii): $\tilde{w} = \mathrm{st}(\tilde{W}_N)$ exists and is continuous by part (i). Also $\tilde{w}_0 = 0$ by definition of $\tilde{W}_N$. The process $\tilde{W}_N$ has independent and identically distributed increments, so, by the central limit theorem (Theorem 2.1.3) and the transfer of equations (5.2) and (5.3), $\tilde{w}_{\circ t}$ is normally distributed with mean zero and variance

$$^\circ\Big(\Big(1-(\frac{\mu}{\sigma})^2\Delta_N t/\beta\Big)\sum_{s<t}\Delta_N t\Big) = {}^\circ t.$$

(iii): Using the notation introduced in Remark 5.1.1 we consider the price process S_N^β on $\mathbb{T}_N^\beta \supset \mathbb{T}_N$. Following the same argument as in the proof of Lemma 3.1(b) in [CKW91] we see that it is sufficient to prove that, for $t \in \mathbb{T}_N^\beta$,

$$E_{Q_N^\beta}\Big[\prod_{s<t}(1+a\Delta_N t/\beta + b\Delta W_{N,s}^\beta)\Big] \quad \text{is finite for } a, b \in \mathbb{R}.$$

Using the independence of the increments of W_N^β we see that the above expectation equals

$$\prod_{s<t}\big(1+(a-b\frac{\mu}{\sigma})\Delta_N t/\beta\big) \approx \exp(\tilde{K}t)$$

for some finite $\tilde{K}$. This implies that, for any $r \in [1,\infty)$, $|S_{N,t}^\beta|^r$ is $\mathcal{S}$-integrable for all $t \in \mathbb{T}_N^\beta$, hence $S_{N,t} \in \mathcal{S}L^r(Q_N)$ for all $t \in \mathbb{T}$. □

By Lemma 5.1.2(i) $W_{N,t}$ is finite, $L(Q_N)$-a.s.. It then follows from (5.1) (or (5.4)) and the proof of Lemma 3.1(a) in [CKW91] that, $L(Q_N)$-a.s.,

$$\begin{aligned} S_{N,t} &\approx s_0\exp\big((\mu-\frac{1}{2}\sigma^2)t+\sigma W_{N,t}\big) \\ &= s_0\exp\big(-\frac{1}{2}\sigma^2 t+\sigma\tilde{W}_{N,t}\big) \end{aligned}$$

for all $t \in \mathbb{T}_N$. Hence, S_N is $\mathcal{S}$-continuous and

$$s_u := \mathrm{st}(S_N)_u = s_0\exp\big(-\frac{1}{2}\sigma^2 u+\sigma\tilde{w}_u\big) \quad \text{for } u \in [0,T], \quad L(Q_N)\text{-a.s..}$$

Since $\tilde{w}$ is a standard Brownian motion, s is indeed the price process in a Black-Scholes model under the unique martingale measure $Q := L(Q_N)$. Hence each claim $h \in L^2(Q)$ can be represented as

$$h = v_0^h + \int_0^T \theta_u^h ds_u$$

for some $\theta^h \in L^2(\nu_s)$ and $v_0^h \in \mathbb{R}$. We now verify assumption (S1) on page 60 by employing the following lemma:

Lemma 5.1.3. *Suppose S_N satisfies*

$$E\left[(\Delta S_{N,t})^2\middle|\,\mathcal{A}_{N,t}\right] = (S_{N,t})^2 K\Delta_N t \tag{5.5}$$

for some finite constant $K \in {}^*\mathbb{R}$ with $^\circ K > 0$. For any nonanticipating process Θ and any $\theta \in L^2(\nu_s)$ we then have

$$\Theta \text{ is a } \mathcal{S}L^2(\nu_{S_N})\text{-lifting of } \theta \quad \Leftrightarrow \quad \Theta S_N \text{ is a } \mathcal{S}L^2(Q_N\times\Lambda_N)\text{-lifting of } \theta s.$$

Proof. By Definition 2.2.7 the internal algebra $\mathcal{A}_{\Omega_N \times \Lambda_N}$ on $\Omega_N \times \mathbb{T}_N$ is generated by the sets $\{A \times \{t\} : t \in \mathbb{T}_N, A \in \mathcal{A}_{N,t}\}$. Using (5.5)

$$\nu_{S_N}(A \times \{t\}) = E\left[1_A(\Delta S_{N,t})^2\right] = E\left[1_A S_{N,t}^2\right] K\Delta_N t$$

for $t \in \mathbb{T}_N, A \in \mathcal{A}_{N,t}$. Since $S_{N,t}$ is non-infinitesimal, $L(Q_N)$-a.s., and square-$\mathcal{S}$-integrable we have

$$\nu_{S_N}(B) \approx 0 \quad \Leftrightarrow \quad (Q_N \times \Lambda_N)(B) \approx 0$$

for $B \in \mathcal{A}_{\Omega_N \times \mathbb{T}_N}$. This implies that Θ lifts θ with respect to ν_{S_N} if and only if ΘS_N lifts θs with respect to $Q_N \times \Lambda_N$. Furthermore,

$$\begin{aligned}
\int_{\Omega_N \times \mathbb{T}_N} \Theta^2 \, d\nu_{S_N} &= E\Big[\sum_{t<T} \Theta^2(\Delta S_{N,t})^2\Big] = E\Big[\sum_{t<T} (\Theta S_{N,t})^2 K\Delta_N t\Big] \\
&= K \int_{\Omega_N \times \mathbb{T}_N} (\Theta S_N)^2 \, d(Q_N \times \Lambda_N)
\end{aligned}$$

by (5.5) and the tower property for conditional expectations. Hence, $\Theta \in \mathcal{S}L^2(\nu_{S_N})$ if and only if $\Theta S_N \in \mathcal{S}L^2(Q_N \times \Lambda_N)$. □

In order to check (5.5) we calculate

$$\begin{aligned}
E\left[(\Delta S_{N,t})^2 \middle| \mathcal{A}_{N,t}\right] &= (S_{N,t})^2 E\left[\left(u^{\omega_t} d^{\beta-\omega_t} - 1\right)^2\right] \\
&= (S_{N,t})^2 \Big(E\left[\left(u^{\omega_t} d^{\beta-\omega_t}\right)^2\right] - 2E\left[u^{\omega_t} d^{\beta-\omega_t}\right] + 1\Big) \\
&= (S_{N,t})^2 \Big(E\left[\left(u^{\omega_t} d^{\beta-\omega_t}\right)^2\right] - 2 \cdot 1 + 1\Big) \qquad (5.6)
\end{aligned}$$

Now

$$\begin{aligned}
E\left[\left(u^{\omega_t} d^{\beta-\omega_t}\right)^2\right] &= \sum_{j=0}^{\beta} \binom{\beta}{j} u^{2j} d^{2(\beta-j)} q^j (1-q)^{\beta-j} \\
&= \left(u^2 q + d^2(1-q)\right)^\beta \\
&= \left(\frac{1}{u-d}\left(u^2 - u^2 d + u d^2 - d^2\right)\right)^\beta \\
&= (u + d - ud)^\beta \\
&= \left(2 + 2\mu \frac{\Delta_N t}{\beta} - \left(1 + (2\mu - \sigma^2)\frac{\Delta_N t}{\beta} + \mu^2(\Delta_N t/\beta)^2\right)\right)^\beta \\
&= \left(1 + \sigma^2 \frac{\Delta_N t}{\beta} - \mu^2(\Delta_N t/\beta)^2\right)^\beta \\
&= 1 + (\sigma^2 + \varepsilon)\Delta_N t
\end{aligned}$$

with $\varepsilon \approx 0$. Substituting into (5.6) we see that S_N satisfies (5.5) with $K = \sigma^2 + \varepsilon$.

We are now in a situation where we can apply the results of Section 4.4, in particular Theorem 4.4.5, so that the valuation of claims and calculation of mean-self-financing trading strategies in the hyperfinite $(\beta+1)$-nomial CRR model is equivalent to the corresponding operations in the BS model.

Convergence Results

As in Section 2.4 we can consider the space $\mathcal{C} = \{x \in C[0,T] : x(0) = 0\}$ with the measure Q together with the finite subspaces $\mathcal{C}_n$ of polygonal paths of W_n and measure Q_n. Using Theorem 2.4.1 together with the definition of D^2-convergence in the context of these spaces we have the following analogue of Theorem 2.4.7:

Theorem 5.1.4. *Let $H_n : \Omega_n \to \mathbb{R}$ be a sequence of claims in the $(\beta+1)$-nomial CRR models and $h \in L^2(Q)$ a claim in the BS model. Then the following are equivalent:*

(i) $H_n \xrightarrow{D^2} h$.

(ii) $\Phi^{H_n} \xrightarrow{D^2} \phi^h$.

(iii) $V(\Phi^{H_n}) \xrightarrow{D^2} V(\phi^h)$.

(iv) $G(\Phi^{H_n}) \xrightarrow{D^2} G(\phi^h)$ *and* $V_0^{H_n} \to V_0^h$.

An important aspect of D^2-convergence is the existence of a discretisation scheme which maps paths in $\mathcal{C}$ back into $\mathcal{C}_n$.

Proposition 5.1.5. *There is an adapted Q-discretisation scheme for the $(\beta+1)$-nomial CRR model.*

Proof. Let $(d_n^\beta)_{n\in\mathbb{N}}$ be the adapted Q-discretisation scheme for the binomial CRR model on Ω_n^β (see Remark 5.1.1 and Theorem 2.4.3), so that $d_n^\beta : \mathcal{C} \to \mathcal{C}_n^\beta$ where $\mathcal{C}_n^\beta := \{W_n^\beta(\omega) : \omega \in \Omega_n^\beta\}$ denotes the path space for the binomial CRR model. Define a map $\tilde{d}_n : \mathcal{C}_n^\beta \to \mathcal{C}_n$ by

$$(\tilde{d}_n(\omega))(t) := \omega(t) \quad \text{for } \omega \in \mathcal{C}_n^\beta \text{ and } t \in \mathbb{T}_n$$

and filling in linearly between points in $\mathbb{T}_n$. So $\tilde{d}_n$ samples paths in $\mathcal{C}_n^\beta$ at points in $\mathbb{T}_n$ and "forgets" what happens between these points. We now define $d_n : \mathcal{C} \to \mathcal{C}_n$ as

$$d_n := \tilde{d}_n \circ d_n^\beta.$$

To see that $(d_n)_{n\in\mathbb{N}}$ is an adapted Q-discretisation scheme we note that d_n is $\mathbb{A}_n$-adapted and Q-measure preserving since d_n^β and $\tilde{d}_n$ are. It only remains to show that

$d_n(\omega) \to \omega$ in Q-probability. Fix $\varepsilon > 0$ and let $\delta > 0$. As $(d_n^\beta)_{n\in\mathbb{N}}$ is a discretisation scheme there exist $n_\delta \in \mathbb{N}$ such that

$$Q\big(\|d_n^\beta(\omega) - \omega\| < \varepsilon/2\big) > 1 - \delta \quad \text{for all } n \geq n_\delta \tag{5.7}$$

(here $\|\cdot\|$ denotes the sup norm on $\mathcal{C}$). Furthermore, $\|\tilde{d}_n(\omega) - \omega\| \leq 2(\beta+1)\sqrt{\Delta_n t/\beta}$ for all $\omega \in \mathcal{C}_n^\beta$. We therefore have $\tilde{n} \in \mathbb{N}$ such that $\|\tilde{d}_n(\omega) - \omega\| < \varepsilon/2$ for all $n \geq \tilde{n}$ and $\omega \in \mathcal{C}_n^\beta$. Hence, for $n \geq \max\{n_\delta, \tilde{n}\}$,

$$\begin{aligned}
Q\big(\|d_n(\omega) - \omega\| < \varepsilon\big) &= Q\big(\|\tilde{d}_n(d_n^\beta(\omega)) - d_n^\beta(\omega) + d_n^\beta(\omega) - \omega\| < \varepsilon\big) \\
&\geq Q\big(\|\tilde{d}_n(d_n^\beta(\omega)) - d_n^\beta(\omega)\| + \|d_n^\beta(\omega) - \omega\| < \varepsilon\big) \\
&\geq Q\big(\varepsilon/2 + \|d_n^\beta(\omega) - \omega\| < \varepsilon\big) \\
&> 1 - \delta \quad \text{by (5.7)},
\end{aligned}$$

which completes the proof. □

Remark 5.1.6. The convergence result in Theorem 5.1.4 could have also been obtained under the minimal martingale measure for the multinomial CRR model as defined in Section 4.2. Therefore our choice of the measure Q_n might seem arbitrary; however, note that the minimal martingale measure depends on the specification of the "physical" probability in the underlying model (we have used the uniform measure in Section 4.2), we may therefore also use a martingale measure from the beginning. Furthermore, our choice of Q_n allows the construction of an adapted discretisation scheme from the already existing one for the binomial CRR model as in the proof of Proposition 5.1.5.

We saw in Theorem 2.4.6 that we can always obtain a sequence of D^2-convergent claims by means of an adapted discretisation scheme (the proof of Theorem 2.4.6 also applies to our situation, cf. [CKW93a]). However, when considering a specific claim h in the BS model a D^2-convergent sequence $(H_n)_{n\in\mathbb{N}}$ approximating h can often be found in a more direct and natural way:

Example 5.1.7. If the claim h only depends on the price of the risky asset at maturity, i.e. $h = f(s_T)$ for some function $f : \mathbb{R} \to \mathbb{R}$ then $H_n := f(S_{n,T})$ is a natural choice of an approximating sequence for h. Indeed, if f is piecewise continuous and satisfies a polynomial growth condition then the same argument as in Section 3.4 shows that $H_n \xrightarrow{D^2} h$.

Example 5.1.8 (Asian options). If $h = \left(\frac{1}{T}\int_0^T s_t dt - K\right)^+$ is an Asian call option with fixed strike price K then the claims $H_n := \left(\frac{1}{T}\sum_{t<T} S_{n,t}\Delta_n t - K\right)^+$ yield a D^2-convergent sequence approximating h. To see this note that, for $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, S_N is $\mathcal{S}$-continuous, i.e. for $\omega \in \tilde{\Omega}_N \subset \Omega_N$ the path $S_{N,\cdot}(\omega)$ is $\mathcal{S}$-continuous where

$L(Q_N)(\tilde{\Omega}_N) = 1$. In particular, $S_{N,\cdot}(\omega)$ is bounded, hence $\mathcal{S}$-integrable with respect to Λ_N. Therefore

$$^\circ\Big(\sum_{t<T} S_{N,t}(\omega)\Delta_N t\Big) = \int_0^T s_t(\omega)\, dt \quad \text{for } \omega \in \tilde{\Omega}_N,$$

which shows that H_N is a lifting of h. By Lemma 5.1.2(iii) $S_{N,t} \in \mathcal{S}L^r(Q_N)$ for any $r \in [1,\infty)$. This implies that also $\sum_{t<T} S_{N,t}\Delta_n t \in \mathcal{S}L^r(Q_N)$ for any $r \in [1,\infty)$. Exactly the same considerations apply to an Asian option with average strike price, i.e. $h = \big(s_T - \frac{K}{T}\int_0^T s_t dt\big)^+$.

Summing up the results of this section we have shown that, for any $\beta \in \mathbb{N}$, the $(\beta+1)$-nomial CRR model has exactly the same convergence properties as the complete binomial CRR model, provided we are using mean-variance hedging for pricing and replicating claims in these models. When using finite models as approximations of the continuous time BS model — especially for numerical purposes — the use of binomial models does therefore not offer any advantage over multinomial models. In fact, we will see in the next chapter that the use of $(\beta+1)$-nomial models with $\beta > 0$ may even preferable.

5.2 Direct Discretisation of the Price Process

In this section we consider a Black-Scholes model on a filtered probability space $(\Omega, \mathcal{F}, P)$ with time-dependent deterministic drift $(\mu_t)_{t\in[0,T]}$ and volatility $(\sigma_t)_{t\in[0,T]}$. We assume that the functions $\mu, \sigma : [0,T] \to \mathbb{R}$ are piecewise continuous and bounded. Furthermore, $\sigma_t > 0$ for all $t \in [0,T]$.

It is well-known that this model is complete (see e.g. [BK98, MR97]), so that there is a unique equivalent martingale measure Q for the price process s, and under this measure s is the solution to the stochastic differential equation

$$\frac{ds_t}{s_t} = \sigma_t\, d\tilde{w}_t$$

where $\tilde{w}$ is a standard Brownian motion under Q, i.e. s is given as

$$s_t = s_0 \exp\Big(-\frac{1}{2}\int_0^t \sigma_u^2\, du + \int_0^t \sigma_u\, d\tilde{w}_u\Big). \tag{5.8}$$

From now on we will work with the measure Q; a filtration $\mathbb{F}$ on $(\Omega, \mathcal{F})$ is generated by $\tilde{w}$. For any $h \in L^2(Q)$ let ϕ^h be the unique self-financing strategy generating h.

We want to define a sequence of discrete time models approximating (5.8) following the "direct discretisation" approach in Section 4.3. However, we first approximate the volatility function σ by a sequence of piecewise constant functions as in [RS95]:

For $n \in \mathbb{N}$ and $\Delta_n t := T/n$ define $\sigma_n : [0,T] \to \mathbb{R}$ by

$$\sigma_n(t) := \sigma(0)\mathbf{1}_{\{0\}}(t) + \sum_{s<t} \sigma(s)\mathbf{1}_{(s,s+\Delta_n t]}(t) \tag{5.9}$$

(with the sum taken over $\mathbb{T}_n = \{0, \Delta_n t, \dots, t\}$). This means that $\sigma_n(t) = \sigma(t)$ for $t \in \mathbb{T}_n$ and σ_n remains constant between points in $\mathbb{T}_n$. So σ_n is left-continuous with right limits, bounded and strictly positive.

We define another Q-martingale $s_n : \Omega \times [0,T] \to \mathbb{R}$ by

$$s_{n,t} := s_0 \exp\Big(-\frac{1}{2}\int_0^t \sigma_{n,u}^2\, du + \int_0^t \sigma_{n,u}\, d\tilde{w}_u\Big)$$

We can then obtain a discrete time process $S_n : \Omega \times \mathbb{T}_n \to \mathbb{R}$ by evaluating s_n at the discretisation points $t \in \mathbb{T}_n$, so that S_n satisfies

$$\begin{aligned} S_{n,t+\Delta_n t} &= S_{n,t} \exp\Big(-\frac{1}{2}\sigma_{n,t}^2 \Delta_n t + \sigma_{n,t}\Delta\tilde{w}_t\Big), \quad t \in \mathbb{T}_n \setminus \{T\}, \\ S_{n,0} &= s_0, \end{aligned} \tag{5.10}$$

where $\Delta\tilde{w}_t := \tilde{w}_{t+\Delta_n t} - \tilde{w}_t \sim \mathcal{N}(0, \Delta_n t)$.

The Discrete Time Model and its Internal Version

We define the *n-th discretised BS model* as follows: Let $\Omega_n := \mathbb{R}^{\mathbb{T}_n \setminus \{T\}}$, $\mathcal{F}_n := \mathcal{B}(\Omega)$ and Q_n the probability defined by $\omega_t \sim \mathcal{N}(0, \Delta_n t)$ and $\omega_0, \dots, \omega_{T-\Delta_n t}$ independent. A filtration $\mathbb{A}_n$ is generated by the process $W_n : \Omega_n \times \mathbb{T}_n \to \mathbb{R}$ with $\Delta W_{n,t} := \omega_t$, $W_{n,0} := 0$. The price process S_n is then defined as in (5.10), with $\Delta\tilde{w}_t$ replaced by $\Delta W_{n,t}$.

The discretised BS model satisfies the assumptions of Section 1.1 (cf. the calculations in Section 3.1), so that any claim $H \in L^2(Q_n)$ can be replicated by a risk-minimising mean-self-financing strategy $\Phi^H = (\Theta^H, \Psi^H)$. Again the variance-optimal strategy ξ^H coincides with Θ^H.

As before, for infinite N, this gives rise to an internal model on $(\Omega_N, \mathbb{A}_N, \mathcal{F}_N, Q_N)$ with associated Loeb space $(\Omega_N, L(\mathcal{F}_N), L(Q_N))$. By Proposition 4.3.1(i) the internal process W_N is $\mathcal{S}$-continuous and $w := \mathrm{st}(W_N)$ is a standard Brownian motion on the Loeb space.

Due to the piecewise continuity of σ we have

$$\sigma_N(t) = {}^*\sigma(t) \approx \sigma({}^\circ t) \quad \text{for } L(\Lambda_N)\text{-a.a. } t \in \mathbb{T}_N,$$

so that σ_N is an $\mathcal{S}$-bounded lifting of σ. Hence

$$\begin{aligned} S_{N,t} &= s_0 \prod_{s<t} \exp\Big(-\frac{1}{2}\sigma_{n,s}^2 \Delta_n t + \sigma_{n,s}\Delta W_{n,s}\Big) \\ &= s_0 \exp\Big(-\frac{1}{2}\sum_{s<t} \sigma_{N,s}^2 \Delta_N t + \sum_{s<t} \sigma_{N,s}\Delta W_{N,s}\Big) \end{aligned}$$

is $\mathcal{S}$-continuous and

$$\operatorname{st}(S_N)_u = s_0 \exp\Big(-\frac{1}{2}\int_0^t \sigma_u^2 du + \int_0^t \sigma_u dw_u\Big).$$

Calculating

$$E\left[S_{N,t}^r\right] = s_0^r \exp\Big(-\frac{1}{2}(r-r^2)\sum_{s<t}\sigma_{N,s}^2 \Delta_N t\Big) \quad \text{for } r \in \mathbb{R}$$

shows that S_N is an $\mathcal{S}L^r$-martingale for any $r \in [1,\infty)$. Finally

$$E\left[(\Delta S_{N,t})^2 \,\middle|\, \mathcal{A}_{N,t}\right] = S_{N,t}^2\big(\exp(\sigma_{N,t}^2 \Delta_N t) - 1\big) = S_{N,t}^2(\sigma_{N,t}^2 + \varepsilon_t)\Delta_N t$$

with $\varepsilon_t \approx 0$ for all $t \in \mathbb{T}_N$, so that we can use the proof of Lemma 5.1.3 to show that assumption (S1) in Section 4.4 is satisfied. Hence Theorem 4.4.5 holds for this internal model and the BS model (5.8); we can now use it to obtain standard convergence results for the sequence of discretised BS models (5.10):

Convergence Results

As in Section 5.1 we assume that $\Omega = \mathcal{C}$ for the BS model and we consider the subspaces $\mathcal{C}_n \subset \mathcal{C}$ of polygonal paths of W_n. We can therefore define D^2-convergence as before and have the following analogue to Theorems 2.4.7 and 5.1.4:

Theorem 5.2.1. *Let $H_n : \Omega_n \to \mathbb{R}$ be a sequence of claims in the discretised BS models and $h \in L^2(Q)$ a claim in the BS model. Then the following are equivalent:*

(i) $H_n \overset{D^2}{\to} h$.

(ii) $\Phi^{H_n} \overset{D^2}{\to} \phi^h$.

(iii) $V(\Phi^{H_n}) \overset{D^2}{\to} V(\phi^h)$.

(iv) $G(\Phi^{H_n}) \overset{D^2}{\to} G(\phi^h)$ *and* $V_0^{H_n} \to V_0^h$.

We will see below how we can obtain an almost trivial adapted Q-discretisation scheme $(d_n)_{n\in\mathbb{N}}$ for these models, so that we also have the alternative characterisation of D^2-convergence in terms of "$L^2(d_n(\cdot))$-convergence" given by Theorem 2.4.4. Note that in this case the spaces $\mathcal{C}_n$ are not finite; however, the proof of Theorem 2.4.4 does not use this assumption (cf. [CKW95, Theorem 5.1]).

Proposition 5.2.2. *The family of mappings $d_n : \mathcal{C} \to \mathcal{C}_n$ defined by*

$$(d_n(\omega))(t) := \omega(t) \quad \text{for } \omega \in \mathcal{C} \text{ and } t \in \mathbb{T}_n, \tag{5.11}$$

with $d_n(\omega)$ filled in linearly between points in $\mathbb{T}_n$, is an adapted Q-discretisation scheme.

Proof. Firstly, d_n is $\mathbb{A}_n$-adapted by definition. Furthermore, $Q(d_n^{-1}(A)) = Q_n(A)$ for all $A \in \mathcal{F}_n$ due to the fact that the finite-dimensional distributions of a Brownian motion are multivariate normal. It therefore only remains to show that $d_n(\omega) \to \omega$ in Q-probability.

Using Proposition 4.3.1(i) once again we see that, for infinite $N \in {}^*\mathbb{N}\backslash\mathbb{N}$, the standard part of the process $d_N : {}^*\mathcal{C} \times \mathbb{T}_N \to {}^*\mathbb{R}$ defined by (5.11) is a standard Brownian motion on $({}^*\mathcal{C}, L({}^*\mathcal{F}), L({}^*Q))$, hence

$$L({}^*Q)\big(\|d_N({}^*\omega) - {}^*\omega\| \approx 0\big) = 1,$$

in particular, for any positive $\varepsilon \in \mathbb{R}$,

$$1 = L({}^*Q)\big(\|d_N({}^*\omega) - {}^*\omega\| < \varepsilon\big) \approx {}^*Q\big(\|d_N({}^*\omega) - {}^*\omega\| < \varepsilon\big).$$

By the nonstandard characterisation of convergence of a sequence in $\mathbb{R}$ this means that

$$Q\big(\|d_n(\omega) - \omega\| < \varepsilon\big) \to 1 \quad \text{as } n \to \infty.$$

□

Remark 5.2.3. For the discretised jump-diffusion model of Section 4.3 we can define a sequence of discrete-time models analogously to the discretised BS model above: We set $\Omega := \mathcal{C} \times \mathcal{D}$ where $\mathcal{D} := \{x \in D[0,T] : x(0) = 0\}$ and define $\Omega_n := \mathcal{C}_n \times \mathcal{D}_n$ where $\mathcal{D}_n$ is the path space for the discretised Poisson process. The measures P_n on Ω_n are given by the joint finite-dimensional distributions of the Brownian motion and the Poisson process. We can then define D^2-convergence for this sequence of models as above.

Proposition 4.3.3 shows that the minimal martingale densities $\hat{Z}_n$ in the discrete-time models are D^2-convergent to the minimal density $\hat{z}$ in the continuous-time model. Furthermore, for any sequence of claims $H_n \in L^2(P_n)$ with $H_n \xrightarrow{D^2} h$ for a claim $h \in L^2(P)$ we have

$$\hat{E}_n[H_n] \to \hat{E}[h] \quad \text{as } n \to \infty.$$

We have therefore obtained an alternative proof of the convergence result in [RS95, Theorem 3] for option prices under the minimal martingale measure[1]. Unfortunately, since the continuous-time model is not complete and assumption (S2) is therefore not satisfied, we cannot use Theorem 4.4.3 in this situation so that a corresponding convergence result for the trading strategies is not available.

[1]The piecewise constant approximation of the time-dependent coefficients in [RS95] is defined as in (5.9).

Chapter 6

Numerical Results

In this chapter we examine the numerical and computational aspects of the discrete time models in Chapters 4 and 5 when these are used to obtain approximations for option prices in the Black-Scholes model

$$s_t = s_0 \exp\Big((\mu - \frac{1}{2}\sigma^2)t + \sigma w_t\Big) \quad t \in [0, T] \tag{6.1}$$

We first consider the case of a European call option and obtain a formula for an approximation in the discretised BS model of Sections 4.3 and 5.2; we can then examine the numerical properties of this approximation. Furthermore, we compare the use of multinomial trees to the standard binomial CRR model and see that there are advantages in using these incomplete models.
Section 6.2 deals with the pricing of path-dependent options in the special case of Asian options. This is of particular importance as these options do not allow for closed-form pricing formulae, so that numerical approximations are required.
The final section of this chapter is an appendix containing the Mathematica source code used for these computations as well as additional graphs and tables illustrating the results.

6.1 European Call Options

We first consider the pricing of a European call option $(s_T - K)^+$ in sequences of incomplete models converging to the complete BS model when prices are calculated according to the minimal martingale measure (cf. Section 4.1). At the end of this section we will see how the convergence is affected if we use the specific martingale measures of Sections 5.1 and 5.2 in the specifications of the models. This analysis is valid for general claims which are only dependent on the price of the risky asset at maturity.

In this section we use the following parameters for the Black-Scholes model (6.1): $\mu = 0.05$, $\sigma = 0.2$, $s_0 = 100$, $T = 0.5$. The option will be priced for different strike prices $K = 90, 100, 110$, i.e. we value in-the-money, at-the-money and out-of-the-money options to see if this influences the numerical convergence results.

Direct Discretisation

We first consider the model introduced in [RS95] (see also Sections 4.3 and 5.2) in the case where the Poisson component is not present, i.e. $\varphi = 0$ in (4.16). It was remarked in [RS95, p.378] that the pricing of options by direct discretisation of the price process and the use of the minimal martingale measure for the resulting incomplete markets provides an alternative approach to obtaining converging approximations of BS prices; we are now examining the practical aspects of this methodology.

For fixed $n \in \mathbb{N}$ denote $\Delta t := T/n$ and $t_i := i\Delta t$ $(i = 0, \ldots, n)$. The discretised price process $S_n = (S_{n,i})_{i=0,\ldots,n}$ is then obtained by setting $S_{n,i} := s_{t_i}$ and $W_{n,i} := w_{t_i}$ in (6.1), so S_n satisfies

$$\frac{S_{n,i+1}}{S_{n,i}} = \exp\left((\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma \Delta W_{n,i}\right) \quad i = 0, \ldots n-1, \quad S_0^n = s_0, \tag{6.2}$$

where $\Delta W_{n,i} := W_{n,i+1} - W_{n,i}$ is a Gaussian random variable with mean 0 and variance Δt. In the following we will work only with the discretised processes S_n and W_n (for fixed n). We therefore drop the subscript n to ease the notation.

From [RS95, equations (2.11)–(2.14)] or the calculations in Section 4.3, page 57, we know that the minimal martingale measure $\hat{P}$ for the price process (6.2) is given by its density

$$\hat{Z} = \prod_{i=0}^{n-1} (1 - \alpha(M_i - 1))$$

where

$$\begin{aligned} \alpha &:= \frac{\exp(\sigma\Delta t) - 1}{\exp(\mu\Delta t)\big(\exp(\sigma^2\Delta t) - 1\big)} \\ M_i &:= \exp(\sigma\Delta W_i - \frac{1}{2}\sigma^2\Delta t) \end{aligned}$$

and the value v_n^H of a contingent claim H is given as

$$v_n^H = E\left[\hat{Z}H\right] = E\left[\prod_{i=0}^{n-1} \big(1 - \alpha(M_i - 1)\big)\, H\right].$$

Following the procedure outlined in [RS95, pp.379–381] we obtain

$$
\begin{aligned}
&\hat{E}\,[H] \\
&= E\left[\prod_{i=0}^{n-1}(1-\alpha(M_i-1))(S_n-K)^+\right] \\
&= S_0\,E\left[\prod_{i=0}^{n-1}(1+\alpha-\alpha M_i)\Big(\frac{S_n}{S_0}-\frac{K}{S_0}\Big)^+\right] \\
&= S_0\,E\left[\prod_{i=0}^{n-1}\big(1+\alpha-\alpha\exp(\sigma\Delta W_i-\frac{1}{2}\sigma^2\Delta t)\big)\Big(\exp\big((\mu-\frac{1}{2}\sigma^2)n\Delta t+\sigma W_n\big)-\frac{K}{S_0}\Big)^+\right] \\
&= S_0\,E\left[\prod_{i=0}^{n-1}(f+g\exp(\sigma\Delta W_i))\Big(\exp(\sigma W_n)-\bar{K}\Big)^+\right], \qquad (6.3)
\end{aligned}
$$

where

$$
\begin{aligned}
f &:= (1+\alpha)\exp\big((\mu-\frac{1}{2}\sigma^2)\Delta t\big) \\
g &:= -\alpha\exp\big(-\frac{1}{2}\sigma^2\Delta t\big)\exp\big((\mu-\frac{1}{2}\sigma^2)\Delta t\big) = -\alpha\exp\big((\mu-\sigma^2)\Delta t\big) \\
\bar{K} &:= \frac{K}{S_0\exp\big((\mu-\frac{1}{2}\sigma^2)n\Delta t\big)} = \frac{K}{S_0\exp\big((\mu-\frac{1}{2}\sigma^2)T\big)}.
\end{aligned}
$$

The product inside the expectation in (6.3) now evaluates to

$$
\prod_{i=0}^{n-1}(f+g\exp(\sigma\Delta W_i)) = \sum_{i=0}^{n}\Big(\sum_{I\in I_i}\Big(f^{n-i}g^i\sum_{k\in I}\exp(\sigma\Delta W_k)\Big)\Big)
$$

where I_i is the set of index sets defined by

$$
I_i = \big\{\{k_1,\dots,k_i\} : k_s \in \{0,\dots,n-1\} \text{ and } k_s \neq k_t \text{ for } s\neq t\big\},
$$

i.e. the inner sums represent all possible selections of exactly i out of n indices. Hence

$$
\begin{aligned}
\hat{E}\,[H] &= S_0\,E\left[\Big(\sum_{i=0}^{n}\Big(\sum_{I\in I_i}\Big(f^{n-i}g^i\sum_{k\in I}\exp(\sigma\Delta W_k)\Big)\Big)\Big)(\exp(\sigma W_n)-\bar{K})^+\right] \\
&= S_0\Big(\sum_{i=0}^{n}\Big(\sum_{I\in I_i}f^{n-i}g^iE\Big[\Big(\sum_{k\in I}\exp(\sigma\Delta W_k)\Big)(\exp(\sigma W_n)-\bar{K})^+\Big]\Big)\Big).
\end{aligned}
$$

However, since the $\{\Delta W_i : i=0,\dots n-1\}$ are i.i.d. random variables the particular selection through the index sets $I\in I_i$ does not affect the expectation. Thus, observing that $W_i=\sum_{k=0}^{i-1}\Delta W_k$,

$$
\hat{E}\,[H] = S_0\sum_{i=0}^{n}\binom{n}{i}f^{n-i}g^iE\left[\exp(\sigma W_i)\,(\exp(\sigma W_n)-\bar{K})^+\right], \qquad (6.4)
$$

so that the problem is reduced to calculating

$$\begin{aligned} B_i &:= E\left[\exp(\sigma W_i)\left(\exp(\sigma W_i + \sigma(W_n - W_i)) - \bar{K}\right)^+\right] \\ &= E\left[e^{G_i}\left(e^{G_i+\tilde{G}_i} - \bar{K}\right)^+\right] \\ &= E\left[e^{G_i}\, E\left[\left(e^{G_i+\tilde{G}_i} - \bar{K}\right)^+ \middle| G_i\right]\right], \end{aligned} \tag{6.5}$$

where, for $i = 1, \dots, n-1$, $G_i := \sigma W_i$ is a Gaussian random variable with mean 0 and variance $\sigma^2 t_i$ and $\tilde{G}_i := \sigma(W_n - W_i)$ is Gaussian with mean 0 and variance $\sigma^2(T - t_i)$. Furthermore, for fixed i, G_i and $\tilde{G}_i$ are independent. In the case of $i = 0$ and $i = n$ we have

$$\begin{aligned} B_0 &= E\left[\left(e^{\sigma W_n} - \bar{K}\right)^+\right] \\ B_n &= E\left[e^{\sigma W_n}\left(e^{\sigma W_n} - \bar{K}\right)^+\right]. \end{aligned}$$

For the inner expectation in (6.5) we have (keeping G_i fixed)

$$\begin{aligned} & E\left[\left(e^{G_i+\tilde{G}_i} - \bar{K}\right)^+\right] \\ &= E\left[\left(e^{G_i}\exp\Big(\frac{1}{2}\sigma^2(T-t_i) - \frac{1}{2}\sigma^2(T-t_i) + \sigma(W_n - W_i)\Big) - \bar{K}\right)^+\right] \\ &= \exp\big(\frac{1}{2}\sigma^2(T-t_i)\big)\,\mathrm{BS}\Big(e^{G_i}, \sigma, \frac{1}{2}\sigma^2, T - t_i, \bar{K}\Big), \end{aligned}$$

where $\mathrm{BS}(S, \sigma, r, T, K)$ denotes the Black-Scholes value

$$\begin{aligned} \mathrm{BS}(S,\sigma,r,T,K) &= E_Q\left[e^{-rT}\left(S\exp(rT - \frac{1}{2}\sigma^2 T + \sigma w_T) - K\right)^+\right] \\ &= S\,\Phi\left(\frac{\log(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\ &\quad - \exp(-rT) K\,\Phi\left(\frac{\log(S/K) - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right). \end{aligned} \tag{6.6}$$

(in (6.6) w_T is $\mathcal{N}(0,T)$-distributed under Q). Therefore B_i $(i = 1, \dots, n-1)$ is given as

$$B_i = \exp\big(\frac{1}{2}\sigma^2(T-t_i)\big)\, E\left[e^{G_i}\,\mathrm{BS}\Big(e^{G_i}, \sigma, \frac{1}{2}\sigma^2, T - t_i, \bar{K}\Big)\right]$$

where $G_i \sim \mathcal{N}(0, \sigma^2 t_i)$. For B_0 we have

$$B_0 = \exp\big(\frac{1}{2}\sigma^2 T\big)\,\mathrm{BS}\Big(1, \sigma, \frac{1}{2}\sigma^2, T, \bar{K}\Big),$$

and a direct calculation (evaluating an integral similar to the one in the original Black-Scholes formula) yields

$$\begin{aligned} B_n &= \exp(2\sigma^2 T)\left(\Phi\Big(\frac{-\log(\bar{K})+2\sigma^2 T}{\sigma\sqrt{T}}\Big) - \exp(\frac{1}{2}\sigma^2 T)\bar{K}\Phi\Big(\frac{-\log(\bar{K})-\sigma^2 T}{\sigma\sqrt{T}}\Big)\right) \\ &= \exp(\sigma^2 T)\,\mathrm{BS}\Big(\exp(\sigma^2 T),\sigma,\frac{1}{2}\sigma^2,T,\bar{K}\Big). \end{aligned}$$

Once $\{B_i : i = 0, \ldots, n\}$ have been evaluated — this has to be done numerically — we obtain an approximation v_n^H for the option value v^H according to (6.4):

$$v_n^H := \hat{E}\,[H] = S_0 \sum_{i=0}^{n} \binom{n}{i} f^{n-i} g^i B_i. \tag{6.7}$$

Remark 6.1.1. Note that $\{B_i : i = 0, \ldots, n\}$ only depend on t_i and the parameters S_0, K, μ, σ and T. In particular, the B_i are independent of n. We can therefore write $B_i = \tilde{B}(t_i)$ where $\tilde{B} : [0, T] \to \mathbb{R}$. Once we have calculated a set of values of $\tilde{B}$ for a given n these values can be re-used for a refinement of the approximation.

The following table lists the approximation values v_n^H for various n together with the absolute and relative errors compared to the Black-Scholes value. These values have been calculated for a strike price $K = 100$ so that the Black-Scholes price (to 4 decimal places) for this option is BS = 5.6372.

$K = 100$, BS = 5.6372			
n	v_n^H	abs. error	rel. error (%)
1	5.4814	0.1558	2.764
5	5.6055	0.0317	0.562
10	5.6212	0.0160	0.284
15	5.6268	0.0104	0.185
17	5.6345	0.0027	0.048
18	8.8097	3.1726	56.278
20	10.5519	4.9147	87.183

We see that the approximations yield good results for relatively small values of n. However, as n increases these deteriorate, so that for $n > 17$ the approximations become unstable; a similar pattern can be observed for different strike prices (see Appendix, page 97). This is due to the coefficients $\binom{n}{i} f^{n-i} g^i$ in (6.7) which increase rapidly in n while the function $\tilde{B}$ remains almost constant on $[0, T]$. Even though $\tilde{B}$ can be evaluated with very high accuracy small errors in this evaluation are magnified by the coefficients which causes numerical instability in the approximation (6.7).

Multinomial Models

We now consider the $(\beta+1)$-nomial model introduced in Section 4.2. Recall that this model was obtained by "skipping" $\beta - 1$ steps in the binomial CRR model ($\beta \in \mathbb{N}$). It is therefore natural to compare the approximation results for various values for β with the corresponding results for the CRR model, i.e. for $\beta = 1$.

In the case of a claim that only depends on the price of the risky asset at maturity, i.e. $H = f(S_T)$ for some $f : \mathbb{R} \to \mathbb{R}$ the price can be calculated by backward recursion through the multinomial tree: An example of a trinomial tree ($\beta = 2, n = 2$) with one-step probabilities $q(0), q(1), q(2)$ is given in Figure 6.1. Once the option value at maturity is known for all possible states (i.e. $H_2(1), \ldots, H_2(5)$ in Figure 6.1, or $H_n(1), \ldots, H_n(n\beta+1)$ for a general n-step $(\beta+1)$-nomial model) then the remaining values $H_t(i)$ can be calculated as

$$H_t(i) = \sum_{j=0}^{\beta} q(j) H_{t+1}(i + (\beta - j)) \tag{6.8}$$

for $t = 0, \ldots, , n-1$ and $i = 1, \ldots, \beta t + 1$.

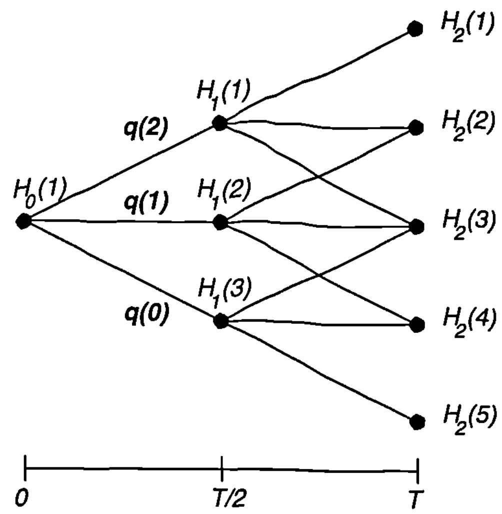


Figure 6.1: Event tree for a 2-step trinomial model

We will again choose the minimal martingale measure as a pricing measure; this is reflected in the probabilities $q(0), \ldots, q(\beta)$ (see pages 91–96 in the Appendix for the implementation of this approach). The results of these approximations for n-step models with $n = 1, \ldots, 100$ for a strike price $K = 100$ are illustrated in Figure 6.2. The continuous horizontal line represents the BS price while the two dotted graphs represent the values obtained by using the binomial CRR model. The results for the trinomial models are given by the dashed line. The corresponding graphs for the strike prices $K = 90, 110$ — showing far more complicated convergence patterns — can be found in the Appendix, page 98.

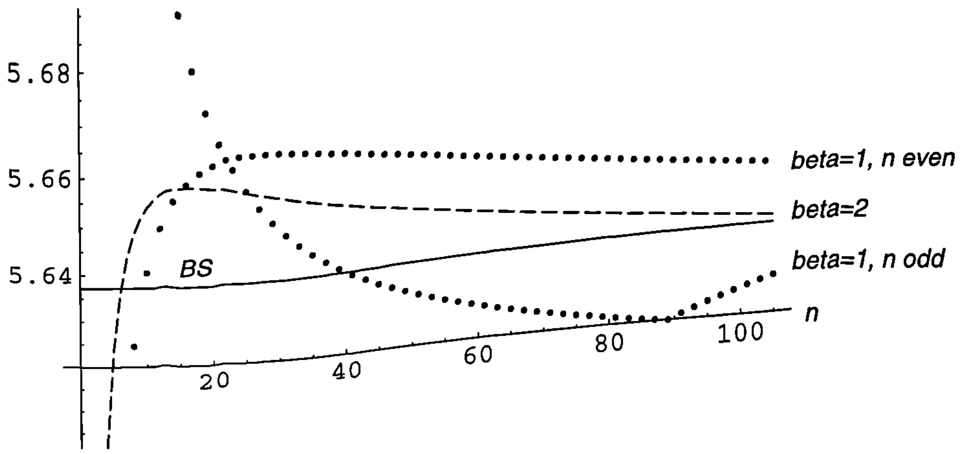


Figure 6.2: European Call Option: Convergence in n for $\beta = 1, 2$, $K = 100$

The most striking feature is the different behaviour of the sub-sequences for odd and even values of n in the case of the binomial model. The trinomial model does not exhibit these "odd-even-ripples". A common technique for smoothing out these ripples is *binomial averaging* where the price is taken to be an average of successive binomial prices V_n (e.g. $\frac{1}{4}(V_{n-1} + 2V_n + V_{n+1})$, see [RS98][1]). However, this requires extra computational effort as V_n has to be evaluated for different n. Furthermore, the resulting average price does not correspond to any pricing methodology: it is usually not the expectation under a martingale measure so that it might allow arbitrage opportunities. These problems can be overcome by using a trinomial model together with the mean-variance hedging approach. We have shown in Section 5.1 that all $(\beta + 1)$-nomial models are equivalent in terms of their convergence to the BS model. We also see from Figure 6.2 that, for fixed n, the prices in the trinomial model usually give better approximations than those for the binomial model. However, we have to take into account that these approximations require more computational operations for the same n since the spatial discretisation of the tree is finer. If we consider multiplications as the computationally most "expensive" operations we have the following formula for the number $C(\beta, n)$ of operations involved in the calculation of an option price in an n-step $(\beta + 1)$-nomial model: It is easy to check that the number $N(\beta, n)$ of nodes in an n-step $(\beta + 1)$-nomial model is

$$N(\beta, n) = (\frac{1}{2}\beta n + 1)(n + 1)$$

[1]This paper also develops an alternative binomial model which is particularly suited for the pricing of barrier options.

and at each node, apart from those at time T, $\beta + 1$ multiplications are required (see (6.8)), hence

$$C(\beta, n) = (\beta+1)N(\beta, n-1) = (\beta+1)n(\frac{1}{2}\beta(n-1)+1).$$

The following Figure 6.3 now shows the option values for trees with $\beta = 1, \ldots, 5$ when plotted relative to $C(\beta, n)$ (see also pages 99–100). We see that the "odd-even" effect which we observed for $\beta = 1$ appears again for $\beta = 3, 5$ — or rather: it *does not* appear for $\beta = 2, 4$ — a possible explanation for these patterns is given below.

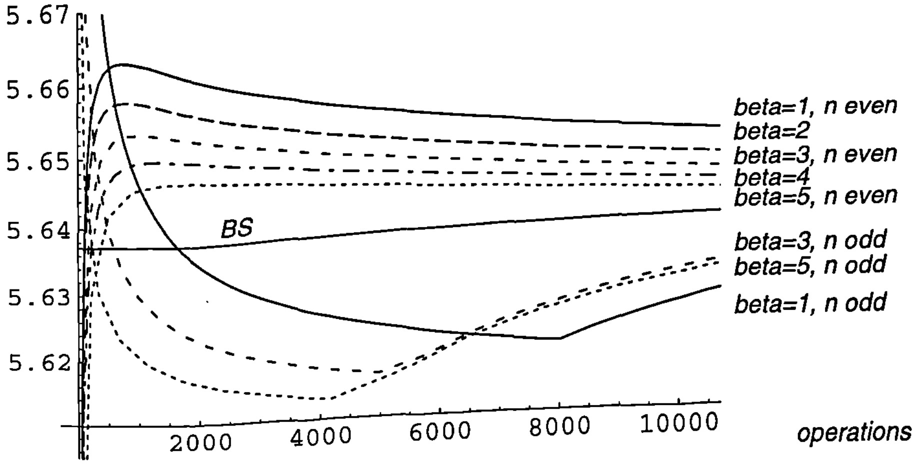


Figure 6.3: Convergence relative to number of operations for $\beta = 1, \ldots, 5$, $K = 100$

We see that even when the results are adjusted for the cost of computation the use of multinomial models with $\beta > 1$ yields better approximations for the option price; however, this effect diminishes with increasing β.

Initial Choice of Measure

So far we have examined the convergence of option prices under the minimal martingale measure for the direct discretisation and multinomial approximation of the BS model. In Chapter 5 we noted that we can also specify the models with a martingale measure for the price process, using the unique martingale measure for the "hidden" model (the BS model for the direct discretisation, the binomial CRR model for the multinomial model). How does this affect the approximation of option prices in these models?

In the case of an option of the form $H = f(S_T)$, i.e. H only depends on the price of the risky asset at maturity, we have the following results:

Proposition 6.1.2. *For the discretised BS model of Section 5.2 we have*

$$v_n^H = v^H \quad \text{for } n \in \mathbb{N},$$

where v^H denotes the BS value of H while v_n^H denotes the value with respect to the discretised model.

Proof. $\log(S_T) \sim \mathcal{N}\big(\log(s_0) - \frac{1}{2}\sigma^2 T, \sigma^2 T\big)$ in the BS model as well as in the discretised model. Since $H = f(S_T)$ we have

$$v_n^H = E_{Q_n}[f(S_T)] = E_Q[f(S_T)] = v^H.$$

□

Proposition 6.1.3. *For the multinomial model of Section 5.1 we have*

$$V_{n,\beta}^H = V_{n\beta,1}^H \quad \text{for } \beta \in \mathbb{N},\ n \in \mathbb{N},$$

where $V_{n\beta,1}^H$ denotes the value of H with respect to an $n\beta$-step binomial CRR model while $V_{n,\beta}^H$ is the value with respect to an n-step $(\beta+1)$-nomial model.

Proof. It is enough to prove the assertion for $n = 1$; the general result then follows by backward induction using the recursion technique in (6.8). Let $H_1(1), \dots, H_1(\beta+1)$ denote the values of the claim at time T in a 1-step model (cf. Figure 6.1). Recall that q is the probability for an "up-movement" in the β-step binomial model on the time line $\{0, T/\beta, 2T/\beta, \dots, T\}$ (cf. Section 5.1), so that $Q(\omega = j) = \binom{\beta}{j} q^j (1-q)^{\beta-j}$ and the price $V_{1,\beta}^H$ in the 1-step multinomial model is given by

$$V_{1,\beta}^H = E_Q[H] = \sum_{i=0}^{\beta} \binom{\beta}{i} q^i (1-q)^{\beta-i} H_1(\beta - i + 1).$$

In the β-step binomial model with the same terminal values as before $H_\beta(1) = H_1(1), \dots, H_\beta(\beta+1) = H_1(\beta+1)$ the price $V_{\beta,1}^H$ is calculated recursively (see equation (6.8)):

$$H_t(i) := qH_{t+1}(i) + (1-q)H_{t+1}(i+1)$$

for $t = 0, \dots, \beta - 1$, $i = 1, \dots, t+1$ and

$$V_{\beta,1}^H := H_0(1).$$

It then follows by induction on β that

$$V_{\beta,1}^H = H_0(1) = \sum_{i=0}^{\beta} \binom{\beta}{i} q^i (1-q)^{\beta-i} H_n(\beta - i + 1) = V_{1,\beta}^H.$$

□

If the binomial CRR model exhibits the "odd-even-ripples" for option prices as observed in Figure 6.2 then it is clear from Proposition 6.1.3 that a $(\beta+1)$-nomial model will "inherit" this effect for odd β while it will not appear for even β.
Furthermore, even though the option price for an n-step $(\beta+1)$-nomial model can also be obtained by using a binomial $n\beta$-step model the former calculation is more efficient as

$$C(\beta,n) = (\beta+1)n\Big(\frac{1}{2}\beta(n-1)+1\Big) < n\beta(n\beta+1) = C(1,n\beta)$$

for $\beta > 1$. This is due to the fact that the products $q^j(1-q)^{\beta-j}$ for the probabilities in the multinomial tree are only calculated once while they are evaluated repeatedly in the binomial tree.

Remark 6.1.4. While we see from Propositions 6.1.2 and 6.1.3 that the option prices in these models coincide the replicating strategies for the claim H will be different: the number of hedging dates differs for different values of n and, while the BS and CRR model allow self-financing replicating strategies, the corresponding strategies in the discretised and multinomial model will only be mean-self-financing.

6.2 Asian Options

For claims of the form $H = f(S_T)$ Propositions 6.1.2 and 6.1.3 show that the prices in the direct discretisation and multinomial models can also be obtained by using the underlying BS and CRR models if we use the "hidden" martingale measure in the specification of the incomplete models. This is no longer true if we consider claims whose values depend on the *entire path* of the risky asset in these models. In this section we consider the case of an Asian call option with fixed strike price, i.e.

$$h = \Big(\frac{1}{T}\int_0^T s_u du - K\Big)^+$$

in the continuous-time BS model. We saw in Example 5.1.8 that a discrete-time approximation for this claim is given by

$$H = \Big(\frac{1}{T}\sum_{i=0}^{n-1} S_i\Delta t - K\Big)^+, \tag{6.9}$$

where $(S_i)_{i=0,\dots n}$ is the price process on $\{i\Delta t : i = 0,\dots,n\}$ with $\Delta t := T/n$.

Direct Discretisation

We now work in the setting of the discretised BS model of Section 5.2, so that the process $(S_i)_{i=0,\dots,n}$ is given by

$$S_i = S_0 \prod_{k=0}^{i-1} \exp\Big(-\frac{1}{2}\sigma^2\Delta t + \sigma\Delta W_k\Big)$$

with $\Delta W_k \sim \mathcal{N}(0,\Delta t)$ for $k = 0,\dots,n-1$. Then

$$\begin{aligned}
E[H] &= E\left[\left(\frac{\Delta t}{T}\sum_{i=0}^{n-1} S_i - K\right)^+\right] \\
&= \frac{\Delta t}{T} S_0\, E\left[\left(\sum_{i=0}^{n-1} \exp(-\frac{1}{2}\sigma^2 i\Delta t)\exp\Big(\sigma\sum_{k=0}^{i-1}\Delta W_k\Big) - \tilde{K}\right)^+\right] \\
&= \frac{\Delta t}{T} S_0\, E\left[\left(\sum_{i=1}^{n-1} a_i \exp\Big(\sigma\sum_{k=0}^{i-1}\Delta W_k\Big) - \tilde{K} + 1\right)^+\right],
\end{aligned}$$

where $\tilde{K} := KT/(S_0\Delta t)$ and $a_i := \exp(-\frac{1}{2}\sigma^2 i\Delta t)$ for $i = 1,\dots,n-1$. Setting $\bar{K} := \tilde{K}+1$ and $G_k := \sigma\Delta W_k$ for $k = 0,\dots,n-1$ we have

$$\begin{aligned}
&E[H] \\
&= \frac{\Delta t}{T} S_0\, E\left[\left(\exp(G_0)\sum_{i=1}^{n-1} a_i \exp\Big(\sum_{k=1}^{i-1} G_k\Big) - \bar{K}\right)^+\right] \\
&= \frac{\Delta t}{T} S_0\, E\left[E\left[\left(\exp(-\frac{1}{2}\sigma^2\Delta t + \sigma W_0)\sum_{i=1}^{n-1} a_{i-1} \exp\Big(\sum_{k=1}^{i-1} G_k\Big) - \bar{K}\right)^+ \,\middle|\, G_1,\dots,G_{n-1}\right]\right] \\
&= \frac{\Delta t}{T} S_0\, E\left[\mathrm{BS}\left(\sum_{i=1}^{n-1} \exp(-\frac{1}{2}\sigma^2(i-1)\Delta t)\exp\Big(\sum_{k=1}^{i-1} G_k\Big), \sigma, 0, \Delta t, \bar{K}\right)\right] \qquad (6.10)
\end{aligned}$$

where $(G_k)_{k=1,\dots,n-1}$ are i.i.d. $\mathcal{N}(0,\sigma^2\Delta t)$. The approximate value v_n^H is now given as an average of Black-Scholes values with respect to a multivariate normal distribution. The integral in (6.10) is difficult to evaluate numerically and is usually computed using Monte-Carlo simulations (cf. [Hul97, TW91]). We will now take a brief look at multinomial trees as an alternative approximation method.

Multinomial Models

For the evaluation of path-dependent options in an n-step $(\beta+1)$-nomial model it is necessary to keep track of all $(\beta+1)^n$ possible paths of the price process. This makes it practically impossible to obtain approximations for large values of n. We list the results of some calculations for binomial and trinomial models in the table below. These are compared to the results obtained by Monte-Carlo simulation in [TW91][2]. The model parameters are as follows: $T = 1/3$, $\sigma = 0.2$, $K = 95, 100, 105$. We also incorporate a constant interest rate $r = 0.09$ in order to achieve comparable results to those in [TW91] where an average over 120 subsequent asset prices was calculated.

[2]This paper also develops an alternative algorithm for the pricing of Asian options which uses an approximation to the distribution of the average of a collection of lognormal random variables. This algorithm produces an accuracy similar the Monte-Carlo method.

n	$K = 95$	$K = 100$	$K = 105$	paths[3]
binomial tree ($\beta = 1$)				
10	5.491	2.412	0.775	512
11	5.505	2.429	0.787	1024
12	5.516	2.440	0.796	2048
13	5.526	2.452	0.805	4069
14	5.534	2.461	0.811	8192
15	5.541	2.469	0.817	16384
16	5.548	2.476	0.823	32786
17	5.553	2.482	0.827	65536
18	5.558	2.487	0.832	131072
trinomial tree ($\beta = 2$)				
6	5.404	2.274	0.677	243
7	5.433	2.321	0.708	729
8	5.461	2.355	0.732	2187
9	5.481	2.379	0.751	6561
10	5.496	2.398	0.766	19683
11	5.509	2.416	0.779	59049
Monte-Carlo simulation				
120	6.80	3.36	1.31	n/a

We see that the results for small values of n cannot be used as acceptable approximations (in contrast to the case of European call options where a relative error of $< 1\%$ can be achieved in binomial models for $n < 15$). In fact, the sequence of approximate values seems to be increasing very slowly in n while the number of paths grows exponentially. Furthermore, the use of trinomial models does not offer any advantage in this case: the results relative to the number of paths are even slightly worse than those in the binomial model.

6.3 Appendix

This section contains the Mathematica 3.0 Notebook (pages 91–96) which was used for the numerical calculations in this chapter as well as for generating the graphs. All calculations were performed on a SPARCstation 10 under SunOS 4.1.
Further numerical results and graphs can be found on pages 97–100.

[3] Note that the number of paths is only $(\beta + 1)^{n-1}$ since the price S_T is not required in (6.9).

Init

```
In[1]:= << Statistics`NormalDistribution`

In[2]:= ndist = NormalDistribution[0, 1]

Out[2]= NormalDistribution[0, 1]

In[3]:= BS[S_, sig_, r_, T_, K_] := S * CDF[ndist,
          (Log[S / K] + (r + (sig^2) / 2) * T) / (sig * Sqrt[T])] - Exp[-r * T] *
        K * CDF[ndist, (Log[S / K] + (r - (sig^2) / 2) * T) / (sig * Sqrt[T])]

In[4]:= BS[S, σ, r, T, K]
```

Out[4]= $-\frac{1}{2} E^{-r T} K \left(1 + \mathrm{Erf}\left[\frac{T\left(r - \frac{\sigma^2}{2}\right) + \mathrm{Log}\left[\frac{S}{K}\right]}{\sqrt{2}\sqrt{T}\,\sigma}\right]\right) + \frac{1}{2} S \left(1 + \mathrm{Erf}\left[\frac{T\left(r + \frac{\sigma^2}{2}\right) + \mathrm{Log}\left[\frac{S}{K}\right]}{\sqrt{2}\sqrt{T}\,\sigma}\right]\right)$

Definition of model parameters (call option)

```
In[5]:= μ = 0.05; σ = 0.2; T = 0.5; S0 = 100;

In[6]:= K = 100;

In[7]:= n := 20;

In[8]:= N[BS[S0, σ, 0, T, K]]

Out[8]= 5.6372

In[9]:= dt := T / n

In[10]:= t[i_] := dt * i
```

"Direct Discretisation" Evaluation

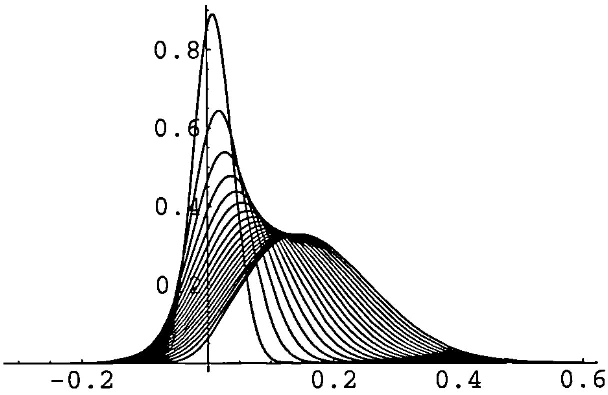
```
In[11]:= Q := (Exp[μ * dt] - 1) / (Exp[μ * dt] * (Exp[σ^2 * dt] - 1))

In[12]:= f := (1 + Q) * Exp[(μ - (1 / 2) * σ^2) * dt]

In[13]:= g := -Q * Exp[(μ - σ^2) * dt]

In[14]:= Kb = K / (S0 * Exp[(μ - (1 / 2) * σ^2) * T]);
```

```
In[15]:= Plot[Evaluate[Table[Exp[x] * BS[Exp[x], σ, (1 / 2) * σ^2, T - vt, Kb] *
             Exp[- (x^2) / (2 * σ^2 * vt)] / (Sqrt[2 * Pi * vt] * σ),
            {vt, t[1], t[n - 1], dt}]],
          {x, -0.3, 0.6}, PlotRange -> All];
```



To calculate B[t] the numerical evaluation of the integral can be restricted to the part of the real line indicated in the graph above. This increases the numerical precision. The following definition of B[t] stores the values once they have been calculated, so that they can be re–used for a refinement of the time line.

```
In[16]:= B[vt_] := B[vt] = Exp[(1 / 2) * σ^2 * (T - vt)] *
             NIntegrate[Exp[x] * BS[Exp[x], σ, (1 / 2) * σ^2, T - vt, Kb] *
               Exp[- (x^2) / (2 * σ^2 * vt)] / (Sqrt[2 * Pi * vt] * σ),
              {x, -0.5, 1}]

In[17]:= B[0] = Exp[(1 / 2) * σ^2 * T] * BS[1, σ, (1 / 2) * σ^2, T, Kb];

In[18]:= B[T] = Exp[σ^2 * T] * BS[Exp[σ^2 * T], σ, (1 / 2) * σ^2, T, Kb];

In[19]:= apprdir[vn_] := Module[{numH},
            n = vn;
            numH = S0 * Sum[Binomial[n, i] * f^ (n - i) * g^i * B[t[i]], {i, 0, n}];
            Print["n=", n, "  approx. value: ", numH, "  absolute error: ",
             Abs[numH - BS[S0, σ, 0, T, K]], "  relative error: ",
             100 * Abs[numH - BS[S0, σ, 0, T, K]] / BS[S0, σ, 0, T, K], "%"];
            res = Append[res, {n, numH, Abs[numH - BS[S0, σ, 0, T, K]],
               100 * Abs[numH - BS[S0, σ, 0, T, K]] / BS[S0, σ, 0, T, K]}];]

In[20]:= res = {};
```

Running the evaluation for n=1,...,25 an storing the results for later use (calculation aborted for presentation).

```
In[21]:= Do[apprdir[i], {i, 1, 25}];

        n=1  approx. value: 5.48137   absolute error: 0.155823   relative error:
         2.7642%

        n=2  approx. value: 5.5581   absolute error: 0.0790986   relative error:
         1.40315%

        n=3  approx. value: 5.58437   absolute error: 0.0528317   relative error:
         0.937198%

Out[21]= $Aborted

In[22]:= res >> resdir90.math
```

Multinomial approximation

Basic coefficients in the multinomial model

```
In[23]:= β = 1;
```

```
In[24]:= dtβ := dt / β
```

```
In[25]:= uβ := 1 + μ * dtβ + σ * Sqrt[dtβ]
```

```
In[26]:= dβ := 1 + μ * dtβ - σ * Sqrt[dtβ]
```

```
In[27]:= mβ[i_] := uβ ^ i * dβ ^ (β - i)
```

```
In[28]:= Eβ := (1 + μ * dtβ) ^ β
```

```
In[29]:= VARβ := ((1 + μ * dtβ) ^ 2 + σ ^ 2 * dtβ) ^ β - (1 + μ * dtβ) ^ (2 * β)
```

Path–independent (backward evaluation)

The one–step probabilities p(i) (original measure) and q(i) (minimal martingale measure) are indexed by the number of up–movements i (i can take values from 0 to β) in the "hidden" binomial model.

```
In[30]:= pstep[i_] := Binomial[β, i] / 2 ^ β
```

```
In[31]:= qstep[i_] := 1 - ((Eβ - 1) / VARβ) * (mβ[i] - Eβ)
```

The possible "states of the world" in the path–independent setting are numbered according to the total number of up–movements j (j can take values from 0 to nβ) in the "hidden" binomial model.

```
In[32]:= mtotal[j_] := uβ ^ j * dβ ^ (n * β - j)
```

```
In[33]:= Tvalue[j_] := Max[{S0 * mtotal[j] - K, 0}]
```

Initialisation of the tree with the terminal values of the option, then backward evaluation of expecatations under the minimal martingale measure. This module changes the global variables n and β:

```
In[34]:= approxβ[vn_, vb_] := Module[{H, numH},
          n = vn; β = vb;
          H = Table[0, {i, 1, (n / 2) * (β * n + β + 2) + 1}];
          Do[H[[(β / 2) * (n - 1) * n + n + 1 + i]] = Tvalue[i], {i, 0, n * β}];
          Do[
            Do[H[[k]] = Sum[pstep[i] * qstep[i] * H[[k + β * l + 1 + i]], {i, 0, β}],
             {k, (β / 2) * (l - 1) * l + l + 1, (β / 2) * l * (l + 1) + l + 1}],
            {l, n - 1, 0, -1}];
          numH = N[H[[1]]];
          Print["n=", n, "  β=",
          β, "  approx. value: ", numH, "   absolute error: ",
          Abs[numH - BS[S0, σ, 0, T, K]], "   relative error: ",
          100 * Abs[numH - BS[S0, σ, 0, T, K]] / BS[S0, σ, 0, T, K],
          "%", "   operations: ", (β + 1) * n * ((β / 2) * (n - 1) + 1)];
          res = Append[res, {n, β, numH, Abs[numH - BS[S0, σ, 0, T, K]],
            100 * Abs[numH - BS[S0, σ, 0, T, K]] / BS[S0, σ, 0, T, K],
            (β + 1) * n * ((β / 2) * (n - 1) + 1)}];]
```

Example of one evaluation. The list of results for a range of values for n can be generated and strored as in the case of the direct discretisation above.

```
In[35]:= approxβ[5, 2]

        n=5  β=2  approx. value: 5.62382   absolute error: 0.0133732
            relative error: 0.237231%   operations: 75
```

Generating graphs

The results of the multinomial approximations are stored externally. These files are available from the author on request. The following intructions generate a plot of the results relative to the number of operations for β=1,...5.

```
In[36]:= res2 = << res2k100.math;

In[37]:= res3 = << res3k100.math;

In[38]:= res4 = << res4k100.math;

In[39]:= res5 = << res5k100.math;

In[40]:= res6 = << res6k100.math;

In[41]:= t2even = Table[{res2[[i, 6]], res2[[i, 3]]}, {i, 2, 105, 2}];
         t2odd = Table[{res2[[i, 6]], res2[[i, 3]]}, {i, 1, 105, 2}];
         t4even = Table[{res4[[i, 6]], res4[[i, 3]]}, {i, 2, 44, 2}];
         t4odd = Table[{res4[[i, 6]], res4[[i, 3]]}, {i, 1, 44, 2}];
         t6even = Table[{res6[[i, 6]], res6[[i, 3]]}, {i, 2, 28, 2}];
         t6odd = Table[{res6[[i, 6]], res6[[i, 3]]}, {i, 1, 28, 2}];
         t3 = Table[{res3[[i, 6]], res3[[i, 3]]}, {i, 1, 60}];
         t5 = Table[{res5[[i, 6]], res5[[i, 3]]}, {i, 1, 33}];
```

```
In[42]:= pl2even =
          ListPlot[t2even, PlotJoined -> True, DisplayFunction -> Identity];
         pl2odd =
          ListPlot[t2odd, PlotJoined -> True, DisplayFunction -> Identity];
         pl3 = ListPlot[t3, PlotJoined -> True,
           PlotStyle -> Dashing[{0.02, 0.01}], DisplayFunction -> Identity];
         pl4even = ListPlot[t4even, PlotJoined -> True,
           PlotStyle -> Dashing[{0.01, 0.02}], DisplayFunction -> Identity];
         pl4odd = ListPlot[t4odd, PlotJoined -> True,
           PlotStyle -> Dashing[{0.01, 0.02}], DisplayFunction -> Identity];
         pl5 = ListPlot[t5, PlotJoined -> True, PlotStyle ->
            Dashing[{0.005, 0.015, 0.02, 0.015}], DisplayFunction -> Identity];
         pl6even = ListPlot[t6even, PlotJoined -> True,
           PlotStyle -> Dashing[{0.005, 0.01}], DisplayFunction -> Identity];
         pl6odd = ListPlot[t6odd, PlotJoined -> True,
           PlotStyle -> Dashing[{0.005, 0.01}], DisplayFunction -> Identity];
         plBS =
          Plot[BS[S0, σ, 0, T, K], {x, 0, 11000}, DisplayFunction -> Identity];

In[43]:= Show[pl2even, pl2odd, pl4even, pl4odd, pl6even, pl6odd, pl3, pl5,
          plBS, PlotRange -> {5.605, 5.67}, DisplayFunction -> $DisplayFunction]

Out[43]= - Graphics -
```

Path-dependent evaluation

Redefining parameters:

```
In[44]:= n = 5; β = 1;

In[45]:= σ = 0.2; T = 0.333; s0 = 100;

In[46]:= K = 95;

In[47]:= r := 0.09;

In[48]:= dt := T / n; dtβ := dt / β;
```

Generating the sample space. Note: since the asset price at maturity does not enter the average each ω has only n–1 components.

```
In[49]:= Ω = Table[IntegerDigits[i, β + 1, (n - 1)], {i, 0, (β + 1) ^ (n - 1) - 1}]

Out[49]= {{0, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}, {0, 0, 1, 1}, {0, 1, 0, 0},
          {0, 1, 0, 1}, {0, 1, 1, 0}, {0, 1, 1, 1}, {1, 0, 0, 0}, {1, 0, 0, 1},
          {1, 0, 1, 0}, {1, 0, 1, 1}, {1, 1, 0, 0}, {1, 1, 0, 1}, {1, 1, 1, 0},
          {1, 1, 1, 1}}
```

Definition of the up- and down-steps; we work with the "hidden" CRR martingale measure from the beginning; we therefore take the interest rate as a drift coefficient.

```
In[50]:= u := 1 + r * dtβ + σ * Sqrt[dtβ]; d := 1 + r * dtβ - σ * Sqrt[dtβ];

In[51]:= qstep := (1 - d) / (u - d)
```

Generating measure on Ω:

```
In[52]:= Qmeas = Table[Product[Binomial[β, Ω[[i, j]]] *
             qstep^Ω[[i, j]] * (1 - qstep) ^ (β - Ω[[i, j]]), {j, 1, n - 1}],
           {i, 1, Length[Ω]}]

Out[52]= {0.0969932, 0.0768093, 0.0768093, 0.0608256, 0.0768093, 0.0608256,
          0.0608256, 0.048168, 0.0768093, 0.0608256, 0.0608256, 0.048168,
          0.0608256, 0.048168, 0.048168, 0.0381444}

In[53]:= mbeta[i_] := u^i * d^(β - i)

In[54]:= pathsum[omega_] := Module[{p},
           p = 1;
           Do[p = p * mbeta[omega[[i]]] + 1,
             {i, n - 1, 1, -1}
             ];
           s0 * p]
```

Calculating the option value for one path ω:

```
In[55]:= value[omega_] := Max[{pathsum[omega] / n - K, 0}] / ((1 + r * dtβ) ^ (n * β))

In[56]:= approxval = Sum[value[Ω[[i]]] * Qmeas[[i]], {i, 1, Length[Ω]}]

Out[56]= 5.34918
```

Everything in one module (this changes the global variables n,β again):

```
In[57]:= approxAsia[vn_, vb_] := Module[{Ω, Qmeas},
           n = vn; β = vb;
           Ω = Table[IntegerDigits[i, β + 1, (n - 1)], {i, 0, (β + 1) ^ (n - 1) - 1}];
           Qmeas = Table[Product[Binomial[β, Ω[[i, j]]] * qstep^Ω[[i, j]] *
                (1 - qstep) ^ (β - Ω[[i, j]]), {j, 1, n - 1}], {i, 1, Length[Ω]}];
           approxval = Sum[value[Ω[[i]]] * Qmeas[[i]], {i, 1, Length[Ω]}];
           Print["n=", n, "  β=", β,
            "  approx. value: ", approxval, "   paths: ", Length[Ω]];
            res = Append[res, {n, β, approxval, Length[Ω]}];]
```

$K = 90$, BS $= 11.7725$			
n	v_n^H	abs. error	rel. error (%)
1	11.6732	0.0992	0.843
5	11.7502	0.0222	0.189
10	11.7610	0.0114	0.097
15	11.7645	0.0079	0.067
16	12.1159	0.3435	2.918
20	18.2353	6.4628	54.898

Table 6.1: European Call Option (strike price $K = 90$): Approximation by direct discretisation

$K = 110$, BS $= 2.2113$			
n	v_n^H	abs. error	rel. error (%)
1	2.0656	0.1456	6.585
5	2.1841	0.0271	1.224
10	2.1978	0.0135	0.601
15	2.2037	0.0076	0.342
18	2.1343	0.0770	3.482
20	6.6003	4.3891	198.490

Table 6.2: European Call Option (strike price $K = 110$): Approximation by direct discretisation

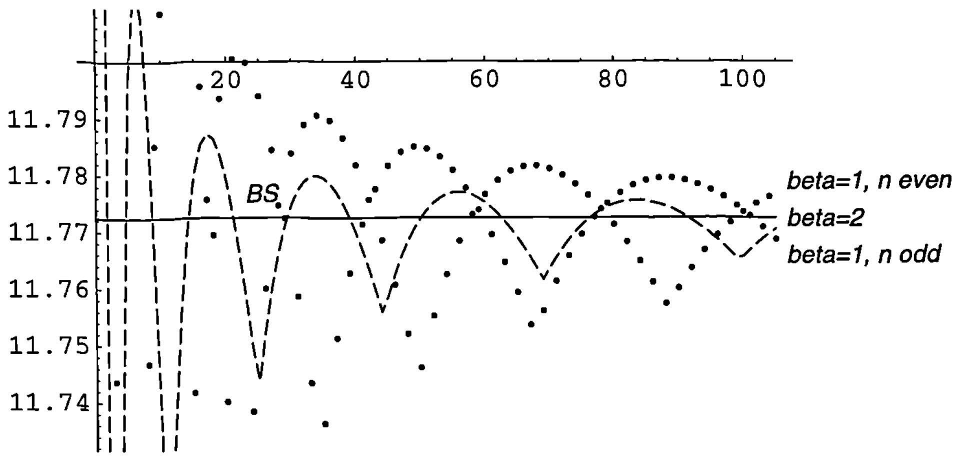


Figure 6.4: European Call Option: Convergence in n for $\beta = 1, 2$, $K = 90$

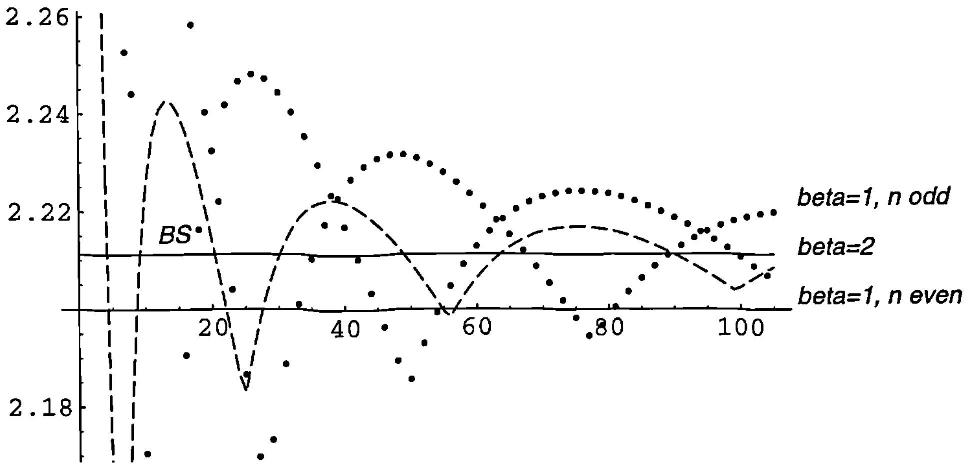


Figure 6.5: European Call Option: Convergence in n for $\beta = 1, 2$, $K = 110$

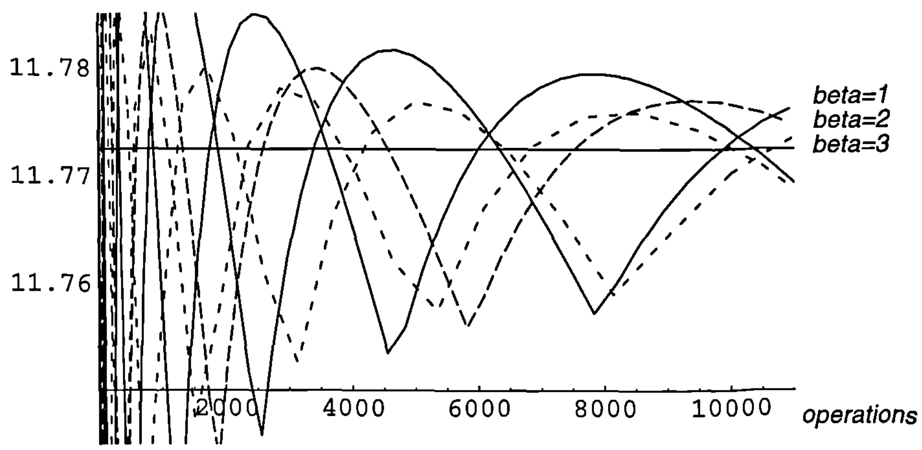


Figure 6.6: Convergence relative to number of operations for $\beta = 1, 2, 3$, $K = 90$

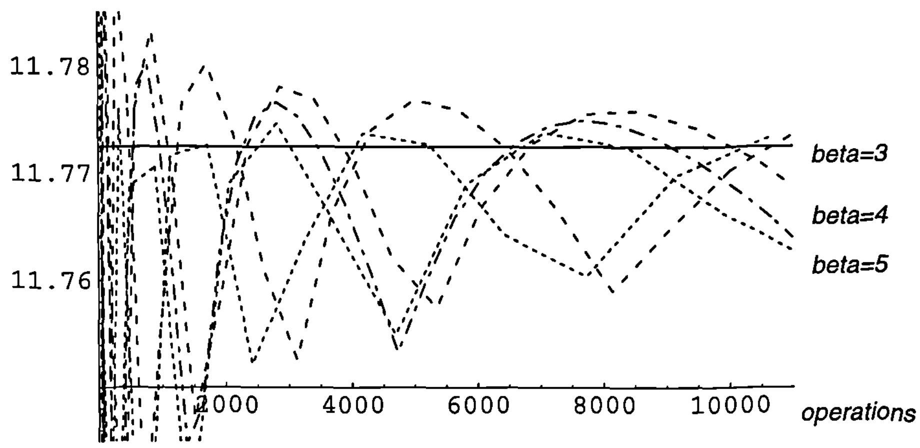


Figure 6.7: Convergence relative to number of operations for $\beta = 3, 4, 5$, $K = 90$

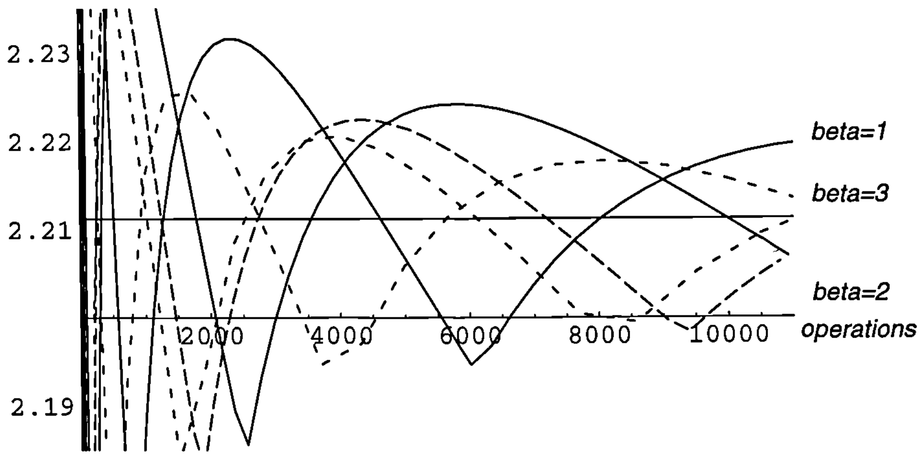


Figure 6.8: Convergence relative to number of operations for $\beta = 1, 2, 3$, $K = 110$

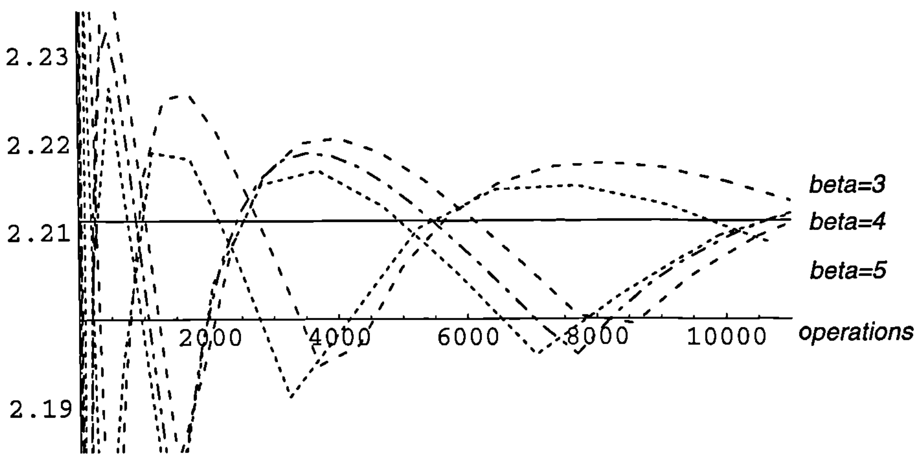


Figure 6.9: Convergence relative to number of operations for $\beta = 3, 4, 5$, $K = 110$

Conclusions

We have extended the applications of nonstandard analysis in mathematical finance to incomplete markets. We have shown that the nonstandard notion of a lifting and its corresponding standard concept of D^2-convergence provide powerful tools in the study of convergence of market models. These allowed us to extend recent results on the convergence of option prices in incomplete market models to trading strategies and value processes. Establishing the convergence of risk-minimal strategies is of particular practical importance in the risk-management of financial instruments.
We have shown that the methodology of mean-variance hedging (and the related method of variance-optimal pricing) in incomplete markets has an appealing stability property under convergence from discrete- to continuous-time trading which should increase its acceptability. At the same time we have given further evidence of the suitability of D^2-convergence as a mode of convergence for financial market models. We have demonstrated the application of these techniques in two alternative approximations of the Black-Scholes model:

Multinomial models combined with the mean-variance hedging methodology have been shown to be equivalent to the usual binomial approach in terms of their convergence properties. When comparing these models under computational and numerical aspects we saw that the use of trinomial models (or indeed general $(\beta+1)$-nomial models for $\beta > 1$) has advantages over binomial models. In fact, trinomial models are widely used for numerical approximations of option prices (see e.g. [Hul97, pp.360,376–378] for a discussion and their relation to explicit finite difference methods for numerical solutions of the Black-Scholes PDE), however without an underlying pricing and hedging theory. Our results now provide a rigorous justification together with explicit formulae for the calculation of risk-minimising replicating strategies which are convergent to the risk-free strategies in the limit model.

On the other hand the "direct discretisation" approach provides a very appealing description of D^2-convergence in terms of a simple discretisation scheme, avoiding the technicalities of mapping the paths of a Brownian motion back into discrete-space random walks. While the resulting pricing formulae may not be suitable for standard numerical integration techniques, direct discretisation is the underlying idea for the use of Monte-Carlo simulations in the evaluation of path-dependent options, and we are again able to provide a theoretical framework for these techniques. In fact, we saw in the case of Asian options that multinomial trees are of limited use due to their computational complexity while direct discretisation methods lead to the numerical evaluation of high-dimensional integrals which are not feasible without the application of some randomisation technique.

Open Problems and Suggestions for Further Work

Regarding the nonstandard results in Chapter 4 there are two major open questions which could be addressed by further research:

(1) Can the assumptions in Corollary 4.1.3 be weakened, i.e. under which assumptions does the lifting property of the price process imply the lifting property of the minimal density? A first step in this direction is an analysis of the Doob-Meyer decomposition

$$s = s_0 + a + m$$

of a semimartingale s that has a lifting S. We can calculate an internal decomposition

$$S = S_0 + A + M$$

of S with

$$\Delta A_t := E\left[\Delta S_t \middle| \mathcal{A}_t\right], \quad \Delta M_t := \Delta S_t - \Delta A_t.$$

How is the internal martingale M related to m (similarly for A and a)? In the case of an $\mathcal{S}$-continuous process S it would be sufficient to show that A is $\mathcal{S}$-continuous; the uniqueness of the standard Doob-Meyer decomposition would then imply that $\mathrm{st}(A) = a$ and $\mathrm{st}(M) = m$. However, even in this special case such a result is not available.

(2) Is Theorem 4.4.3 still true if assumption (S2) is no longer satisfied, i.e. if the claim h has a Kunita-Watanabe decomposition

$$h = v_0 + \int_0^T \theta_u ds_u + l_T$$

with a non-zero martingale l orthogonal to s? The natural way to extend the proof of Theorem 4.4.3 is by constructing a lifting L of s which is internally orthogonal to S. The resulting term in the extension of equation (4.25) would then again be zero. A suitable candidate for such a lifting L could be found by first choosing a general lifting $\bar{L}$ of l and then decomposing

$$\bar{L}_T = \sum_{t<T} \Gamma_t \Delta S_t + L_T, \quad \text{where} \quad \Gamma_t = \frac{E\left[\Delta \bar{L}_t \Delta S_t \middle| \mathcal{A}_t\right]}{E\left[(\Delta S_t)^2 \middle| \mathcal{A}_t\right]}.$$

Then L is orthogonal to S as required. It then has to be shown that the integral $\sum \Gamma \Delta S$ is infinitesimal. Even though $\langle \mathrm{st}(\bar{L}), \mathrm{st}(S)\rangle = \langle l, s\rangle = 0$ it is not clear if this implies that the integrand Γ is infinitesimal. This problem is therefore related to question (1) above where we are trying to relate the internal Doob-Meyer decomposition of an internal process to its standard counterpart.

The pricing of American options in incomplete markets is a topic of ongoing research. Given the successful application of D^2-convergence to the pricing of American options in the CRR and BS models in [CKWW97] one could examine whether it is possible to extend these results to incomplete markets by using the results in [Sch88, Chapter III]. Prices for American options and their optimal exercise times are usually calculated by means of binomial tree models. An extension of our results on multinomial trees could then provide more efficient approximations as demonstrated in the case of European style options.

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