THE UNIVERSITY OF HULL

THE EDUCATIONAL VALIDITY OF VISUAL GEOMETRY

being a Thesis submitted for the Degree of

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in the University of Hull

by

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M.Litt (Aberdeen, U.K., 1986)

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OF VISUAL GEOMETRY

BY

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PREFACE

It seems inevitable that a student of mathematics has much computational work to do. Even at primary school level, the arithmetic or geometrical notions are accompanied by a host of computational exercises. At secondary school level, Pythagoras' theorem is discussed. What is the use of that theorem? To the student it seems to be nothing else but the computation of the lengths of the edges of a right-angled triangle.

In advanced geometry, a thorough investigation of geometrical concepts may require computational skills at a high level. One only has to open textbooks on mathematics or geometry, to notice that many pages of these books are crowded with computations and formulae.

In a way the science of geometry is focused on an arithmetic approach. For some students this could become a block in the progress of their study. To fail in computations and in the applications of formulae means: to fail in geometry.

However, weakness in computational skills does not imply that the student has no talent at all to study geometry. I will explain this now.

My field of research is visual geometry and I was amazed to notice, that computations were not a great help for my investigations. For instance, I tried to assess the visual straightness of a straight line. To that end I scrutinised philosophical texts. In the topic of visual straightness and its philosophical background the validity of computational work even turned out to be uncertain !

This opens the door to an alternative approach to geometry, far from computations and the application of formulae. My research also comprised Projective Geometry and this branch of geometry appeared strongly related to visual art.

So there it is. A new branch of geometry emerges, based on philosophy and visual art. I discovered that visual arts, like painting and architecture, together with some philosophical items, are suitable as basic material from which geometrical concepts can be extracted without the necessity of doing computational work or applying formulae.

It is this approach I have elaborated on in my thesis and I want to apply it to geometry education.

The special nature of this geometry, developed in my thesis, justifies a new name. I have called it 'Educational Geometry', because the educational needs of students prevailed: computations and formulae have been banned from this curriculum.

Educational Geometry gives a cultural background, rather than a technical basis.

I suggest that Educational Geometry could be taught at secondary schools in Great Britain and in The Netherlands. However, the secondary school situation in Great Britain differs considerably from the Dutch situation in secondary education. In Great Britain it only seems possible to integrate new ideas like my ideas into the existing curricula as additional or replacement topics. In the Netherlands it even might be possible to establish a separate course within the present structure. It would be called 'Wiskunde-C', which in English means 'Mathematics-Cultural'. It may be supplementary to existing school curricula and it might help pupils, who are in difficulties because of their weakness in computational skills.

At university level things are less complicated and I expect that 'Educational Geometry' as a 'module' in a degree course would be suitable to be lectured at British universities and also at Dutch universities.Typically a module would involve 3 lectures a week for 12 weeks with associated reading and written work.

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Next I will describe the position of 'Educational Geometry' in the Dutch education system which would result from the research in this thesis. Here follow some preconditions.

Before entering the study of Educational Geometry, a student should have attended at least two years of the customary mathematics education of the 12-14 years old age group to certify that basic notions have been studied. I think that an Educational Geometry school curriculum can be designed for the age group of 14 -18 years old. It should fit in with existing school curricula in mathematics and it could be a valid and valuable alternative for either pupils with weakness in computationa l skills or for those who feel attracted by the cultural background.

Closely connected to the development of a secondary school curriculum the issue of the preparation of teachers, who are going to teach Educational Geometry, has to be considered. In the conclusion of the thesis (Chapter XVIII) I will expand on both issues, applied to the Dutch situation: the development of a school curriculum for the age group of 14 -18 years old and the preparation of teachers.

Now I want to emphasise that Educational Geometry is more than just a school geometry. It was designed to offer interesting geometrical material to pupils of 14 years old as well as to professional university students of mathematical sciences and to other interested adult students without a suitable computational background. The high level of geometrical understanding at which Educational Geometry is aiming exceeds secondary school purposes and it requires a higher level textbook. In the final conclusion of the thesis I will expand on this, as far as applicable to the Dutch situation.

I also emphasise that the teaching of Educational Geometry should not be restricted only to the highlighting of the cultural background. A study of backgrounds does not enhance the student's proficiency and understanding of geometry. Without exercises the student's geometrical level will not go beyond the starting level, not even after thorough reading of background literature. However, these exercises will not comprise the use of formulae or the carrying out of computations.

The following example will illuminate this. Consider the topic: duality. In Chapter VIII of the thesis it is demonstrated that it is relevant material for the Educational Geometry curriculum and it is genuine geometry. So a student could do exercises on 'duality' and at the same time, as background literature, study the life and works of Girard Desargues, the founder of Projective Geometry. Thus, while exercising geometry without doing computations, the student will also become aware of the historical background.

Further, visual art provides excellent material for geometry education. I selected a picture, produced by M.C.Escher, a Dutch graphic artist, which conveys the notion of non euclidean geometry readily by its powerful composition. No computations are required. The picture is called 'Cirkellimiet' and it is shown and discussed in the thesis.

Educational Geometry is not in opposition to normal and customary geometry education, only, it highlights an up to now undeveloped issue: visual geometry. Moreover, it offers an alternative to those students who otherwise might be in difficulties because of computational problems.

So far my sketch of Educational Geometry, pictured in the Dutch education system.

Finally: the thesis has been split up into two parts. The first part is dedicated to the development of theoretical views. In the second part background literature and applications are discussed.

All quotations have been printed in italics, except some photocopied items. Material taken from other sources has duly been acknowledged as being a quotation.

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PART II

Introduction

Geometry education has been a matter of tradition for a very long period. Until the first half of the twentieth century the approach of the ancient Greek geometer Euclid (300 BC.) was still in use in education. Does this mean that Euclid's geometrical display has never been equalled, let alone surpassed? Or did he possess a most efficient but secret method to let his ideas trickle into the minds of the students?

Reading Euclid's texts, one has to admit that his work is a mixture of a simple, direct appeal to the reader and a demonstration of a brilliant command of the material. My investigation of Euclid's 'Optics' (in Part II of the thesis) shows furthermore that his concepts, although more than two thousand years old, can be related directly to the topic of research in my thesis.

Let us now focus on the visualisation of geometric issues. What we see when we look into the world should have a deep impact on the way we handle geometry as a science.

However, the connection between our eyesight and our geometrical understanding bears a heavy burden of prejudice and undue dogmas. One is not allowed to believe what the eye perceives, but it is, unfortunately, considered good practice to consult the textbooks on geometry first to extract from these how to explain the eye's observation.

Much of the spontaneity of watching reality is lost in this way. Moreover, it puts a prejudice into the mind of the student which in due course will create a block to proceeding on the way to the understanding of visual matters and of geometry as a science. It is unnecessary to say that it is of great importance to lift those dogmas and remove the block on the way to a better understanding so that the student is enabled to move much more quickly along the path of learning.

My research has been aimed at the detection of educational barriers. Moreover, I have tried in such cases to find a way out once a bar was found.

Besides the dogmas and the prejudices, there are the more common pitfalls. We might consider the knowledge of analysis and computational skills, required to gain a proper understanding of geometry, as a major stumbling block. In my research I have made a sharp distinction between geometry which has to be studied with the help and the application of analytical and computational skills and the branch of geometry which is more or less independent of arithmetic. That last branch has become the topic of my research. Firstly one may wonder if anything is left after dropping the arithmetical aid and secondly if any material remains: has it any educational value? So it is my concern to provide geometrical material that can fruitfully be used in education without requiring the display of arithmetical abilities.

The question is whether laborious work on a geometric issue yields insight. I am convinced that lengthy computations and long series of exercises can be used as a (poor) substitute for real understanding. Proceeding along the path of doing numerous exercises seems a tiresome and tedious method; however, working on the exercises has at least the result that the material is committed to memory, unfortunately often with a minimum of understanding. I have observed that process in the classroom. My research provides a method to offer a path of learning which is an alternative to the wearisome procedure sketched above.

To that end I was searching for a presentation to the student which has an instant impact and yields understanding in a short time. I found such an alternative in the application of visual art. There is a famous example of a non-Euclidean geometry, portrayed by M.C.Escher, a Dutch graphical Artist. The picture is artistically so powerful and at the same time so instructive about geometry, that I choose it as a model of non-verbal geometrical education.

Summarised, my geometry is visual, essentially non-verbal, and free of computations.

The label 'non-verbal' should not be taken too strictly; it is the essence of the sort of education that is non-verbal, but quite a lot of words have been used to clarify the visual presentation.

I do not expect that this kind of geometry will be easier to study than the customary courses on the subject. That is partly due to the traditional practice which is difficult to abandon; especially by teachers of mathematics, words are very often used in the traditional sense. This is of course very understandable because many of the teachers have been teaching the material for years and they expect their pupils to use the same terminology as presented by them in front of the classroom.

Sometimes a comprehensive understanding of the material is not even required. I will demonstrate this by an example: somebody driving a motorcar need not know all the mechanical secrets of his vehicle. Especially for professional geometric applications it could in some cases be more useful to know how to apply the right computations than to have an understanding at a theoretical level of the science of geometry.

The students I have in mind for my geometry are people at university level; some of the material could even be seen as postgraduate. In that respect I deviate from the habit of writing educational work at the level of primary school or secondary school or vocational education. Moreover, I think that the regions of higher geometry should not be exempted from the impact of education. It is my opinion that the application of educational issues should not be seen as degrading the level of the material. What is really necessary is to make higher level geometry more accessible to those who are gifted enough to study it so that no talents are wasted.

Let's take the example of a scientist who was extremely gifted at geometry but mediocre at analysis. I am talking about Jacob Steiner, a nineteenth century Professor of Geometry, who contributed essential materials to Projective Geometry but had to consult his colleagues when trivial formulae had to be applied. He was very embarrassed at those moments.

To be specific: it is not my aim to separate analysis and geometry. The help of arithmetical skills is indispensable for the development of geometry. Most of the geometrical knowledge we have is lodged in the analytical apparatus. But it is my aim to highlight the underdeveloped realm of visual geometry and I am convinced that the best approach to reach that goal is the visual, non-verbal, and non-computational approach. Visual geometry is underdeveloped just because of the abundant application of computational issues in traditional geometry education. So we have to exclude analysis and computations to save at least something of what might be called: 'geometry of sight'.

To emphasise the visual character of my geometry I have produced drawings which can be seen as artistic and geometrical at the same time. I think that to be educationally acceptable these designs have to be geometrical in the first place and artistic in the second place. If these products of visual Art are to convey geometrical knowledge then the geometrical properties should be the first and most important topic; artistic beauty should rather be an incentive to study the picture and it should be so impressive that it will stick in the mind.

Continuing with my artistic geometry, I found fifteen people kind enough to read my work and about a month later to answer questions about what they had read. They were not informed beforehand about the questions they would have to answer. It appeared that they had not had any major difficulties in understanding the questions and giving an answer. This already shows that visual geometry at a comparatively high level can be understood by people, who generally have no other background than a secondary school education which seldom comprised more than three years of geometry. All those interviewed were adults with a job, or housewives.

The thesis has been split up in two parts. In Part I a view of visual geometry has been developed. It is the more theoretical part of the thesis. I avoided the use of axioms on purpose but in only one case I have consulted a definition taken from a Mathematics Dictionary which probably is based on the use of axioms.

Why did I not use a system of axioms? Such a system is customary and almost compulsory when non-analytical or non-differential geometry is involved. A rigorous discussion of axioms might scare off interested students. Another reason is that I have extracted geometrical properties from what the eye perceives and not from the intellectual activities of the mind. This does not imply that my visual geometry is anti-intellectual. However, it should be possible to use exclusively the visual phenomena as a source of study and understanding of the science of geometry. So, I avoided the use of analytical and computational skills as well as the difficulties of a strictly axiomatic approach. The axioms and the study of it are considered by Professor Barrau (who will be quoted several times further in the thesis) as a thorn-hedge through which one has to struggle or, alternatively, which one has to keep a healthy distance from.

Part I also comprises a Chapter on Intuition in which the philosophers Descartes and Spinoza are quoted to develop a concept of intuition that is fruitful for a better understanding of visual geometry.

Part II is more concerned with the realm of applications. A lesson on the subject "Duality" has been worked out in detail in Chapter VIII, section 8.2. to demonstrate how this kind of geometry can be taught. The teaching of a subject like visual geometry is an important issue because it can carry the ideas further in the minds of the students. Some suggestions are made for lessons on Group Theory, Triangulations and the Euler Characteristic. These topics can be fruitfully taught and studied and will contribute to an understanding of geometry at a higher level without the interference of computations, analysis and axioms.

In Part II Euclid's 'Optics' is being considered and there is an example of an application of Euclid's theory to my artistic drawings. Furthermore the Van Hiele method on secondary school geometry education is scrutinised. A short history of geometry education in Dutch primary schools and an outline of a newly introduced curriculum of geometry education in Dutch secondary schools is discussed. An aspect of Plato's cosmology serves as a model for 'Educational Geometry' in a discussion between members of a group (Chapter XVI).

In Part II applications of geometry have been reviewed from ancient times to the present day. Sources can be found in the works of Euclid and Plato. From Plato an early cosmology stems which is described in his book "Timaeus" of which I have used a quotation. Plato refers in his work to geometrical figures, which are being transformed and they are moulded in gold. "It is rather gold than geometry", he states.

We know that in Greek architecture the question of visually distorted statues was a controversial topic. Nevertheless visual geometry was at least a subject in antiquity on which a point of view had to be taken.

As in antiquity, visual geometry appears nowadays to be a topic that can be studied well in connection with other views of the material, for instance artistic views.

From the Van Hiele method, treated in Part II, we learn that pupils at secondary schools are proceeding from lower levels to higher levels by going through the following phases: information, guided orientation, explicitation, free orientation and integration. The word 'explicitation' was invented by Dr Van Hiele and it does not mean the same as 'explication'. In my view 'explicitation' means: to make explicit.

In the concise chapters on the history and development of primary and secondary school geometry a move towards visual geometry can be noticed especially where so-called 'lines of sight' are going to be a considerable part of the whole of geometry education at schools.

Chapter I

1.1 Visual Geometry

Visual geometry will be at the centre of the coming investigations. It will thus be necessary to determine when a geometry is called 'visual'. Also the question arises whether not all kinds of geometry are essentially visual so that it is unnecessary to underline that aspect. Certainly the word 'geometry' reminds us of a survey of the surface of the earth so that we have at once a picture in mind of engineers, measuring area, distances, altitudes and so on. Alternatively one might think of the construction of beautiful buildings, monuments and other kinds of architecture and sculpture. Is it not this we have in mind, when 'visual' geometry is mentioned?

However, when we remember the days we attended lessons on geometry in primary and secondary schools, no memories of surveying arise, although sometimes there were references to area of meadows or the height of a tower had to be computed.

The above examples can all be seen as applications of practical geometry, which means that geometry is used for the construction of buildings or for the measurement of area and more of such purposes. One could use the term applied geometry. But is that precisely the same as 'visual' geometry?

By 'visual' we generally mean: that which is perceived by the eye. So the restriction of the definition of 'visual' geometry to a geometry perceived by the eye should help us to progress in the direction of what we consider 'visual geometry'. But now the question arises: is not all geometry essentially perceived by the eye so that 'visual' geometry is a natural fact which needs no further justification? Moreover, some curiosity arises about a geometry which is not 'visual'. What is the meaning of that?

It is a well-known fact that in geometry there is a lot of reasoning. Take for instance the Pythagorean theorem. A proof has to be provided and computations have to be made. Such a proof and those computations are no longer a matter of what is perceived by the eye but rather reflect intellectual processes which are not visible. Geometry as a science does not only contain visible images but it also comprises a lot of analytical and computational formulae and, moreover, much rational thinking about proofs and so on.

So now the concept emerges of a geometry, which is organised as follows: there is a visual image, which serves as a focus point to carry out computations, all kinds of reasoning and proofs, and to apply analytical formulae. It is the kind of geometry we all remember very well from our primary and secondary school periods. The importance of the visual image was usually reduced. The proofs, the computations, the final answers of these computations and the application of the right formulae seemed to be the ultimate goal of our geometric studies. A firm knowledge of theorems, like those of Pythagoras, was needed to promote these achievements.

Let us now consider the limited function of visual images in the whole of geometry. These visual images seem to provide an initial view of the problem on which our attention is focused but the problem eventually has to be solved with help of reasoning, computation, and so on.

Is there something wrong with these visual images? Is there a reason to end their functioning at quite an early stage? Are they unreliable or, even worse, do they mislead us?

Indeed could we state that the appearance of a visual image may demonstrate a lot but can also conceal much? Not every solid, for instance, can be rotated about the centre as a sphere can, with the result that its rotation is actually invisible. The shadow of a cube, which rotates in the beams of the light of the sun, changes constantly following the rotation. Moreover, the cube can be drawn in such a way that it is not clear whether a cube is depicted or a square.

Besides this there is also a problem of the distortion of images. Already in antiquity Euclid (300 BC.) stated that a right angle, seen from a distance, is sometimes visually deformed.

In this way our visual geometry is nothing more than a set of visual appearances which yield unreliable and sometimes distorted material. So it seems sensible to abandon these visual images as soon as possible and restrict oneself to the reliable and firm grounds of computation, reasoning and the application of formulae. Why stick to the realm of what the eye perceives?

The point is that one can learn a lot from these visual images even before consulting computation, Part II is more concerned with the realm of applications. A lesson on the subject "Duality" has been worked out in detail in Chapter VIII, section 8.2. to demonstrate how this kind of geometry can be taught. The teaching of a subject like visual geometry is an important issue because it can carry the ideas further in the minds of the students. Some suggestions are made for lessons on Group Theory, Triangulations and the Euler Characteristic. These topics can be fruitfully taught and studied and will contribute to an understanding of geometry at a higher level without the interference of computations, analysis and axioms.

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The point is that one can learn a lot from these visual images even before consulting computation, reasoning or formulae. The mere appearance of what the eye perceives can be subject of a very interesting debate and, moreover, such investigations provide a lot of understanding and insight to the student. In order to start such an investigation, it is interesting to note that there are apparently at least two different kinds of visual geometry which exist alongside each other almost without noticing each other's presence.

What I am referring to is the difference between geometry in the microcosm and geometry in the macrocosm. Geometry in the microcosm is usually denoted as "local" geometry and geometry in the macrocosm is usually called "global" geometry. When we look at the differences between these two kinds of visual geometry it will turn out that these differences can not be studied without the help of visual images.

As an example of local geometry I would like to have a look at the bulk of the geometry we all remember from the geometry textbooks we used in our secondary schools. By local I mean that the figures depicted never exceeded a limited range. We studied images of triangles, squares, cubes, and none of these geometrical configurations reached the horizon. These images will never go beyond the range of our desk.

The situation of global geometry is quite different. In this kind of geometry we have to deal with lines running towards the horizon. One may think of paintings of the sea, where almost always a horizon can be seen and lines of very great length are depicted. The scenes demonstrated are so large that they not only do not fit on our desk but the desk itself is reduced to little more than a small spot in the universe.

Now we reach an interesting point. Is it attractive to study the differences between 'local' and 'global' geometry? Does it provide new perspectives? How does it relate to what we initially called 'visual' geometry?

One of the most conspicuous points is that the two kinds of geometry seem to be almost completely separated. Consider the following example: a cube, which can contain only one litre, has to be depicted from a distance of 500 metres. It will take an excellent artist to manage to draw the cube so that it can be recognised as a cube. In most cases it is reduced to a dot in the painting. So the demands of drawing a small object in a wide surroundings from a distance demonstrate the differences between 'local' and 'global' geometry.

'Local' geometry sometimes is described as 'geometry in the small', and 'global geometry' may be called 'geometry in the large'.

There is, however, a much more serious difference between the appearances 'in the small' (locally) and the appearances 'in the large'(globally).As we all know, when we have to draw objects, certain distortions are sometimes inevitable. Now it happens to be that the distortions occurring in geometry of 'the small' are not the same as the distortions, which occur in 'the large'. So a 'global' picture demonstrates other distortions than a 'local' one. We might demonstrate this with the above given example of the 'one litre' cube depicted from a distance of 500 metres in a landscape. The image of the cube is affected badly by the distance and it has shrunk to a dot but the landscape in which the cube is placed is not affected by such distances but, on the contrary, will provide an excellent picture.

Another example is that straight lines, if prolonged indefinitely, are not longer visually straight but seem to become curved. This phenomenon will be discussed further in the thesis. So a straight line running to the horizon will have to be depicted in many cases as curved.

For these reasons it is attractive and even useful to study the different distortions of 'local' and 'global' geometry. It is clear, however, that such different distortions do not appear from computations or formulae. And this is rather strange. If different visual images are distorted visually in different ways, why does that not follow from the computational and analytical approach?

One might say that these differences have been reasoned away. Although a straight line, running towards the horizon, seems to become curved when it approaches the horizon, nothing of this can be found in the relevant calculations and formulae. So very conspicuous visual differences are simply ignored and can nowhere be found in the analytical apparatus, in which almost all of our geometrical knowledge is lodged. This ignorance should be and actually has been a reason for me to investigate the apparent differences in distortions which occur in 'local' and 'global' geometry.

These are the stones of which our building of 'visual geometry' will be constructed. However, there are even more conspicuous facts, which also have to be investigated. Let us take again the example of the 'one litre' cube at a distance of 500 metres. At such a distance the cube can no longer be recognised as a cube, at least by people with normal perception. Nevertheless, it still is a cube as we suppose and it still contains one litre. From this example we learn that the form of an object changes when the distance changes. But this has severe consequences. Take in mind a square with length and breadth one metre, and lay it on the ground. If we walk away, say 10 metres, and we observe the square again, it will no longer look like a square but it has become something like a 'visual' trapezoid. Now the objection will be that although it looks like a trapezoid, it still remains a square. Yes, that might be true according to reasoning, but not according to visualisation. So, according to my interpretation of visual geometry, a square can never visually be seen as a square; that is to say that it is impossible to stand somewhere at a point, so that from that point the quadrangle has four right-angles visually. It means that visually a square does not exist but that by reasoning a quadrangle is constructed with satisfies the demands of being a square. This interpretation of visual geometry might seem very unscientific at first glance. However, as we will see, it provides unexpected chances to explain a lot of higher geometry which will otherwise remain obscure.

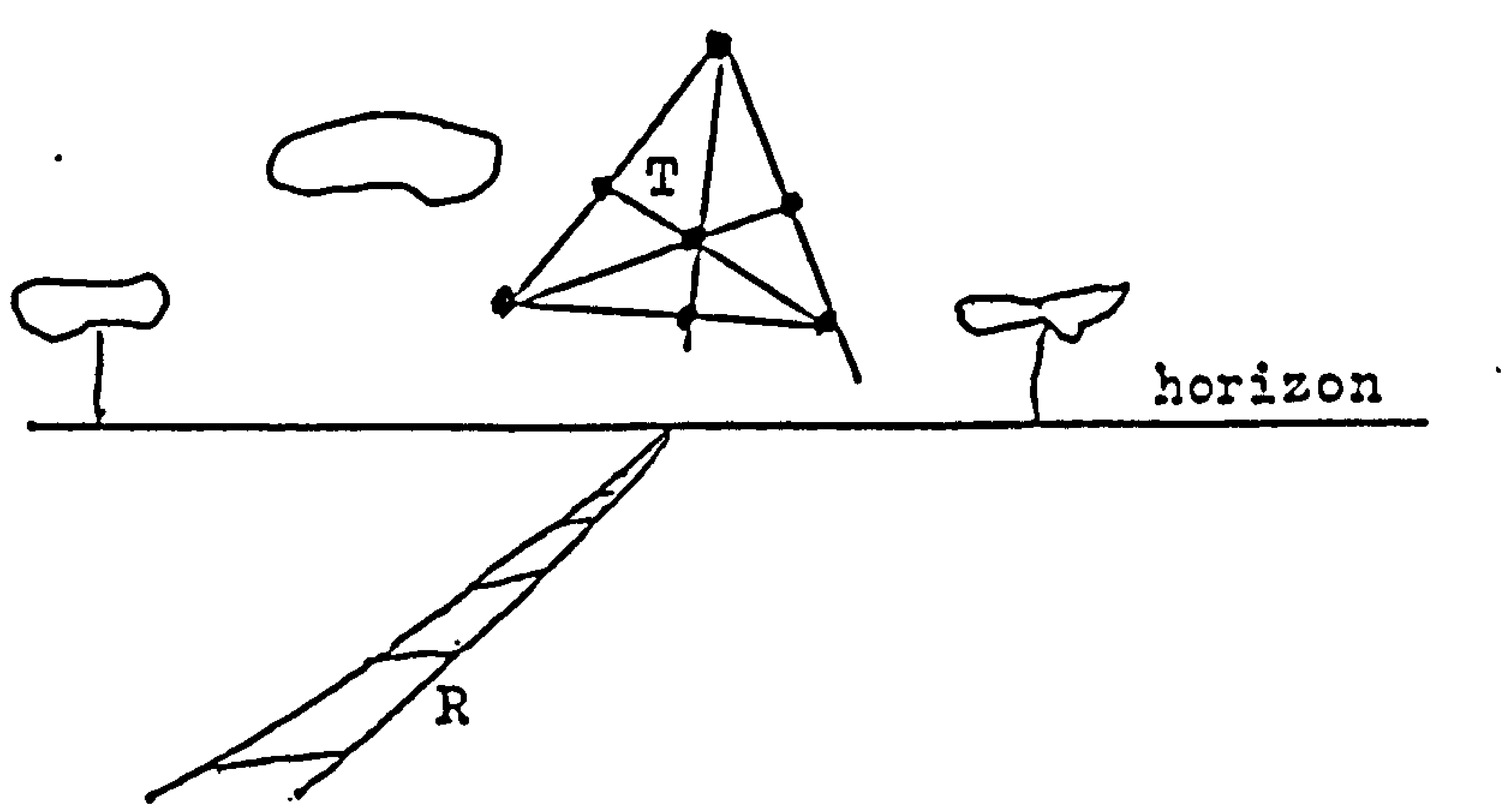
My approach, which drops the application of analysis and computations, makes geometry much more accessible to students for whom the application of formulae and computations is a deadlock in their studies. In other words it becomes possible to understand much geometry at a comparatively high level, without carrying out a single computation. This, I expect, will be a great advantage for many people who are interested in geometry, but fear that their ability in analysis and computation is too limited to study geometry successfully. It has to be acknowledged that a comprehensive study of geometry is inconceivable without the help of analysis and computations but much of this can be replaced by the visual geometry as I have sketched above.

Of course, to provide the students with adequate material, visual pictures have to be produced which are attractive and designed so that they convey the right concept of the geometrical topic. Such powerful images can be found in the realm of Art. A very important example will be demonstrated by the so-called " Escher's Pond " which conveys the concept of a non-Euclidean geometry. Other examples of artistic pictures, produced by me, will demonstrate properties of projective geometry or illustrate spatial views of Euclid and Plato.

1.2 Visual Geometry and Education

It has become fashionable to look around to detect geometrical objects rather than to consult geometry textbooks. What can one expect? The geometric objects have been identified as such by the theorems of the textbooks. Is it then reasonable that, independently of these textbooks, valid geometry will be offered by the environment outside the strictly limited world of those textbooks? Let us have a look at figure 1.

figure 1



It shows a straight railway R heading for the horizon and a triangular configuration T. The triangle T has been taken from a page of a textbook; the railway R shows a straight line in natural surroundings. Moreover, the railway has not been drawn along a ruler but nevertheless it is supposed to be straight.

The two images T and R do not combine with each other in figure 1. Apparently different geometric worlds are demonstrated. Somehow the triangle does not represent a natural form in the landscape. Of course one may assume that also in a natural landscape triangles will be present but seldom have the form of T. The triangle T seems to belong to a mathematical world but the straight railway R belongs to a landscape and will not at first sight be identified as a mathematical object. It seems difficult, if not impossible, to make those two universes cohere. This is a weak point in the new approach of geometry. If you look around you, you find a geometric world which differs widely from the universe of the images of the textbooks on geometry.

Painters of course are very much aware of the situation. The horizon was invented (or detected) in the sixteenth century (Albrecht Dürer, 1471-1528) to structure paintings. It was a breakthrough in painting and centuries later, a comparable breakthrough in mathematics took place. Visual methods, in use by painters, had foreshadowed developments in nineteenth century mathematics to display new types of geometry, the so called non-Euclidean geometries. One may not say, however, that a painting equipped with a horizon is a model of visual geometry simply because the painter did not intend to portray mathematics.

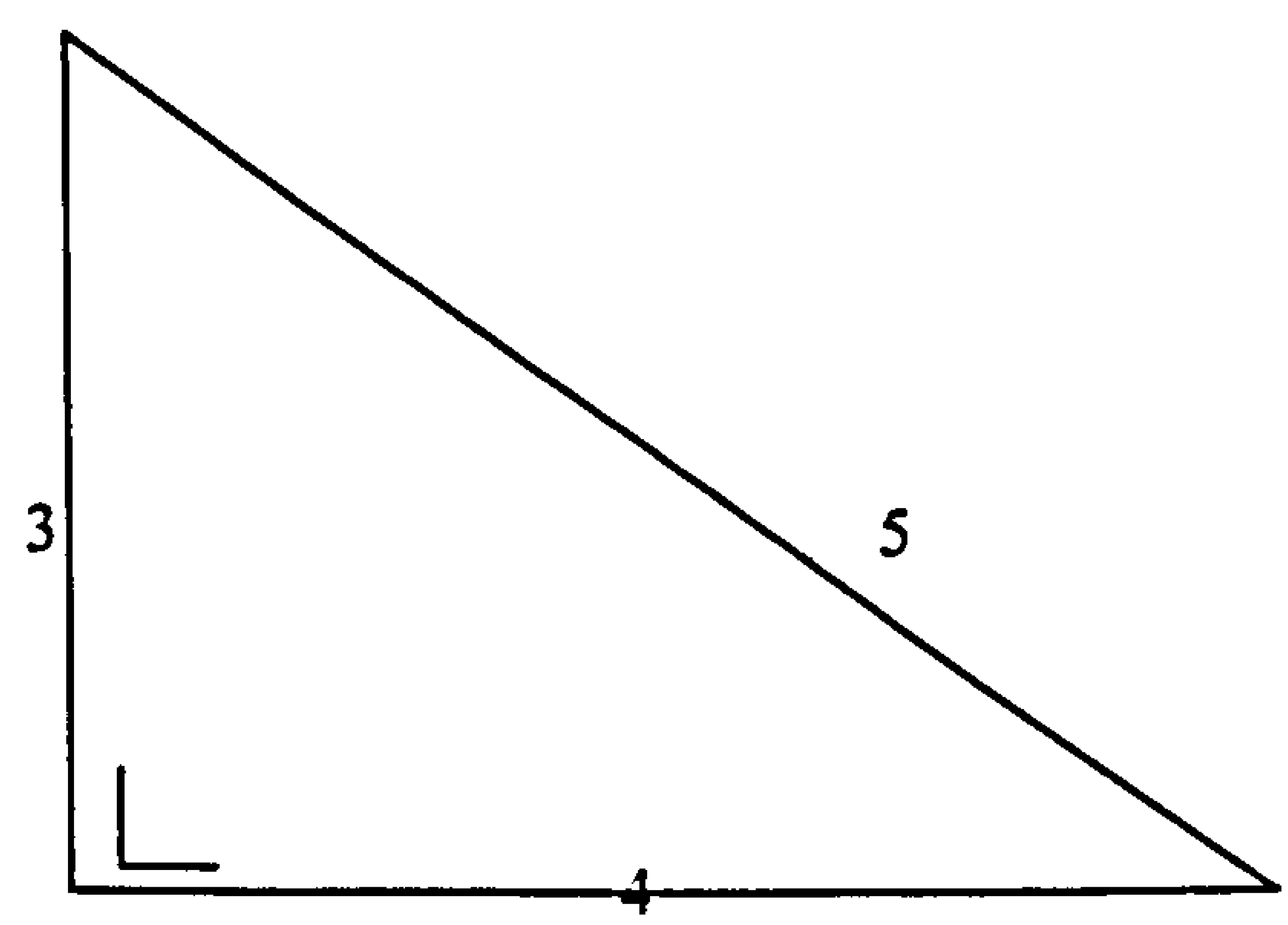
Nevertheless it is possible to find geometric concepts in such paintings (Houckgeest, 1600 - 1661). Moreover, for the study of geometric concepts with help of paintings or, in general, products of visual art, no knowledge of analysis or computational skills is needed. That is extremely valuable for people who have never managed to cope with analysis or arithmetic. The customary blockades are lifted to allow them to enter the realm of geometry. I think this is a basic issue in education in visual geometry with the help of visual art. With visual geometry we denote R in figure 1 rather than T. An accurate definition of visual geometry will be provided within a few pages. Furthermore, the Pythagorean theorem will be scrutinised and it turns out that it does not belong to visual geometry. The question now arises whether visual geometry can teach the students valuable things with respect to the science of geometry. In other words: the educational value of visual geometry is at stake. Is there something we can learn from it?

To answer this question we have another look at figure 1 in which the railway R is considered as visual geometry. This would mean that a construction like triangle T must not be seen as visual geometry but rather as 'mathematical' geometry. R is assumed to belong to another kind of geometry than the one T belongs to. Nevertheless, in practice, theorems valid for triangle T are sometimes automatically but unjustifiably applied to the different kind of geometry to which R belongs. Such a practice will be denoted as 'educationally invalid'. The notion of 'educational validity' will be accurately defined in the next Chapter. Our first concern is to investigate the meaning of the notion 'visual geometry'.

What is visual geometry? It might be interesting to look at the following example. It is Pythagoras' (572-490 BC.) theorem.

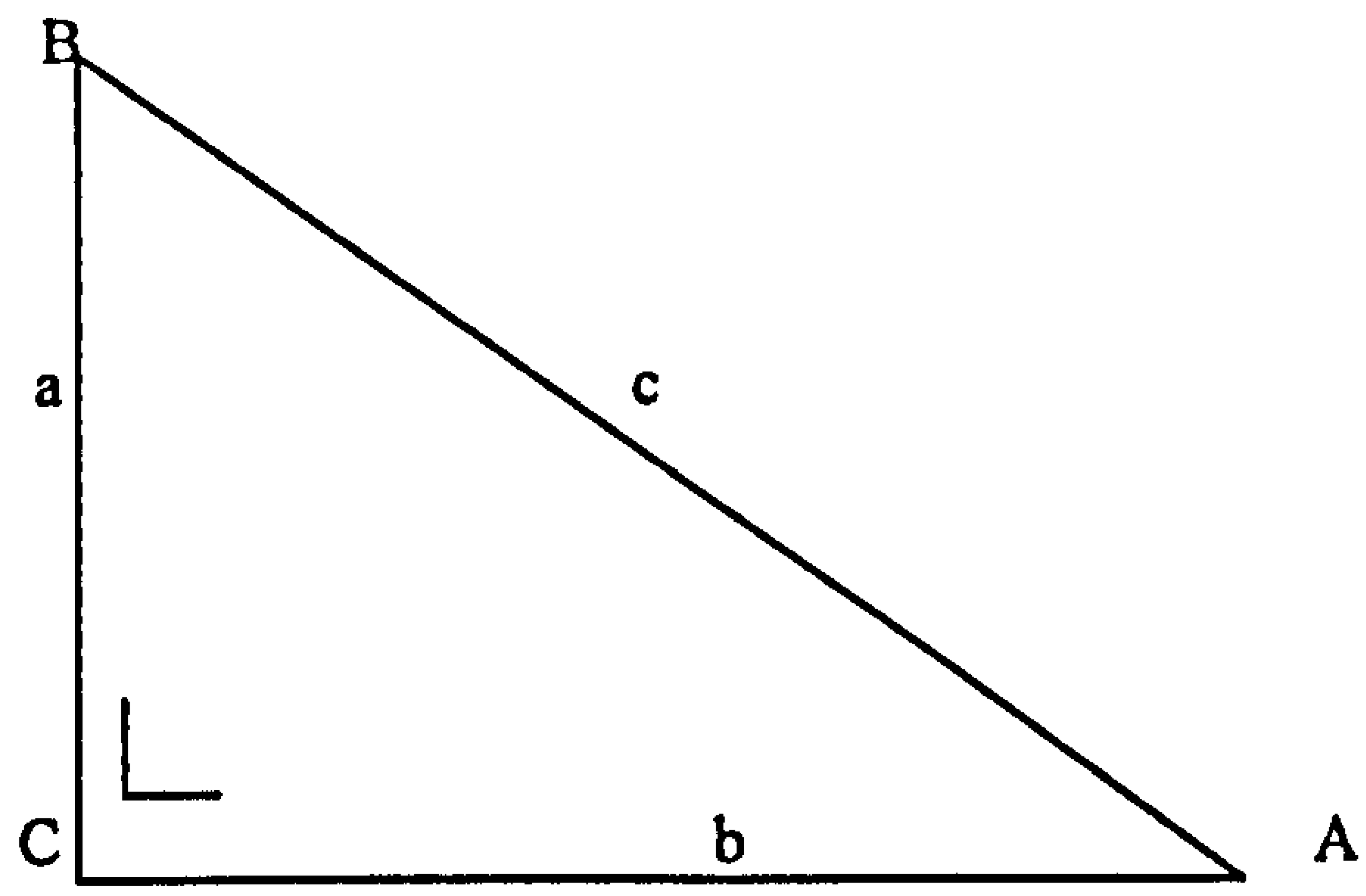
Consider Pythagoras' theorem in the right-angled triangle (figure 2). There is a relationship between the length of three sides of the triangle: 3.3+4.4=5.5 or 9+16=25.

figure 2



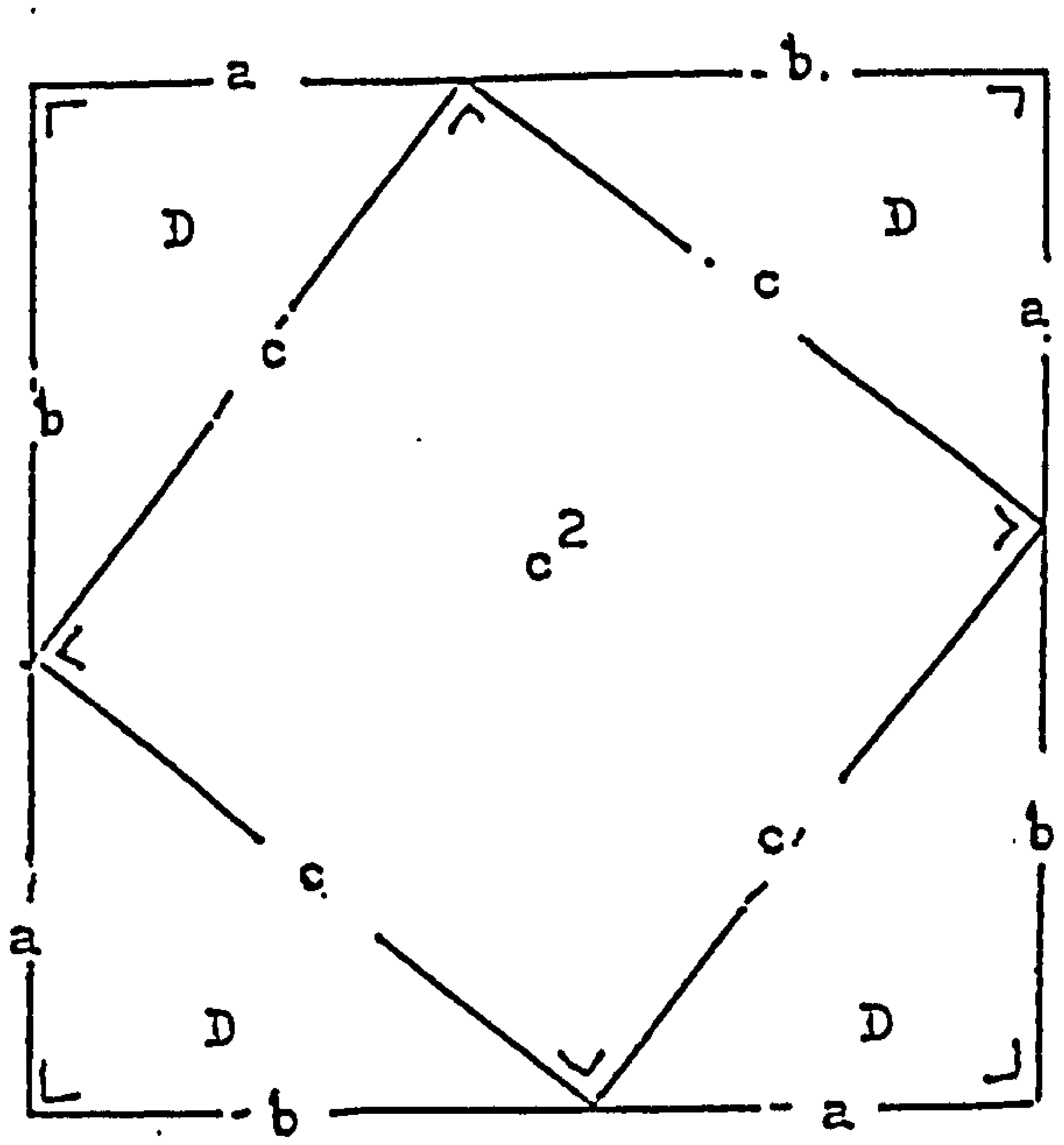
In the next figure we see the general case in which the numbers are replaced by characters (see figure 3).The edge opposite vertex A is called: a. Likewise we find b opposite B and c opposite C. According to Pythagoras' theorem there is a relation: a.a + b.b = c.c

figure 3



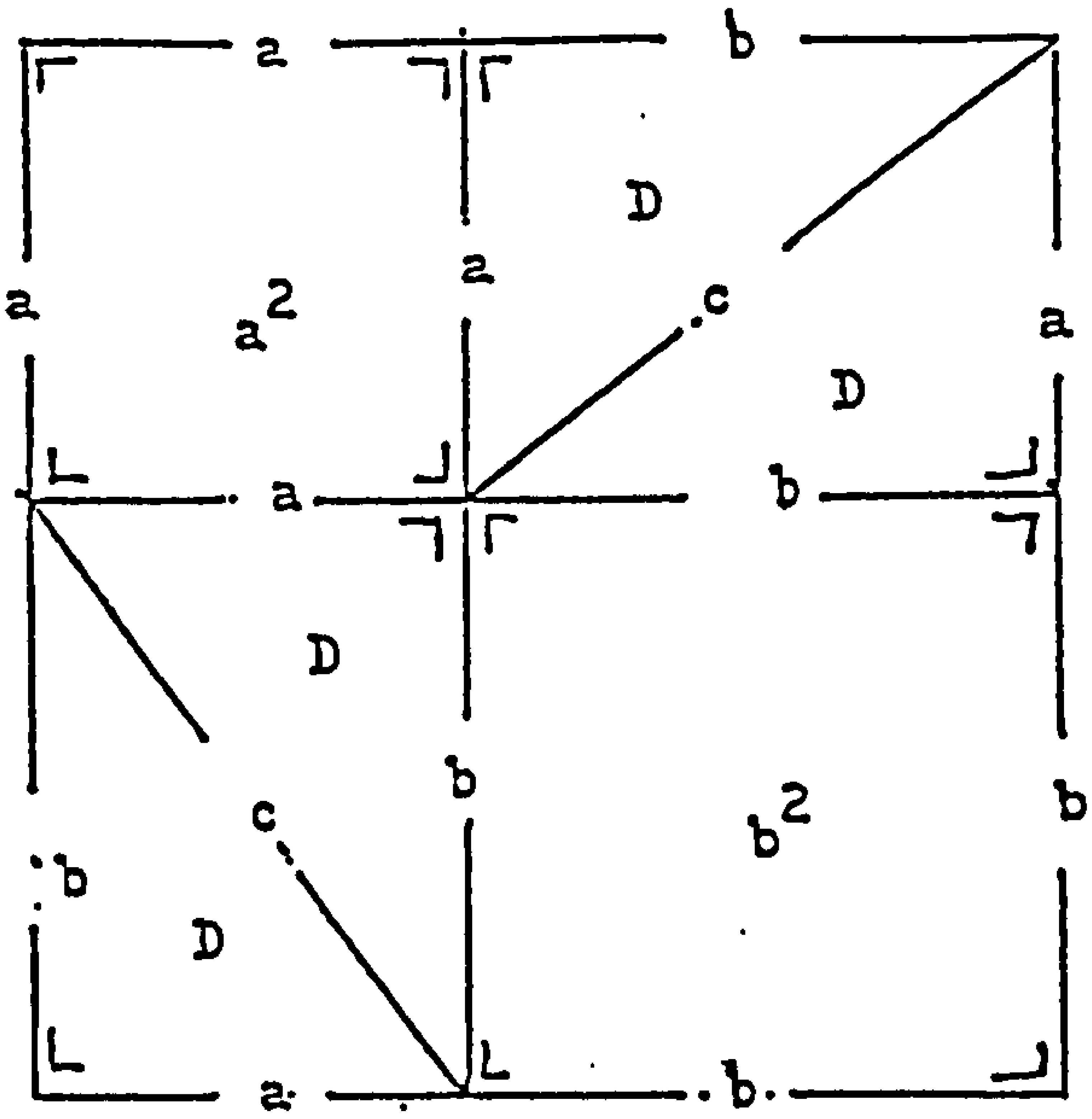
There is a "visual" proof for this theorem: in figures 4 and 5 there is a square with each side a + b, so the areas of both squares are equal. The proof is a general proof; it holds good for every number, substituted for a and for b. D is the area of the right-angled triangles which are all congruent. Because the areas of the squares of the figures 4 and 5 are equal, the equality c.c = a.a + b.b is correct. This proves the theorem.

figure 4



Area = $c^2 + 4D$

figure 5

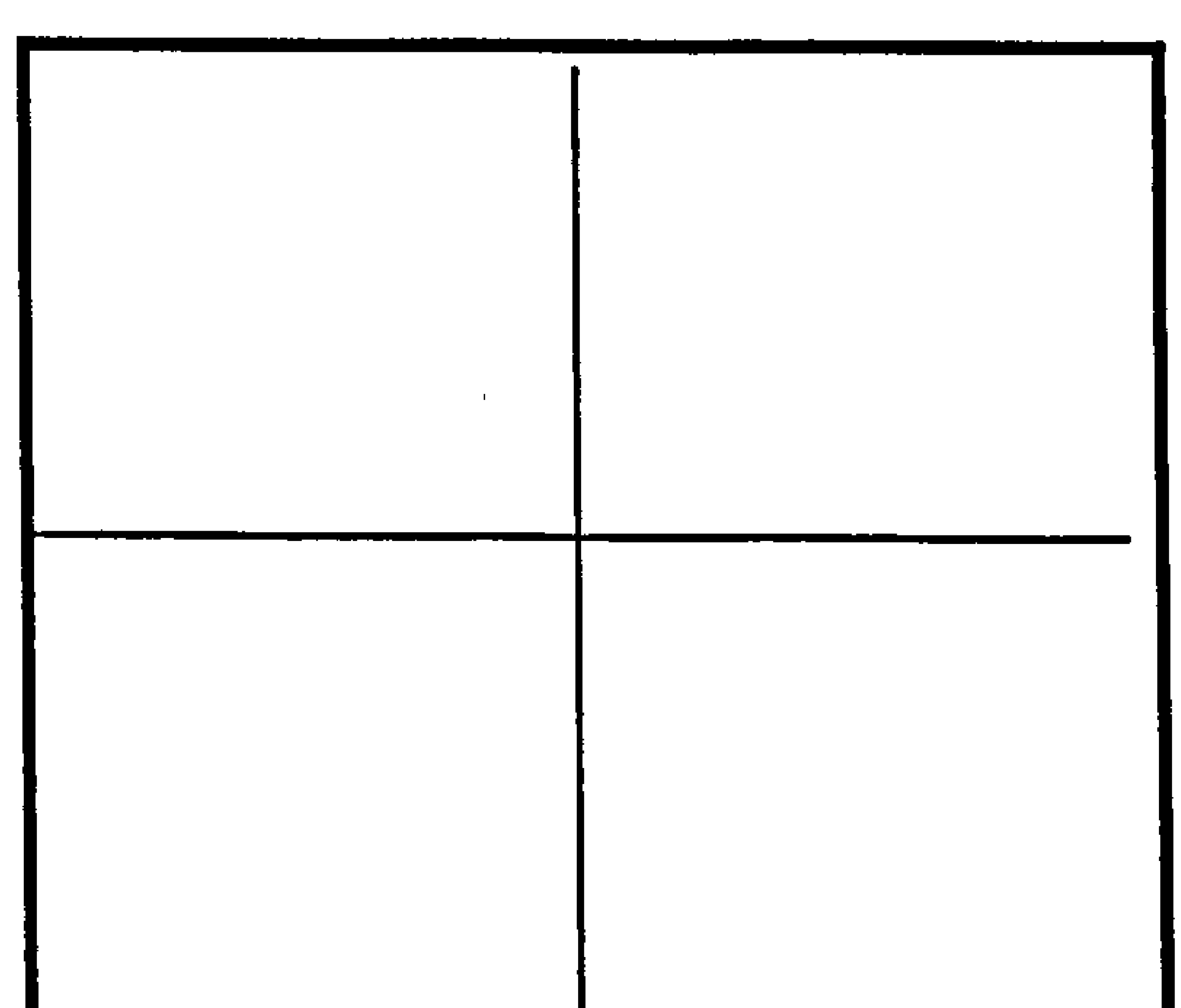


$$\text{Area} = a^2 + b^2 + 4D$$

In visual geometry proofs have a different background. Were the squares in the figures 4 and 5 squares visually?

Normally a quadrangle like the one in figure 6 will be seen as a square.

figure 6



Mathematical Square

The square of figure 6 is supposed to have four right-angles. In visual geometry one may investigate whether these four angles are right-angles visually. The magnitude of visual angles is dependent on the position which the observer takes. The angles of the 'square' of figure 6 are certainly not all visually right. Look at figure 7. An observer P is watching the angle BAD. The angle BAD however is visually equal to the angle BED, seen from P. We take the square's sides to be 2 and the altitude PC = 1 and then the angle BED is 120 degrees and so the visual angle BAD, seen from P, is 120 degrees.

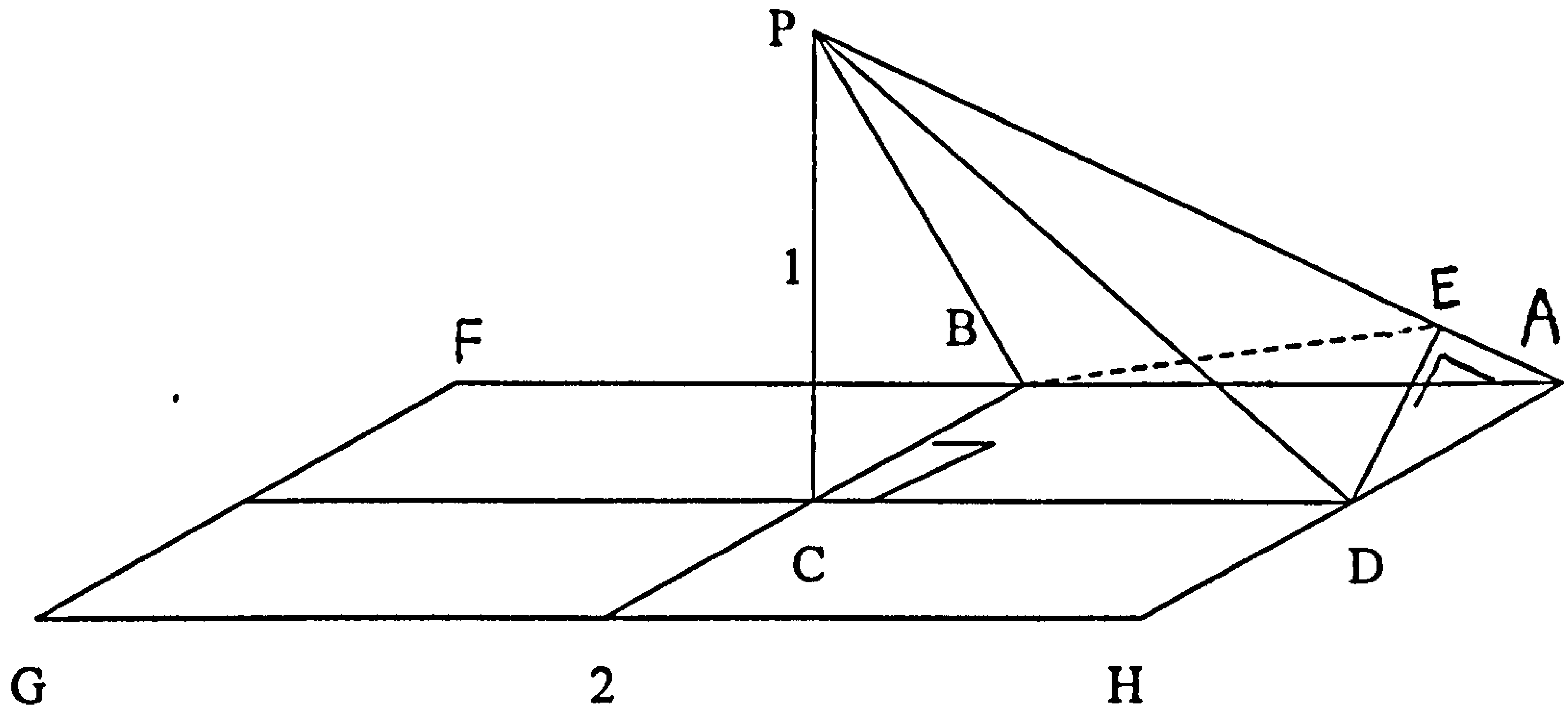
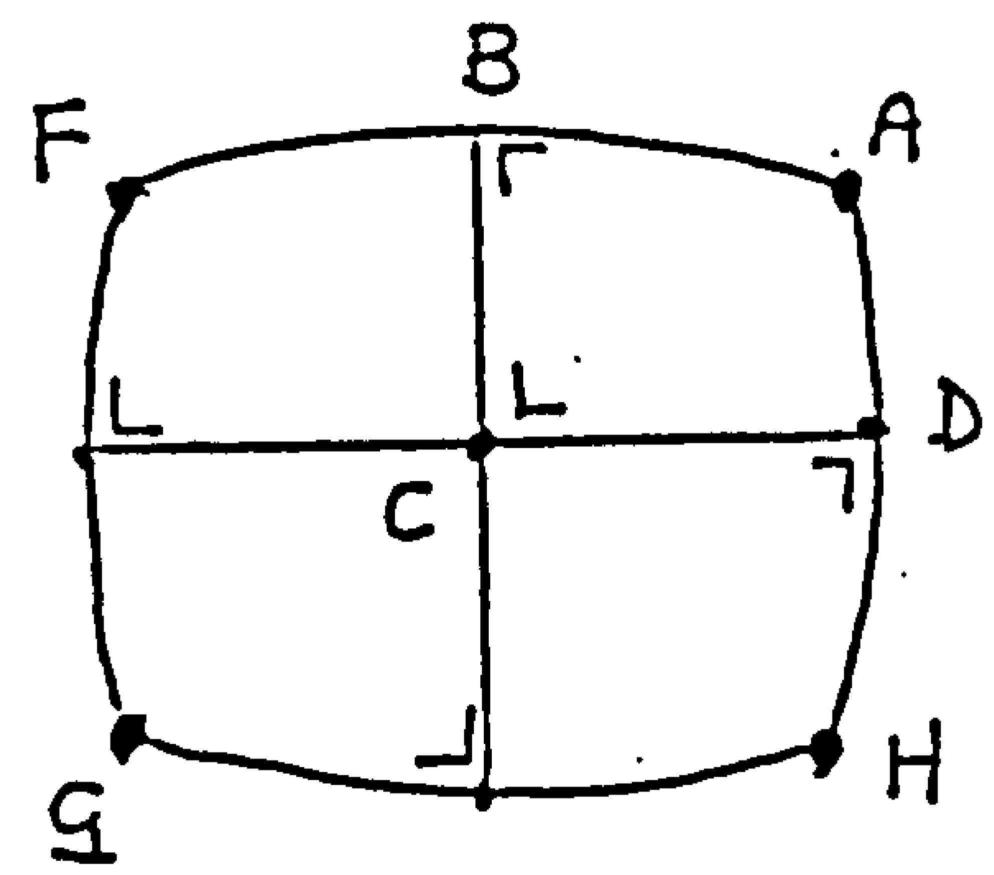


figure 7

Thus the angles BAD, DHG, HGF, and GFB are visually 120 degrees, as seen by P. Actually P will perceive the square AHGF as follows (figure 8):

figure 8



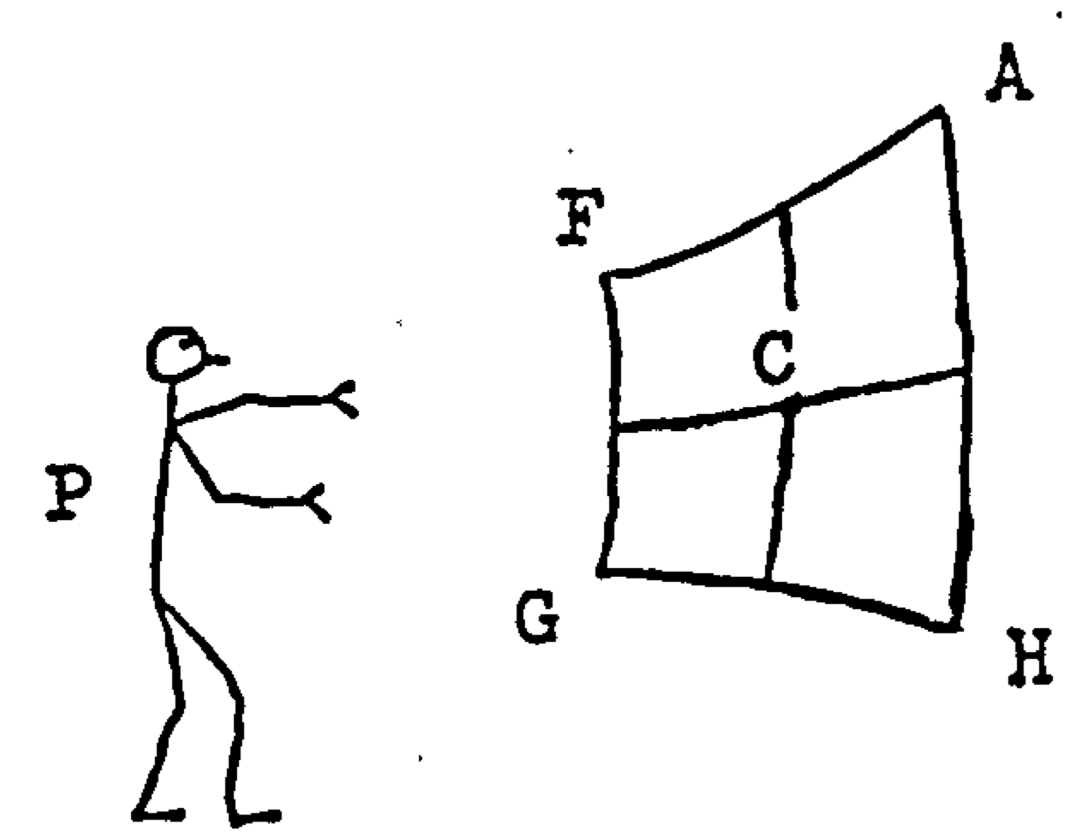
Visual Square

Let us call the visualisation of figure 6 a "mathematical square", abbreviated in the following by 'm.sq '.

The picture of figure 8 will be called a "visual square", abbreviated in the following by 'v.sq '.

A m.sq cannot be perceived in reality. By 'reality' I mean that somebody (P) is watching a wire model of a square while this person P is standing in front of it (figure 9)

figure 9



The wire model, perceived by P, looks very similar to the v.sq of figure 8. The picture of figure 9, observed by P, seems to be something like the panoramic screen of a motorcar. Point C (figure 9) is closest to P and the corners A,F,G and H are more remote.

A m.sq is rather a mathematical working scheme than a natural image of reality (in the sense of the situation of figure 9). In a m.sq the relevant mathematical properties are summed up: it has equal right-angles, equal straight edges,symmetry. These properties are much more difficult to recognise in a v.sq.

The construction of a m.sq is such that it is visually more related to mathematics than to the visual reality (in the sense of figure 9). It demonstrates a list of mathematical demands and for that purpose the depiction of reality (the reality of figure 9) has been distorted. One may state that a m.sq reflects the reality of figure 8 and of figure 9 in a visually distorted way on behalf of the application of mathematics (analysis, computations, distances, angle measuring).

In a m.sq the edges of a square are not only supposed to be mathematically equal (of equal length) but these edges are also supposed to be visually equal. And that is obviously not the case but the observer is supposed to ignore that visual reality (the visual reality of the figures 8 and 9).

However, configurations like a m.sq provide pictures of outstanding beauty. The square shown in figure 6 may be seen as a product of fine Art belonging to "Magic Realism" This is discussed in Chapter II, section (2.3). It is a good reason for not rejecting it but it would be wrong to state that it is an accurate image of reality (in the sense of figure 9).

We come to the following definition:-

(3) Definition: Visual geometry is a geometry which is not distorted by mathematical demands

By mathematics here we mean analysis and arithmetic. A visual square is an example of visual geometry according to Definition (3).

It should be admitted now that in the foregoing pages a most difficult subject in geometry has been touched upon: the subject of elementary geometry education.

The proof of Pythagoras' theorem (figures 4 and 5) has to be rejected because it is based on the use of visually distorted squares so that it can not be seen as visual geometry. I stated that the demonstrated proof was "visual", but I used quotation marks. It looks like a visual proof. Now we have seen that Definition (3) does not permit us to accept the given proof (figures 4 and 5) in a visual geometry where Definition (3) is valid.

Elementary geometry education is a difficult subject because it does not solve the problem of the difference, for instance, between visual squares and mathematical squares. The problem whether Pythagoras' theorem really has been proved by the demonstration of the figures 4 and 5 must remain open.

There are scientists who firmly believe in the supremacy of analysis and arithmetic above the reality of visual geometry. For instance, Hendrik de Vries' concept of geometry contradicts my approach.

"There has been a time that one was not quite conscious of the fact that notions like 'point', 'straight line', 'plane', can not be defined without help of the notion 'number' and that geometry which is not based on the notion of 'number' actually is a building that hangs in the air." (de Vries, 1923, page 60)

Hendrik de Vries is late Professor of Mathematics at the University of Amsterdam. Following his concept of the definition of elementary notions like point, straight line and plane, one can understand that the visual aspects of geometry may have become a taboo in mathematics.

These visual aspects contradict the analytical foundation of geometry according to the above-developed ideas by Hendrik de Vries. If it is no longer possible to talk about, for instance, a square in terms of equal angles, equal sides and a computed area, how is it then possible to speak even of geometry? That word "geometry" itself refers to measurements and computations. This is where de Vries' view leads.

This analytical approach of geometry by Hendrik de Vries denies access to geometry to many people who can not cope with analysis. Is that necessary?

A famous counterexample is from Jacob Steiner (1796-1863) who was a genius at Projective Geometry but had a very mediocre talent for Analysis. Thanks to his labours, Projective Geometry has become more valuable. Due to him the notion of duality has been developed to its present level.

In a way, however, Hendrik de Vries is right. The geometry of the 'mathematical squares' is indeed fully dependent on the notion of number. That kind of geometry has deliberately been designed to apply numbers (for instance to compute the length of line segments belonging to the configuration of the mathematical square). Some people call the 'mathematical squares': abstract squares, to distinguish them from visual squares or other constructions. These 'abstract' squares are, of course, visual too but I do not call it visual geometry in the sense of Definition (3). The mathematical (or abstract) square is so peculiarly adapted to the demands of arithmetic and analysis that its existence in the reality of a natural landscape (figure 1) has vanished.

Some scientists admit there is a problem:

"The geometrical plane. In this Chapter we will take for granted that plane geometry and trigonometry have been treated as customary at secondary school level. Elementary geometry has to be treated in a cursory way because the beginning of geometry is the most thorny part. Once beyond that it is developed in a rigid way which is considered as a model of rigidity. A few thorny points are the definition of a straight line, the notion of similarity of figures, line segments, angles, triangles and so on. All very plausible, but there is little logic in it. Elementary geometry education has to work itself through this thorn-hedge." (Barrau, 1918, page 11)

J.A.Barrau is late Professor of Mathematics at the State University Groningen, The Netherlands.

Barrau wrote a textbook on Analytical Geometry and with some relief he states that it is not his concern to find a way through the thorn-hedge of elementary geometry education. He is glad to be able to evade the difficulties of a pure geometrical system of axioms because what he builds is a purely analytical geometry.

I can understand what Barrau means when he states that the notions in elementary geometry are thorny. There even looms a danger: when visual geometry (according to Definition (3)) is separated from analysis, the identity of geometry is at stake. What may be considered as belonging to the realm of geometry and what not? There is a danger that all kinds of irrelevant material will be brought in.

Let me provide an example of this. As will be demonstrated in Chapter II, the 'mathematical square' can be seen as a product of art, belonging to Magic Realism. So Magic Realism plays a role in Mathematics as an illustration of what is denoted as 'abstract' or 'mathematical' configurations. May we now allow Magic Realism to enter the building of geometry and consider it as belonging to Mathematics? The Science of Mathematics seems not to be too eager to admit that visual Art dwells in the building of Mathematics.

I think that the most important question is whether visual art can contribute to the development of geometrical knowledge and I will demonstrate that the answer is positively yes. Therefore it is my intention to use visual art to pave the way to a faster and better understanding of geometry which applies especially for those students who wish to drop the use and help of analysis and formulae because it is a deadlock for their geometrical studies. Further in this thesis visual art as a means for geometry education will be discussed and developed.

This is not the only motive. The reduction of visual images to a series of formulae and computations makes geometry incomplete. It may even discourage students. However, some students dislike the visual aspects of geometry and greet the analytical approach as a salvation. It saves them from geometrical extinction. This, of course, should never be a reason to disfavour and reject spatial images.

There is also the possibility of describing geometry as the study of visual structures. Some people dislike the rigid mathematical demands which are predominant in the science of geometry. They might assume that there are visual structures which may have non-mathematical aspects.

The looming danger I mentioned above might appear to be a confusion about what precisely geometry is. May Magic Realism be considered as a kind of geometry or is it rather a model of visual structures, still belonging to geometry? Is Definition (3) not contradicting itself? Can undistorted images serve to develop geometrical notions?

The kind of geometry I aim to elaborate is higher geometry. Images like 'mathematical' or 'abstract' squares are excellent means to be used in practical applications but they become a bar when students want to extend their views and deepen their knowledge. It is, for instance, an obstacle to further study of geometry when a student assumes that straight lines are visually straight. We saw in figures 6 and 8 that the sides of a 'mathematical' square seem bowed, seen from a certain distance. According to the associated computations it became clear that this is not a matter of optical illusion. In the next two Chapters we will learn that a straight line, sufficiently prolonged, becomes visually curved. This means that our concept of perspective has to be revised. The customary notions about perspective satisfy practical needs perfectly but fall short when the differences of local and global appearances of straight lines have to be discussed.

The following "parable" may clarify my intentions. Let me compare geometry to an estate with buildings. The building of geometry is a huge castle in which we find rooms, dedicated to Analytical Geometry, Differential Geometry, and so on. The surrounding grounds are identified as Visual Geometry. The area of the grounds is large, even reaching the horizon. There are woods, lakes, gardens, hills, ponds. At some points the views are so brilliant that one may talk about 'beauty spots'. Somewhere we find Barrau's thorn-hedge and also a labyrinth called 'educational invalidity', and many more objects like Escher's Pond which we will visit in Chapter III. One may get lost in the labyrinth and at the entrance there is a warning sign. Round the estate there is a wall with a gate. Admission is prohibited for unqualified people and in- and outgoing persons are checked rigorously. End of the parable.

My intentions are to make the beauty spots wholly accessible to the general public with a public footpath leading across the estate. (See the parable).

Chapter I will now be summarised and the summary is called: "Basic Paper". It gives an outline of the research to be carried out. Chapter II is a continuation of Chapter I and some notions will be defined accurately in that Chapter.

1.3 Basic Paper

SUMMARY OF SUBJECTS

It is the aim of my research to define education in higher geometry which is based on visual art. From history, examples of geometry to be observed in fine paintings have come to us. In my view, geometrical concepts have roots in intuitive knowledge. The use of analysis has been dispensed with.

PROBLEMS OF THE CURRENT VISUAL GEOMETRY.

The bulk of recorded geometrical science is lodged in an analytical apparatus which is a kind of shadow-geometry. Some textbooks on high level geometry do not even contain a single visual image. Scientifically visual geometry is largely ignored because of the self-contradictory or half-true presentations of its concepts. I call this presentation "educationally invalid" because the underlying concepts are presented in such a way as to hinder subsequent developments in using and understanding them. This is especially the case in customary textbooks on geometry where usually the relationships between local and global geometry are ignored. The application of ordinary visual geometry is often necessary on behalf of practical geometrical solutions.

SOLUTION OF THESE PROBLEMS

Visual Art is a valid alternative to analysis which can be used to enhance the student's geometrical level and to lift the inner contradictions of ordinary, customary visual geometry. As a means to study geometry it is supplementary to analysis but it can not quite replace it because analysis serves as the 'bookkeeper' of geometry. In the above parable Analysis could be denoted as the 'gardener' of the grounds of visual geometry. By these comparisons I want to indicate that analysis and arithmetic should not have the final say in geometrical matters but that nevertheless their role is an important one.

Chapter II

2.1 Educational Validity

In this Chapter closer attention will be paid to the differences between 'local' and 'global' geometry because that can deepen the concept of 'educational validity'. However, there is also the issue of "intuition" in geometry. Ephraim Fischbein has written about intuition in his book 'Intuition in Science and Mathematics'. According to his concept of intuition, the intuition of people is linked to their experience. Consequently he states that for a physicist a straight line corresponds to a light beam. To a pupil, a straight line is a line drawn on a sheet of paper. To a traveller, a straight line means going straight ahead. In this way Fischbein supposes that the concept of a straight line is formed by a physicist, a pupil and a traveller respectively. Their concept of what a straight line is, is strongly linked not only to their experience in the past, but also to the situation they are in at the moment. So the pupil, supposedly having studied physics at a later stage, shifts his notion of a straight line towards what in his profession normally should be the image of a straight line: a light beam.

From these examples Fischbein concludes that there is no 'straight line' in reality and that it is obviously an abstraction (Fischbein, 1987, page 20).

However, with such a characterisation of 'straight line' we are not able to progress in the direction of 'visual geometry', let alone towards a discussion of the differences between 'local' and 'global' geometry. When the straight line appears to be nothing more than an abstraction, a product of the mind, the actual visual images can in no way influence the study of the subject 'straight line' because the concept has already been settled 'by abstraction'. In the meantime the actual images appear, according to Fischbein's concept, to be no more than a personal experience with the abstract phenomenon 'straight line'.

If something has to be achieved in connection with visual geometry, we can not hold a view like Fischbein's. The difference between 'global' and 'local' geometry is not dependent on what the occasional person has ever experienced with straight lines. A global straight line should be global, not only with respect to a physicist, a pupil or a traveller; the notion of "globality" should be generally valid. Fischbein's book is discussed in more detail in 'Review', Chapter VI, section 6.3.

It is interesting how we can define the globality generally and specifically in the case of a straight line. The following example will serve to get an idea of what globality means. Ages ago it was assumed that the surface of the earth was flat. Why did people think in this way? The following must have been the case: when somebody looks around him, it seems that the soil on which he is standing, is flat. When the area is extended beyond, for instance, the horizon, it is understandable that nobody suddenly suspects that the total surface we have at our disposal might be curved and that the earth is a globe. So it was generally assumed that the totality of our soil (the global surface of the earth) had to be flat because a small part of it looked flat. This is a kind of misunderstanding which might have severe consequences for the concept of what the totality of a geometrical figure looks like.

Analogously, in the case of a straight line, it is understandable that people assume that the image of a straight line will not change when it is prolonged indefinitely. An additional problem is that nobody is able to observe such an image; it is simply too large to be surveyed by us. Here the problem arises of what a global straight line looks like; in what way can it be presented visually?

Popularly one would characterise a global straight line as a line without end; it will continue beyond every limit which one could have in mind but this does not imply that its visual appearance changes. In visual reality, as the observer perceives it, visual appearances change contrary to popular expectation. That is indeed the case as other examples will demonstrate. I took a photograph, standing on the rail of a railway, and three of the four rails running towards the horizon appear to be curved in the resulting image. Now one could remark that a photograph could be deformed by photographic gadgets. However, the rail, assessed by the eye, remains visually curved.

Now it is of course possible to talk about optical illusion because with the help of computations it is not possible to determine the visual deviations of the visual form of the rail. The only valid argument would be that the earth is a globe and that on such a globe straight lines simply cannot occur. This argument indeed is fully valid and it will be treated further in the thesis. However, it does not diminish the fact that a global straight line is often largely deformed if observed by the eye.

Further it will be necessary to have an adequate terminology. Without such a terminology one has to refer to all the reasoning preceding the acceptance of new terms. These new terms, once introduced, reflect the whole of the reasoning which has led towards this terminology. It has become necessary to denote a straight line segment by the words: "visual straight line". This means that mainly local geometry is involved.

Now we arrive at the educational component of the distinction between 'local' and 'global' geometry. As we might conjecture: our everyday geometry for 99 percent is a matter of local character. Normally one has little to do with what happens at the 'infinite distance'. Our daily concerns are mainly determined by travelling, architecture, directions which all can be depicted in terms of local geometry. The house in which we live seldom has aspects of infinite distance; nor have the places we are travelling to, for instance by train. So what is so special about the existence of global geometry?

We have already seen that the automatic application of local properties has led to an inaccurate concept of the form of the earth. And at this point we have to note that the ignorance of global straight lines may result in a wrong concept of what a straight line is or what it could be. We should note that the representation presented in Fischbein's book on intuition, quoted above, is not only incomplete but it offers a wrong image of the phenomenon 'straight line'. The concept of straight line is too limited and too restricted and from that point of view it is understandable that one arrives at the conclusion that a straight line is an 'abstraction'. Such a point of view is merely the result of a lack of information. The student has not been informed about the fact that a global straight line often drops the property of being visually straight. That is why the terminology 'visual straight line' has been chosen. Such a 'visual straight line' can not be anything else than local because the visual straightness ceases when it is prolonged sufficiently. So local geometry is considered as the domain of 'local straight lines'. This, of course, can be only relatively true. A line segment, sufficiently prolonged but finite, already starts to look curved. So the terminology 'visual straight line' refers only to the general idea of a configuration of limited range. A popular notion says that the shorter the straight line segment is, the less deformed it will look.

The matter of educational validity is now very near. It is quite clear that global geometry should be handled in another way than local geometry. However, that is seldom the case. A straight line in a picture, which is supposed to run towards the horizon, is often depicted as a visual straight line to denote that it is a straight line that is involved. Unfortunately it is supposed to be indefinitely long so it should not be drawn along a ruler; and this procedure leads to the unacceptable practice that global geometry is visualised by means which may only be applied in local geometry. When that happens one can talk about 'educationally invalid' presentations. These educationally invalid presentations are a severe blockade for the continuation of the study of geometry because the student is provided with incorrect visual images which cannot lead to a correct interpretation of what a global straight line is and thus stop further study.

We have already seen that the depiction of objects inevitably results in distorted drawings. Is not every visual image a distortion of reality?

It is quite evident that different pictures can be made of the same object. May we conclude that one of these pictures is the only right one and that all the others are wrong? Not at all. As everybody will agree, it is possible to have excellent alternative pictures of the same object, all of which are correct. Can it nevertheless be possible to draw distorted images? Take for instance a square. The opposite edges of the square are considered to be parallel but in a drawing these opposite edges might meet at the horizon and thus visually they are not parallel. That is why I do not assume that a square belongs to visual geometry. If a square did belong to visual geometry, its opposite edges would be parallel visually whichever way the square was placed. That is not the case and one has chosen for the visual presentation of a square as an 'abstract' model which approaches as closely as possible the supposed appearance of a real square. Such a practice is of course understandable but it yields a presentation of a square which should be called educationally invalid. It is assumed that the opposite edges of the abstract square do not meet at a visual horizon and thus the existence of such a visual horizon is ignored. But that means that global geometry is swept away because it does not serve the local presentation of the abstract square. Thus, if the abstract presentation of a square has to be called educationally invalid, its use has severe disadvantages, namely that it is a blockade for the geometrical understanding of the student. Not many students understand that the abstract presentation of a square is largely wrong from the point of view of visual geometry. So the presentation is considered to be fully adequate and valid but that keeps the student from developing further and on to higher geometrical notions. These new notions comprise, for instance, the introduction and understanding of non-Euclidean geometries which will be discussed in the next chapter.

We saw that the customary presentation of squares is educationally invalid but the same is true for the usual presentation of a cube. These educationally invalid presentations are a compromise between what geometrical abstract theory demands and the practical need to show acceptable images, which preferably should be beautiful as well. Now that last issue is of great importance. A beautiful presentation, which is also practically useful, appears to be attractive. Such perfect images have the disadvantage that the common thought is that the presentation and the concept are not different. One is inclined to accept that the demonstrated square is the square itself and no difference is distinguished between image and concept. That is wrong but it is the consequence of the fact that the presentation is so skilfully chosen, so beautiful, so powerful and so useful.

In the case of the above sketched geometrical forms, the square and the cube, the presentation is, at least in my opinion, extremely beautiful. It is true that there are possible alternatives but that has not prevented the fact that the most elegant models were chosen to present squares, cubes and other geometrical figures. Such a presentation of spatial forms as demonstrated in the case of the square and the cube can be abundantly found in visual art and especially in the visual art called Magic Realism. It is very understandable that the presentation of spatial figures is linked to forms of visual art which in a certain epoch were current. I think the same is the case here and the so-called 'abstract' squares and cubes are nothing else but examples of Magic Realism. The matter will further be discussed in Chapter II, section 2.2.

2.2 EDUCATIONAL VALIDITY DEFINED.

In figures 1-4 of this section four configurations are depicted. It is my purpose to focus attention on the phenomenon 'straight line'.

The figures 1,2, and 3 have been drawn but in figure 4 we observe a photographic picture taken by me when I was standing on the second rail from the left. Figure 4 is supposed to show a straight piece of railway. Observing the picture of the railway closely (figure 4) one detects something very remarkable. If we take a ruler and we place it along the second rail from the left (on which I am standing) it appears that this rail looks perfectly straight. However when the ruler is placed along the first rail from the left we notice that the ruler does not fit the rail. Compared to the straightness of the ruler, the rail on the extreme left seems to be slightly curved. The same happens when we try to investigate the two rails on the right side. Both rails, which supposedly are straight, do not fit the ruler we place along them. They seem to be slightly curved too.

These observations may cause doubts and uncertainties when we look at figures 2 and 3. In both pictures a horizon has been drawn and in both a cube has been depicted. In figure 2 the line ABF has been drawn along a ruler and we assume that it proceeds as a straight line towards the horizon. In figure 3 the line ABF has not been drawn along a ruler and it does not fit one. However, it is supposed to proceed towards the horizon as a straight line.

That is a contradiction. The straight line ABF in figure 3 has been drawn according to the model of figure 4, contradicting the model of figure 2.

This is what I mean when I say that geometrical issues may become self-contradictory when applied to visual geometry. The geometrical concept of a straight line leads us to contradictory visual images (the straight line ABF in figures 2 and 3). Moreover: at least one of the presentations in the figures 2 and 3 seems to be half true.

figure 1

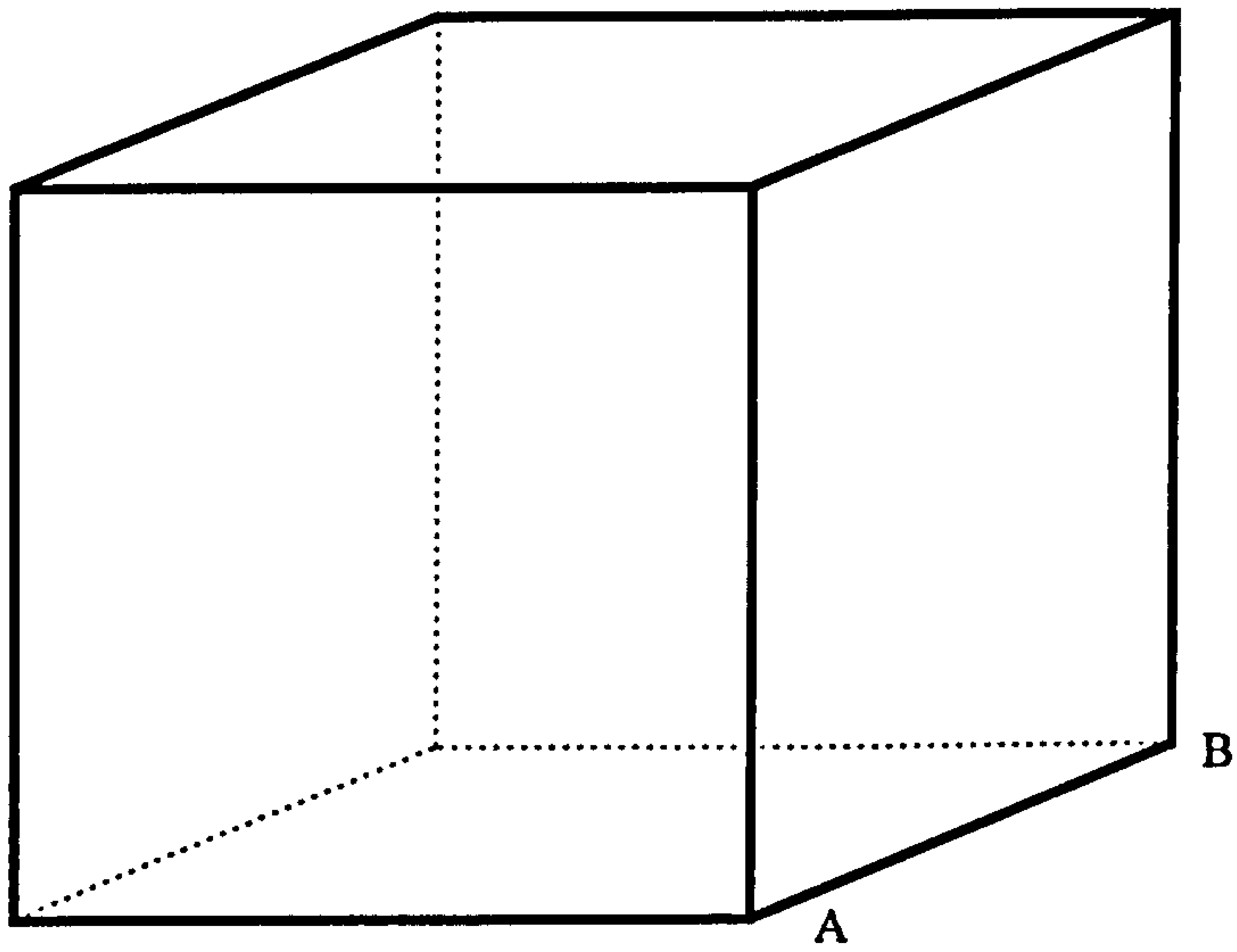


figure 2

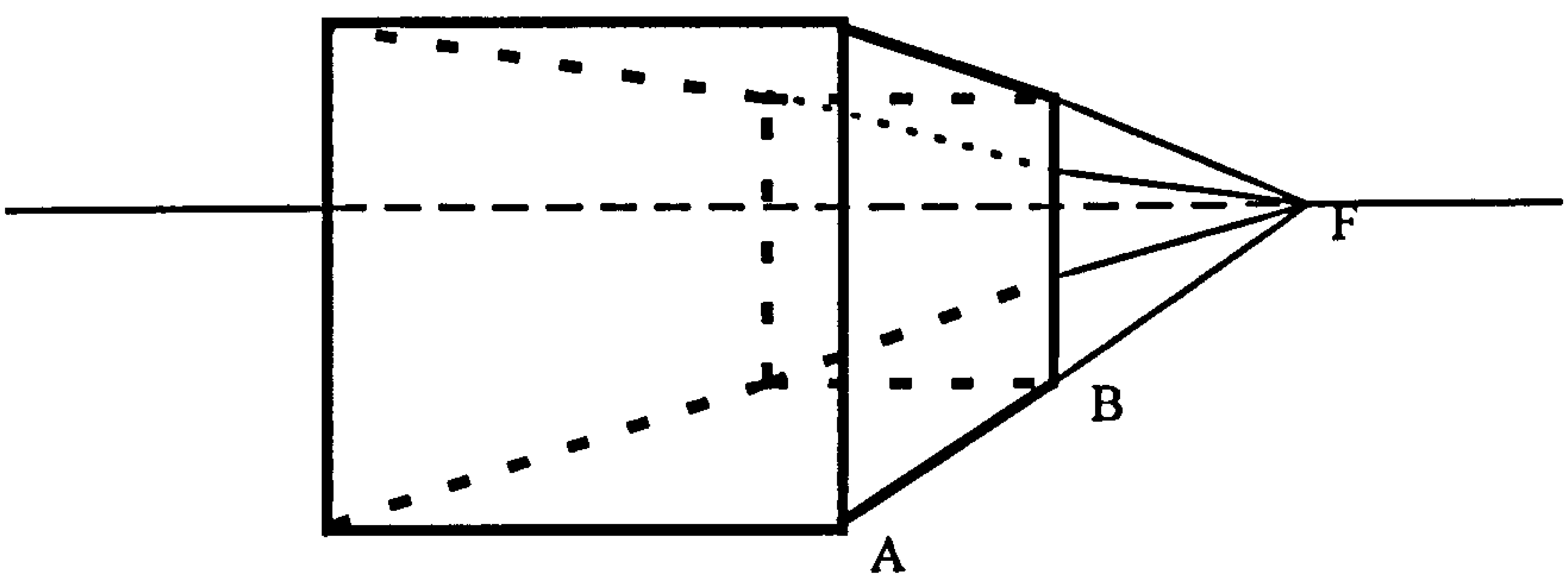


figure 3

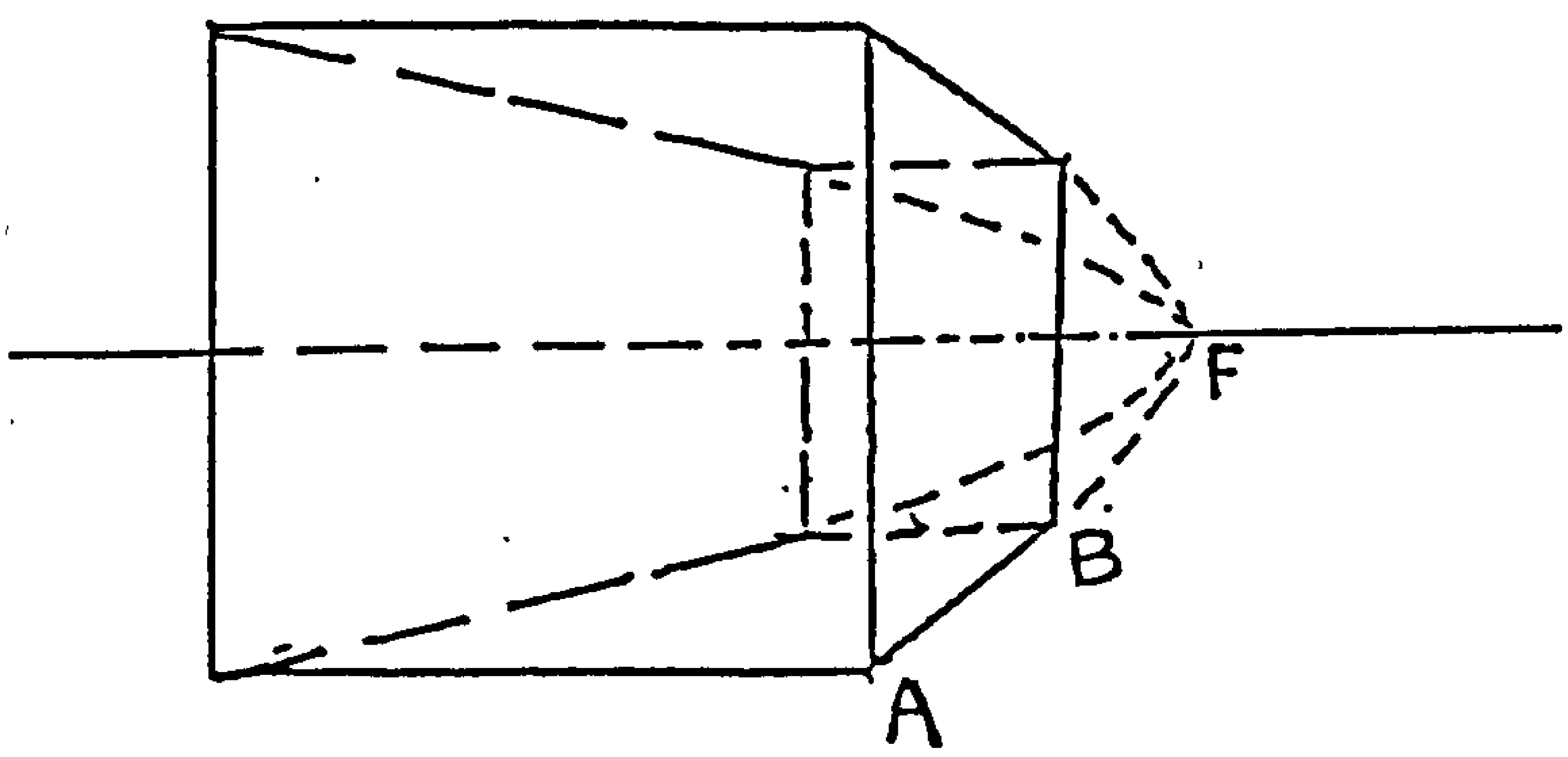


figure 4

These statements need further justification. One can argue that both presentations of the straight line ABF in figures 2 and 3 are valid but that the one in figure 2 is more suitable for technical purposes, such as to carry out computations. The models of figures 2 and 3 are not contradictory but rather complementary (with respect to each other). The idea of different models of a straight line has been worked out by Ephraim Fischbein:

"*Let us analyse another example, the notion of a straight line. It, is obviously, an abstraction. There are no" straight lines" in reality. For a physicist, a straight line corresponds to a lightbeam.To a spupil, a straight line is a line drawn on a sheet of paper. To a traveller, a straight line means going straight ahead.*" (Fischbein, 1987, page 20).

Despite all these examples, I maintain my statement that the presentations of the straight line ABF in the figures 2 and 3 are contradictory with respect to each other and that at least one of them is half true. The fact is that such examples as given by Ephraim Fischbein provide only a superficial notion of a straight line.

Fischbein confuses lines and line segments. What he is talking about is line segments. The line drawn on a sheet of paper does not exceed a limited length. The case of the traveller is even worse. Going straight ahead refers to a movement at a certain point and it looks rather like taking a direction standing on a spot, and this is within limited range. These line segments can be compared to the edge AB of the cube in figure 1. The edge also has a limited length.

However, the straight lines ABF in the figures 2 and 3 are assumed to be indefinitely prolonged. They are no line segments of limited length. At this stage I think it will be necessary to talk about terminology. What precisely do we mean by, for instance, " a visual straight line " ?

(1) Definition: A visual straight line is a line segment drawn along a ruler not exceeding a length of 40 centimetres.

So in figure 1 all the edges of the cube are visual straight lines. Fischbein's example of the pupil and the traveller also refer to visual straight lines.

Further the notions of '<u>locally</u>' and '<u>globally</u>' have to be introduced. The line segments AB in figure 2 and figure 3 are only a very small part of the total line AB which can be said to be indefinitely long. As soon as we consider line segments of small length we deal ' locally ' with that line. One has to recognise that the line segments AB in figure 2 and figure 3 have a length far less than 0.00000000001 of the total length of the line ABF.

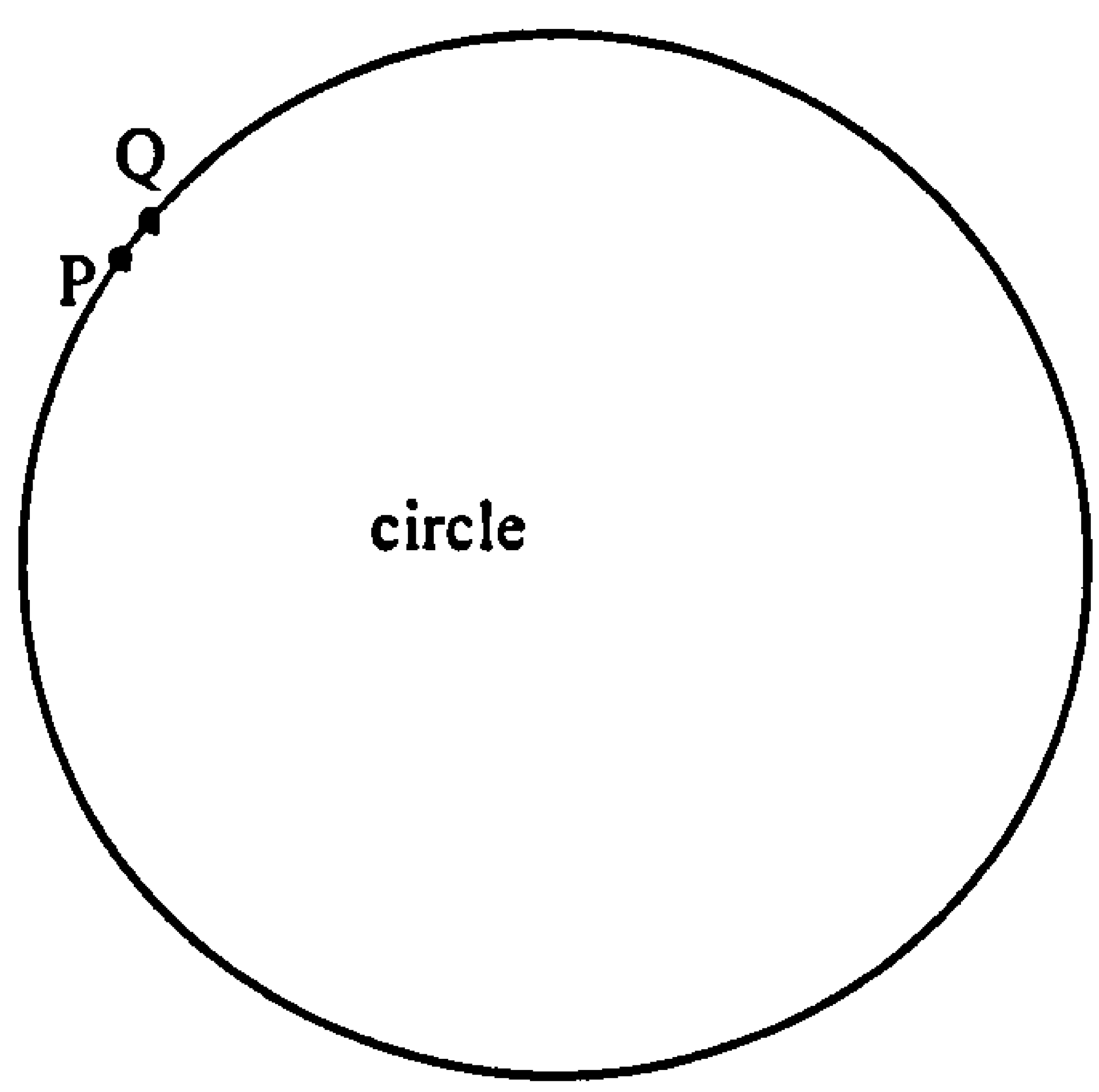
Most of the geometry we need in practical situations is local. A carpenter or an architect works within limited distances. The above-mentioned limit of 40 centimetres for visual straight line is arbitrary and refers mainly to the standard length of an average ruler. A length of, say, 100 meters, observed from a distance, may very well be seen as a visual straight line. But if you are too close to the line with a length of 100 metres it may seem no longer perfectly straight. In figure 4, for instance, 100 metres along the left rail already yields a curved image.

Generally one might state without taking the limits too strictly:

<u>**Local geometry can be depicted with help of visual straight lines.**</u>

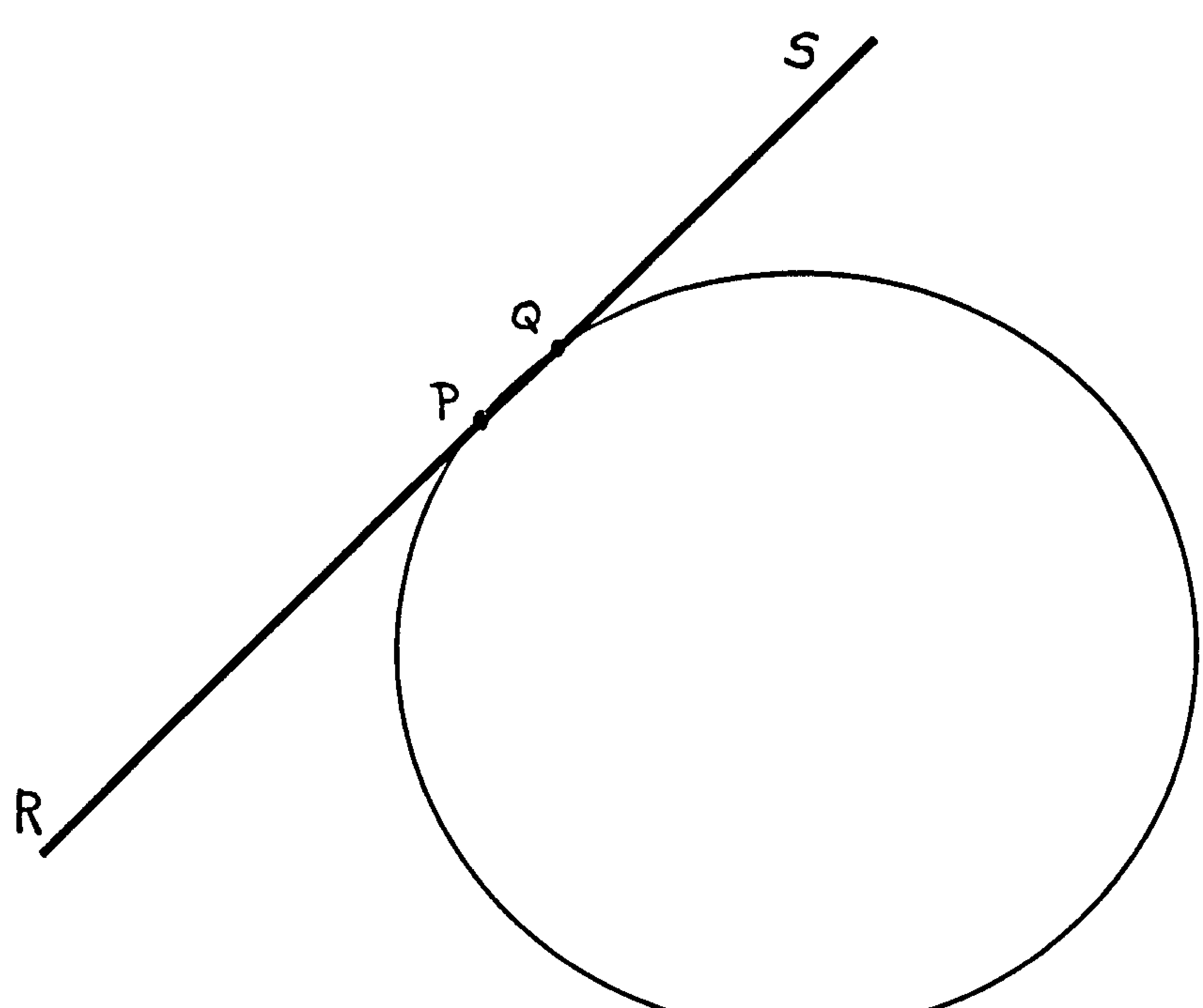
Next we come to the notion of 'global' geometry . Take for instance a circle (figure 5). The part PQ of the circumference is local but the circle as a whole is 'global' geometry.

figure 5



The example of figure 5 refers to a historical misunderstanding, already discussed in Chapter II, section 2.1. In former ages it was believed that the earth was flat. Let us represent the earth by a circle and assume somebody living in the local part PQ of it (figure 6).

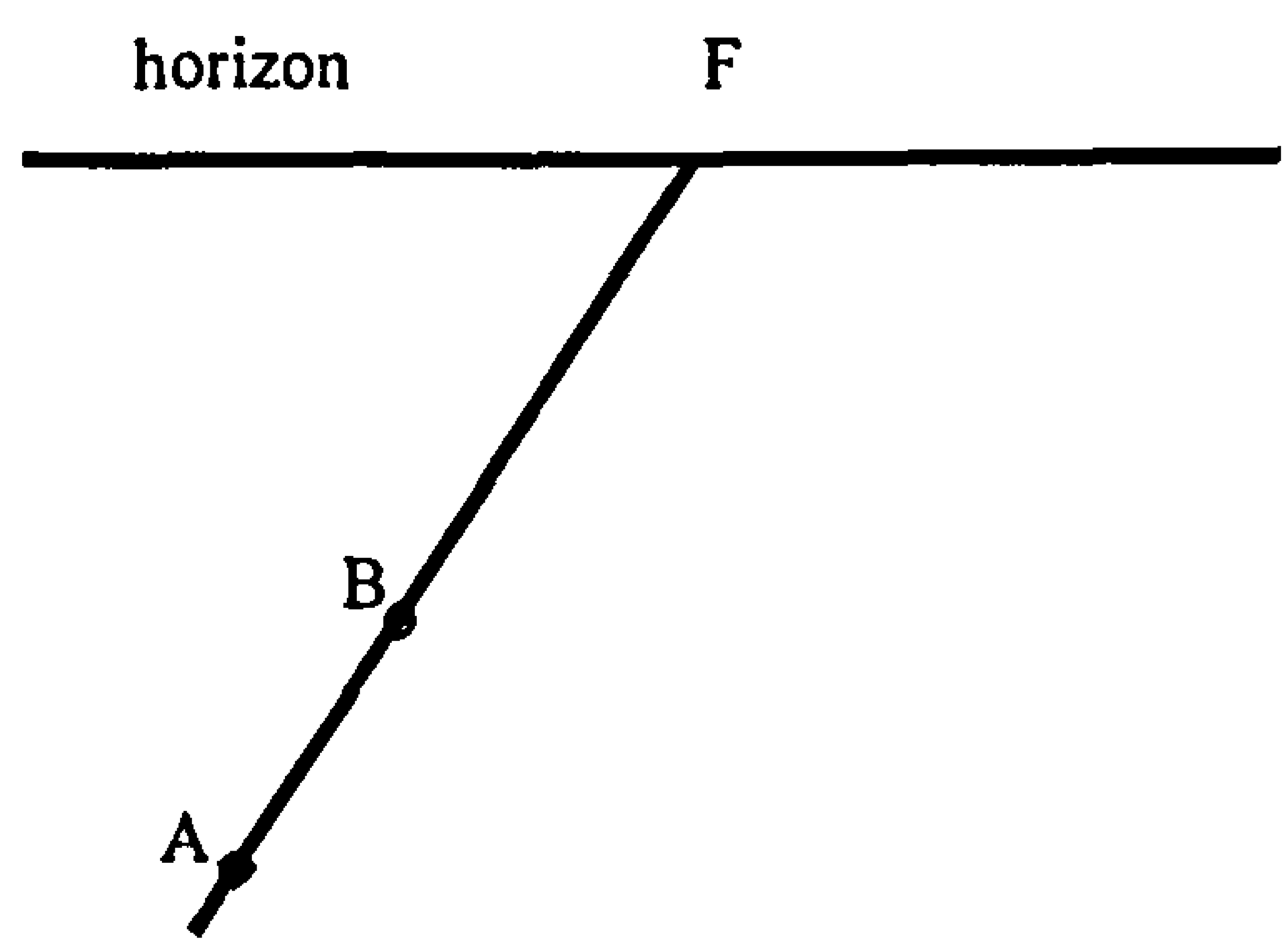
figure 6



People, living in the area PQ consider the soil on which they live to be flat. So in their opinion the totality of the earth is an enlarged copy of their own spot. In this way they believe that the totality of the earth is represented by the flat line RPQS and not by the circle.

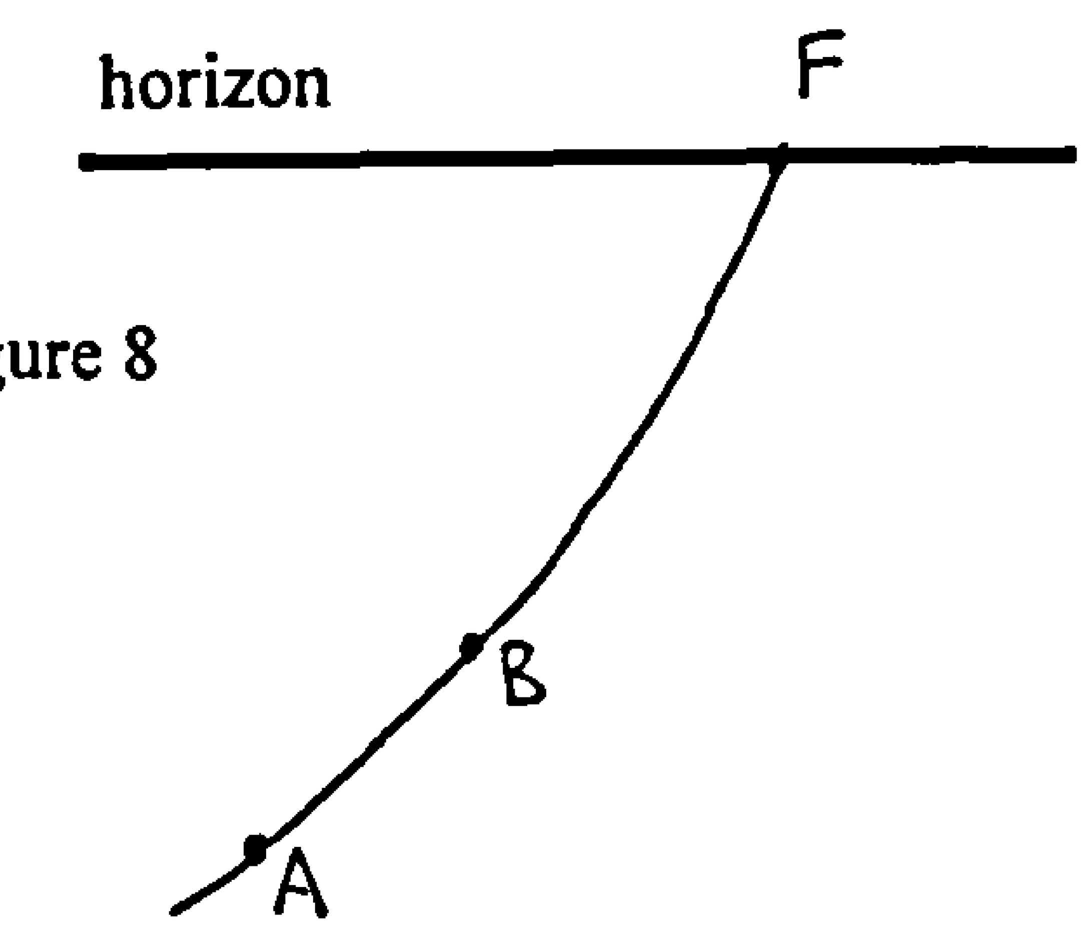
This is precisely the same sort of mistake we observe with people who think that the totality of a straight line is nothing more than a very large line segment. Superficially one assumes that a straight line proceeding towards the horizon is represented by the enlarged line segment ABF (figure 7). That is wrong. A straight line of indefinite length (on earth) is represented in figure 8 by the arc ABF and the point F.

figure 7



Wrong image of a line of indefinite length.

figure 8



Correct image of a line of indefinite length.

line = straight line

Now I can sustain my statement that geometrical assertions may become self-contradictory and half true when applied to visual geometry.

In figure 7 the line-piece ABF is an enlarged line-piece and so it can not depict a global straight line. However, it is supposed to do so. This is self-contradictory. In textbooks the straightness of the line-piece ABF in figure 7 is used to underline that a straight line of indefinite length is denoted. That is half true because ABF cannot possibly represent a straight line in figure 7.

I consider the representation of a global straight line in figure 7 as educationally invalid because it is erroneously portrayed by a local straight line. It also yields a distorted image of a landscape with a horizon because straight lines are represented wrongly. And worst of all: wrong presentations like the one in figure 7 hinder subsequent educational developments in using and understanding geometry. The representation of figure 7 may be seen as a block of the understanding of the image of figure 8 and generally of the development of the concept of a global straight line.

It will be our target to investigate the representation of a global straight line. It may be a help to improve erroneous images like the one shown in figure 7. The use of analysis will be dispensed with, except to support my view in Chapter III, section 3.4. page 49. It is my aim to keep my investigations focused on visual geometry, also with regard to students who will never come to terms with formulae and computations.

I have replaced the role of analysis by the application of visual art. A powerful artistic picture is able to focus on the essence of geometry. It displays instantly what is going on and thus speeds up the process of understanding. This is displayed in Chapter III and Chapter IV.

The following statement can be made:

(2) Definition: A representation is called educationally invalid when items of local geometry are erroneously applied to global geometry.

Next we will consider the relation between " Magic Realism " and visual geometry. It is a prelude to the display of global straight lines.

2.3 Magic Realism

In the following, 'Magic Realism' will be mainly considered as a notion of visual art, especially of the art of painting.

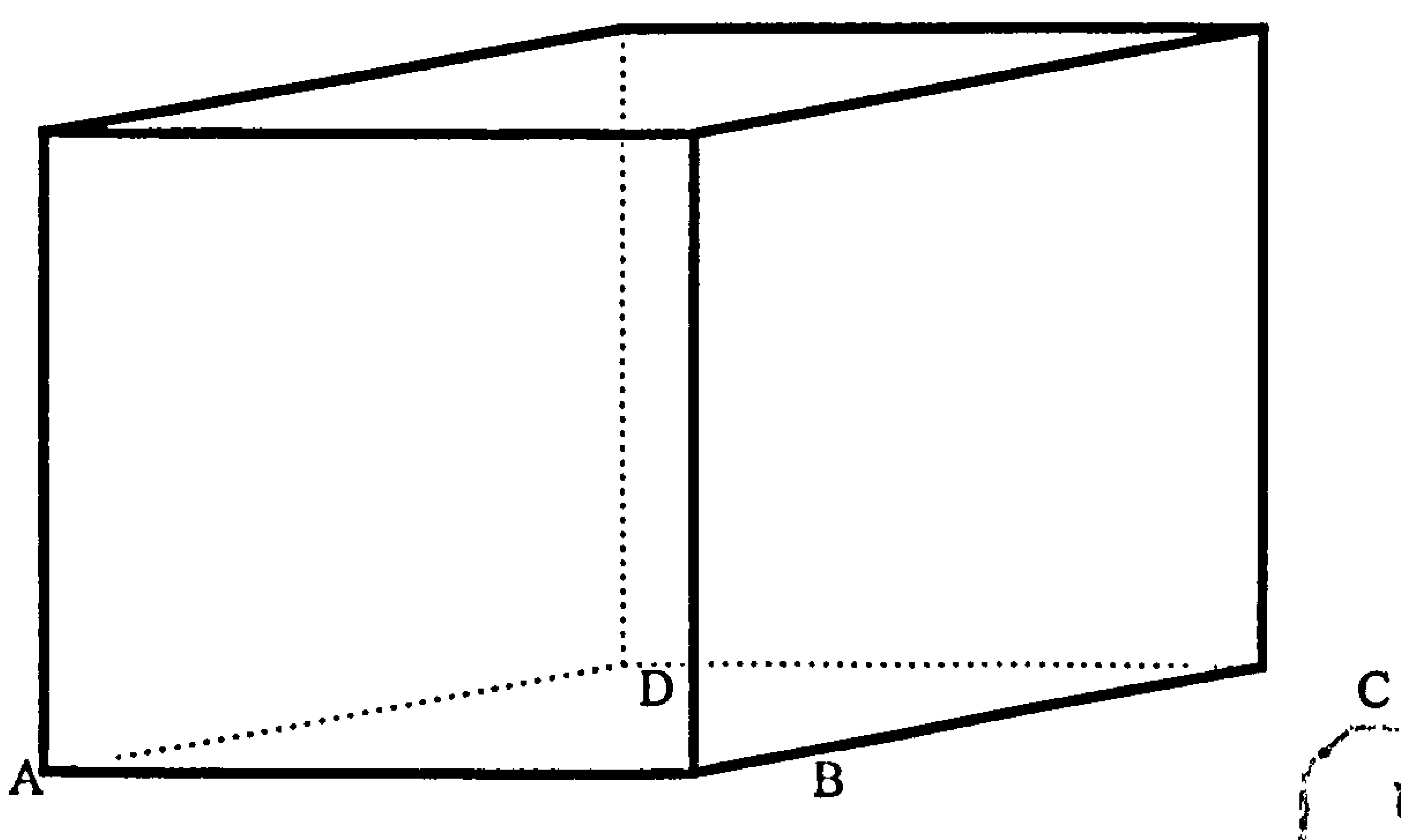
The term "Magic Realism" was formulated by Franz Roh and he gave a description of it in 1925. There is a comparable term "New Objectivity" which denotes the same movement in art but is more in use in the field of literature. Nevertheless these terms will from now on be considered as interchangeable to denote the artistic movement. New Objectivity started directly after the end of World War I and it reflected the feelings of the war-weary people of Europe and the United States in 1918. There are characteristics of Magic Realism in the art of painting which can remarkably be applied to the way images of visual geometry are drawn.

We have seen before that there seems to be a parallelism in development between the Art of painting after 1500 AD and nineteenth century geometry. This was the non-Euclidean geometry foreshadowed in the sixteenth and seventeenth centuries by painters (Houckgeest, Vredeman de Vries and others; see Chapter IV, section 4.2.) who applied perspective and artificial horizons. Their labour preludes the work of nineteenth century geometers such as N.Lobatchefsky.

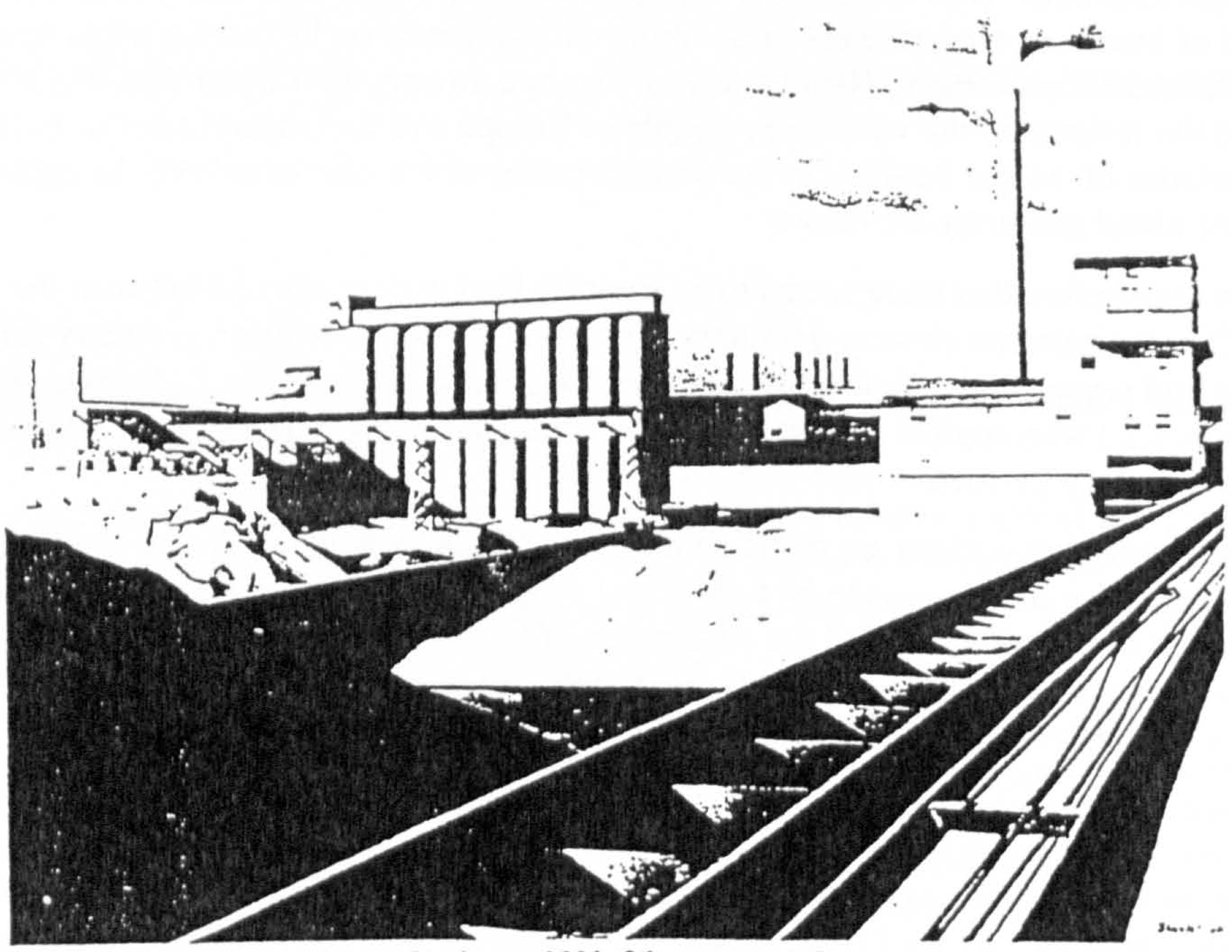
The way geometrical notions are depicted may well be dependent on the way painters look at these images. In our day there seems to be a shift in the display of geometrical notions. This is also due to the development of the computer and the emergence of the 3-D pictures to serve, for instance, medical purposes. It is a way to support the work of surgeons by demonstrating spatial images on the screen which give a cross-section of, say, the brain.

Let us take the example of the cube (figure 1) without perspective and the cube (figure 2) with perspective. The cube of figure 1 has served during decades as the best model to be demonstrated in secondary schools. In our day a model like the one in figure 2 seems to have become fashionable for secondary school use. As we know, the distances are more difficult to determine in the cube of figure 2 so that we may say that there is a move do decrease the amount of computations and analytical work and focus more on the so-called geometry of sight which emphasises the more visual part of geometry.

figure 1

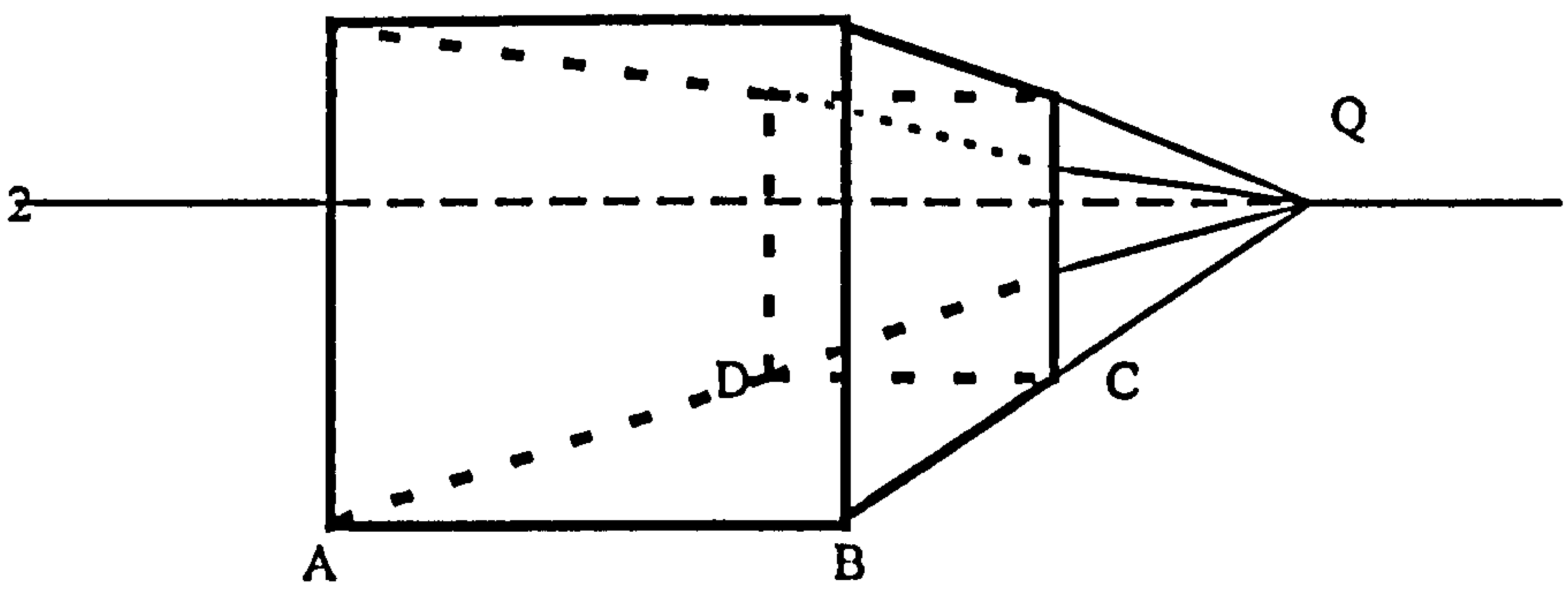


MAGIC REALISM



Charles Sheeler, *Classic Landscape*, 1931. Oil on canvas, 25" × 32¼". Collection of Mrs. Edsel B. Ford.

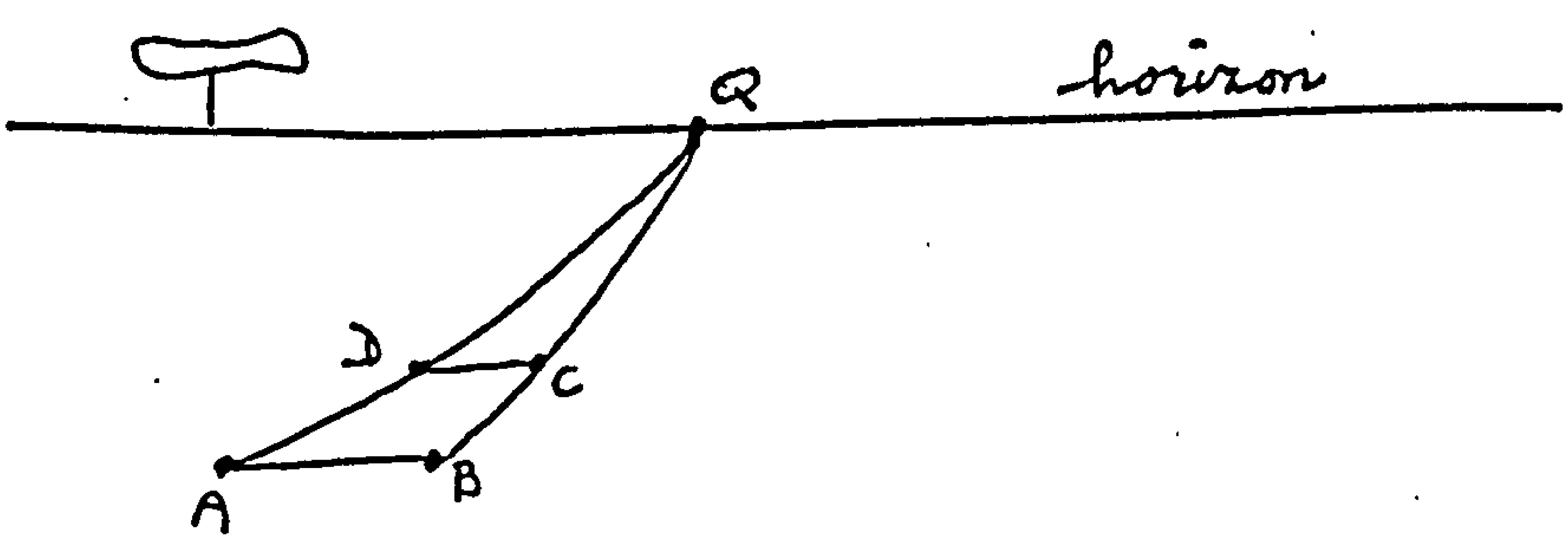
figure 2



It will now be demonstrated that both figure 1 and 2 are related to the visual art which is called 'Magic Realism'. We read: "*Ultrasharp focus is probably the single most dominant feature of magic realist painting. In and of itself, it produces a strange effect on the viewer. Whereas realistic painters in general follow the human eye's normal vision in gradually moving distant objects out of focus, most magic realists tend to paint all objects in the picture with an equally sharp focus*". (Menton, 1983, page 20)

What does 'ultrasharp focus' mean in this case? Let us start with figure 2. The straight lines AD and BC of the cube meet on the horizon at Q. As we observe in figure 3, these parallel lines AD and BC follow in reality visually curved lines and meet somewhere at a hazy distance probably at some point which we could call Q. In reality such a point Q cannot be located precisely. The lines AD and BC move out of sight when they approach the horizon.

figure 3



This is what we would normally perceive. However, the picture of figure 2 does not show a hazy distance around Q but a clear configuration of horizon and straight lines AD and BC. We have focused the infinite, far-away space 'ultrasharp'. This indeed, as Menton stated above, produces a strange effect on the viewer and one can say that the reality around Q in figure 2 is depicted in a 'magic' way. Here we observe Magic Realism in visual geometry.

The picture of figure 1 is also a subject of ultrasharp focus. Because of technical demands, all parts of the cube of figure 1 have to be displayed extensively so that we do not only look at a cube but also get a survey of the details. The cube of figure 1 has no 'dark corners'. All parts are highlighted equally sharply and that is a characteristic of a magic realist painting. Actually ultrasharp focus is a way to express an artistic feeling about reality. Considered in this way, our scientific images are not at all value-free pictures but reflect the cultural and artistic values of a certain epoch. A further feature of Magic Realism is 'objectivity'. The objects (landscapes, people, still lifes) are portrayed *"with an apparent objectivity that eliminates the presence of the artist from the canvas"*. (Menton, 1983, page 21). This is what happens in geometry textbooks. We are not supposed to know who actually depicted the images. Sometimes the artist is quoted in the preface but he never signs his creation which indeed generates a feeling of objectivity. Another meaning of 'objectivity' might be an almost obsessive interest in objects or things. The cubes of figures 1 and 2 could be considered as still lifes from which the artist who has drawn them has been eliminated.

Coldness is purposely chosen as an element of 'new objectivity'. By this is meant that, for instance, products of magic realistic visual Art appeal more to the intellect than to the emotions. The pictures of the figures 1 and 2 can in a way be compared to 'naive' pictures:

" *In the primitive picture there is no unifying aerial perspective, no naturalistic lightning, no approximation of the appearance of observed reality. The spectator's eye travels from one portion of the picture to another, accumulating bit by bit the represented content, for it was in this way that the primitive artist constructed this picture.*" (Menton, 1983, page 22)

Now this is precisely the way the cubes of the figures 1 and 2 have been constructed. Every minuscule part of them is equally important and also of equal importance to the totality which is supposed to be the image of the cube.

The image of figure 2 is not only naive in an artistic sense. The cube and the lines prolonged towards Q consist of visual straight lines (definition (1)) and so the drawing, except for the horizon, belongs to local geometry (figure 2).

The location of point Q on the horizon is the result of a clumsy, primitive, naive and even emotional attempt to link local and global geometry (figure 2). Q as point of intersection of the visual straight lines BC and AD has a limited distance from the cube. However as a point on the horizon point Q is indefinitely far away (figure 2). The childish presentation of the cube in figure 2 may be suitable for immature pupils but it has to be dropped when a deeper knowledge is wanted.

The cube of figure 1 is a different case. I consider it as a distorted image of reality. By 'reality' I mean the following: if you place a wooden cube in front of you on the table, you are looking at a 'real' cube. In this case the wooden cube will never be depicted accurately by the cube of figure 1. In figure 1 there is no drawn horizon but the wooden cube is equipped with a horizon. I think that the distortion of the image in figure 1 with respect to the 'real' wooden cube is the cause of the great beauty of the image in figure 1. In geometry, aesthetic presentations are very important. We saw that the cube of figure 1 may be seen as belonging to Magic Realism. It is a product of fine art.

Actually the presentation of the cube in figure 1 is educationally invalid. The edges AD and BC are drawn parallel. This is a result of a local property. AD and BC will not intersect after they are prolonged because they are parallel. The global consequence of parallel straight lines, however, is that they meet visually at a point on the horizon. There is no horizon drawn in the picture of figure 1 and the horizon has been dropped because it would affect the local demands of visual geometry. The edges of the cube are visual straight lines (according to the definition (1)) so a horizon is not even possible. Thus the horizon has been omitted erroneously to allow local properties to prevail.

It means that the presentation of figure 1 is educationally invalid (according to definition (2)).

Anyhow in my opinion the presentation in figure 1 is an example of Magic Realism and it is a product of art of outstanding beauty. We will now proceed towards the presentation of global straight lines.

Chapter III

3.1 Global Geometry

It is only natural when people are confined to their immediate environment that they do not demonstrate much interest in territories which are remote and far away. So it is not surprising that generally geometry means local geometry, and that interest in, let alone knowledge of, global issues is limited. However, as a result of the study of astronomy and the possibility of travelling in space, the subject has been shifted more to the centre of common interest. At the same time investigations have been made about the true character of the space we live in, with amazing results. Nevertheless, there are also simple examples of global geometry which do not suppose the notion of elements infinitely far away. Take for instance the notion of a circle. Considering the circle as a visual image, it seems very much a local image because the distances from one point of a circle towards an arbitrary other point do not exceed the circle's diameter. But actually the circle is a global figure. Here globality is used in the sense of 'complete'. Looking at the visual image of a circle, one gets the idea that nothing can be added to it so that it is finished off; it can not be prolonged. Now the natural extension of figures is a rather complicated matter. If you have, for instance, a straight line segment at your disposal, so to say a visual straight line, then you might get the intention to extend it. In what way should such a straight line segment be continued? Normally one takes a ruler to continue the segment. A ruler, however, is little more than a gadget to draw; but the computation is not dependent on the qualities of the ruler. So the continuation of a straight line segment by use of a ruler is a symbolic act but does not guarantee at all that the line produced is accurately straight. This problem also appears when the initial line segment is drawn. The line is drawn visually straight but the use of a ruler does not imply that the result is an impeccable straight line segment. And this is one of the issues of visual geometry. Visually it seems correct but when the result is scrutinised it appears that the image has no more than symbolic value. To be specific: when a drawn straight line is investigated with help of a magnifying glass, the line is reduced to a set of wildly scattered ink spots which no longer remind us of a straight line.

The accurate representation of a straight line is a problem in its own right but the continuation of a straight line is another one. Again, in what way can we prolong an existing line segment? Actually there is not such a method. In practice the prolongation is carried out with help of a ruler but theoretically no means are available to produce the straight line segment further. Analytically all line segments are given by formulae, but by the use of these formulae one already accepts that an accurate continuation of a straight line exists. This is not the case. This may be a warning that the image of a straight line might possibly not be visually straight.

We are now moving towards the concept of a non-Euclidean geometry. But first it might be helpful to show what a Euclidean geometry looks like. This can be best shown with help of the well known co-ordinate system. There is a 2-dimensional space R^2 which can be seen as a flat plane equipped with two mutual orthogonal co-ordinate axes; the X-axis and the Y-axis. We also know a 3-dimensional space, denoted as R^3, which can be seen as the space we live in, equipped with three mutual perpendicular axes: the X-axis, the Y-axis and the Z-axis. These spaces R^2 and R^3 are very commonly used in secondary schools. R^2 and R^3 are examples of so called Euclidean spaces.

It is not difficult to provide an example of a non-Euclidean space. Think of our earth which is a sphere. We are living on the surface of that sphere. Now the surface of the earth can be considered as a 2-dimensional space and every surface of any sphere will look like a copy of the surface of the earth. The type of 2-dimensional surface which can be identified as the surface of a sphere will be denoted by S^2. It is a 2-dimensional space but it is conspicuously different from the space R^2, which is Euclidean. Take the issue of straight lines. In R^2 the co-ordinate axes are what we call visual straight lines but such lines do not fit to the surface of our earth or to the surface of any sphere. However, also on the surface of our earth lines can be defined which may be identified as truly straight. To find such a line we have to take in mind the equator of the earth. This equator is a circle in R^3, the surrounding Euclidean space in which the earth is situated. But considered as a part of S^2 (a 2-dimensional non-Euclidean space), the equator has to be seen as a genuine straight line. Analogously, all the meridians, emanating from the North Pole and reaching the South Pole, are straight lines on the surface S^2. In this way we have not only a new kind of space (S^2), but also a new kind of straight line.

Now the sphere is such a common phenomenon that it will take some effort to convince the observer that the surface of it is geometrically something special. One is used to think of a football and why should the surface of a football be considered as a non-Euclidean, and thus extraordinary, space? The new geometries have to be presented in a more conspicuous way. This is what happens in the case of Escher's Pond which is a drawing produced by the Dutch graphical artist M.C.Escher. Escher's Pond is a creation in which the properties of non-Euclidean spaces are demonstrated in a very clear and obvious manner. It will be shown and discussed in Chapter III, section 3.2. and section 3.3. Escher's Pond provides a model of a 2-dimensional space which may be seen as a universe in its own right. It has its own type of straight lines and the residents of this universe are flat, of course. These residents are fishes which swim through the Pond from one point of the horizon to another point, following straight lines. These straight lines are presented by white, visually curved, lines. The rim of the drawing is a representation of the horizon.

We have now several examples of 2-dimensional spaces. First of all there is the Euclidean space R^2, then we have the spherical surface S^2, and finally we have the 2-dimensional universe, denoted as Escher's Pond. These three 2-dimensional surfaces will be compared to each other in Chapter III, section 3.2. and section 3.3. It turns out that locally they are not really different. If we take a very small part of each surface, we get three small pieces which are clearly comparable. Globally, however, these three universes differ widely. The straight lines, for instance, on a spherical surface do not visually end in a point but the straight lines of Escher's Pond do. Here we have obvious cases in which geometries may be locally equal, but globally they differ widely. This again emphasises that global geometry is of great importance in any geometry, Euclidean or non-Euclidean, and that the ignorance of global properties leads to unacceptable identifications of spaces which seem to be equal, but in reality are quite different. The ignorance of global differences, again, should be denoted as: educationally invalid.

There is a further issue about how a global straight line must be depicted. In the Euclidean space R^3, in which we live, such a global straight line cannot be perceived because it disappears in the distance. Already Euclid (300 BC) noted that geometrical figures will be deformed when observed from a far distance. So how do we have to depict a global line which is assumed to be straight in R^3 ? I took a Mathematical Dictionary and with help of the definition of a straight line from that Dictionary I arrived at the idea that a global straight line in R^2 and R^3 can best be represented by just two points. That may be amazing but any other representation has great objections to it. I think that the whole matter is that we, human beings, are organised in such a way that our direct surroundings are of more importance than the territories at far distances so that it is ultimately difficult to find an adequate way to depict those remote places. Analogously, I have wondered how somebody, standing on an indefinitely produced flat plane, will perceive that plane. It is a problem that was already discussed in the Middle Ages. That can be learned from a note in Euclid's Optics (Euclid, 1959, page 9) where it is stated that a flat plane, extended indefinitely, no longer will be flat visually. This assertion is said not to be due to Euclid, but seems to have been inserted in Medieval Times in Euclid's book which demonstrates that the subject has been a subject for discussion at that time. I have come to the same conclusion as the medieval author. A flat plane, in my expectation, will be perceived as a saucer by a person who is standing on it. The rim of the saucer is the visual image of the horizon.(Chapter III, section 3.4., page 45)

Actually no such large planes can be perceived in reality. But I assume that, even if such a large plane could be constructed somewhere, the visual appearance will not be as simple as an uncomplicated flat piece. Instead of that, the plane will be heavily deformed at the horizon, that is to say far from the observer.

So the horizon is some kind of a virtual line, not a real one. Sometimes I think that straightness can only be found with help of these kind of virtual lines. Imagine the following: an observer is standing at the centre of a large sphere, say of the magnitude of the earth. Then, when this observer looks at the equator, he is watching a straight line. I find some support for my view in Spinoza's philosophy (see Chapter VI). As far as I can see straightness is a notion that has roots in intuition in the sense of the third kind of knowledge, indicated by Spinoza. Like a horizon, the notion of straightness can not be pinpointed somewhere, in a real line for instance. Going back to the observation of the equator, the equator of such a large sphere as the earth is, observed by someone standing at the centre, can not be distinguished visually from the horizon. So I might be inclined to assume that the only genuine visually straight lines can be identified as virtual lines serving as a horizon. This consideration follows the results of computations which I carried out. These computations showed that in general,a straight line, running to the horizon, can not be straight visually. The computations can be found in Chapter III, section 3.4. There is one problem with these computations: the use of the formulae with the help of which the computations were carried out, is based on the assumption that the plane is impeccably flat but the result turns out to be that the plane visually is not flat. So the instrument I used to demonstrate that the plane looks like a saucer may be affected by the conclusions of the investigations. If the plane would not appear to be flat at all, then the use of the formulae would become a problem.

3.2. Different Global Geometries

M.C. Escher (1898 - 1972)

A picture produced by M.C. Escher will display the notion of a global straight line. It means that complete straight lines are portrayed.

The picture was called " Cirkellimiet" by the artist and he produced it after consulting a book, written by H.S.M. Coxeter, who is an authority on Geometry. Coxeter was satisfied with Escher's drawing and he wrote to Escher that he found " Cirkellimiet" very interesting.

About his own picture " Cirkellimiet" Escher wrote:

> " *No component of all these series of fishes, which emerge like fire arrows perpendicularly from the limit and get lost in it again, will ever reach the borderline".*(Bool, 1993, page 153)

In figure 1 the picture which I took standing on the rail of the railway is displayed again. Issues from this picture are used to explain Escher's drawing.

The 4 rails (figure 1) are all supposed to be following a straight line. However, three of the four rails shown seem slightly curved. (which can be checked with a ruler).

Often pictures show curved lines, which supposedly are straight (see figure 2).

figure 1

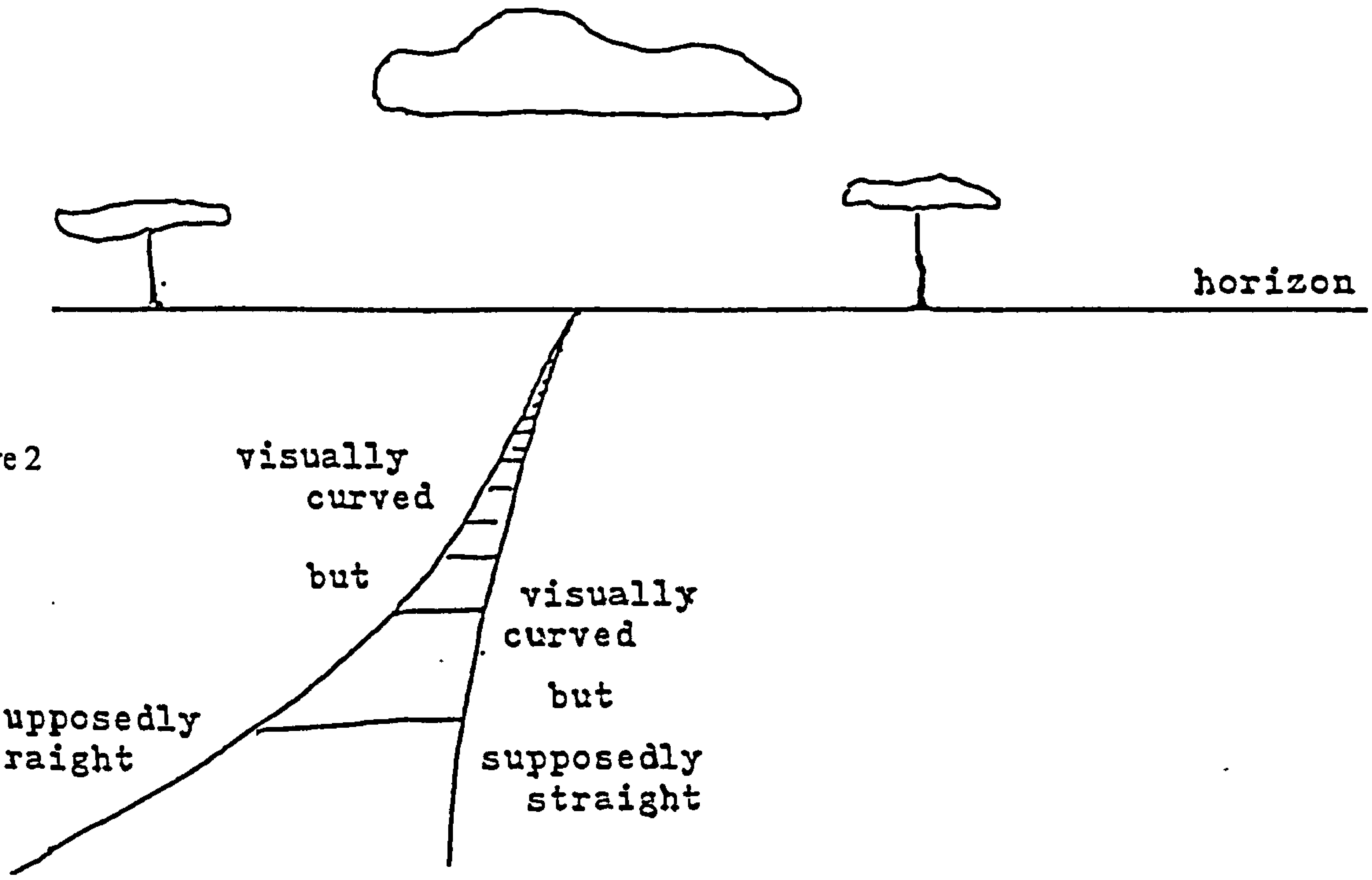


figure 2

It is very remarkable that most of the straight lines which we observe in our environment should be drawn curved. That is what space does. Straight lines, observed from a distance are visually deformed to curved lines. Now look at the 'cross' quadrangle in figure 3.

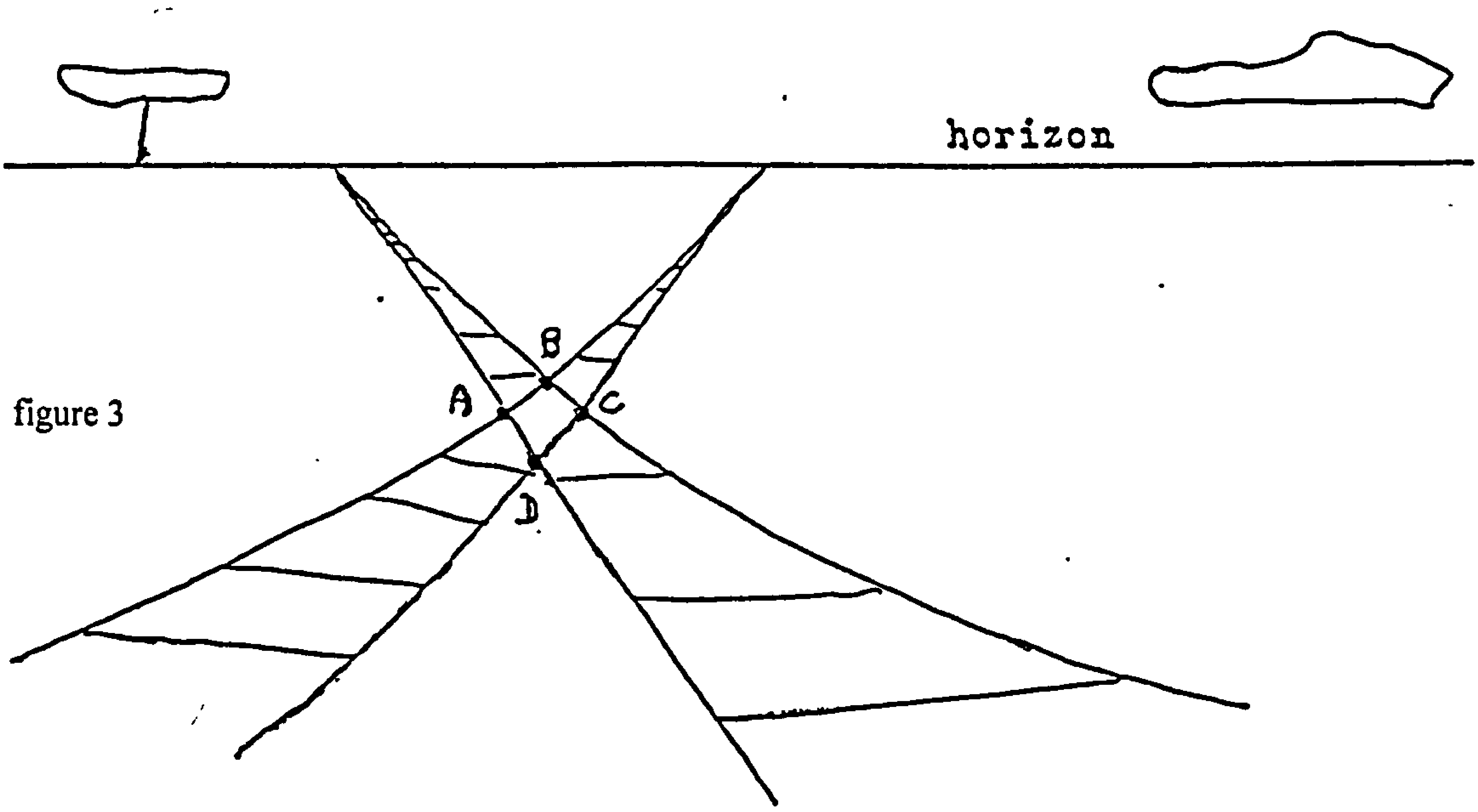
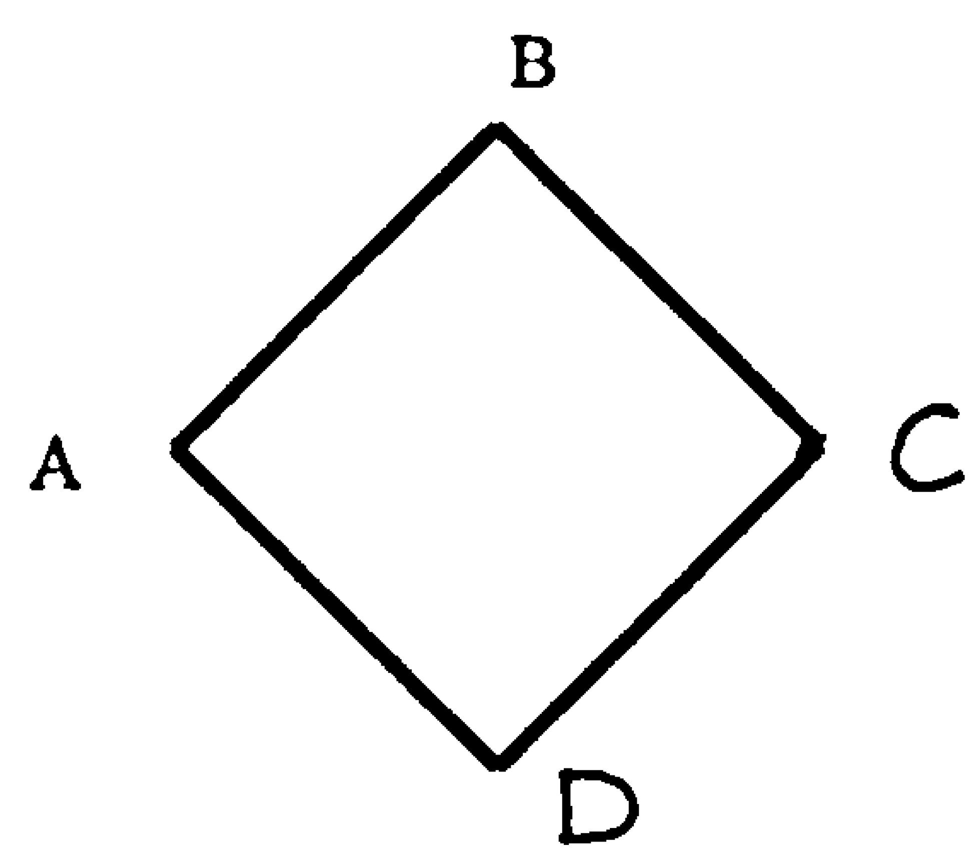


figure 3

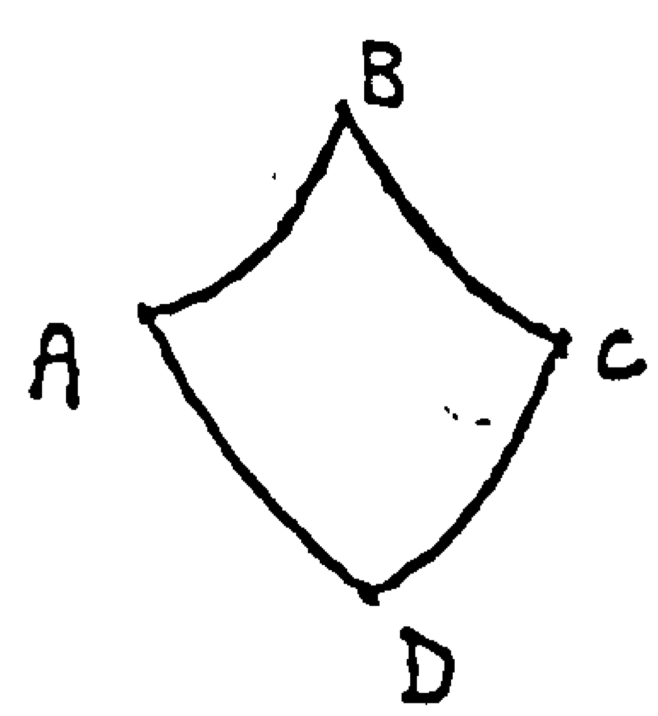
The edges of quadrangle ABCD are straight lines (the rails in figure 4).

figure 4



However, the 'cross' quadrangle of figure 3 shows 4 curved sides (see figure 5)

figure 5



An object, travelling towards the horizon, not only seems to become smaller, but its shape is visually deformed.

The foregoing is confirmed by the stories of travellers, lost in the desert. They try to escape by following a straight line. But a visual straight line turns out to be curved: see figure 6.

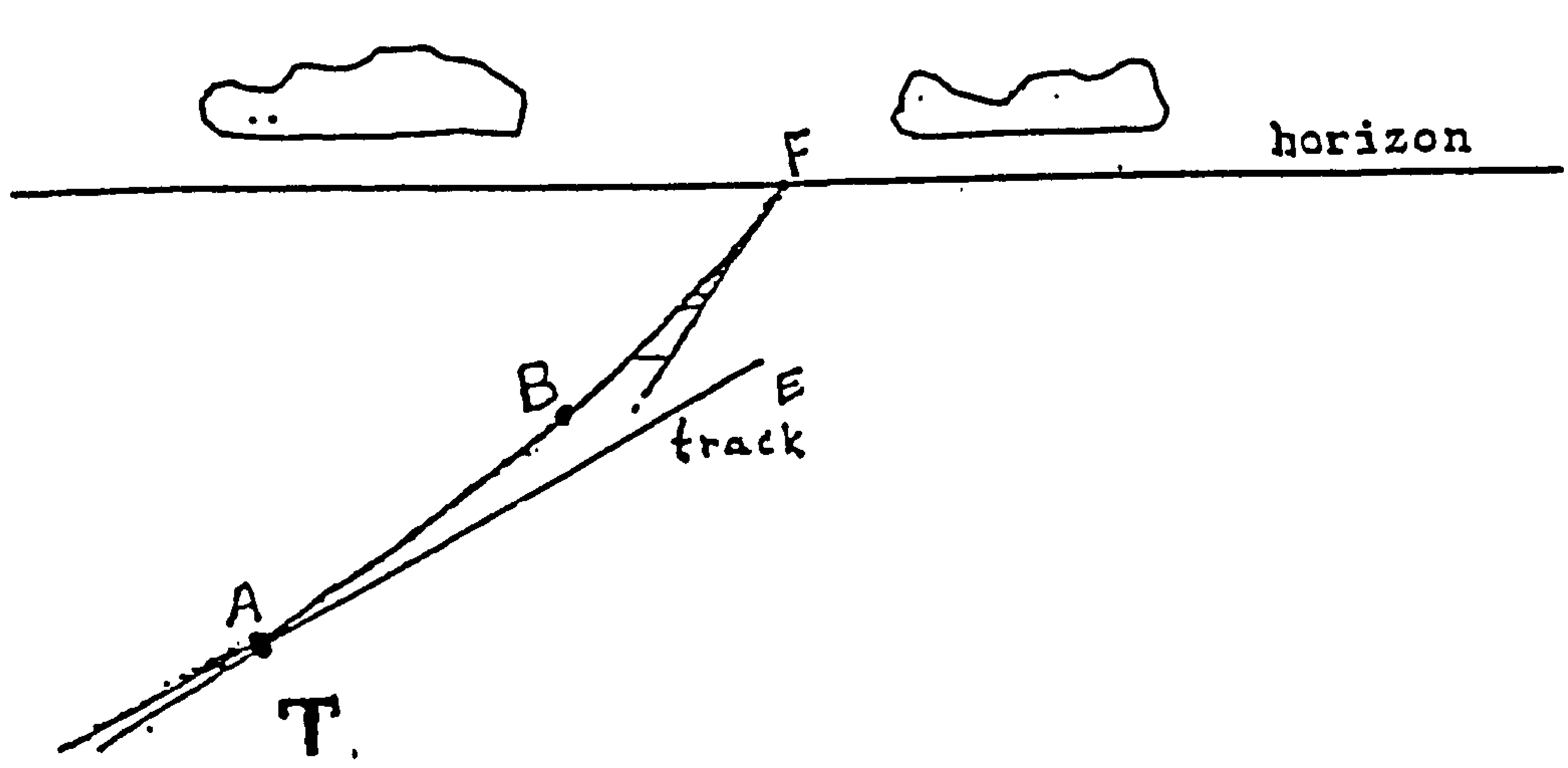
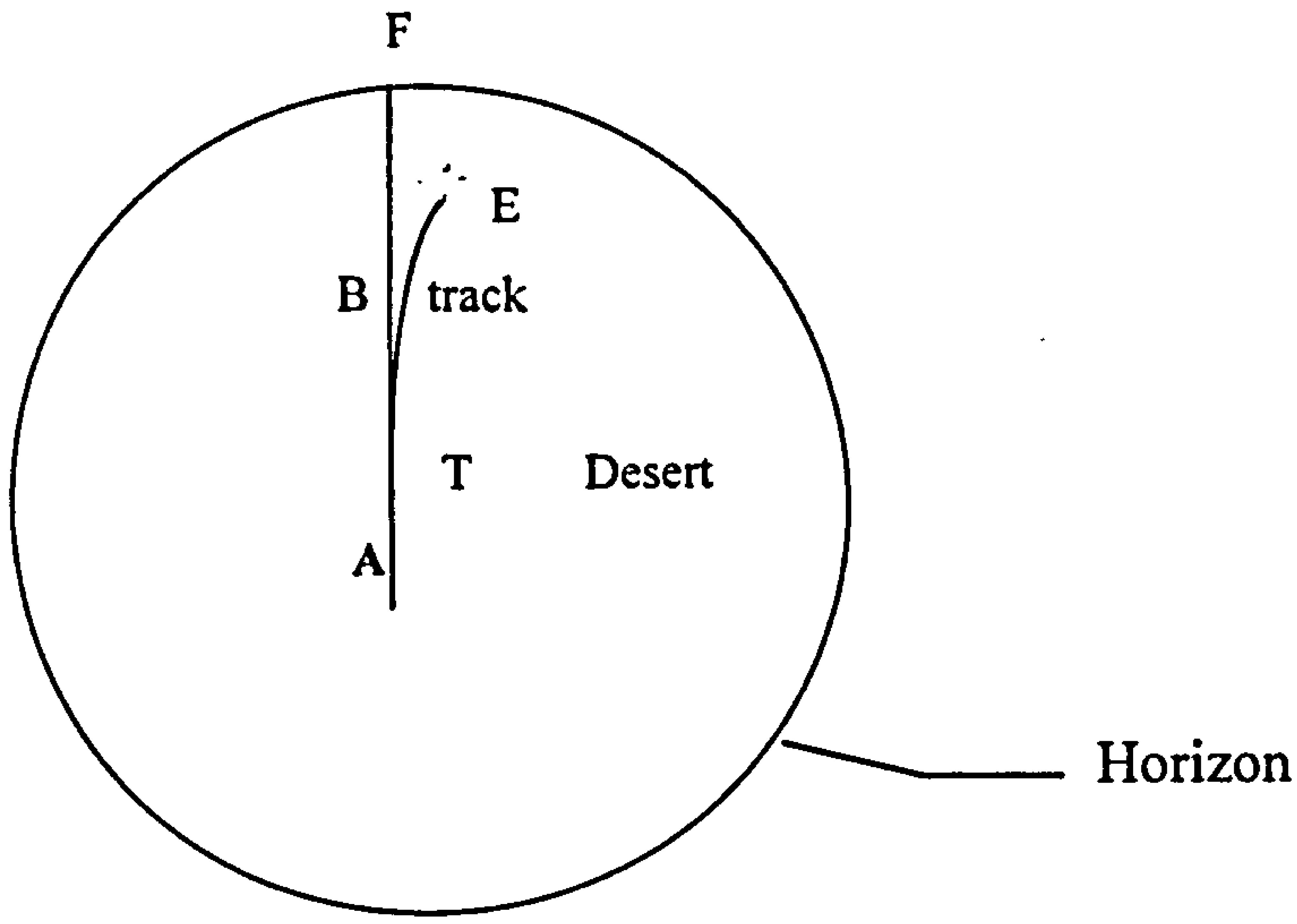


figure 6

The line ABF is the straight line: AE is curved. So, if you follow a visual straight track AE, you are travelling along the circumference of a circle, as the travellers often have noticed. After days of wearisome trudging along a seemingly straight track they are back at the starting point. The situation is sketched in figure 7.

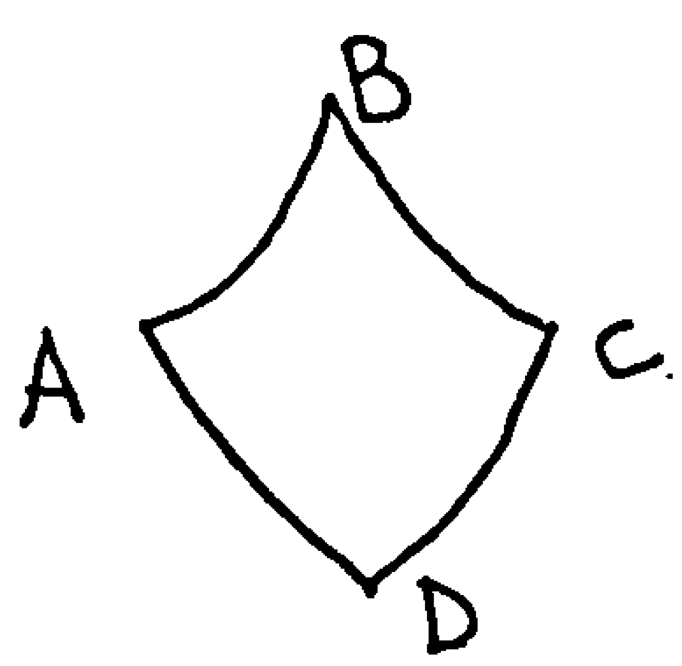
figure 7



Let us assume we are looking down from an aeroplane that has taken off to rescue the traveller T, in figure 7. As shown in figure 6, T thinks that he follows a straight line AE, but in figure 7 we see, that AE is curved, so that T moves along a circular track, which does not lead him out of the desert. The track in figure 7 is seen by T as straight.

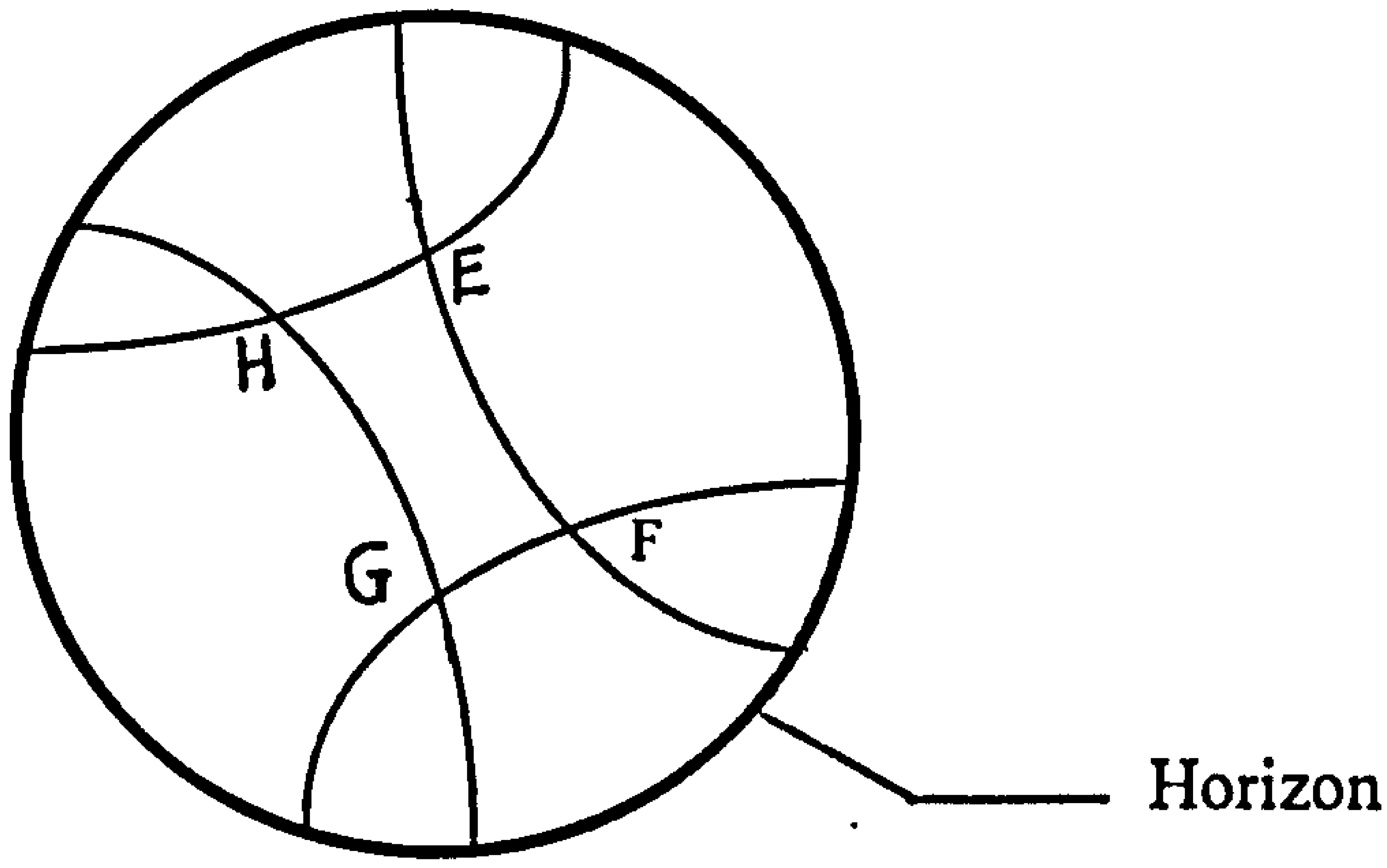
With this knowledge, complicated games can be played. For example, recall the quadrangle of figure 5, which shows 4 curved edges. These sides are supposedly straight (see figure 3).

figure 5



Now take the horizon as a circle and draw quadrangle EFGH in the interior of that circle (see figure 8)

figure 8



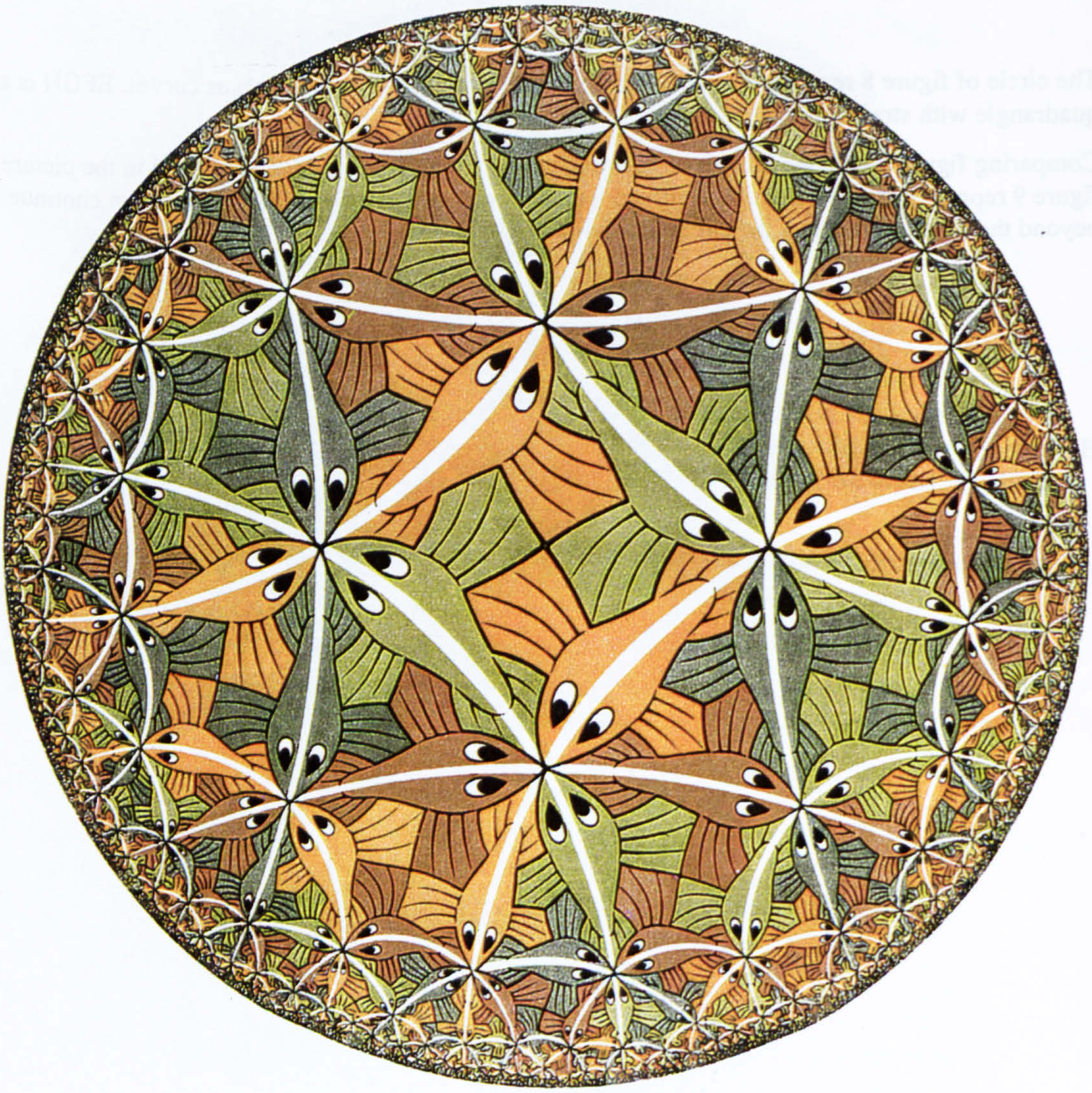
The circle of figure 8 represents a plane, in which the straight lines are drawn as curves. EFGH is a quadrangle with straight edges.

Comparing figure 8 with the picture of figure 9, we notice that the white curved lines in the picture of figure 9 represent straight lines, The curved lines in the pictures of figures 8 and 9 do not continue beyond the circumference of the circle (horizon).

3.3 Escher's Pond

Escher's drawing (figure 9)

It shows a pond, in which an infinite number of fish are swimming. The circumference of the pond is a circle which represents the horizon. Fish swimming towards the horizon are drawn smaller than those in the centre, but their sizes are supposed to be equal. The white curved lines represent straight lines, although they are drawn curved. The fishes proceed along the white lines.



CIRKELLIMIET by M.C. ESCHER FIGURE 9

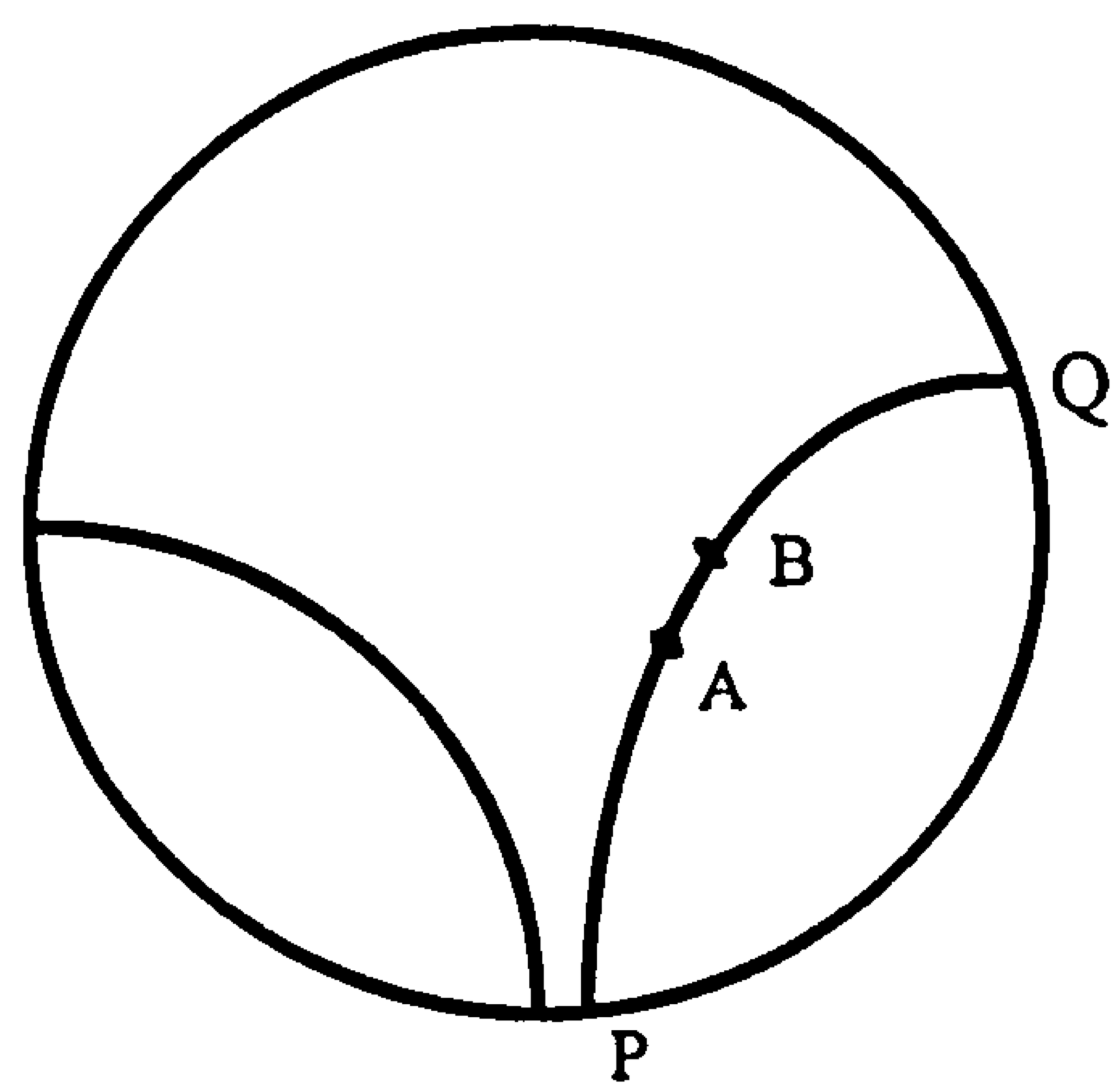
Escher's Pond is a geometrical universe in its own right.

It is possible to define distances, angles, areas and so on. Its great advantage is that it has been drawn on such a scale that one can lay it on a desk and study it.

A straight line of indefinite length can be assessed at a glance. It is a simple arc. In figure 10 the arc PABQ represents a global straight line of infinite length and the small part AB may be seen as a visual straight line in the sense of Definition (1).

Actually AB is a ruler in the universe of Escher's Pond.

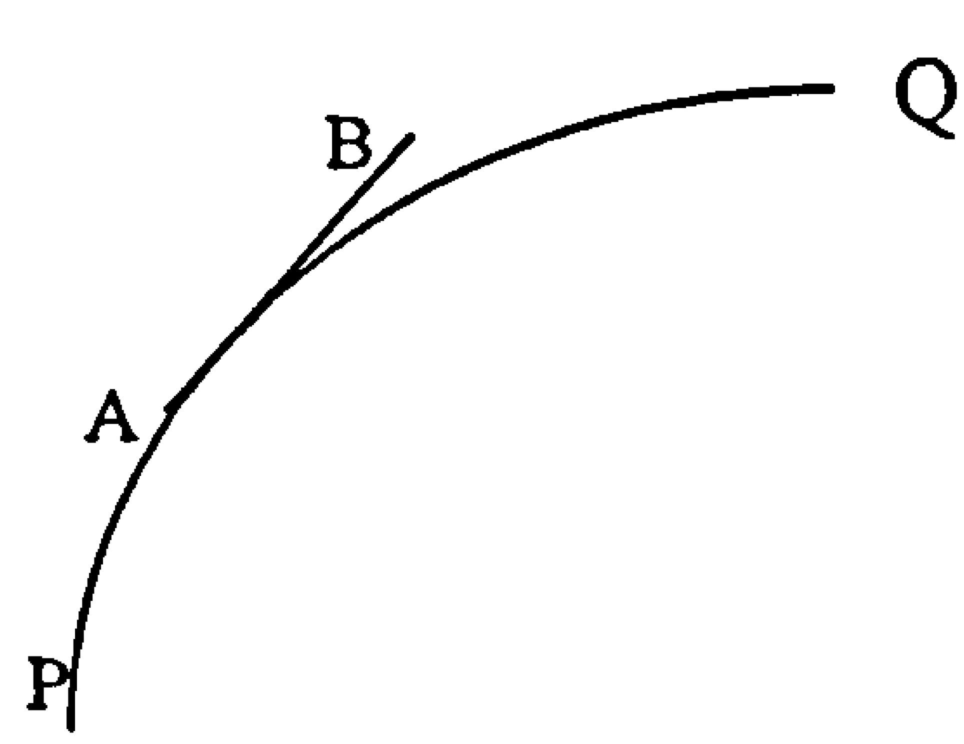
figure 10



Escher's pond

With the help of Escher's Pond, it becomes possible to study the relationship between local and global geometry. Escher's Pond is educationally valid because the global properties of straight lines are not erroneously influenced by local considerations (Def (2)).
The ruler AB is more or less a straight line piece in the same sense of straightness as defined in Definition (1). Its shape is obviously different from the shape of the global straight line PABQ. The ruler AB seems similar to the real one that lies in front of me on my desk at this moment.
The global straight lines of Escher's Pond are curved just like three of the four rails of the railway I photographed.
It may sound strange, but one could see the local straight line AB as a tangent to the global straight line PABQ (figure 11).
In the space in which we live, global geometry is so remote that it is good to have a model like Escher's Pond to remind us of it.

figure 11



Global geometry around us.

As I wrote about global geometry being so remote in the space in which we live, I noticed a globe standing on top of my book-case. Here was another geometrical universe!
The world in which we live is called R^3 because it has three dimensions: length, breadth and height. There is also an R^2 which has two dimensions: length and breadth.
Escher's Pond is a 2-dimensional universe: the fishes are flat. Normally, R^2 is identified as a plane. Now Escher's Pond may be seen as a rather artificial geometric universe. However, the sphere is a quite natural form (figure 12).

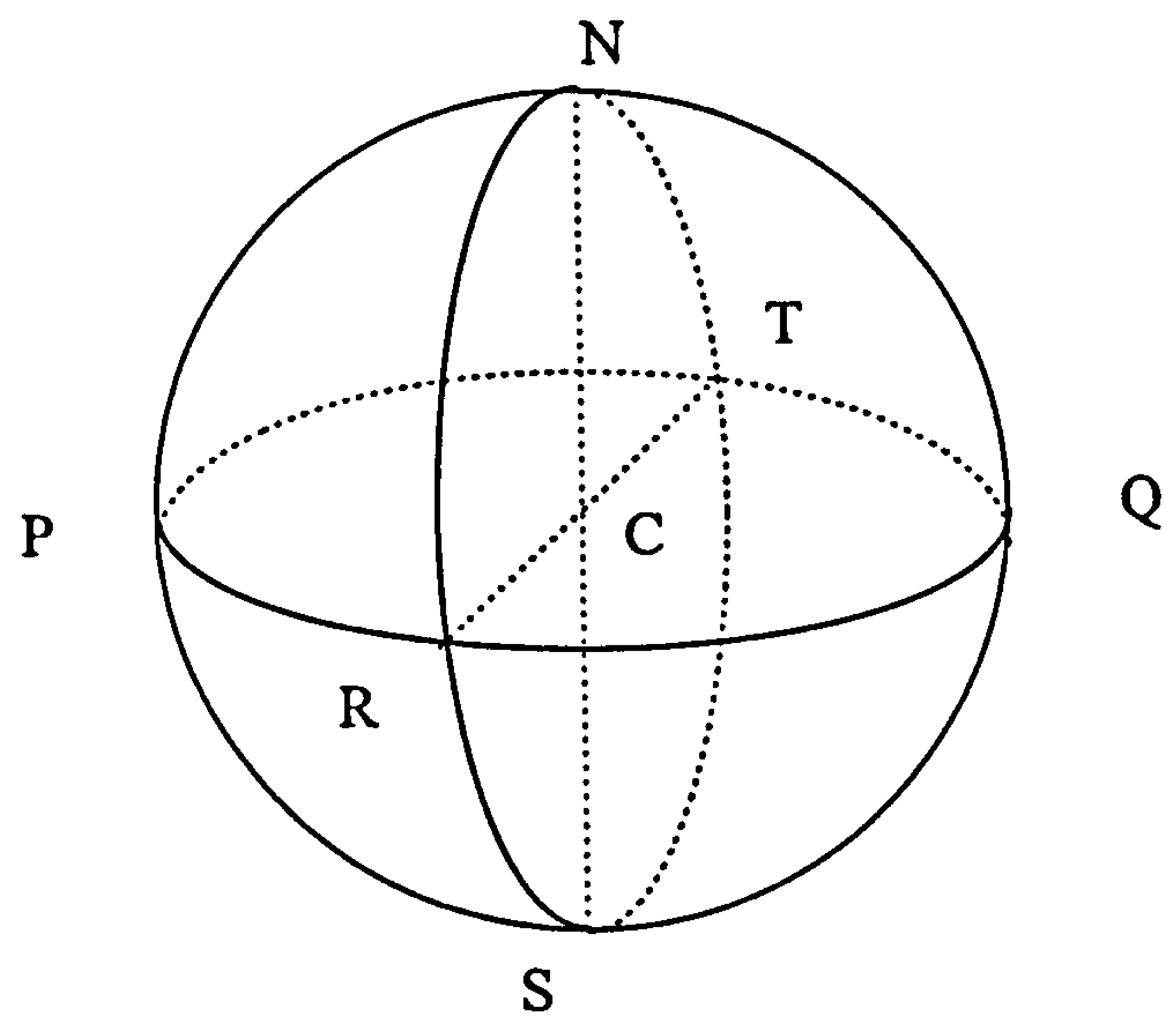


figure 12

Our new geometric universe is the surface of a sphere. Neither the interior of the sphere, nor the exterior of the sphere, counts.
In figure 12 our sphere has a North pole, a South pole, meridians, like NRST, and an equator, PRQT. The surface of the sphere can be seen as 2-dimensional but it is not a plane. Straight lines in this universe are circles on the surface of the sphere of which the centre is C, the centre of the sphere. Thus the circles PSQN, PRQT, RSTN and so on are considered as straight lines of the universe.
The surface of a sphere as a geometrical universe will be denoted by S^2.

R^2 as a geometrical universe.

We consider R^2 as a part of R^3; we denote R^2 as a subspace of R^3.
R^3 is portrayed in figure 13 by the three axes and the choice of a unit length.

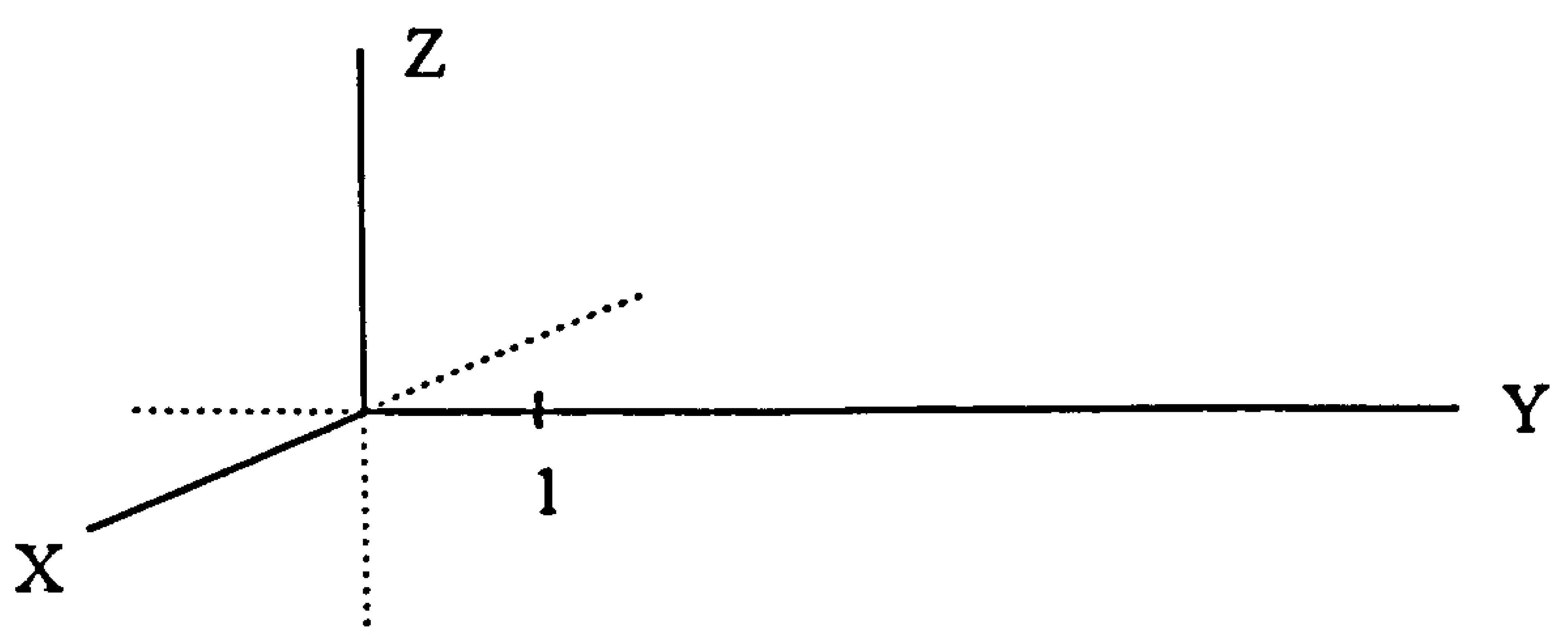
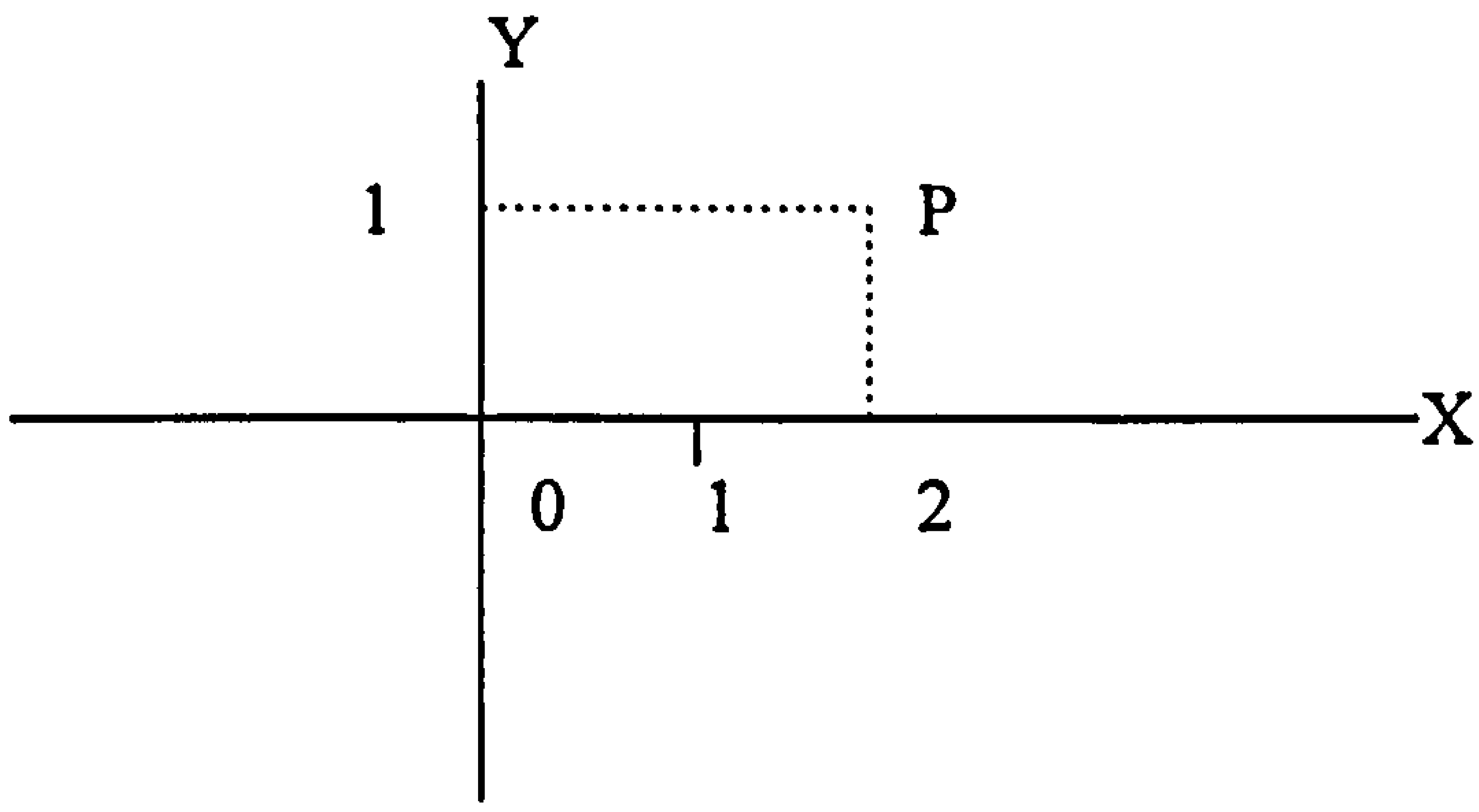


figure 13

The space R^2 is portrayed in figure 14 by the two axes and the choice of a unit length. Points have co-ordinates: the point P in figure 14 evidently has the co-ordinates (2,1).

figure 14



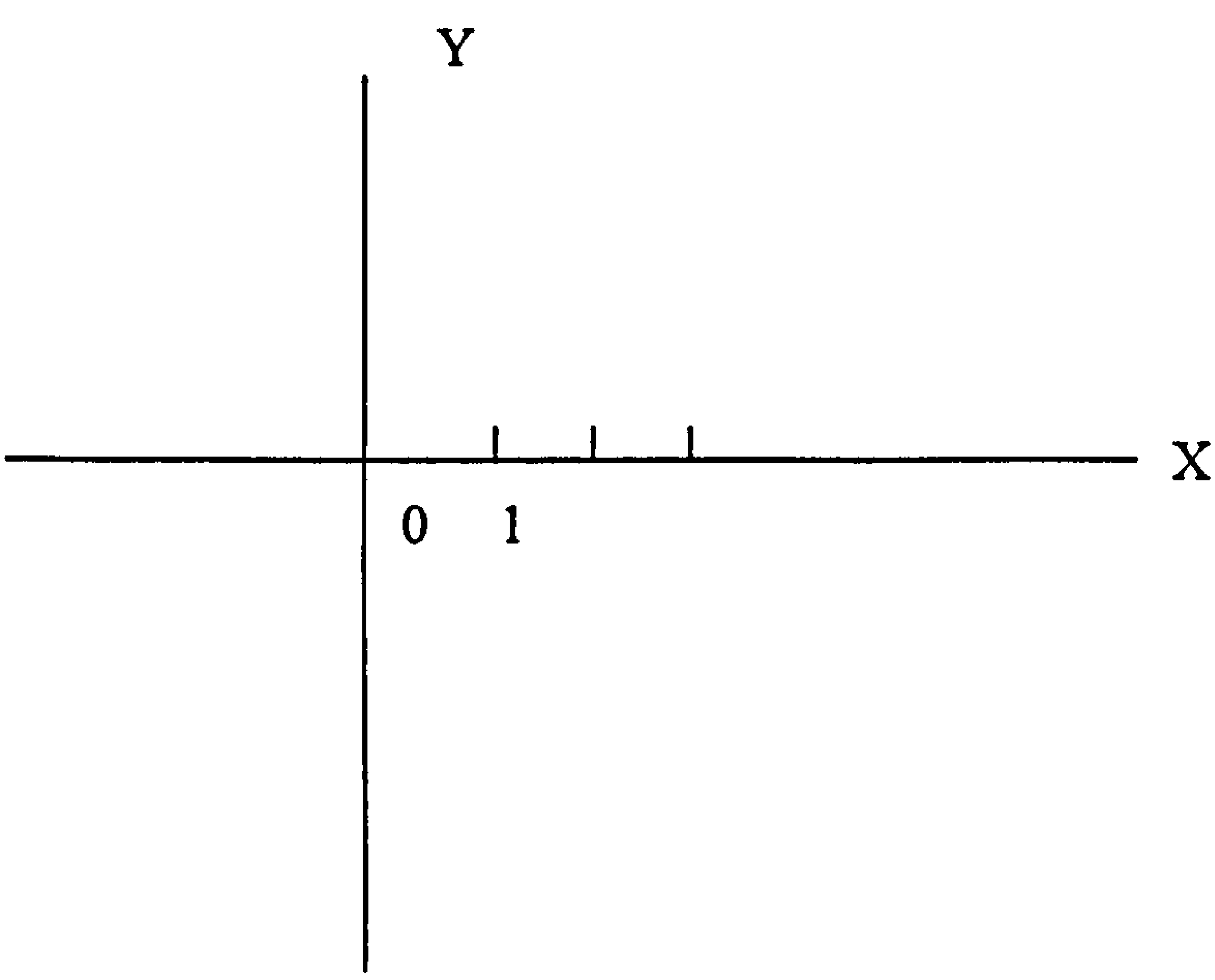
In R^2 and R^3 visual straight lines are in use (Def (1)).

Local and global geometry

When we compare local and global geometry, our investigations will be restricted to 2-dimensional universes. This means that we are confined to R^2, Escher's Pond, and S^2. In the following paragraph, "Escher's Pond" will be denoted by the abbreviation 'EP'.

The three universes R^2, EP and S^2 are not too different locally, with respect to each other. R^2 and EP can be considered as 'flat', but S^2 has a curvature seen from the three-dimensional space which it surrounds. Nevertheless a mapping is possible so that S^2 may be represented locally by the co-ordinate system in figure 1. The other universes, R^2 and EP, can be demonstrated locally in figure 1, also.

figure 1



Talking about local geometry means that the use of visual straight lines is necessary. (Def (1)).
So, in figure 1, the co-ordinate axes are represented by line segments of limited length.
In local geometry there are statements which are valid for the three universes R^2, EP and S^2.
Two such statements are: (1) Two points are connected by one straight line.
(2) Two straight lines intersect at one point.

Statement (1) is demonstrated in figure 2; statement (2) is demonstrated in figure 3.

figure 2

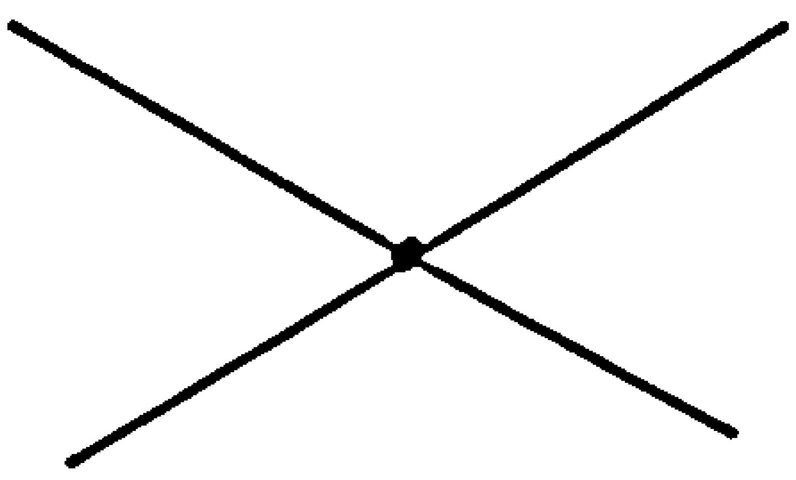


figure 3

The statements (1) and (2) are not automatically valid in global geometry. Consider statement (1): 'Two points are connected by one straight line'. A counterexample is yielded by the universe S^2 which shows that statement (1) is only locally valid. In figure 4, the points N and S are connected by the two straight lines, NPSQ and NRST, and many more besides.

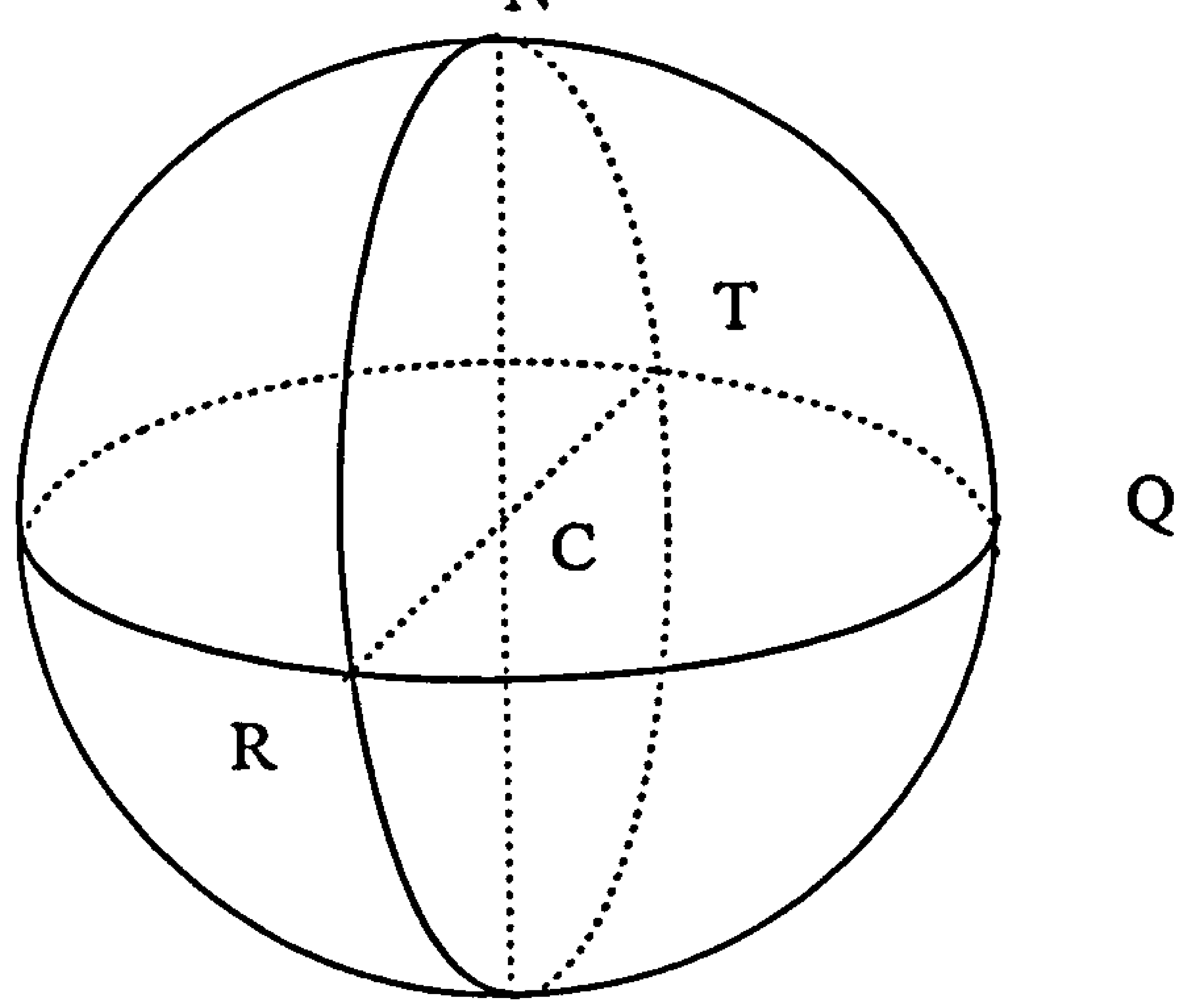


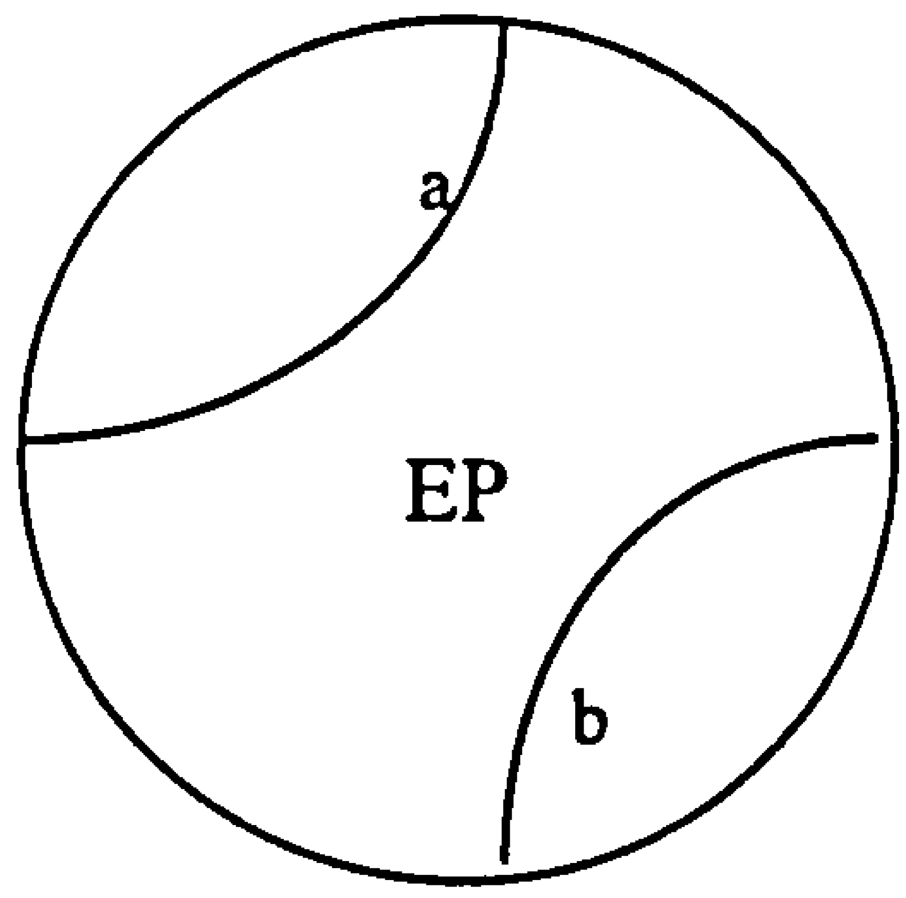
figure 4

The universe S^2

Now consider statement (2): 'Two straight lines intersect at one point'.

Again the universe S^2 provides a counterexample. The straight lines PRQT and PSQN intersect at two points: P and Q (figure 4).
A second counterexample is provided by EP (figure 5). The two straight lines shown, a and b, do not intersect at all (figure 5).

figure 5



These examples show, again, that properties of local geometry may be invalid in global geometry.

DICTIONARY

It is interesting to compare the foregoing example with the definition of a straight line given in a Mathematics Dictionary. The definition given in the dictionary will be abbreviated by 'De'.

De: *"A line such that if any part of it is placed so as to have two points in common with the other part, it will lie along the other part; a straight line is usually called simply a line".* (James & James, 1949, page 215).

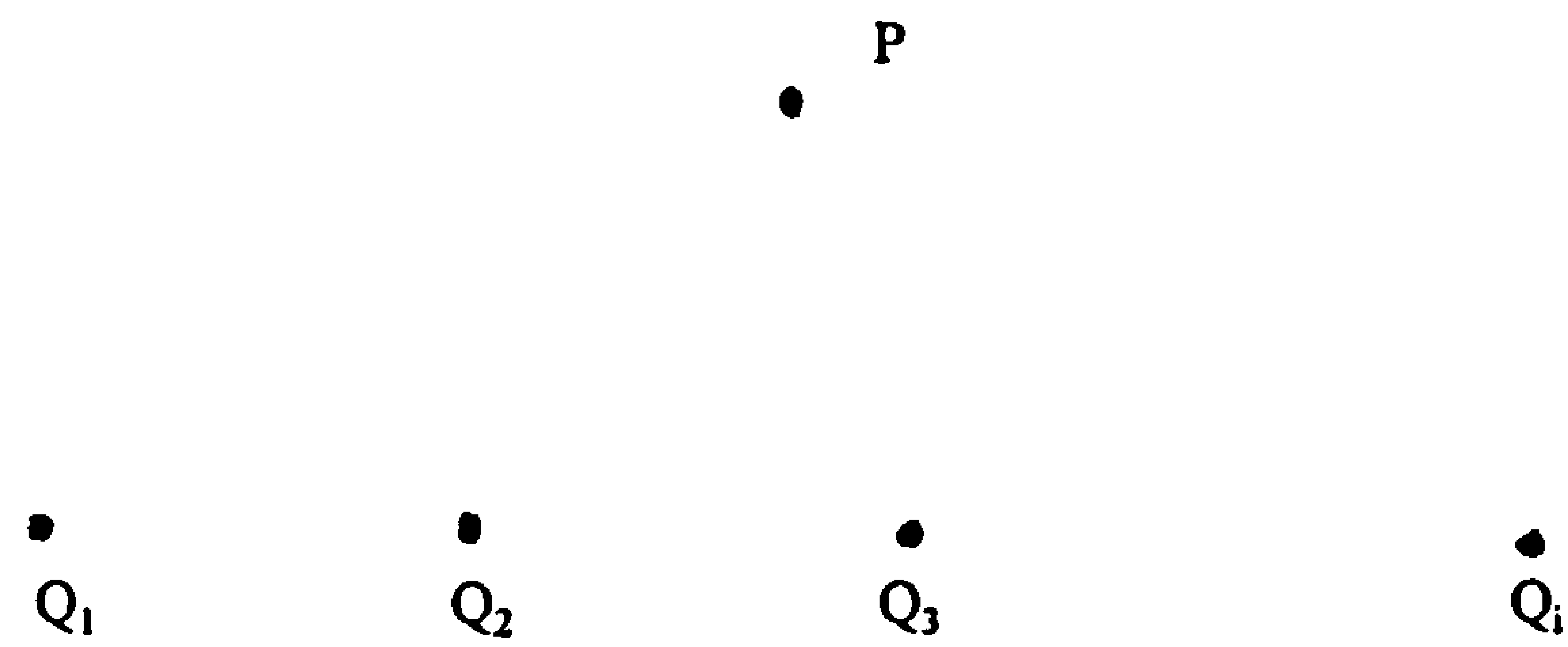
This definition of a straight line is certainly not valid for the global straight lines of S^2. Taking the straight line sections NRS and NPS (figure 4), which have the points N and S in common, it is not true that PSRN is a straight line.

Locally, 'De' can be applied in S^2.

We conclude that 'De' is only locally valid in S^2. One may not say that 'De' is educationally invalid for S^2 because statements about straight lines are generally assumed to concern only R^2 and R^3, unless otherwise stated.

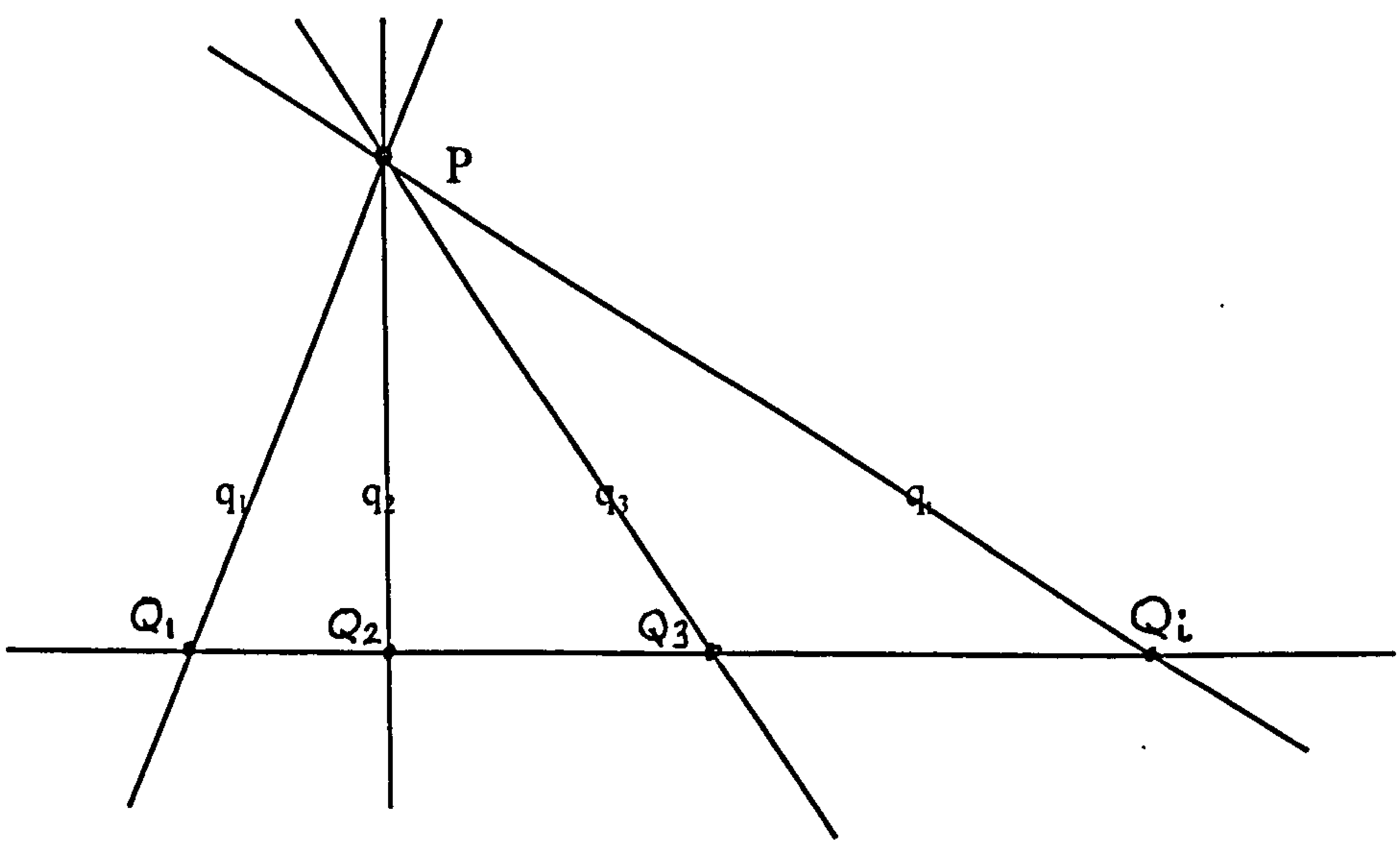
So, one may assume that 'De' can be applied to the local presentation of a straight line in R^2, and also to the global presentation of it.
However, R^2 is actually local and its lines are visual straight lines (Def (1)).
According to 'De', the global version of a straight line in R^2 is fully determined by two points. So it seems quite natural to represent a global straight line in R^2 by just two points and no more. At any rate, it seems to be impossible to depict the global straight line totally in a local picture, as far as we are talking about R^2. It would be the same as trying to draw a complete map of Great Britain within the map of London.
In the figures 6 and 7, the differences between the presentations of global and local straight lines are demonstrated (R^2). In figure 6 we have the global straight lines PQ_1, PQ_2, PQ_3, and PQ_i. Each line is represented by just two points.

figure 6



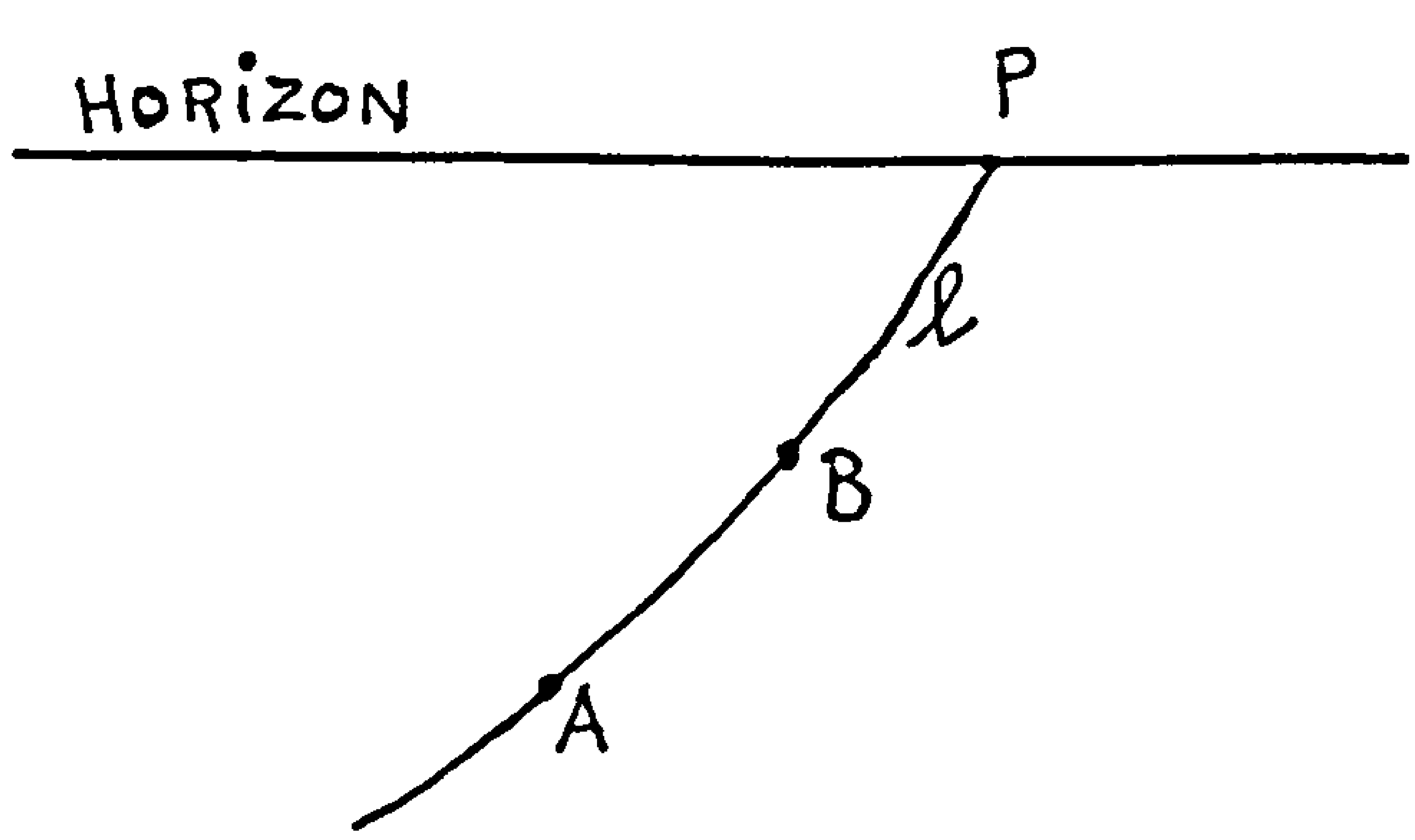
The associated visual straight lines, tangential to the global straight lines, are demonstrated in figure 7.

figure 7



In figure 6, global geometry is portrayed. In figure 7, local geometry is portrayed.
It is educationally invalid to consider figure 7 as global.
Is it acceptable to depict a global straight line by just two of its points? Do we not omit almost everything? Let us recall the presentation of a global straight line in R^3 (figure 8).

figure 8



If a person is observing the line ℓ from point A, then the total line will be reduced to a point. The observer, looking from A, gets a global picture of it just by observing point B. In the same way, when the observer looks from B and perceives A, he gets a global presentation of the straight line as if it were contracted in A. So the global presentation of a straight line in R^3 may be reduced to two points without dropping any of the points.

To finish off this chapter on global geometry, I will provide a proof that a straight line can be visually curved. It is necessary to use analysis to sustain the eye's perception. It will be demonstrated that the perception of a straight line as being visually curved is not an optical illusion. In figure 9 we look again at the photograph I took standing on the second rail from the left.

figure 9

A ruler was placed along the left rail in the picture and the image appeared to be curved. This action is shown in figure 10.
The left rail, however, was already curved because it lies on the surface of the earth which is a spherical surface. Straight lines on earth are straight lines like the equator and essentially curved in the space surrounding the earth.

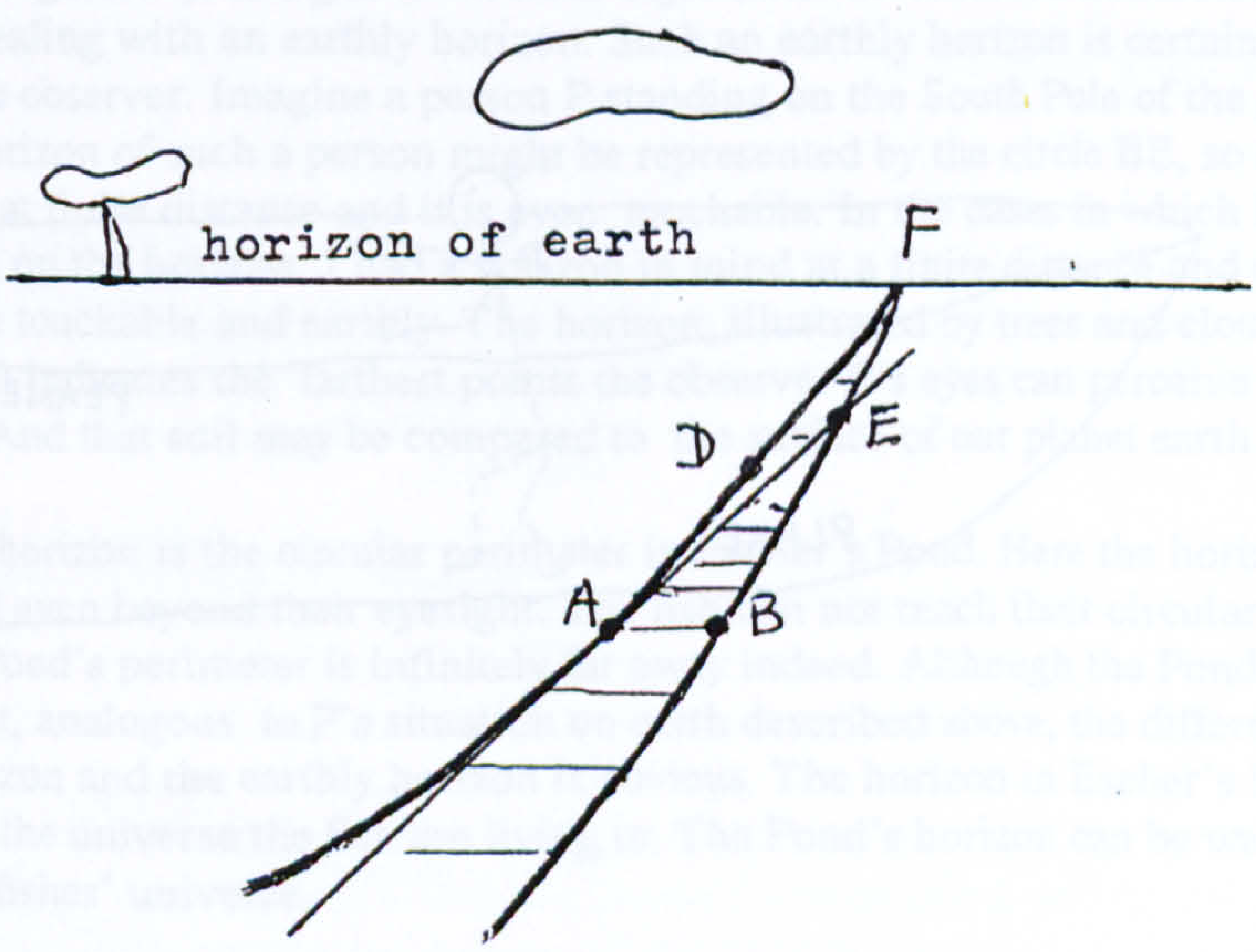


figure 10

The situation in figure 10 is drawn in figure 11 on the surface of the earth. Up to now we have taken the horizon of the earth as the natural horizon of our geometry. In assuming this, we are making the same mistake as people did ages ago when they assumed that the surface of the earth was flat because the small part on which they were standing seemed to be flat. That is educationally invalid.
If one admits this error then the following reaction is customary: "It is evident that a straight line should be drawn along a ruler even if you portray infinitely long lines. The deviation is due to the fact that the earth is a sphere".
In figure 11 we look at the situation of figure 10 but from a further distance. The distance AB in figure 11 is exaggerated to underline the actual situation. From this, it appears that it will be necessary to prove that straight lines are visually curved on a plane, too, and not only on the surface of the earth.
Therefore, we need a very large plane, far surpassing the size of the earth.
Imagine that one is standing on such a large plane. The plane will end in a kind of horizon (see figure 12) and the observer will get the idea of standing on the bottom of a saucer and looking at the rim of the saucer.

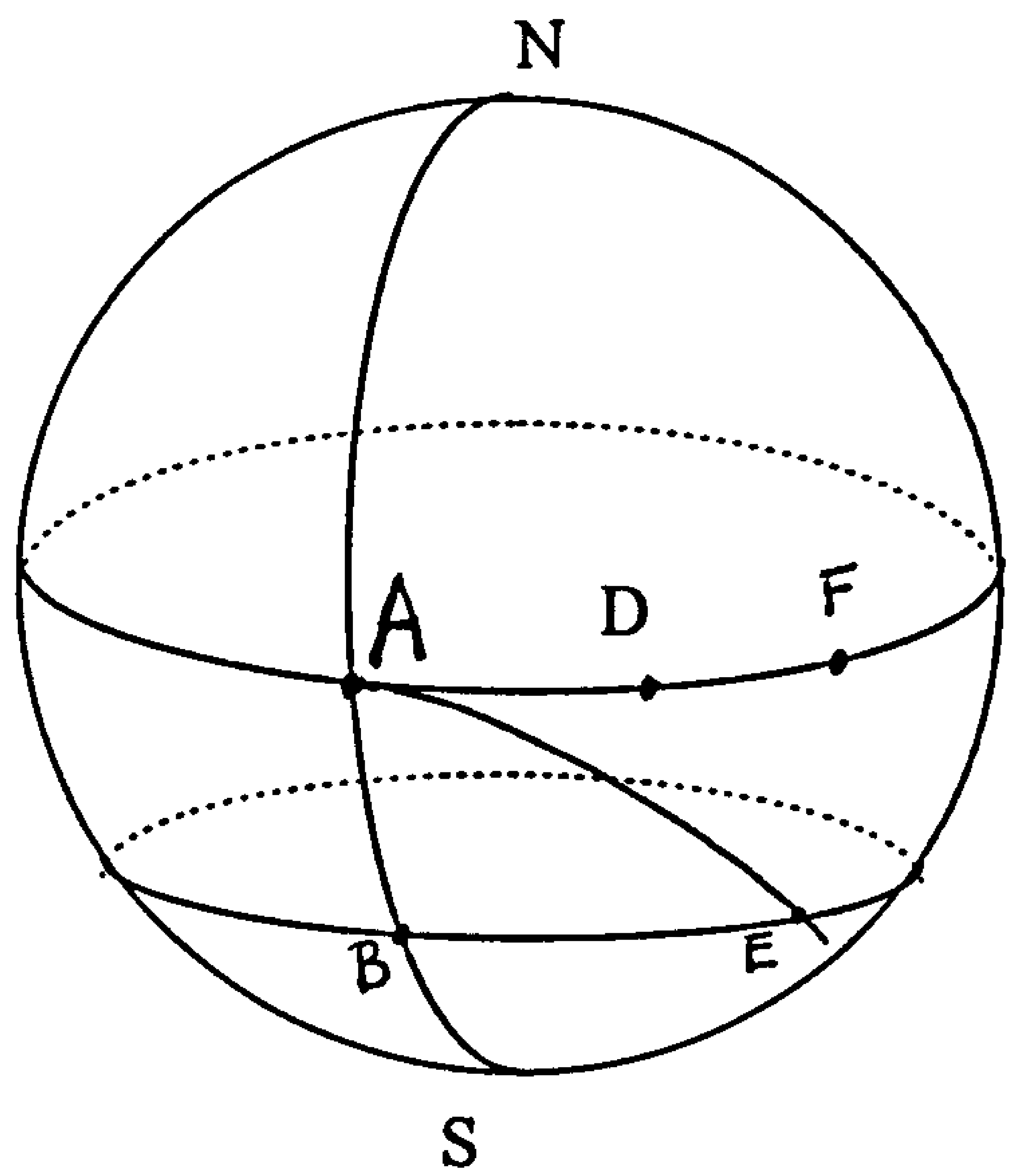


figure 11

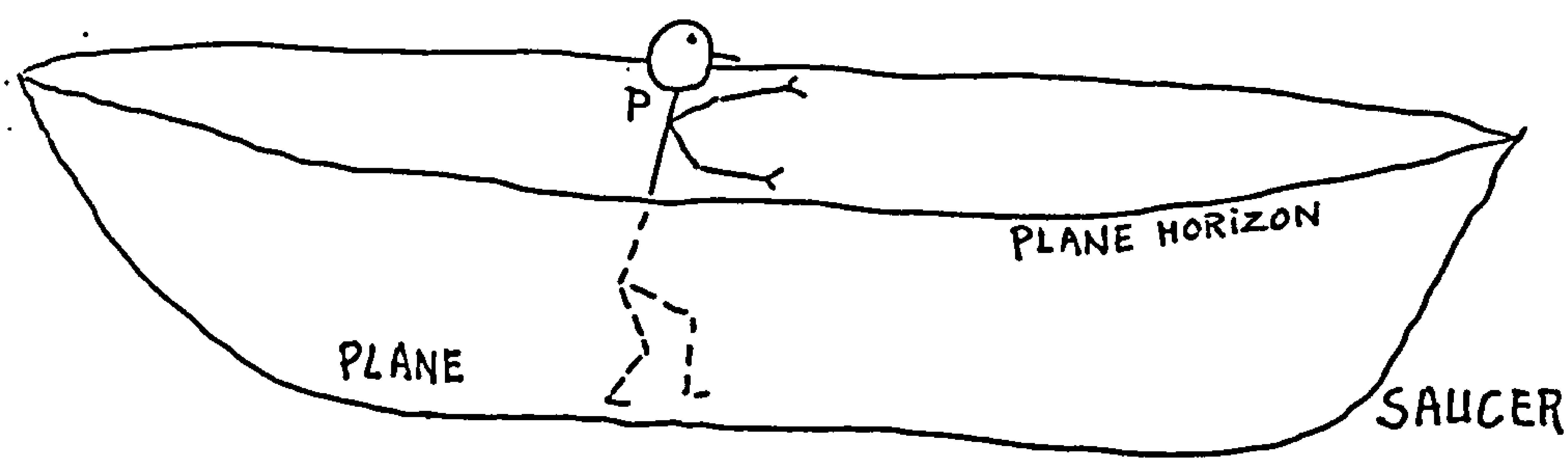


figure 12

In figure 13 we recognise the customary picture of a railway running to the horizon along ruled straight lines. This is educationally invalid because it prevents the student from imagining the situation in figure 12.

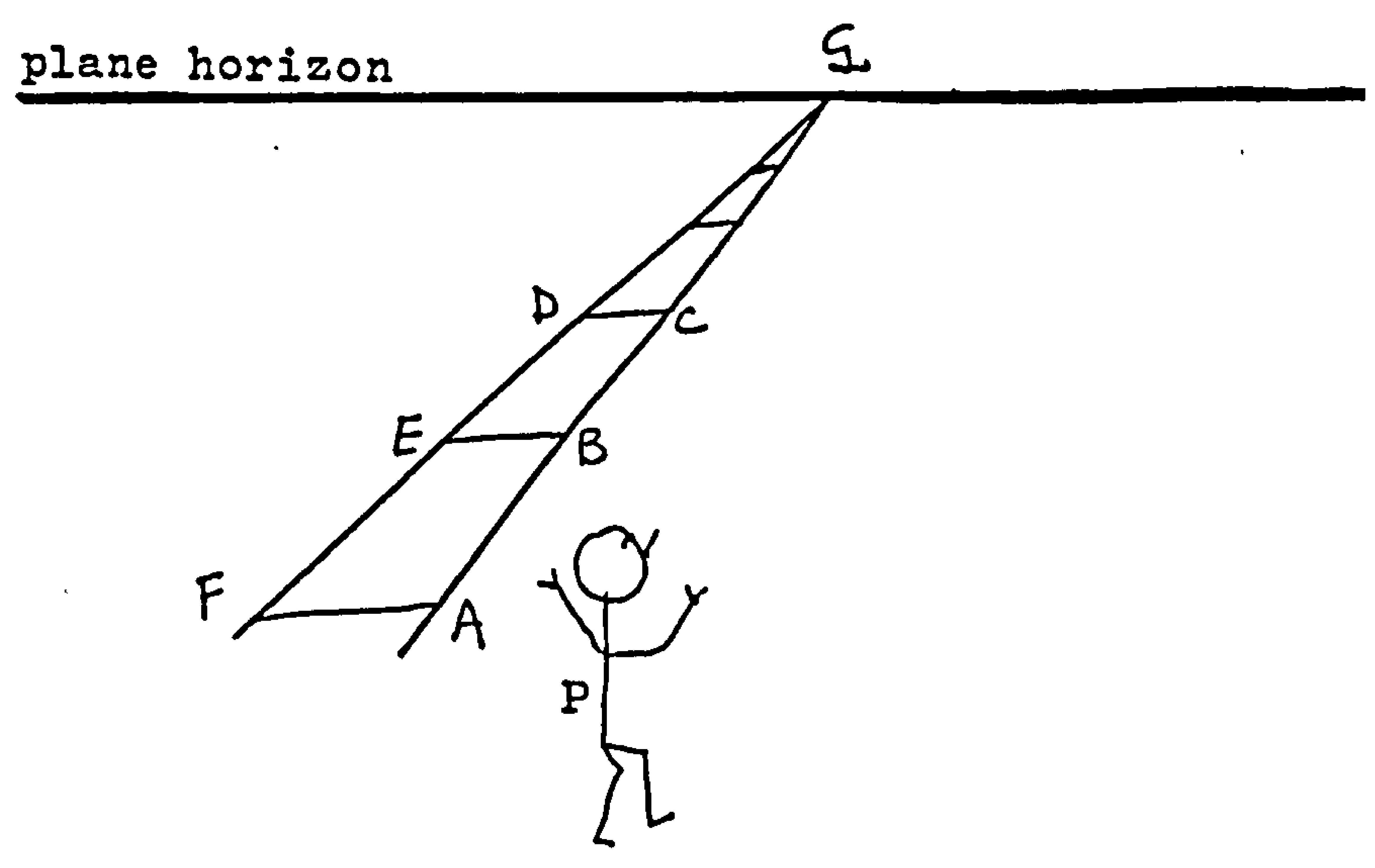


figure 13

3. 4. THE PLANE HORIZON

Different kinds of horizons have been demonstrated in the text above. In some pictures trees were drawn, apparently standing on the horizon and clouds in the skies were sketched above such a horizon (see for instance figure 10). In figure 9 various objects can be observed at the horizon. In these two cases we are dealing with an earthly horizon. Such an earthly horizon is certainly not at an infinite distance from the observer. Imagine a person P standing on the South Pole of the globe in figure 11. The earthly horizon of such a person might be represented by the circle BE, so this horizon, P's horizon, is at finite distance and it is even touchable. In the cases in which I have pictured trees and clouds on the horizon, I had a horizon in mind at a finite distance and essentially it was a horizon which is touchable and earthly. The horizon, illustrated by trees and clouds is meant to be concrete. Actually it indicates the farthest points the observer P's eyes can perceive to the soil on which P is standing. And that soil may be compared to the surface of our planet earth.

A quite different type of horizon is the circular perimeter in Escher's Pond. Here the horizon is not touchable by the fish and even beyond their eyesight. The fish can not reach their circular horizon in any way. For them the Pond's perimeter is infinitely far away indeed. Although the Pond's horizon limits the fishes' eyesight, analogous to P's situation on earth described above, the difference between the Pond's horizon and the earthly horizon is obvious. The horizon in Escher's Pond symbolises the border of the universe the fish are living in. The Pond's horizon can be watched only by us, living outside the fishes' universe.

Now we arrive at the horizons of figures 12 and 13. These kinds of horizon are called: 'plane horizon'. The planes of figures 12 and 13 are supposed to be situated somewhere in space, they are not attached to the earth. So the planes of the figures 12 and 13 are not tangent to our earth and the planes may even be considered as abstract ones. For that reason I have drawn a 'mathematical' plane in figure 13, which may be seen as educationally invalid, because the straight lines FEDG and ABCG, touching the horizon, are erroneously represented by 'visual straight lines' (Definition 1). Moreover, figure 13 does not represent an earthly situation, although a person P is watching the scene. The presence of P in figure 13 only indicates that we are supposed to have a look at an abstract plane, like the person P. On the horizon of such an 'abstract' plane, as shown in figure 13, no trees or clouds can be observed, firstly because these objects are assumed to be infinitely far away and thus visually infinitely small, and secondly because ' invisible trees on an abstract horizon' seem to be an impossible combination.

In figure 12 the plane on which person P is standing, is supposed to be an abstract plane, and it is not related to, or tangent to our earth; it is just a plane somewhere in space. I have drawn a saucer to indicate, what the observer P might perceive. So the picture of the saucer in figure 12 only shows P's assumed visual impression. The situation of figure 12 is, of course, purely imaginary, because nobody has ever seen such a plane in space. The size of it must appear to the eye as enormous, let alone that somebody has ever been standing on such a plane. Moreover, the light, illuminating the plane of figure 12, is supposed just to be there, without any assumption about the light source. The observer P, although supposed to be human, is no longer acting in 'earthly' circumstances.

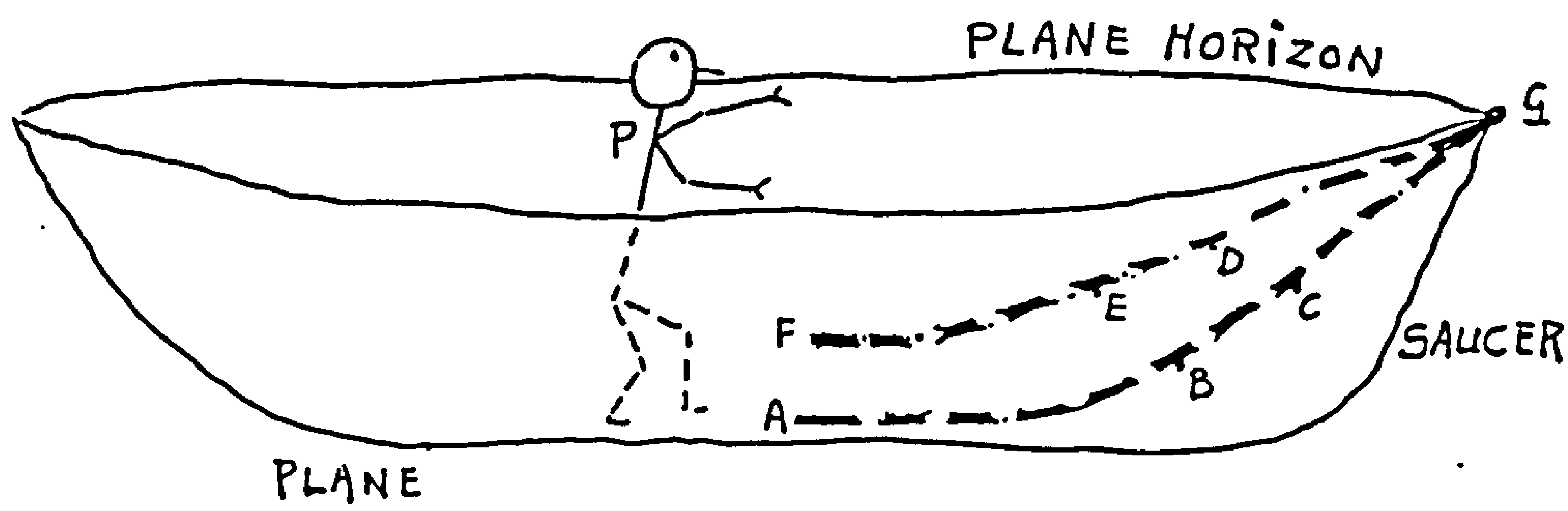
The horizon in figure 12 is represented by the rim of a saucer. This horizon is called the 'plane horizon', because it represents the border of the observer's perception of the plane, and moreover, the 'plane horizon' is touched by indefinitely produced straight lines, which lie entirely in the plane. This is, of course, different from the situation on earth, where no straight lines at all can be observed on the surface on which we live. The situation in figure 12 is less abstract than it is in figure 13. There is a probability that P's real view of the plane will resemble the view we have in figure 12. However, we do not know for certain, because such large planes do not exist in reality, as far as we know. Figure 12 is no more than an imaginative reflection of P's perception. The concept of P's perception in figure 12 however, shows less distortion than P's perception in figure 13. Anyhow, the display of trees and clouds 'near' the plane horizons of the figures 12 and 13 seems unacceptable, because such objects are supposed to be visually infinitely small, observed by P, as a consequence of the assumed 'infinite distance between observer P and the horizon'. The scenes in figures 12 and 13 are far away from earthly circumstances, so, no trees will grow on these 'abstract' planes and no clouds will hover above the 'abstract' horizon.

From now on, I will not draw trees and clouds near horizons, when abstract planes are pictured, which can hardly be considered as earthly. These objects (trees and clouds) at the horizons on such purely abstract and imaginative planes are bound to become visually infinitely small, when observed from a very large distance (see figure 19). So it would be better not to try to draw them.

Now, let us look again at figure 13.

What the person P, in figure 13, observes in reality is the view of figure 14. The straight lines ABC and FED, which are parallel, are observed by P as curved lines, meeting at a point G at a far distance and above the level of the bottom of the saucer (figure 14).

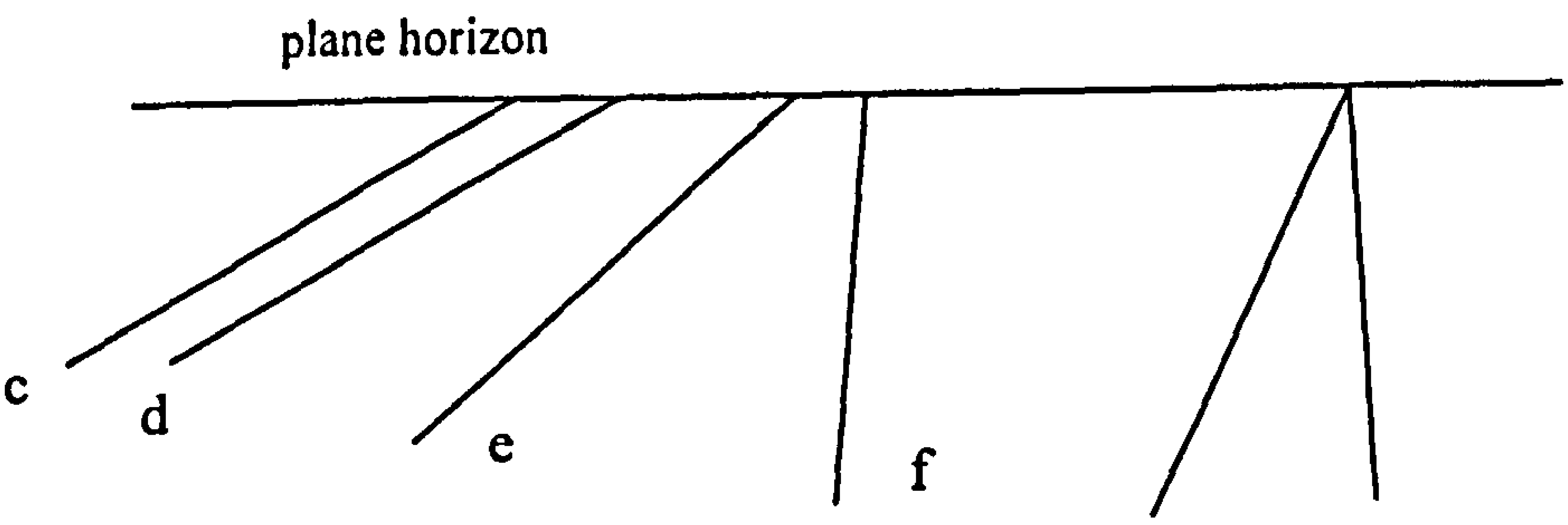
figure 14



So the situation on an indefinitely prolonged plane is not quite as is suggested in figure 13. The picture of figure 13 is a 'mathematical', or an 'abstract' picture. The rails are depicted straight, that is to say, as visual straight lines. But in figure 14 we see that those rails follow the slope of a visual saucer.
It is possible that some of the straight lines observed by P in figure 14 do indeed look like visual straight lines; these are the straight lines emanating from his feet and following the surface of the saucer. But what exactly seems straight: the lines themselves or their projection on the surface of the saucer, seen from P's eyes?

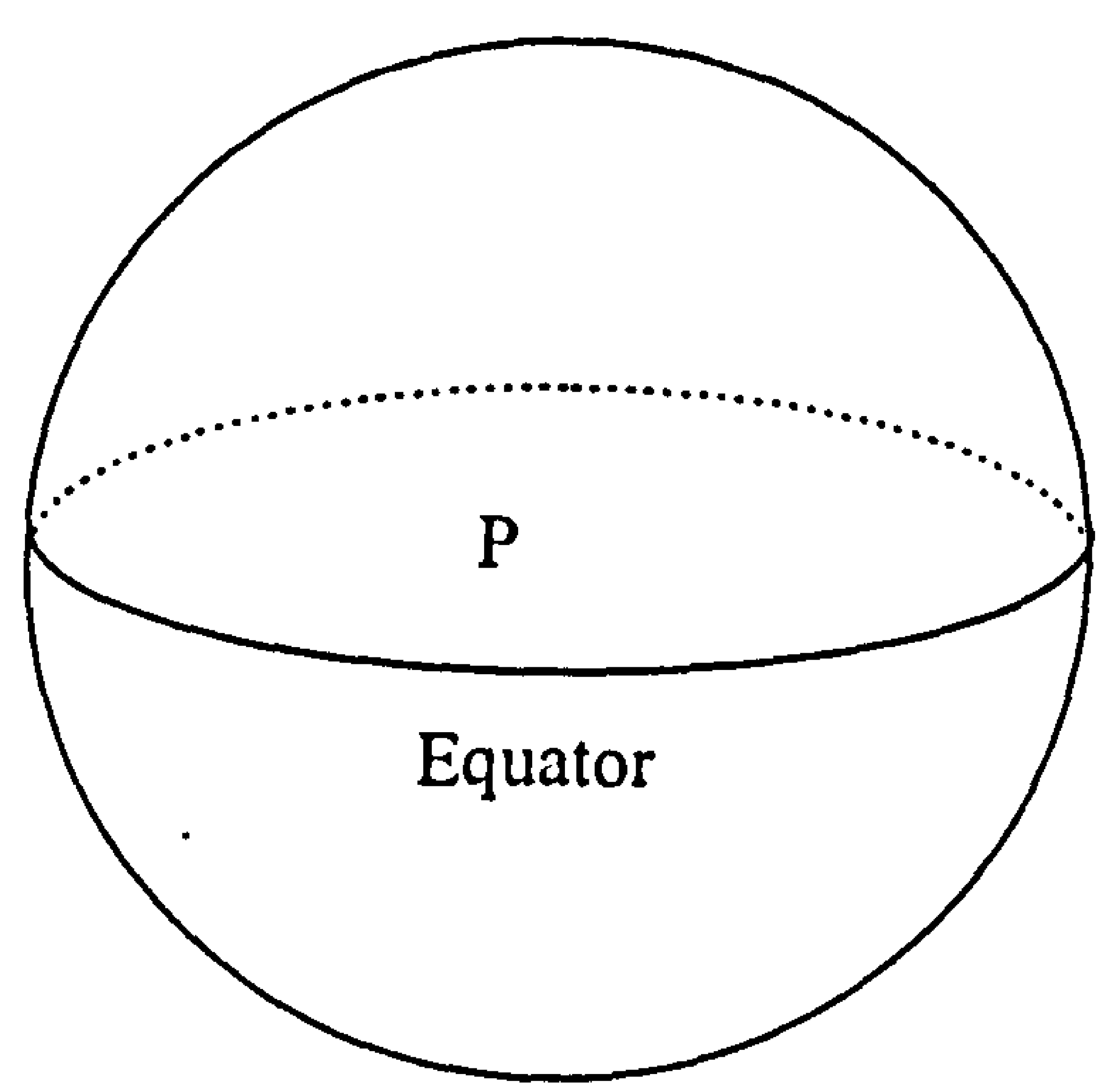
Let us have a look at figure 15.

figure 15



In this picture (figure 15) we observe that the representation of straight lines, running along the edge of a ruler towards the horizon, must be wrong. If we scrutinise the straight lines c and d in figure 15, we see that these two lines apparently will not intersect because they are evidently visually parallel. The problem is that they approach different points on the horizon and that means that they are not parallel. This provides a contradiction. Also it is evident that the straight lines e and f, although not parallel, do not intersect. However, beyond the horizon there is a point of intersection, but are we allowed to walk behind the horizon? I have used these positions of straight lines in a plane to remove the horizon because it leads to contradictions, and I have replaced the horizon by just one point. This is worked out in my comment on Euclid's 'Optics' (see Chapter XIV).
If a straight line can not be found on a plane, let us look at a sphere. In figure 16 a person P is standing in the centre of the earth looking at the equator. This equator will be seen by him as a straight line.

figure 16



From the foregoing, we have seen that it is not quite clear how best to depict global straight lines. Here is another method to depict them: the saucer in which the person P was standing (see figure 14) will be flattened, rectified. The slope of the saucer disappears and the view the person P has is depicted with the help of visual co-ordinates in figure 17. Figure 17 can be compared to the view of an abstract plane (figure 13) but seen as the copy of a visual reality. On the next page the visual co-ordinates are computed.

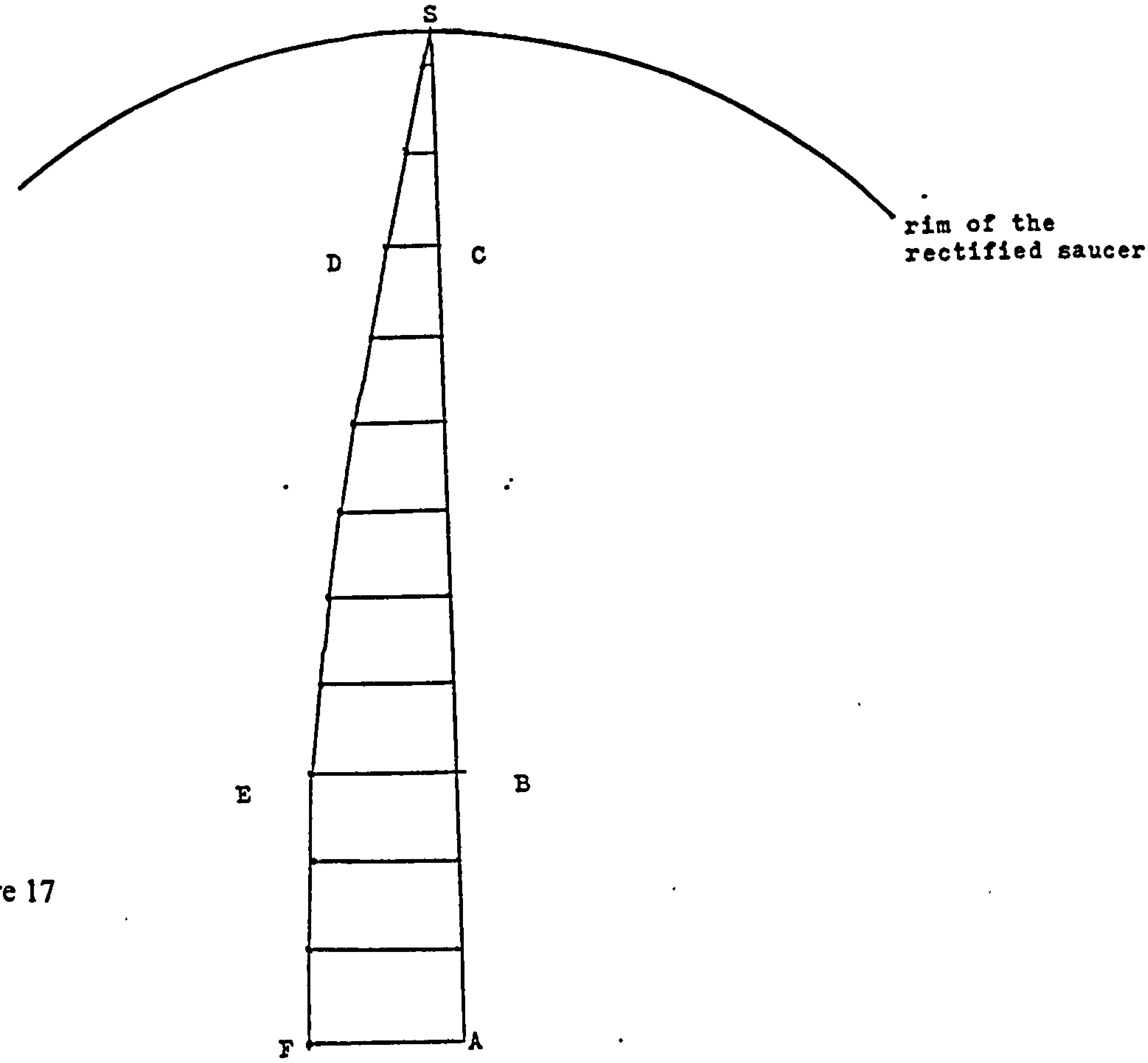
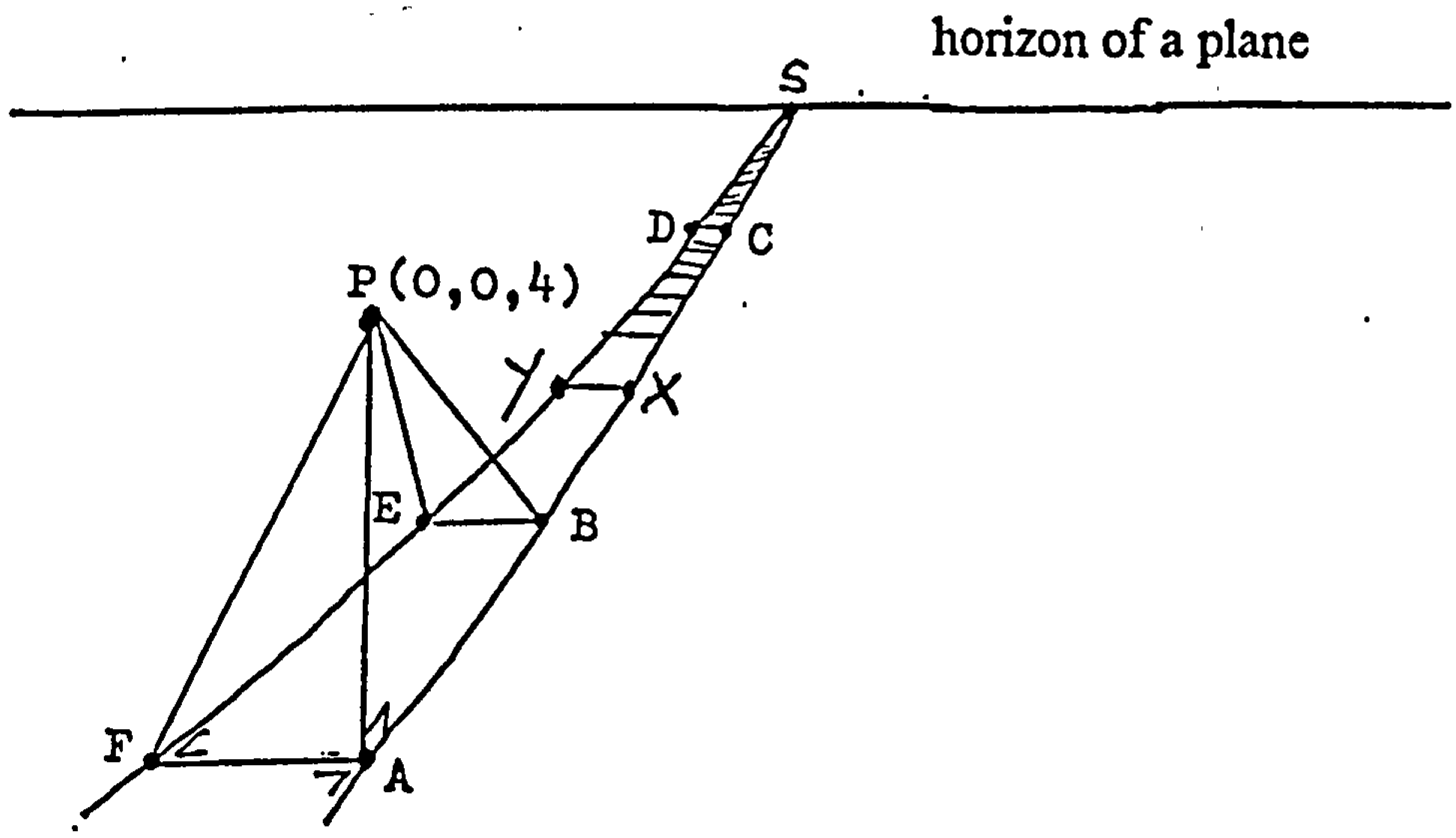


figure 17

Some computational work follows: a person P is watching a railway (figure 18). There is a co-ordinate system. The X-axis is along the visual straight line AB, the Y-axis is along the straight cross-sleeper FA, and the Z-axis points vertically to the sky along AP. Distances will be measured in angles, perceived by P. So the distance AB is the angle APB. The angle APF represents the length of the cross-sleeper AF. The work will be carried out as follows: take for instance the angle APB to be 45 degrees.

figure 18



We take for P the co-ordinates (0,0,4) and AB = 4 , so we have B(4,0,0) and E(4,1,0), taking F(0,1,0).
Thus:

$$\cos \angle BPE = \frac{(4,0,-4)(4,1,-4)}{\sqrt{32}\ \sqrt{33}} = 0.98473 \quad \text{and} \quad \angle BPE = 0.174969$$

We knew already that the angle $\angle$ APB = 45 degrees = 0.7854 radials.
The visual co-ordinates of E are (0.7845, 0.1750). Visual means: perceived by P.
So these visual co-ordinates are (angle APX, angle XPY).

Here follows a list of visual co-ordinates of points of FS

(0, 0.2450)	(0.8378, 0.1657)	(1.5359, 0.0087)
(0.14, 0.2427)	(0.9774, 0.1389)	(1.5533, 0.0043)
(0.28, 0.2358)	(1,1170, 0.1092)	(1.5621, 0.0022)
(0.4188, 0.2245)	(1.2566, 0.0771)	(1.5690, 0.0004)
(0.5585, 0.2089)	(1.3963, 0.0433)	
(0.6981, 0.1892)		

From these computations and their meaning, it may be obvious that we cannot simply apply a ruler to draw a line that is called straight. The panoramic screen (figure 19) shows what the person P (of figure 13) might have observed.

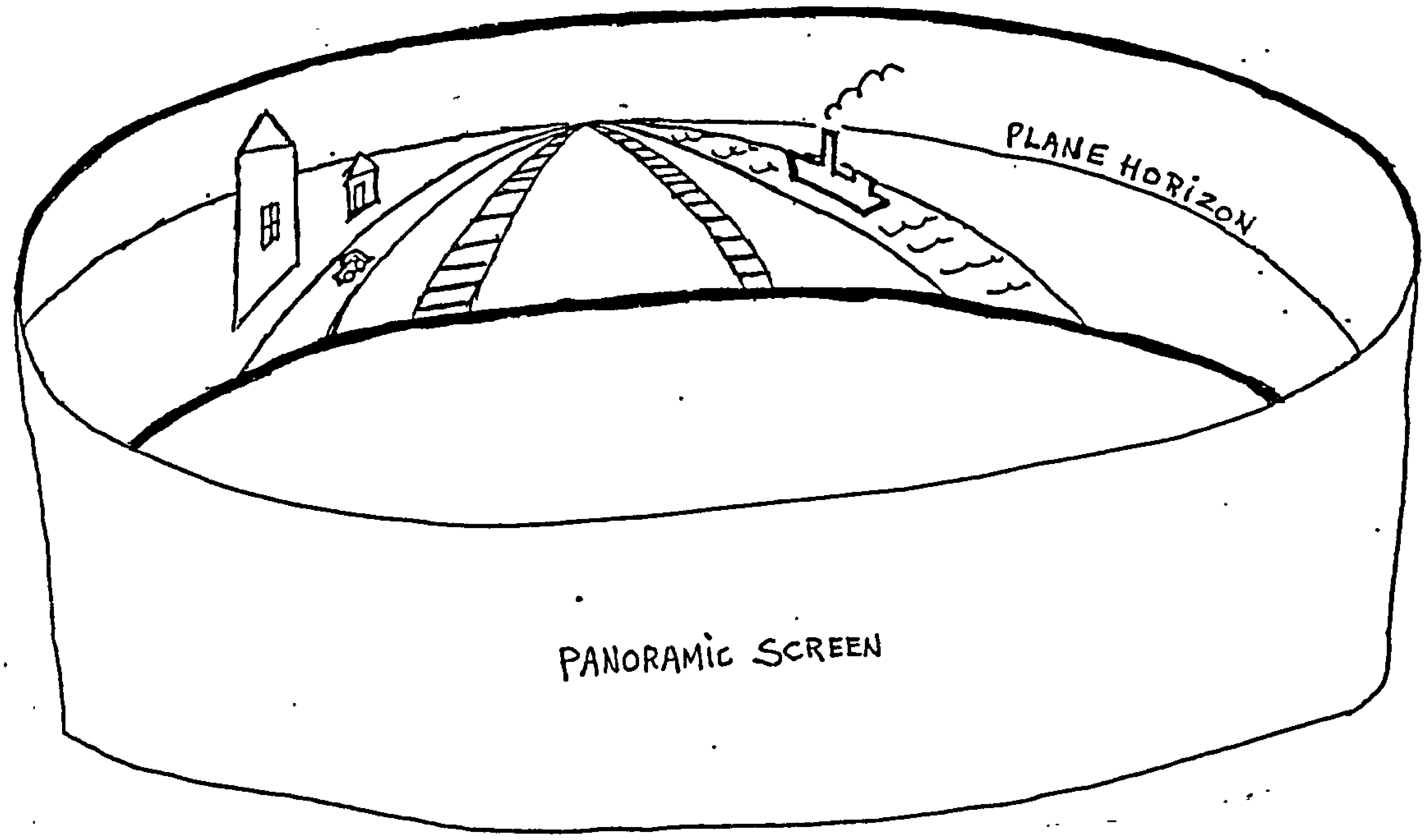


figure 19

In the next chapter attention will be paid to local geometry. An introduction on Duality is the first step into the world of Projective Geometry. In Part II of the thesis, a lesson is designed on the subject of Duality and it is demonstrated that global items ('ideal' points and lines at infinity) can be quite natural combined with local geometry. This is in the interest of the educational validity of the subject.

Chapter IV

4.1. Local geometry

It is possible to pull the horizon from a far distance to the desk you are writing on. As might be expected, that horizon will be greatly transformed: it changes from a virtual, untouchable straight line in the far distance to an ordinary visual straight line which can be produced with the help of a ruler.

The phenomenon I am referring to is the introduction of an artificial horizon. Actually the drawing of a virtual straight line is already a transformation. In fact the horizon marks the limitation of what the eye perceives of the plane but is not a straight line. The painting and drawing of it transform it to a real straight line on the paper but this real line is a symbol that visualises the invisible virtual horizon. And now, from the status of a supposedly infinite far-away line towards a line which can be grasped, is a further step. Nevertheless it is possible. An arbitrary straight line, drawn on a piece of paper, may be treated as if it were a horizon. This means that lines, which supposedly are parallel, have to meet each other (intersect) on that 'artificial' horizon.

As I will indicate in Chapter IV, section 4.2., games can be played with such an artificial horizon. An example will be produced by me in Chapter IV, section 4.2. It is a prelude to the discussion of what seventeenth century painters achieved. They produced paintings with more than one horizon so that the action happened in more than one space at the same time. Further, an example is given of a horizon which is supposed to be in the interior of a church (Gerard Houckgeest, 1651). By that artificial horizon a corner of the church is appointed to become a separate space in its own right. A perspective analysis of a painting by Houckgeest is demonstrated in Chapter IV, section 4.2.

This method of deforming space by introducing an artificial horizon has foreshadowed the emergence of non-Euclidean geometries in the nineteenth century. Actually the drawing 'Escher's Pond' possesses such an artificial horizon; it is the circle circumference which limits the portrayal of the depicted universe. In this way the seventeenth century painters could be seen as forerunners of the nineteenth century mathematicians, who constructed non-Euclidean spaces.

It should be emphasised that those artificial horizons, although presented as a 'visual straight line', may not be seen as local geometry. Local geometry means : the geometry of a relatively small and limited part of the space. No horizon may be called 'local' because indefinite prolonged lines are involved. A very important example of such a local geometry is demonstrated in the text on 'Duality'. The concept of duality provides a deeper understanding of the things that happen in geometry.

Duality makes relations explicit between points, lines, planes. It is a phenomenon in geometry that is a riddle on the one hand but extremely useful on the other. With the help of duality, the notions of points and lines seem to become interchangeable. Let me give an example: we should consider a triangle. It has three vertices and three edges. In that last statement: 'a triangle has three edges and three vertices', it is possible to interchange the words 'vertices' and 'edges' and the result is that the statement remains precisely the same. Another example is a tetrahedron which has four faces and four vertices. In the statement : 'a tetrahedron has four faces and four vertices' the words 'vertices' and 'faces' may be interchanged again without affecting or even changing the statement.

Now the application of duality is not so simple that we have only to interchange two words and that then the result is that the statement does not change. That would not yield very much. The key notion of duality is that, if a statement is true, we are allowed to interchange some words. After this action a new statement appears which, however, is automatically valid because the first statement was valid. That is a beautiful result without any effort, thanks to duality. An example follows (in R^2):

Let us assume that the notions 'point' and '(straight) line' may be interchanged because they are dual with respect to each other. Take statement (a): Many lines share a point. We must choose a point P through which many lines go. These lines share the point P (in R^2). The dual statement is (b) : Many points share one line. This second statement (b) is valid because it's dual to (a) which has appeared to be valid. So, as a result of the duality and statement (a) there is a line 'p' in R^2 which is shared by many points. By saying that points share a line we denote that these points are lying on that line. Moreover, point P from the above example is dual to line 'p', mentioned above.

It will be evident that duality is not easy to grasp. In the nineteenth century there was a bitter fight between two French Mathematicians about the accurate meaning of duality. It was Jacob Steiner who brought duality to a higher status. This is discussed in Chapter IV, section 4.3.

The subject is so important that in Part II of this thesis I have dedicated a lesson to it (Chapter VIII). It means that many more examples are provided and that the implications of duality for geometry are highlighted more than has been done in Chapter IV, section 4.3. In that lesson in Chapter VIII, section 8.2. not only the notion of duality will be explained but there are also exercises so that students can check their ability on the topic. Furthermore, the answers to the questions are worked out so that a comprehensive treatment of duality emerges. It is of course little more than basic but it provides some introduction. The impact of global geometry to duality is discussed too. By the introduction of so-called ideal points, a solution has been found to deal with global items without affecting the local geometry in which we are working.

In the 3-dimensional space there are also dual polyhedra. A point in R^3 is dual to a plane in R^3 and straight lines are dual to straight lines. Take the example of a cube and an octahedron: a cube has 8 vertices and 6 faces. An octahedron has 8 faces and 6 vertices. The number of edges of both solids are equal, that is 12. This means that both cube and octahedron are dual solids . The numbers 6 and 8 are interchanged when we compare cube and octahedron and the number of edges keeps equal . This is all demonstrated in Chapter IV, section 4.3.

Looking at the customary representations of cube and octahedron, we notice that these appearances are educationally invalid because the existence of a visual horizon is totally neglected. Is that an objection to presenting these images ? I am inclined to say no. The presentation of cube and octahedron in the traditional way is so beautiful that it would be damaging to abandon it. Moreover, for the application of duality the visual form of the solid is less important, so why not choose the most elegant visualisation ?

The same remark is true for the topology, which can be found in Chapter IV, section 4.4. I have inserted the subject because the Euler Characteristic can often be computed with help of a visual representation. In this case visual geometry has a function. To compute the Euler Characteristic in many cases one has to use the method of so-called triangulation. The plane figure is decomposed into triangles and to these triangles Euler's formula must be applied. The same method can be applied to solids. It is useful to note that the Euler Characteristic under certain conditions fully determines the character of the solid or plane figure. A class of spatial figures is determined by one number. Also noteworthy is that a spatial figure like a torus can be triangulated and its Euler Characteristic computed. A torus is comparable to the tyre of a bicycle.

The examples of Duality and the Euler Characteristic have been given to demonstrate that a lot of geometry can easily be studied with the help of visual images. There are also more abstract approaches to visual geometry. Think of a circle. It can rotate freely in itself so that its appearance does not change during the move. Here we have a geometrical phenomenon that can be described with the help of Group Theory. We also know that, for instance, an equilateral triangle may rotate 120 ° or 240 ° without changing its visual appearance. All the moves which allow certain geometrical figures to keep visually unchanged, can be described by group theory. Even examples of translations can be provided. Normally a configuration changes visually when it's translated. However, when a system of rectangles, which covers the whole visual plane, is translated into the direction of certain vectors, then the whole system might remain visually unchanged. Hence figures have to remain visually unchanged under the moves given by the elements of the group.

In Part II a lesson will be designed involving these groups; see Chapter IX, section 9.2. The subjects of duality, topology, triangulation and group theory are very suitable to enable the student to study geometry without the burden of having to apply complicated formulae and computations. These subjects, provided that the explanation is adapted and restricted to visual geometry, may be seen as educationally valid. They do not bar the road to further understanding and continued study.

In the next Chapter (V), a piece of Projective Geometry will be demonstrated, designed by myself. It is an artistic approach to a purely scientifically subject. The configurations are art and geometry at the same time. Fifteen people have been asked to read my design and afterwards answer questions about it. My aim with that design was to bring real geometry, with the help of art, to people who generally have made no special study of the subject. It appears that almost all of the students, who were chosen arbitrarily, had no difficulty in understanding what the questions are about but that the same material is also interesting for teachers of mathematics; and almost all those interviewed stated that geometry appeared more interesting after they had studied 'Projections', as the design is called.

This material has to be chosen very carefully. It has to be geometry, disguised in an artistic appearance. There are artists who create artistic works based on mathematical issues. Such products of art can be very beautiful but the mathematics has to be prominently present. Escher's Pond demonstrates that an artist can produce a beautiful drawing which is also helpful for the teaching of mathematics.

It might be contested whether such drawings can still be considered as art. Of those interviewed, almost every one stated that my design was considered by them as art. However, the most important result would be that the students, attracted by the beautiful drawings, in a short time arrive at a higher geometrical level.

4.2. Games with Local Geometry

Local geometry is the geometry of visual straight lines (definition (1)). We identified R^2 and R^3 as such; small parts of the universes S^2 and EP are also local geometry. Sometimes in local geometry a simulated horizon is used. A visual straight line is chosen to be appointed to play the role of horizon. One may talk about a local horizon, often applied by painters. Such a horizon could also be called an artificial horizon. Straight lines, meeting on such an artificial horizon, are assumed to be parallel. One may obtain beautiful visual effects. Actually a horizon is simulated by a visual straight line (definition (1)). Straight lines, approaching an artificial horizon, will not become curved, but continue along the ruler which had produced the line. This emphasises the fact that the horizon is only artificial. One can also play games with artificial horizons. I have produced such a game which is demonstrated on the following pages.

The pages shown are somewhere between global and local geometry. The global geometry is simulated by local means. In an earlier stage this practice was condemned as educationally invalid. However, when it is obviously a simulation, not pretending to reflect the actual situation, there is nothing against it. On the contrary, it becomes evident what kind of an instrument the horizon is and what you can do with it.

Many geometrical activities refer to the presence of a horizon. For instance, drawing parallel lines in local geometry is not possible when the global notion of an indefinite prolonged line segment is not involved. These parallel lines are supposed not to intersect, not even when indefinitely prolonged. And that means that there is a horizon. A subject which can be carried out completely locally is duality. There is the presentation of a global straight line by two points but two such points can always be connected by a visual straight line. So following my game with the horizon, duality will be the next target.

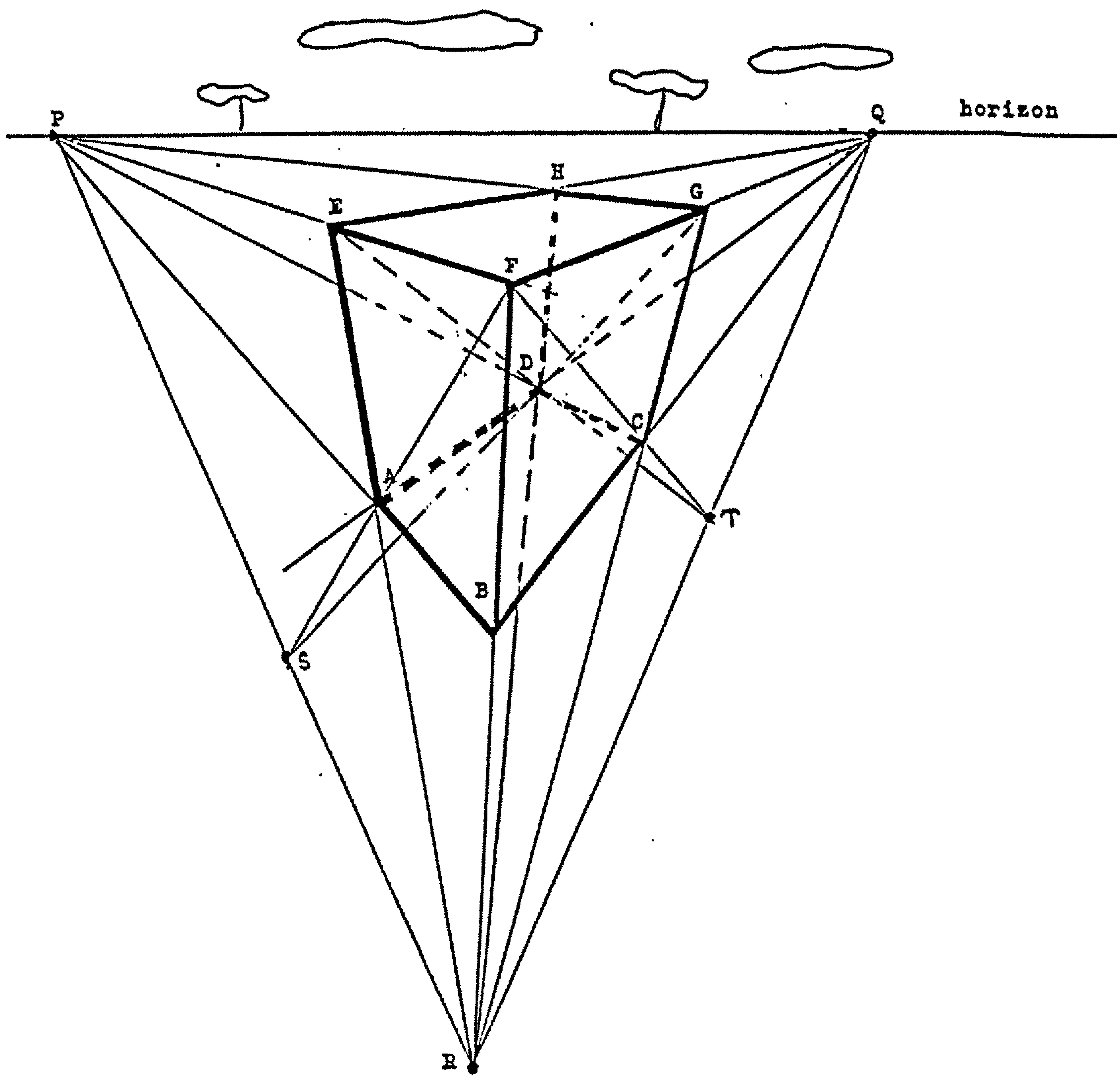
A somewhat remote subject is the Euler Characteristic which actually is a topological issue. It is demonstrated after duality has been dealt with.

GAMES WITH THE HORIZON

In figure 1 we see a 'cube' ABCD.EFGH drawn in perspective. The horizontal lines meet at points P and Q; the vertical lines meet at R. The horizon goes through points P and Q. We know that horizontal parallel lines meet each other at a point on the horizon. Along the edges of the cube, there are three directions possible: two horizontal (towards P and Q) and one vertical (towards R). Since the invention of the horizon, by the end of the middle ages, artists have played with the horizon, and sometimes the results were marvellous. Normally one has a horizon, which is supposed to be indefinitely far away (figure 1). Horizontal parallel lines meet each other at the horizon.

Looking at figure 1, we observe that the line PR can also be seen as a horizon. For instance AF and DG are considered as parallel and their 'horizontal point of meeting' is S on PR.

figure 1



Similarly the lines FC and ED will meet at a point T on QR. Consequently the line QR can be considered as a horizon. If QR is serving as a horizon, then P is the vertical direction. If PR is the horizon, then Q is the vertical direction.

It follows that when you draw a cube, you are also assuming three horizons. Of these three, one is accepted as the 'real' horizon and there remains a vertical direction, perpendicular to the horizon. In the case of figure 1 the points P,Q and R may be indefinitely far away, or at least, almost indefinitely far. For point R one could easily take the centre of the earth and for P and Q respectively Belfast and Oslo, while B is in The Hague. However, one can go further and take an artificial horizon, not so far away. Many painters assume a horizon at a distance of, for instance, 10 metres. In that case you get a distorted drawing, but the view may be brilliant. As an example there is a painting by the Dutch painter Gerard Houckgeest, who in 1651 depicted the interior of the New Church at Delft in the Netherlands, and assumed that the horizon was in the building itself.

Now I will demonstrate a product of my own imagination. It delivers a 'cube' whose vertical direction points at the horizon (figure 2). So point R (figure 2) is a point of the line PQ (figure 2) and this yields figure 3.

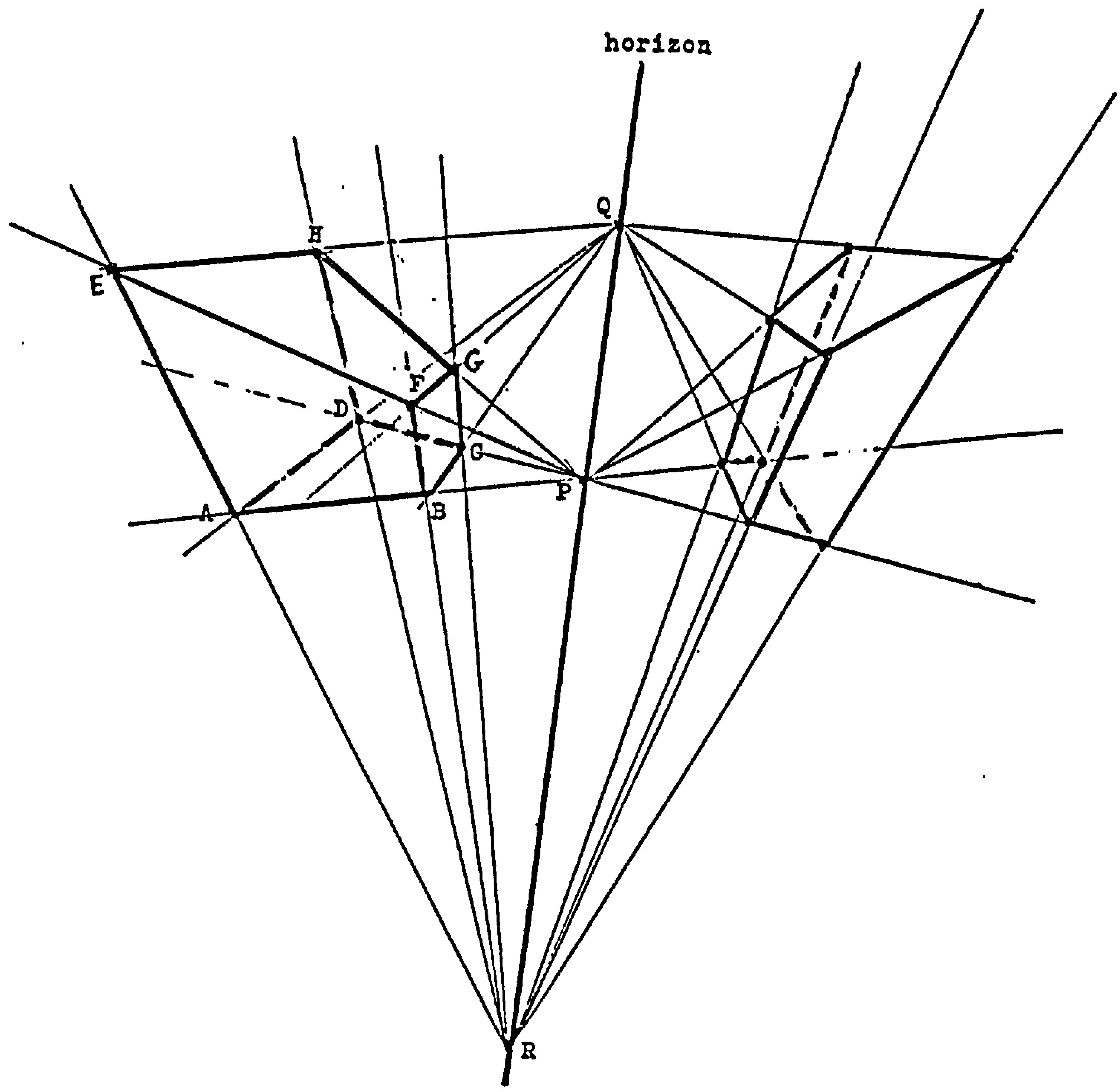
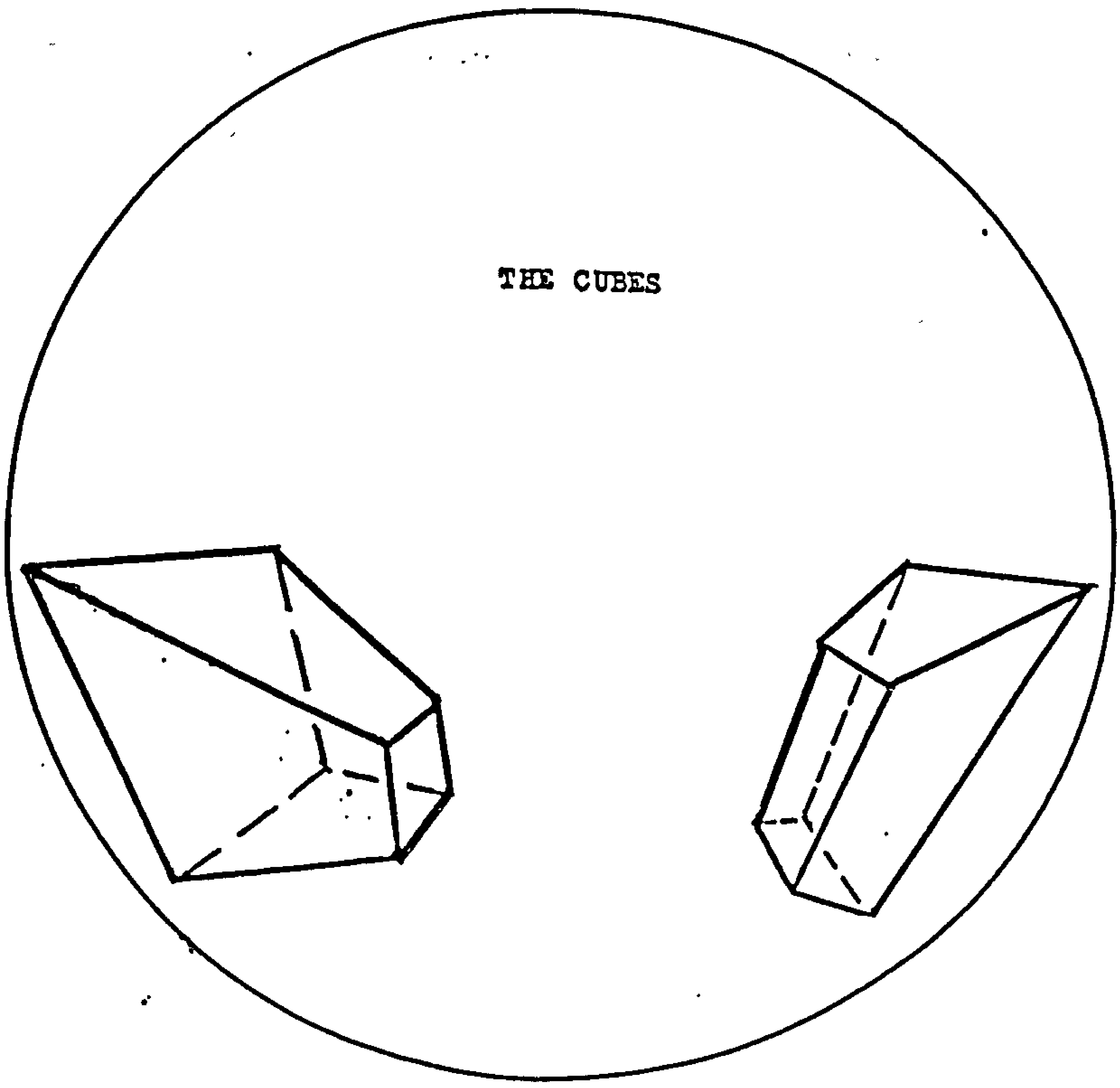


figure 2

A SPATIAL JOKE

figure 3



Further applying artificial horizons, we may have a look at a painting of the Dutch seventeenth century artist, Gerard Houckgeest. In the interior of the New Church in Delft, a horizon is pictured. The perspective analysis is also shown. Houckgeest plays with the horizon.

Following the issue of the artificial horizon, some pages will be concerned with Duality and Jacob Steiner. In Projective Geometry, duality is a basic subject. Although it is presented as a local item, it is certainly possible to involve the horizon in the theory of duality. However, I will stick to the local presentation.

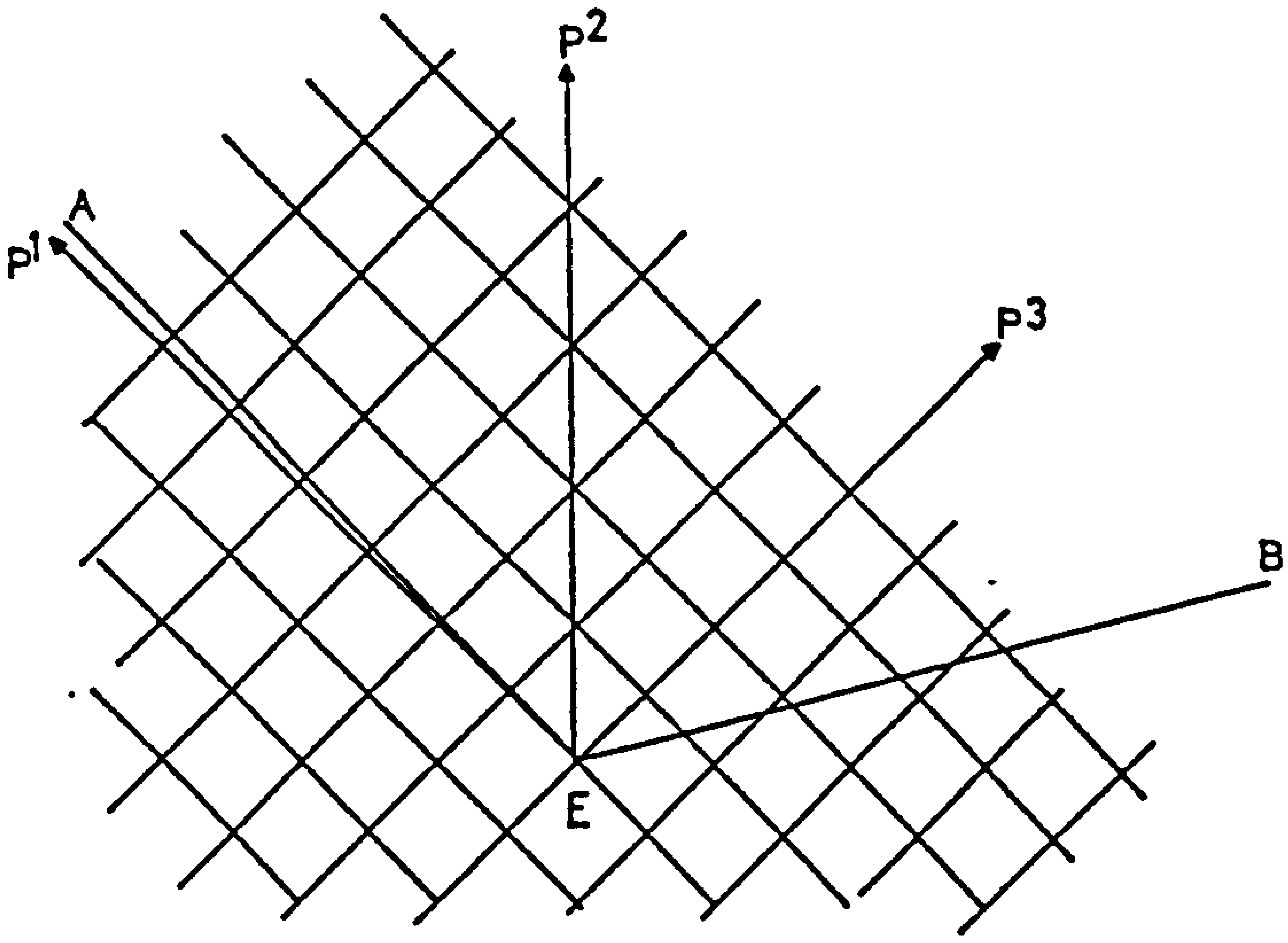
Gerard Houckgeest:

Interior of the New Church at Delft

with

the Tomb of William the Silent, 1651

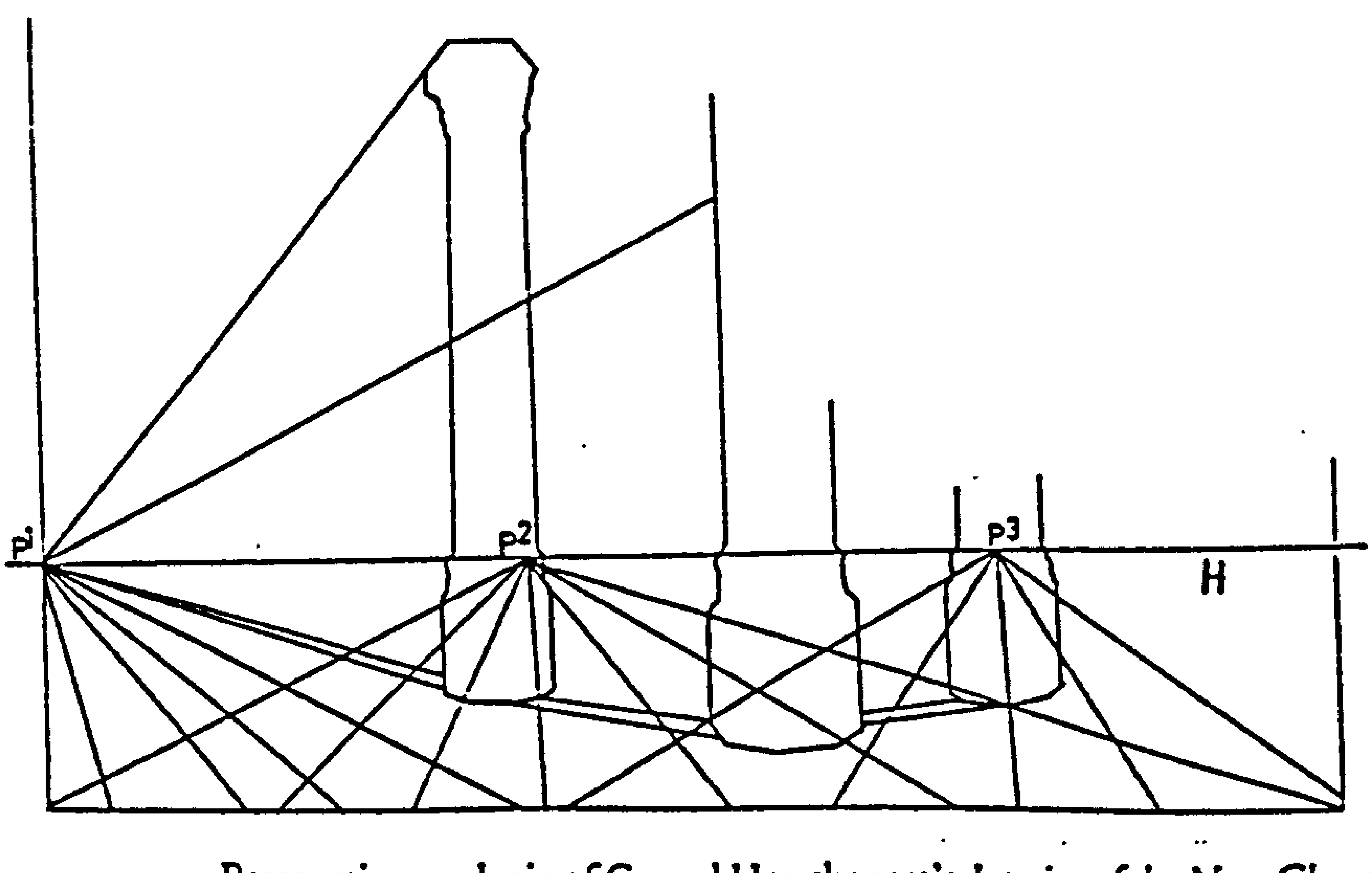
The Hague, het Mauritshuis



Demonstration of the viewing angles in Houckgeest's *Interior of the New Church at Delft.*

E—observer
P1, P2, P3 as in below.

AB—visual angle



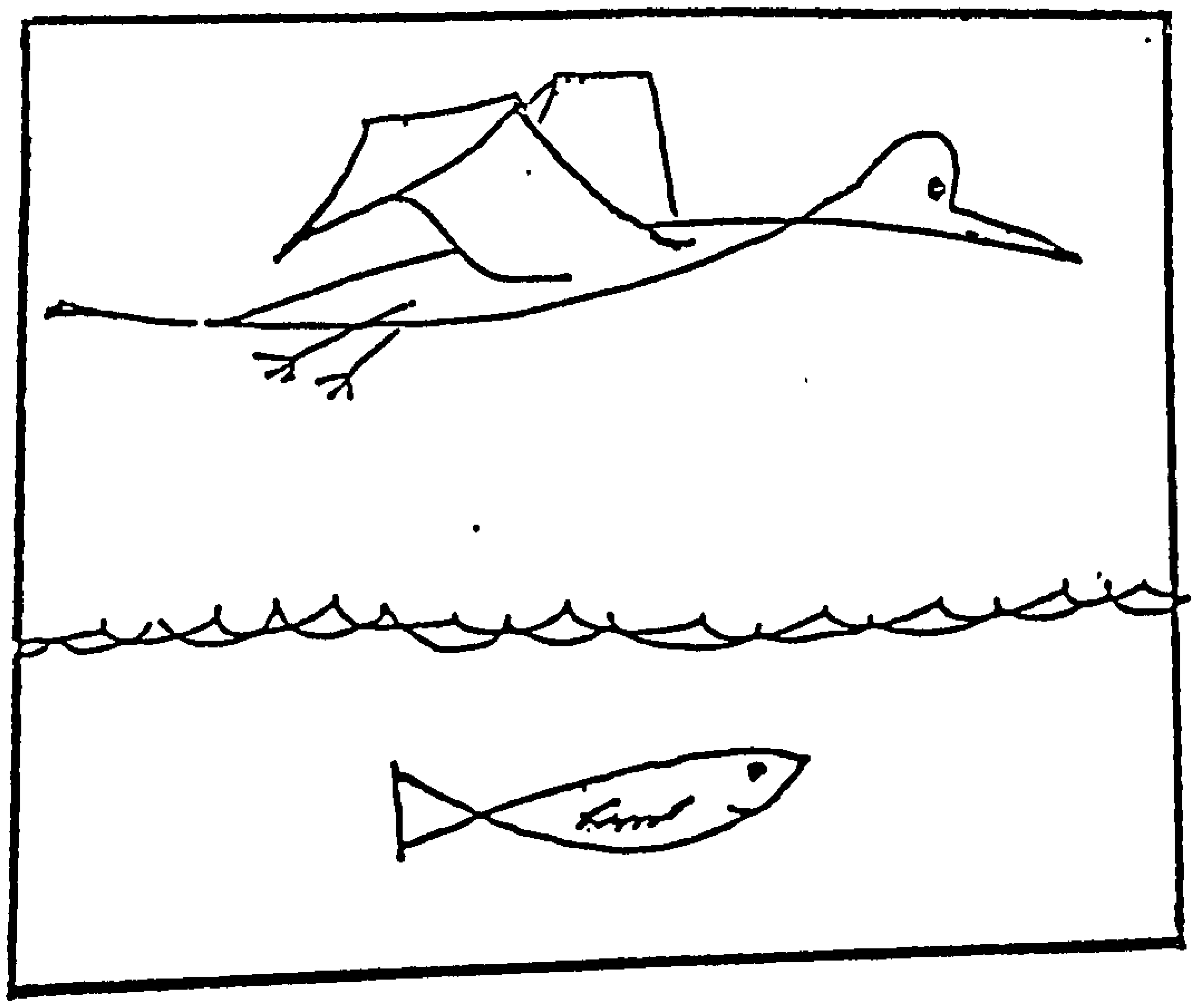
. Perspective analysis of Gerard Houckgeest's *Interior of the New Church Delft.*

P1—point of convergence of orthogonals of floor tiles looking down the aisle
P2—point of convergence of diagonals of the floor tiles
P3—point of convergence of the orthogonals looking across the choir
H—horizon

N.B. Text and pictures on this page (60) are photocopies taken from Kemp, 1990, page 117.

4.3. Duality

figure 1



Two statements can be made with regard to figure 1:

(1) The air is above the water

(2) The water is below the air.

In a sense, sentences (1) and (2) present opposite views of the same picture. Statements (1) and (2) are considered as dual to each other.

Statement (1) contains words (or notions): air
above
water

Statement (2) also contains notions: water
below
air

Placing the notions of (1) and (2) next to each other:

(1)		(2)
air	________	water
above	________	below
water	________	air

we consider	air	⟷	water
and	above	⟷	below

as dual notions.

In the case of figure 1 more dual notions can be found:

fish	⟷	bird
swim	⟷	fly
jump out of	⟷	dive
warm	⟷	cold
up	⟷	down
air	⟷	water
above	⟷	below
wet	⟷	dry

and many others.

Now any statement about the bird, composed with the above notions, can be immediately translated into a statement about the fish, simply by replacing each notion with its dual.

For example, one may state :

(a) The flying bird dives from the warm air into the cold water.

Replacing the underlined notions by their duals we get:

(b) The swimming fish jumps out of the cold water into the warm air.

Sentences (a) and (b) are called dual statements: one can be derived from the other.

Duality in Geometry

In plane geometry, the notions

point ⟷ line

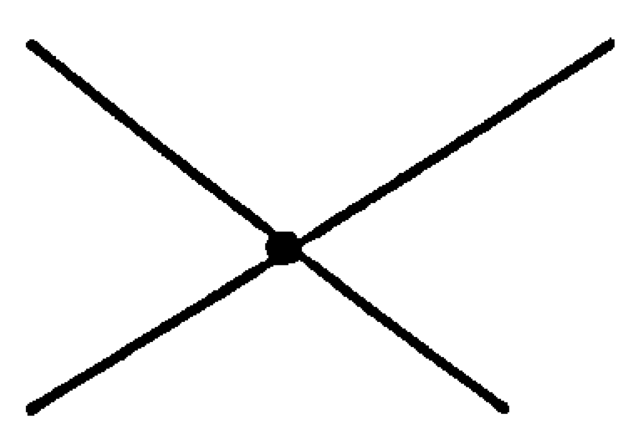
are connected by ⟷ intersecting

appear to be dual.

We state :

(c) Two points are connected by one line

replacing the underlined notions by their duals, we get:



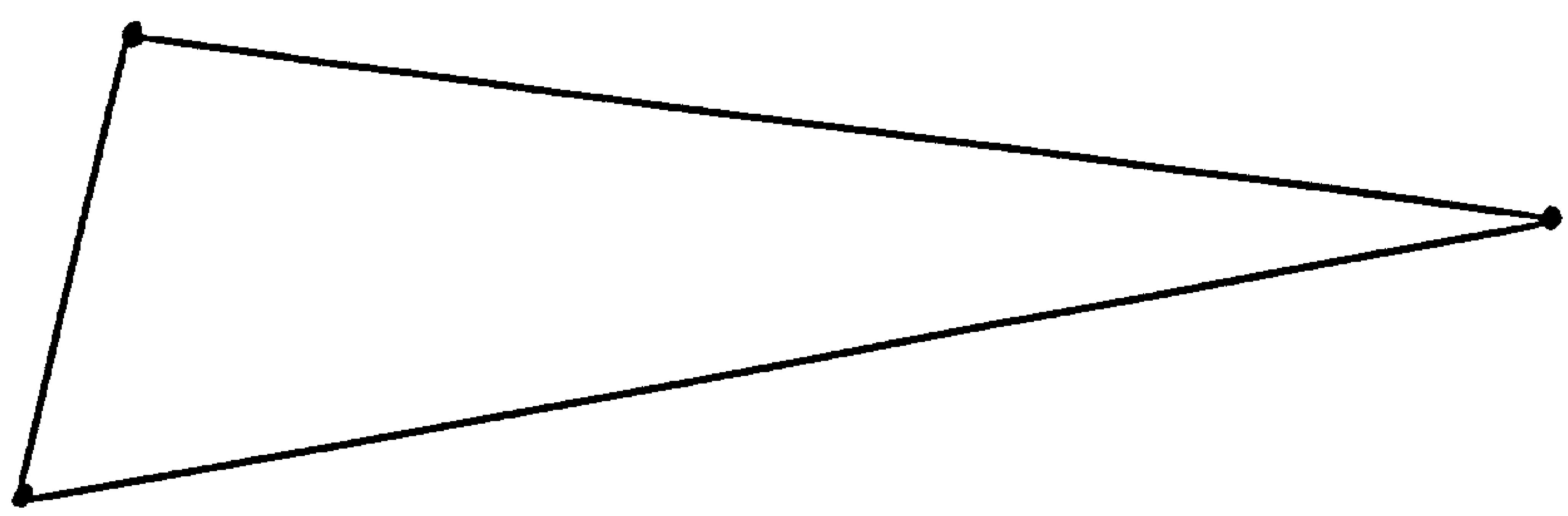
(d) Two lines intersect at one point

Both statements are true and they are dual.

If a statement in plane geometry, involving the above notions, is true, then the dual statement is automatically valid.

Another example: (e) Three lines intersect at three points

dual: (f) Three points are connected by three lines



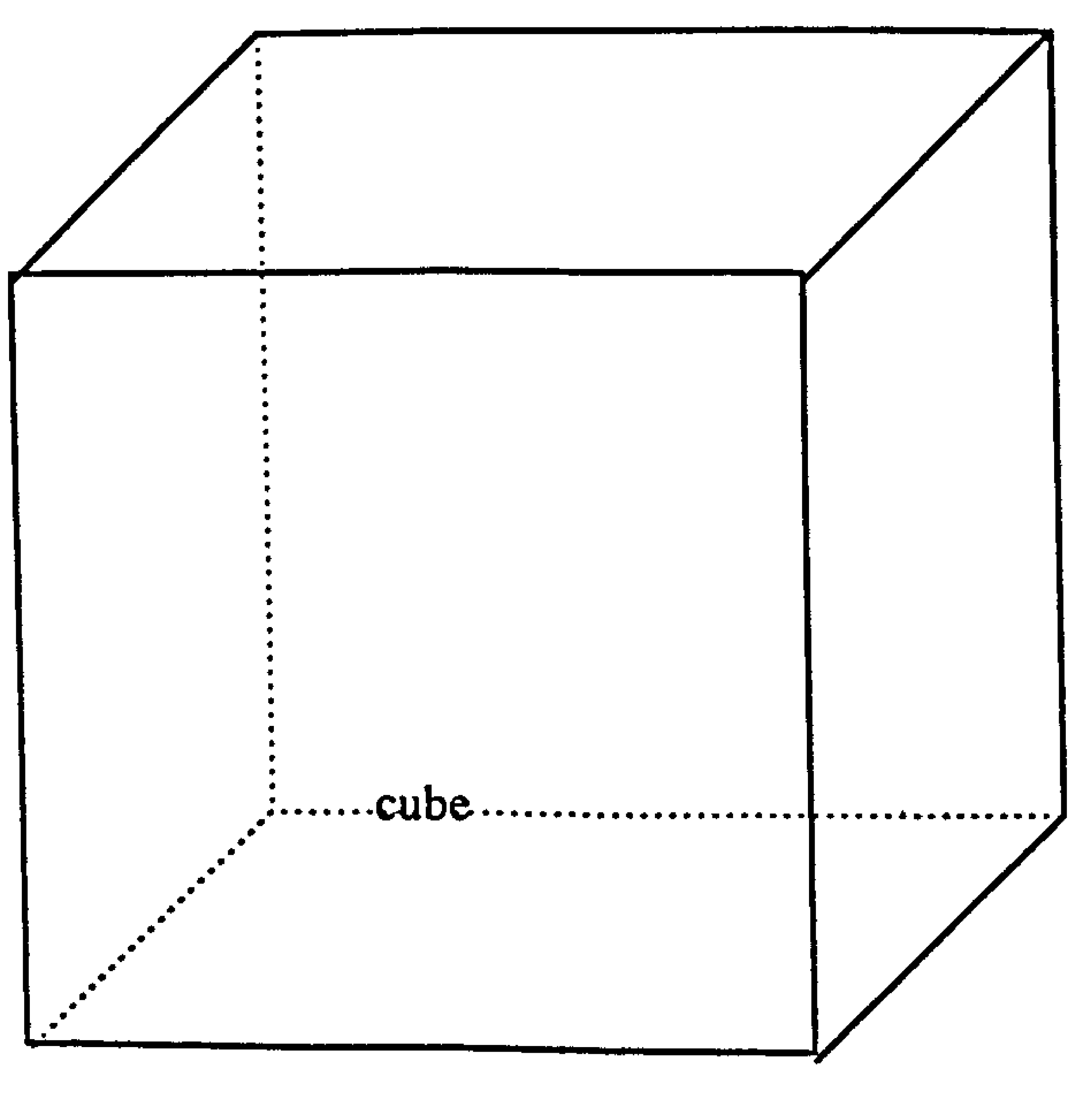
From (e) follows (f) and the reverse.

So, if one has found a theorem with the above notions, then one has found a second (dual) theorem which is valid too. It simplifies geometry.

DUAL POLYHEDRA

In solid geometry 'point' and 'plane' are dual notions. Important examples of this may be found in the Platonic polyhedra, of which we see two in the following pictures:

figure 2



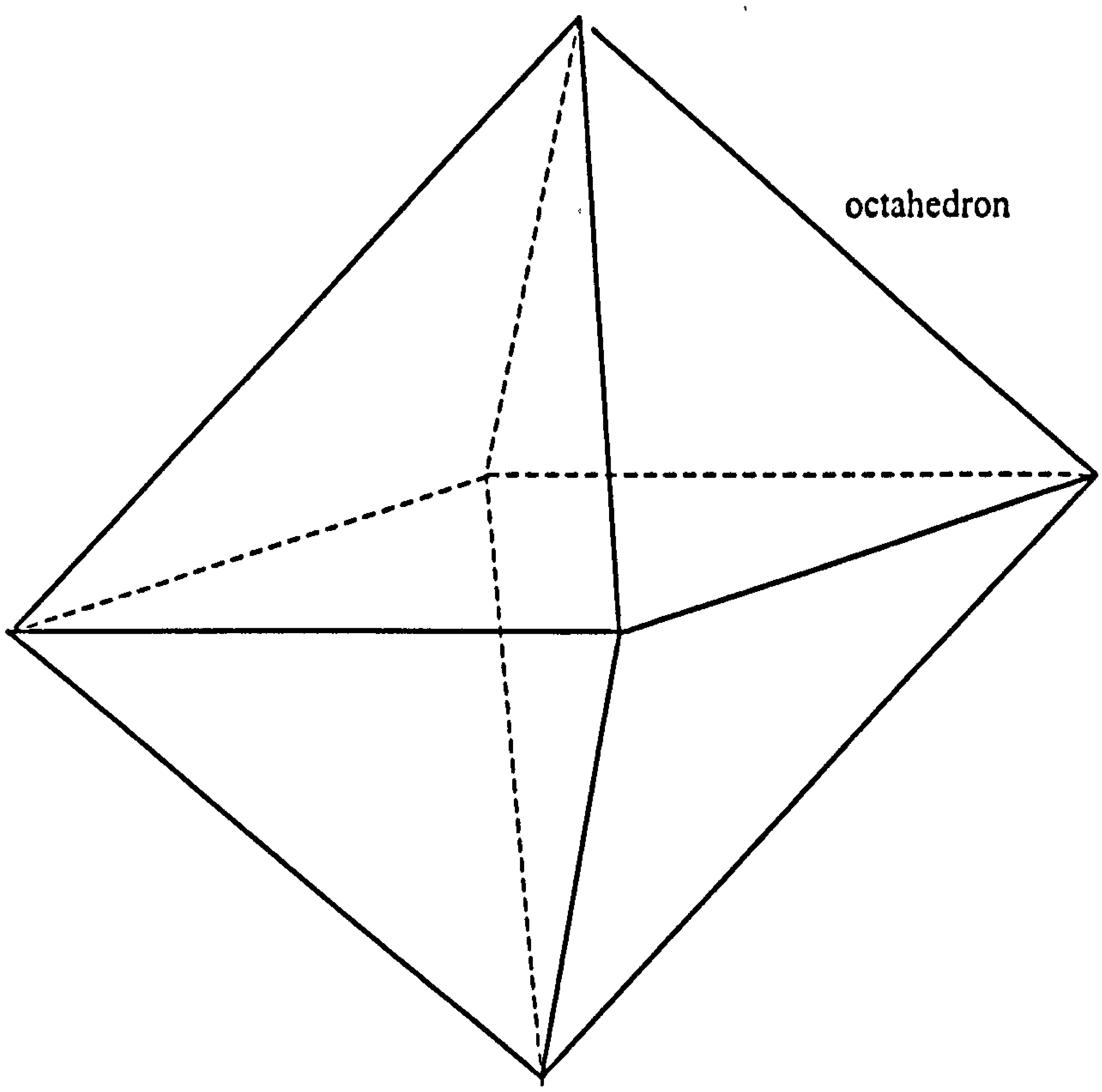


figure 3

We can state:

(3) A cube has 8 vertices and 6 faces.

(4) An octahedron has 8 faces and 6 vertices.

Placing the notion of (3) and (4) next to each other:

(3)		(4)
cube	________________	octahedron
vertex	________________	face

face ——————————————— vertex

We consider cube ⟷ octahedron

and vertex ⟷ face

as dual notions.

Now take sentence (3): A cube has 8 vertices and 6 faces.

Then replace in (3) each notion by its dual.

We get (4) : An octahedron has 8 faces and 6 vertices.

Conclusion: By dualising the notions we dualise the whole statement. (3) and (4) are considered as dual statements.

Cube and Octahedron are dual polyhedra .

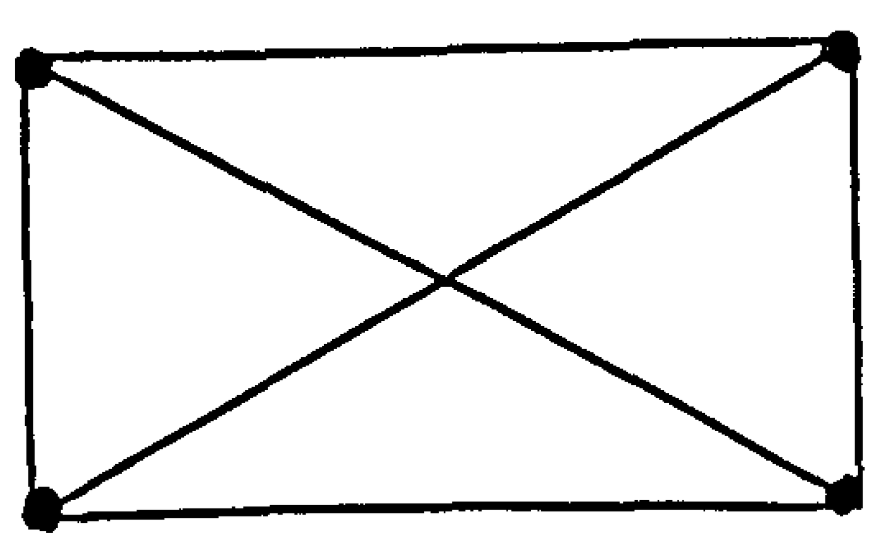
Next some examples of duality will be demonstrated.

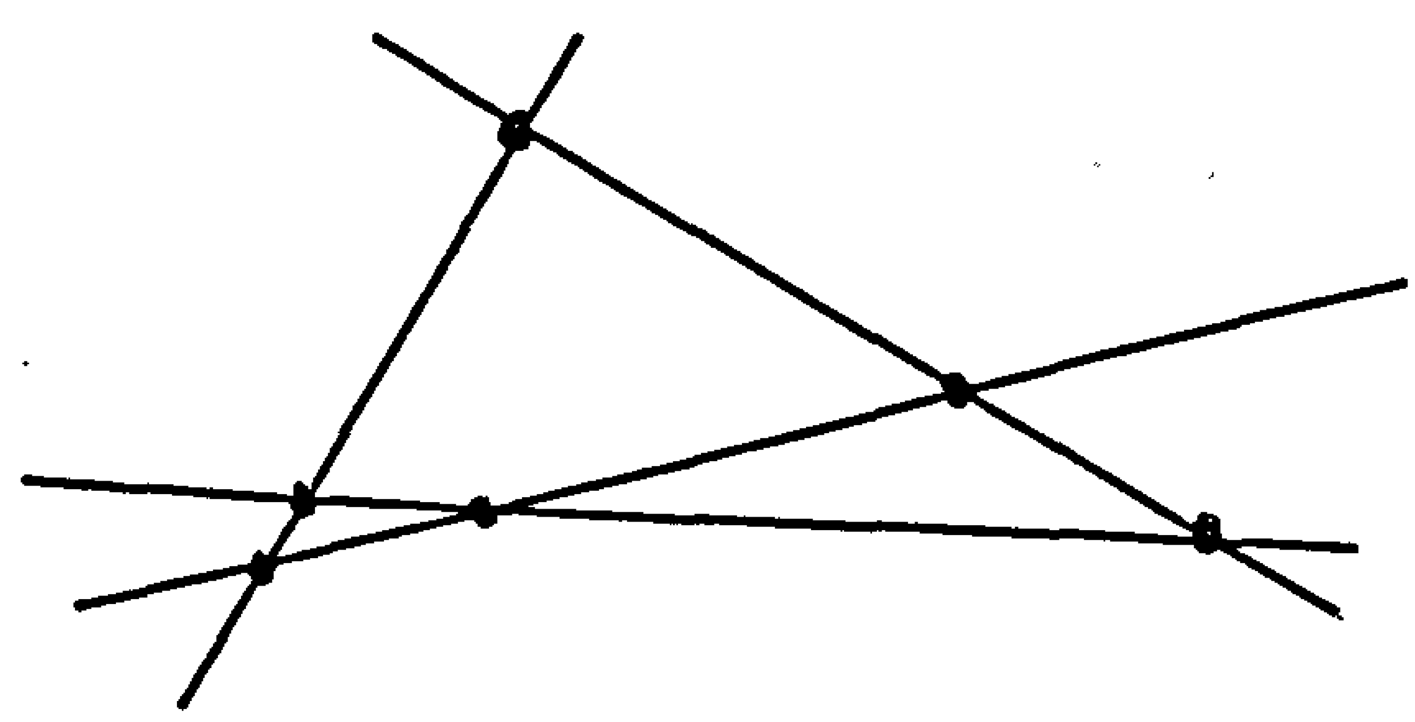
Now that we have the dual notions point ⟷ line

are connected by ⟷ intersection

in a plane we can give examples of dual statements.

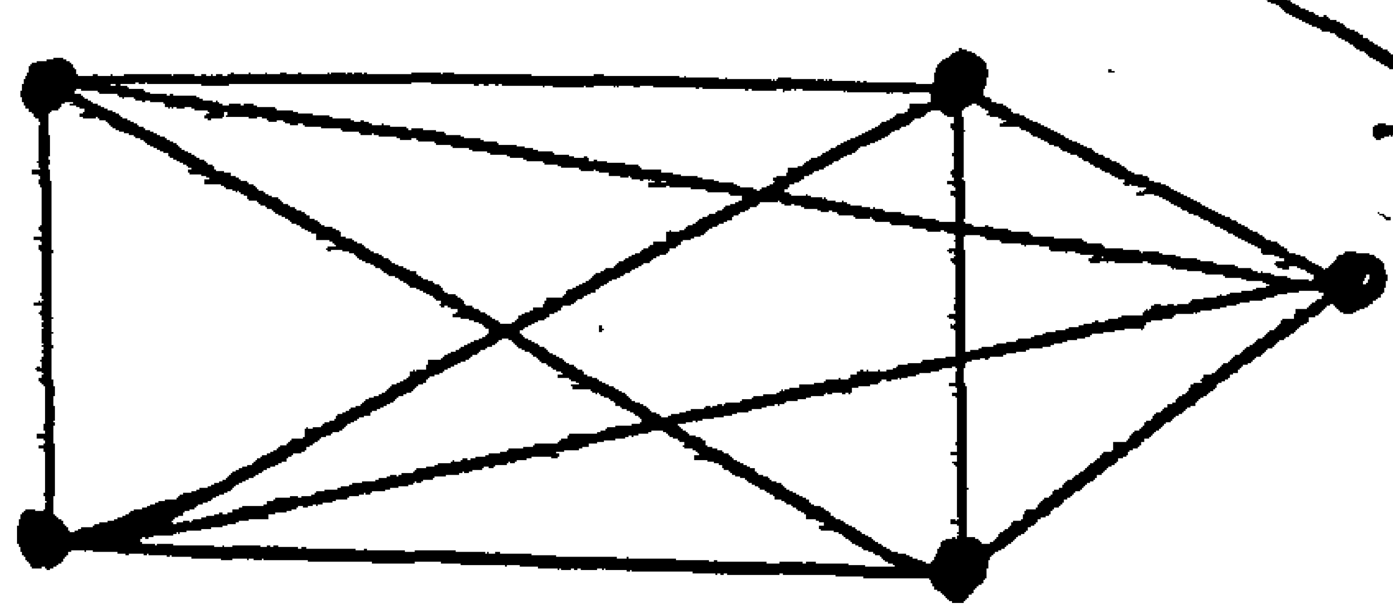
STATEMENT	DUAL STATEMENT
A. 4 points are connected by 6 lines.	A. 4 lines intersect at 6 points

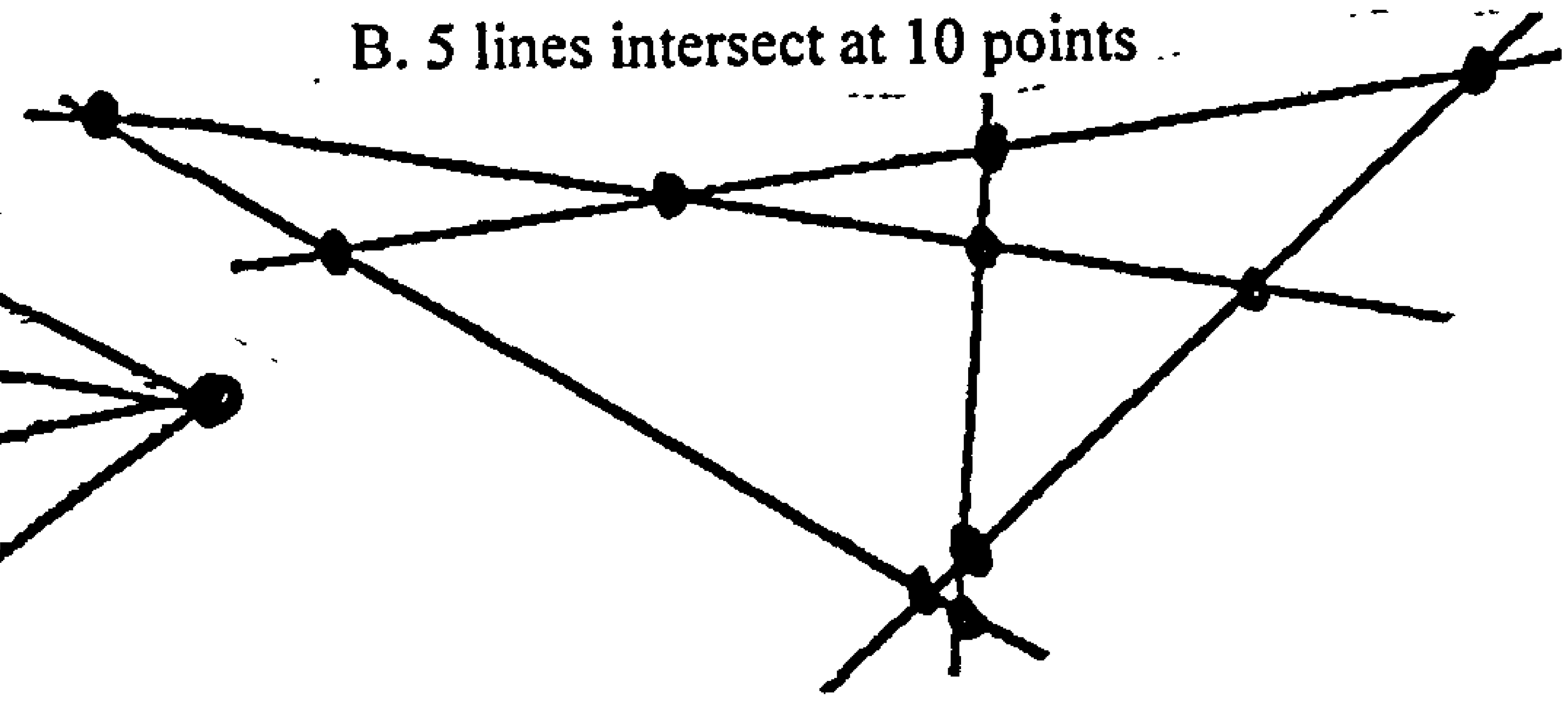




B. 5 points are connected by 10 lines

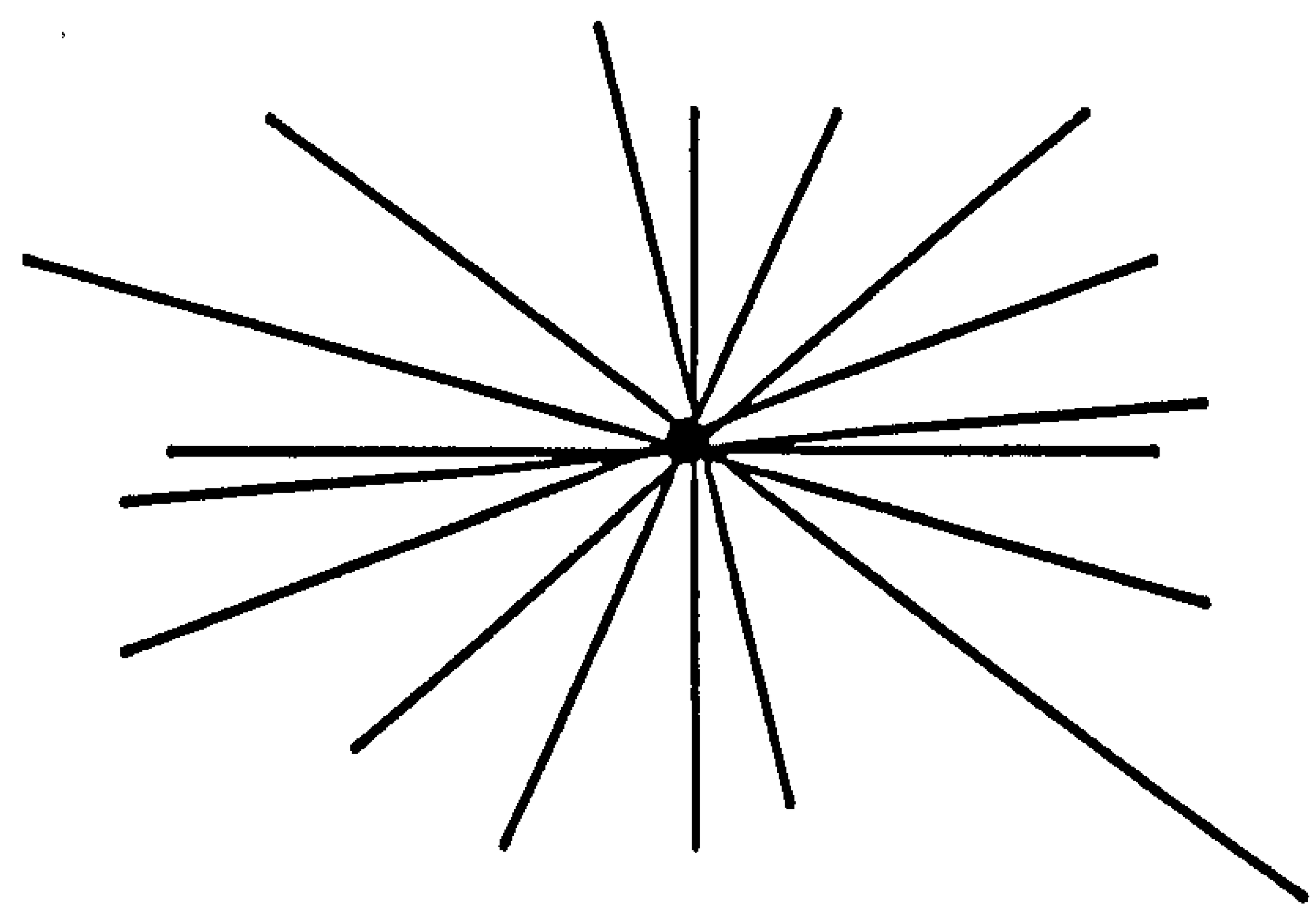
B. 5 lines intersect at 10 points





C. Many points lie on a line

C. Many lines go through a point



Duality is a word denoting that points and lines are in a sense equivalent and interchangeable. It is a basic notion in Projective Geometry and the theory of conic sections. Jacob Steiner has played an important role in the development of Duality and the following section concerns him.

Jacob Steiner (1796 - 1863)

Born in Switzerland, he was educated at a village school; by his 18th year he was, apart from having reached proficiency in arithmetic, little more than illiterate. As a boy he herded his father's cattle on the Jura slopes. Years later, he would ironically say that he had learned over there to 'recognise the stupid cattle at the furthest distance'. He was skilled in computing mentally and earned money at the market place by displaying it. In 1814 he left his parental home to be educated at Yverdon on the banks of Lake Neufchatel and entered Pestalozzi's school, which became a great success. Steiner, a single-minded geometer, appreciated Pestalozzi's methods and even taught geometry at the school. In 1829 he was appointed as a senior teacher at a Berlin grammar school (Oberrealschule); however he showed little patience with his pupils, scolding them and losing his temper.

In the classroom he displayed resignation and was depressed because of the stupidity of his students. Although this was resented, pupils, colleagues and the principal noticed that he was an extremely gifted man as far as geometry was concerned. He did have a weak point: analysis. His competence in this area was very limited, which annoyed him intensely. His main work, "Systematische Entwickelung der Abhängigkeit Geometrischer Gestalten von einander", appeared in 1832. It was a breakthrough in Projective Geometry.

In the same year he received a honorary doctorate at the university of Königsberg and in 1834 he was elected a member of the Prussian Academy of Science. Also in 1834 he was appointed as professor of higher geometry at Berlin Polytechnic. His health in the second part of his life was very weak; he needed many leaves of absence to recuperate. What makes Steiner exceptional is his splendid competence and achievements in Projective Geometry alongside his weakness in analysis.

Looking through the pages on duality of polyhedra one may wonder whether the use of a cube and a octahedron is not educationally invalid, because it refers to the disapproved secondary school model. I do not agree. It should be recognised that the Platonic bodies show outstanding beauty when depicted in the educationally invalid way. It is a new example of Magic Realism and as such one should not reject it but recognise it as such. This is also true for the polyhedra and plane figures like squares in the following treatment of the Euler Characteristic.

4.4. Euler Characteristic

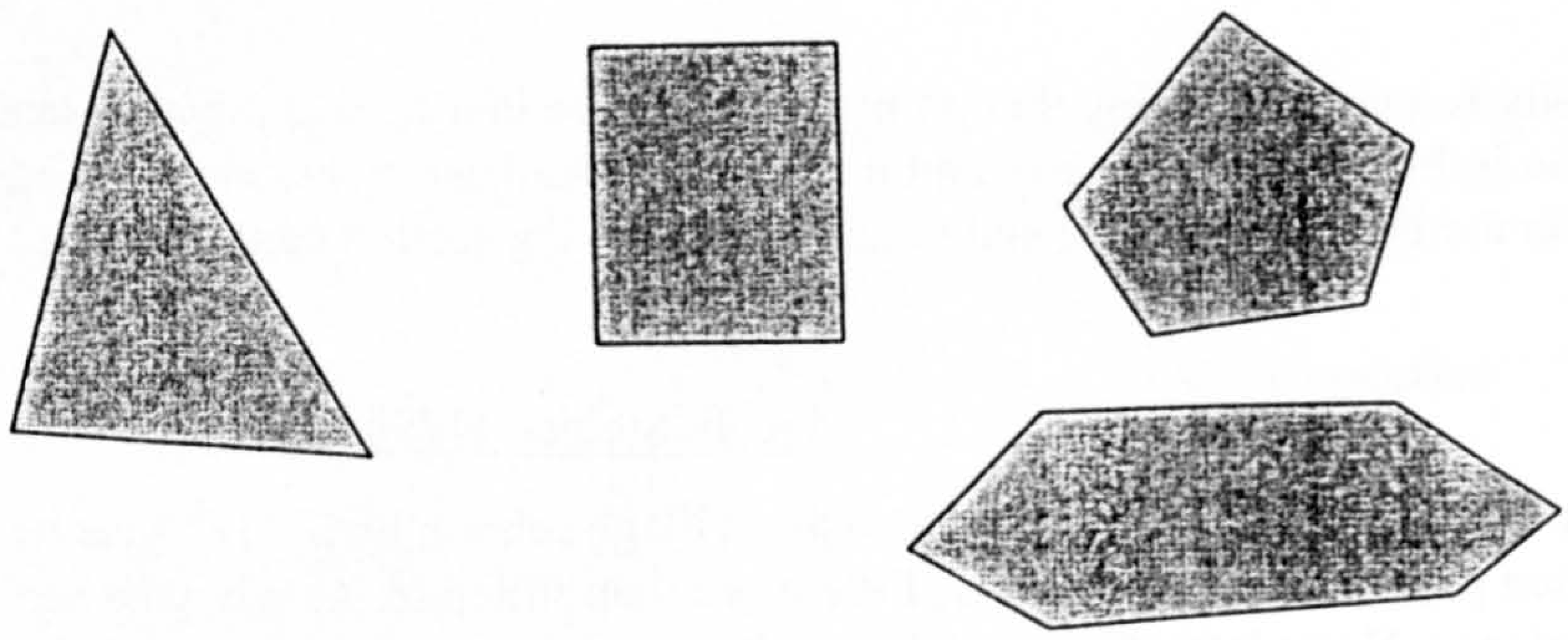
The Euler Characteristic assigns whole numbers to certain types of geometrical figures, so that each specimen of the type gets the same number.

We may consider triangles, quadrangles, pentagons, circles and so on as a category of plane figures, to which the number 1 is attached by the Euler Characteristic (see figure A).

The Euler Characteristic is denoted by the symbol: χ

figure A

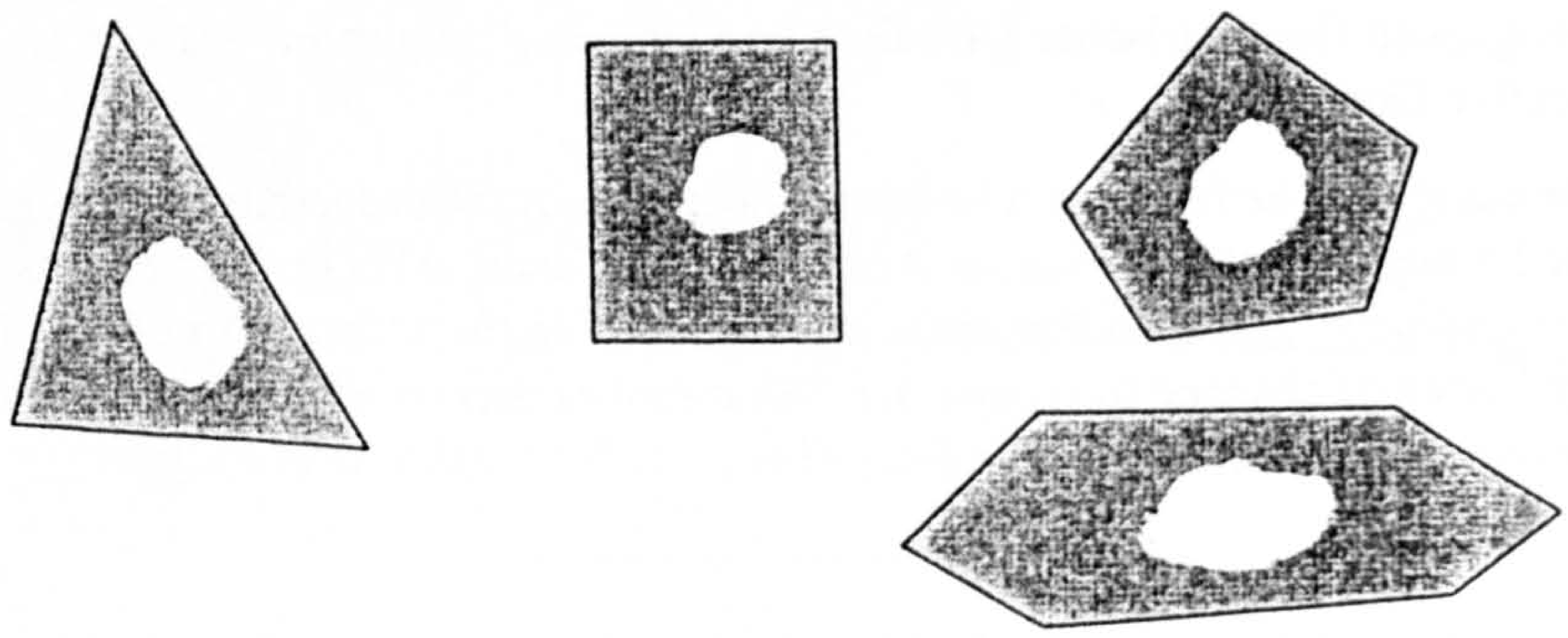
Euler Characteristic 1



A different type of geometric figure is obtained when in the interior of each figure a hole is punched. Figures in this category get the Euler Characteristic zero (see figure B).

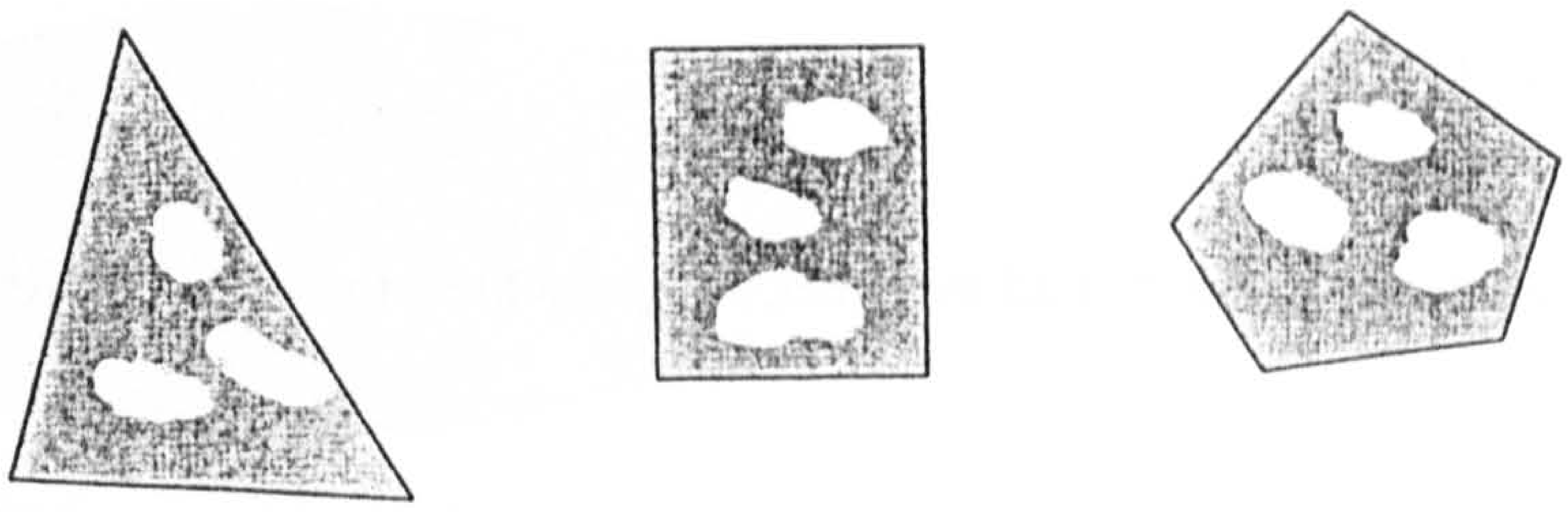
figure B

Euler Characteristic 0



For each additional hole, punched in the interior of the figure, the Euler Characteristic decreases by one. So, in figure C, the Euler Characteristic is -2 figure C

Euler Characteristic -2



So the Euler Characteristic assigns a whole number to a category of figures, of which the actual visual appearance is not decisive. It works like an 'equaliser'; only some essential properties of the figure are relevant. In some respect the figure seems to be 'de-visualised'. A whole branch of mathematics, **Topology,** deals with such essential properties of geometrical figures.

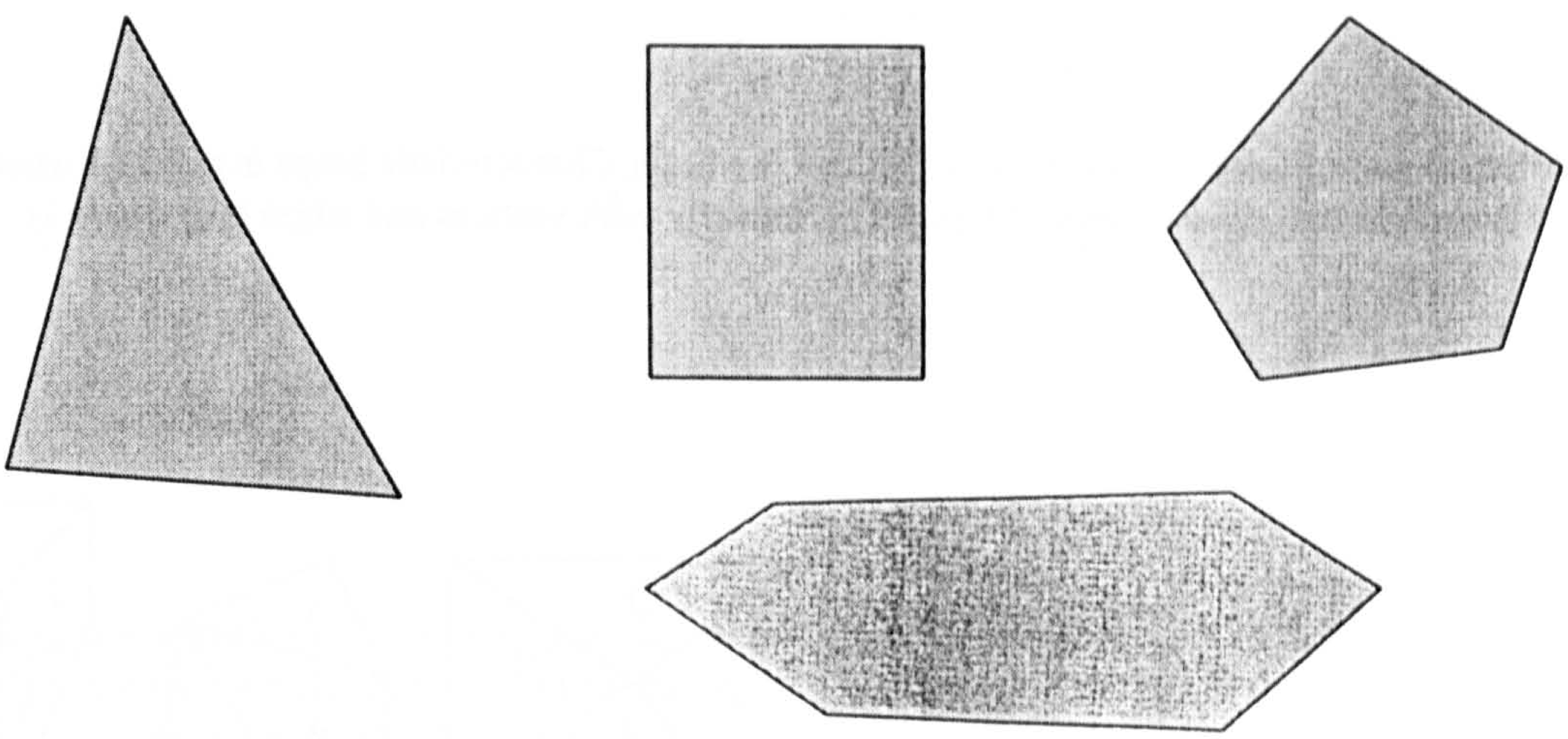
Topological properties are not quite dependent of visual appearances. For instance a cube has Euler Characteristic 2, but a sphere has the same Euler Characteristic. It means, that topologically there is no difference between cubes and spheres. A cube with a hole through its interior is topologically equal to the tyre of a bicycle and both figures are called: **torus.**

In a way the Euler Characteristic ignores the visual appearance of geometrical figures but just therefore it is eligible to be part of the curriculum of Educational Geometry. It is fascinating that categories of geometric figures which are very much different visually, may be topologically equal.

The actual values of the Euler Characteristic will now be discussed in detail.

In figure 1 several polygons are shown: there are a triangle, a quadrangle, a pentagon, a hexagon.

figure 1



The shaded area of each configuration is considered.

Do all these images have something in common ? Yes, there is a formula, which can be applied, and the result of the use of that formula is called 'Euler Characteristic', and it is a number. Euler Characteristic is computed according to the formula :

$\chi \quad = \quad V + F - E$ or alternatively: ($= V - E + F$ as applied in Chapter X)

in which V = the number of vertices
F = the number of faces (shaded area)
E = the number of edges.

Let's apply the Euler Characteristic to the polygons, shown in figure 1 (see figure 2).

A triangle has 3 vertices, 1 face and 3 edges. So, Euler Characteristic for a triangle reads:

$\chi = 3+1-3 = 1$

We will now demonstrate, that all the other polygons have the same characteristic : 1.

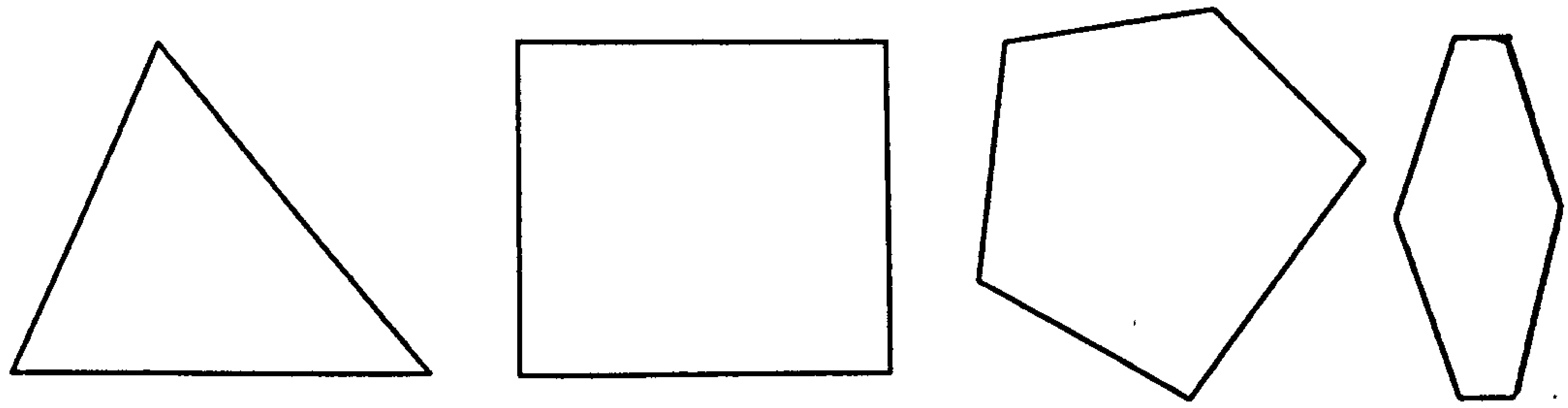
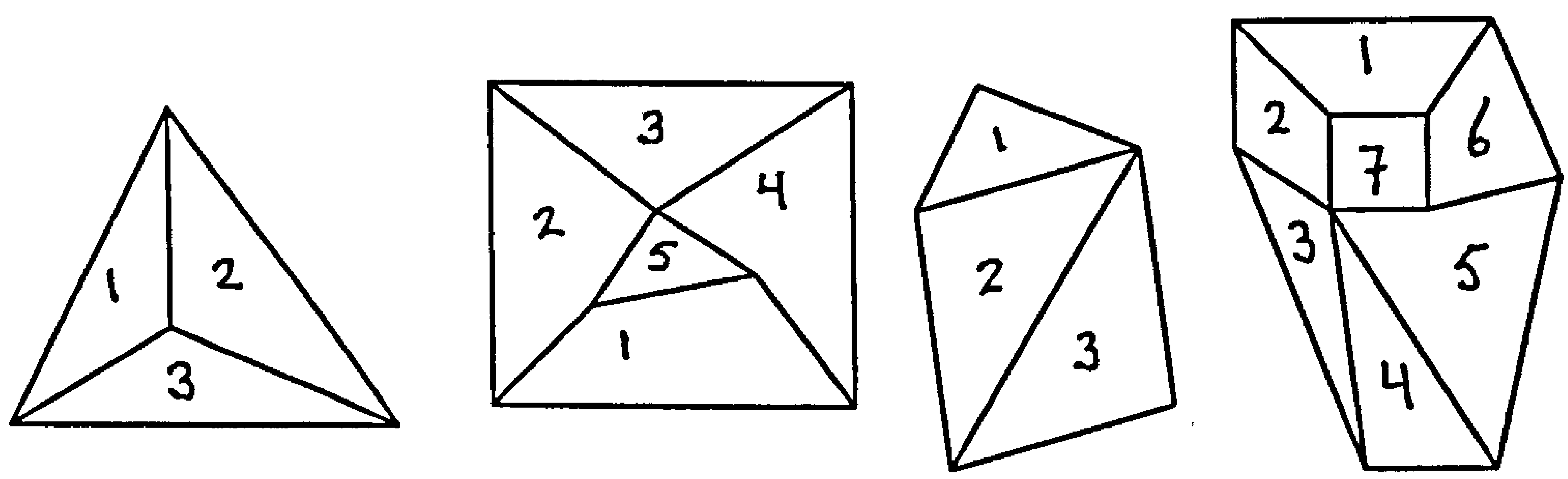


figure 2

Polygon	triangle	quadrangle	pentagon	hexagon
vertices	3	4	5	6
faces	1	1	1	1
edges	3	4	5	6
χ = V+F-E	1	1	1	1

So far, so good. However, it is remarkable that the Euler Characteristic keeps invariant 1 when we cover the inner area (the shaded areas of figure 1) arbitrarily with vertices and edges (see figure 3).

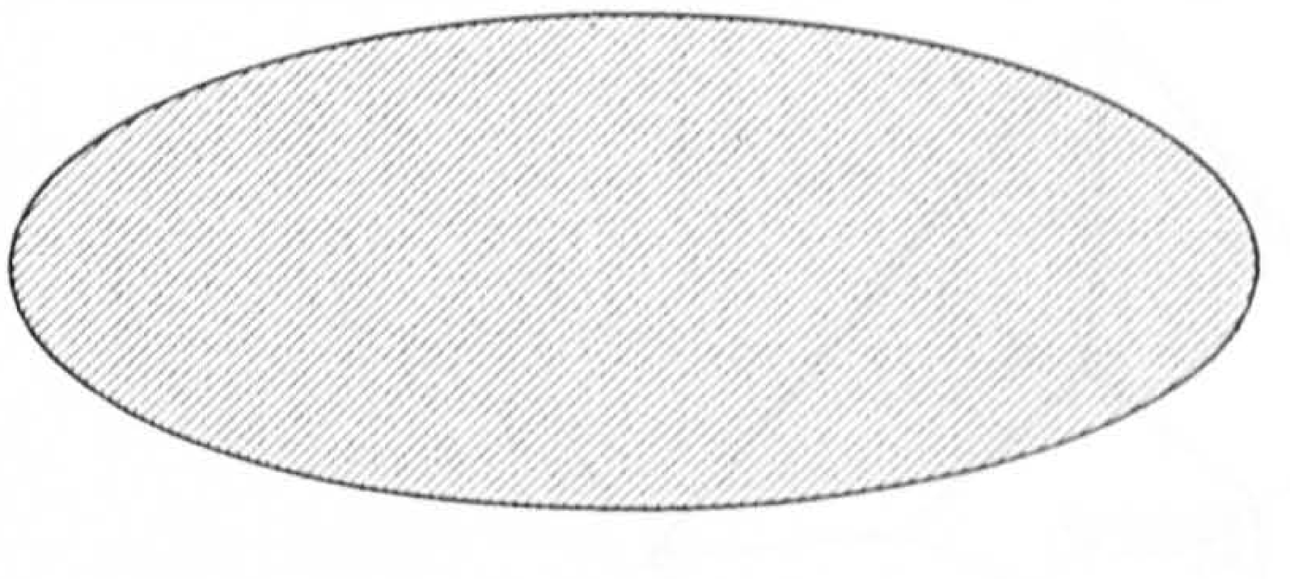
figure 3



Polygon	triangle	quadrangle	pentagon	hexagon
vertices	4	7	5	10
faces	3	5	3	7
edges	6	11	7	16
χ = V+F-E	4+3-6=1	7+5-11=1	5+3-7=1	10+7-16=1

In each polygon the faces have been counted. The covering vertices and edges are arbitrarily chosen. How can a polygon be characterised ? We might say that it looks like a part of a plane, enclosed by a closed curve (the circumference), and that it has no holes. Alternatively these polygons can be seen as a plane figure with Euler Characteristic 1. A visual representation of a polygon is given in figure 4; it is an oval with shaded inner area.

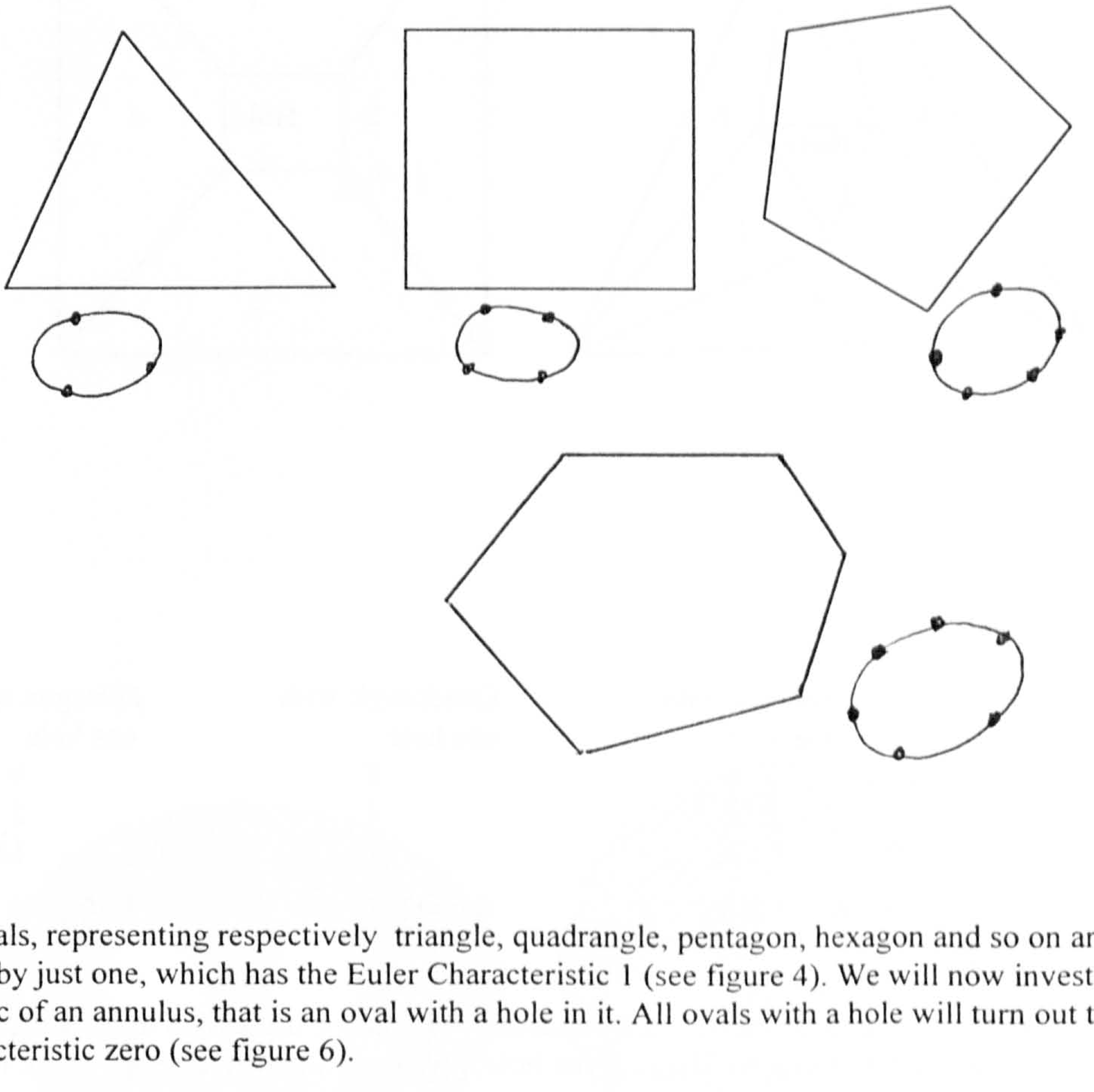
figure 4



$\chi = 1$

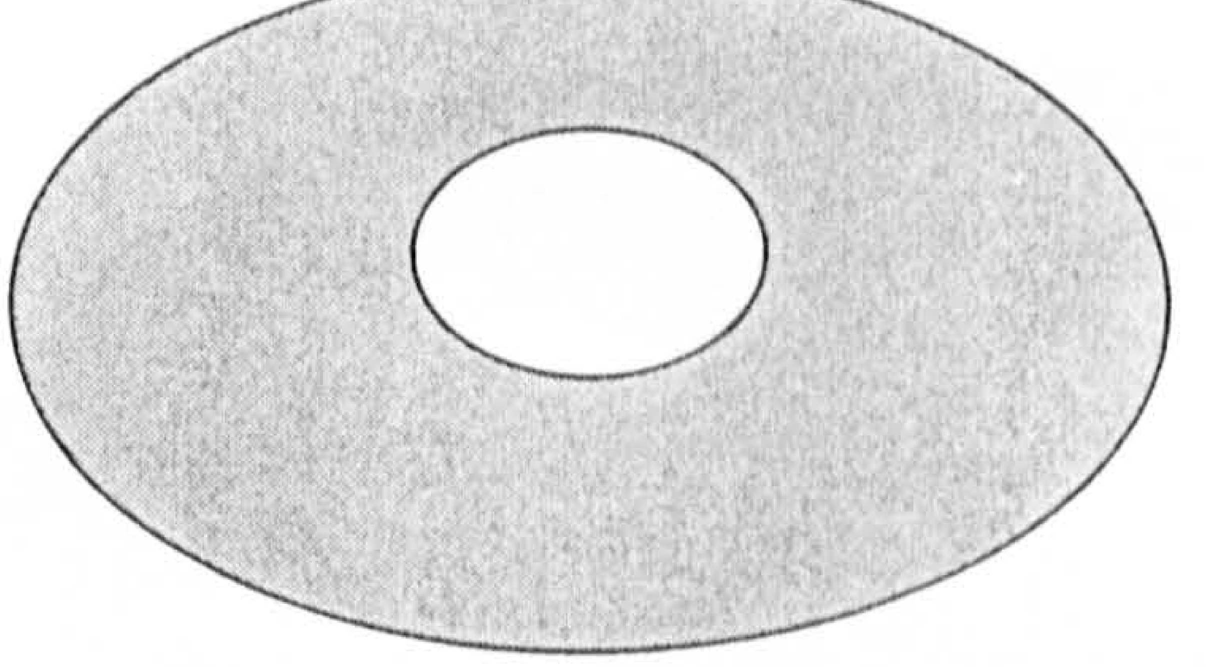
This representation is an example of an abstraction. We can compare triangle, pentagon, hexagon, and so on, to ovals with respectively 3,4,5 or 6 vertices on their circumferences (see figure 5).

figure 5



All these ovals, representing respectively triangle, quadrangle, pentagon, hexagon and so on are represented by just one, which has the Euler Characteristic 1 (see figure 4). We will now investigate the characteristic of an annulus, that is an oval with a hole in it. All ovals with a hole will turn out to have the Euler Characteristic zero (see figure 6).

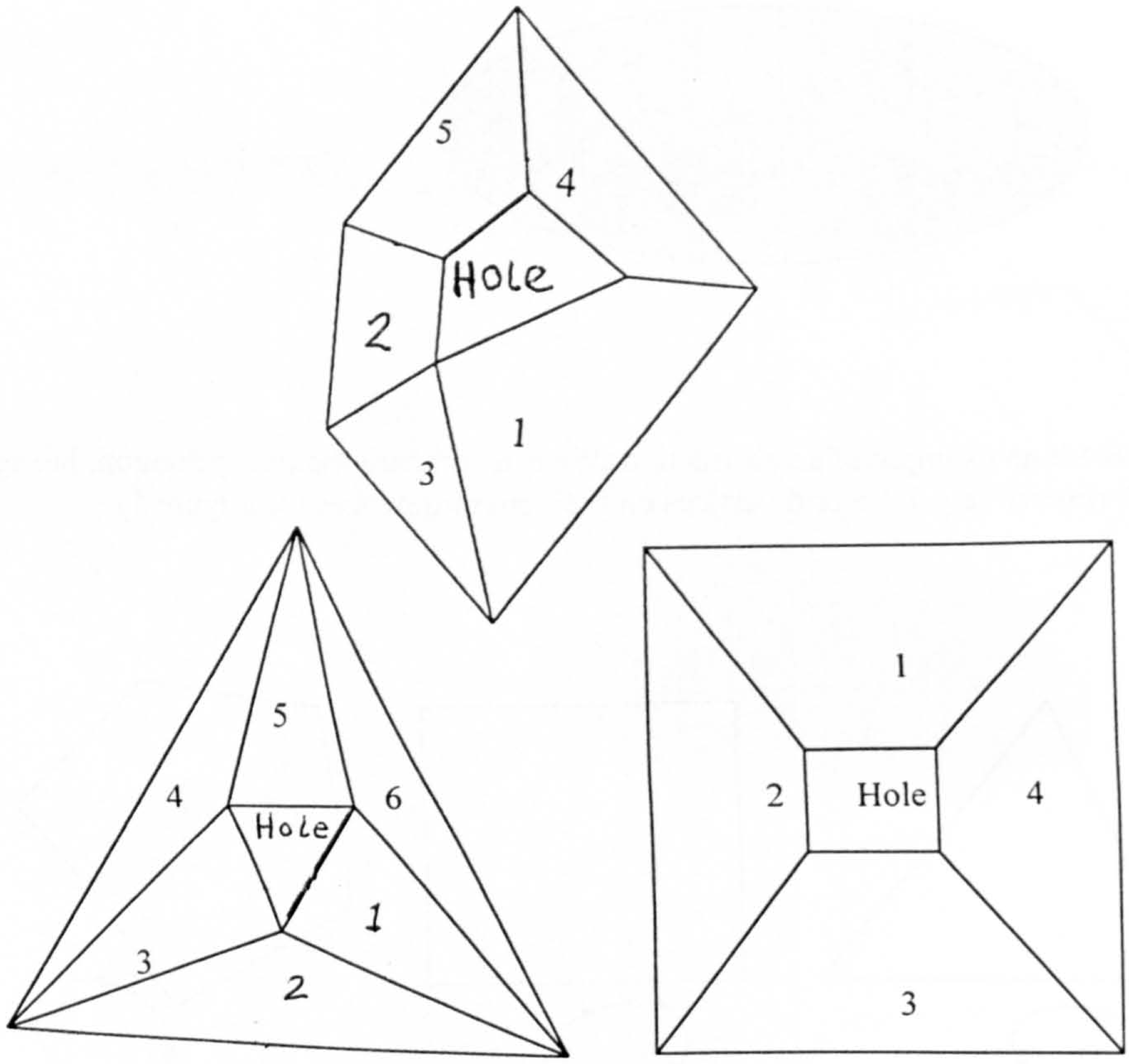
figure 6



$\chi = 0$

Here are a few examples (see figure 7).

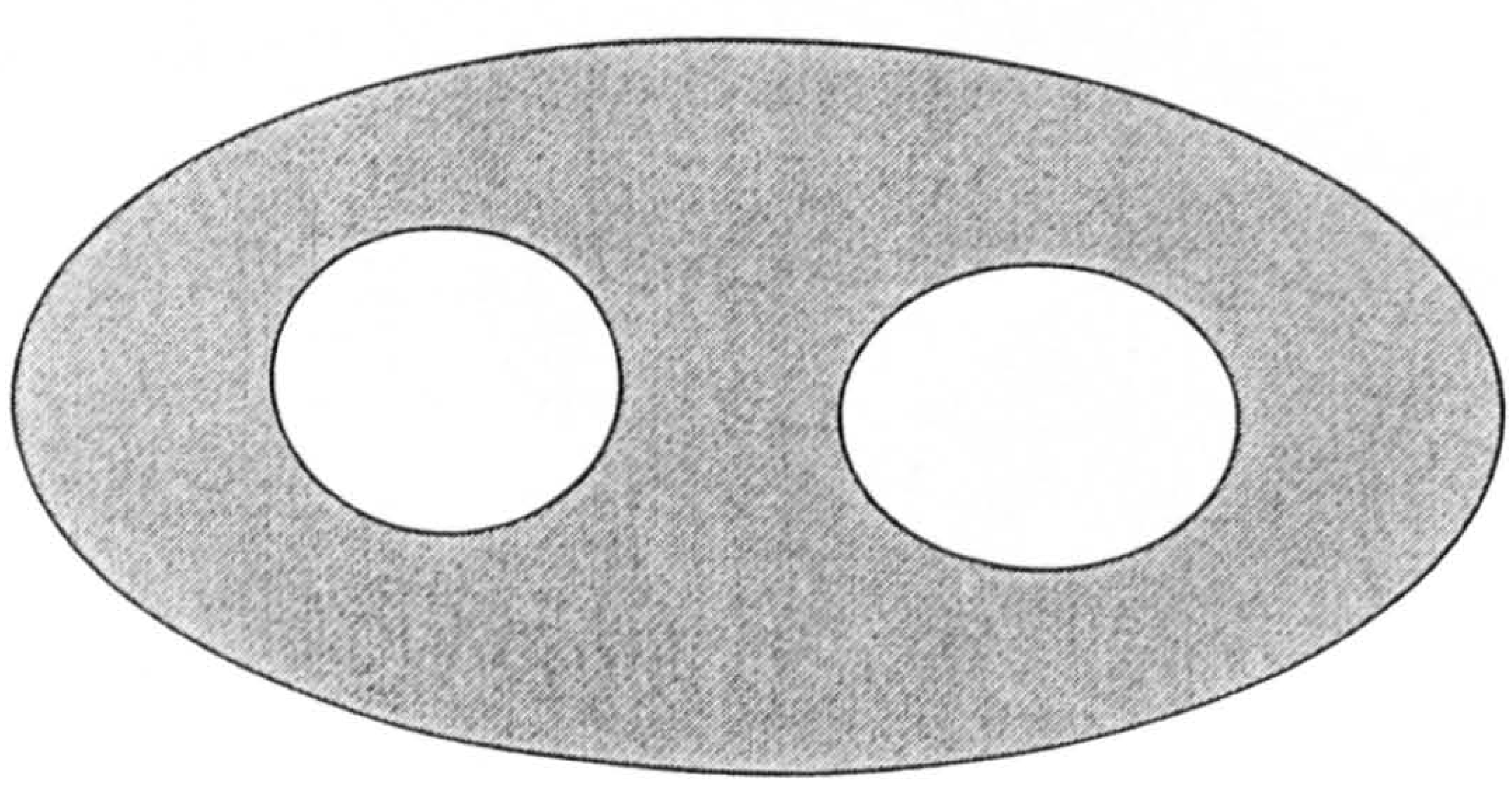
figure 7



Polygon with	Triangle with one hole	Quadrangle with one hole	Pentagon with one hole
vertices	6	8	9
faces	6	4	5
edges	12	12	14
χ = V+F-E	6+6-12=0	8+4-12=0	9+5-14=0

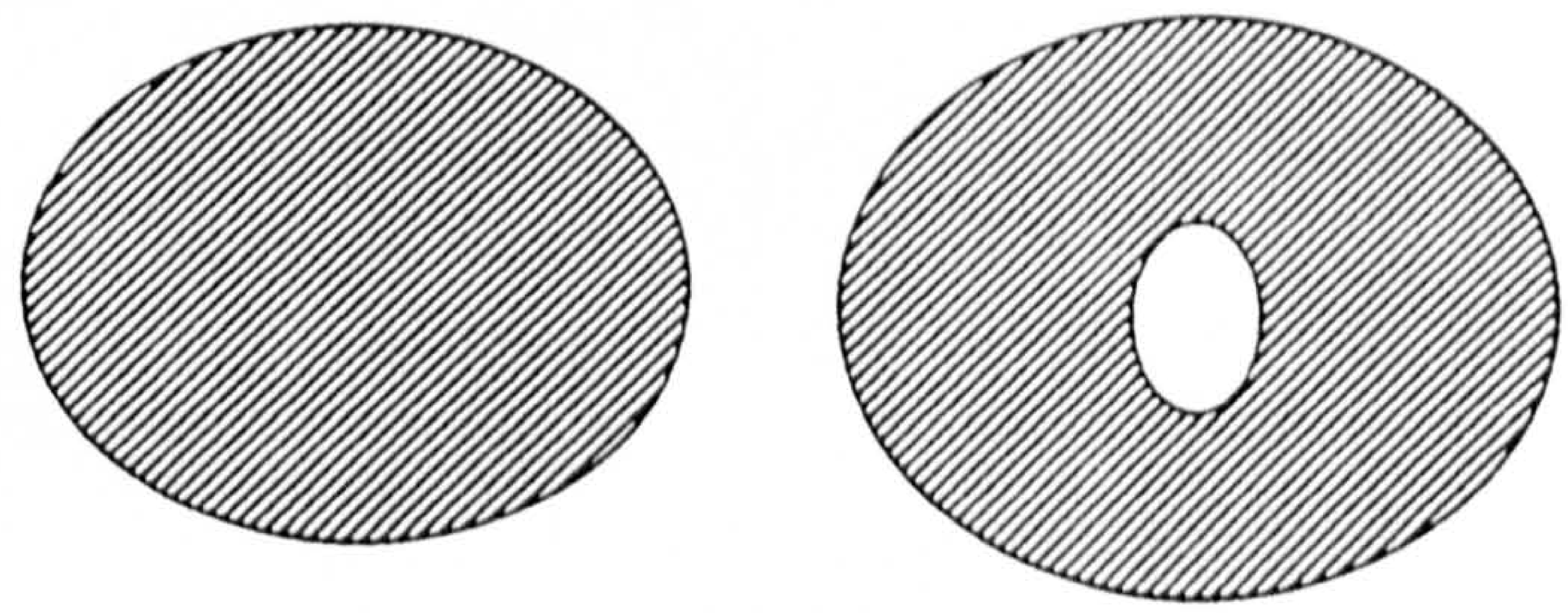
The covering edges and vertices are arbitrarily chosen. From the configuration in figure 7, it follows that areas like those enclosed by polygons, having one hole, give the Euler Characteristic zero. Proceeding in the same way, one can demonstrate, that the inner area of a polygon which has two holes yields the Euler Characteristic -1 (see figure 8).

figure 8



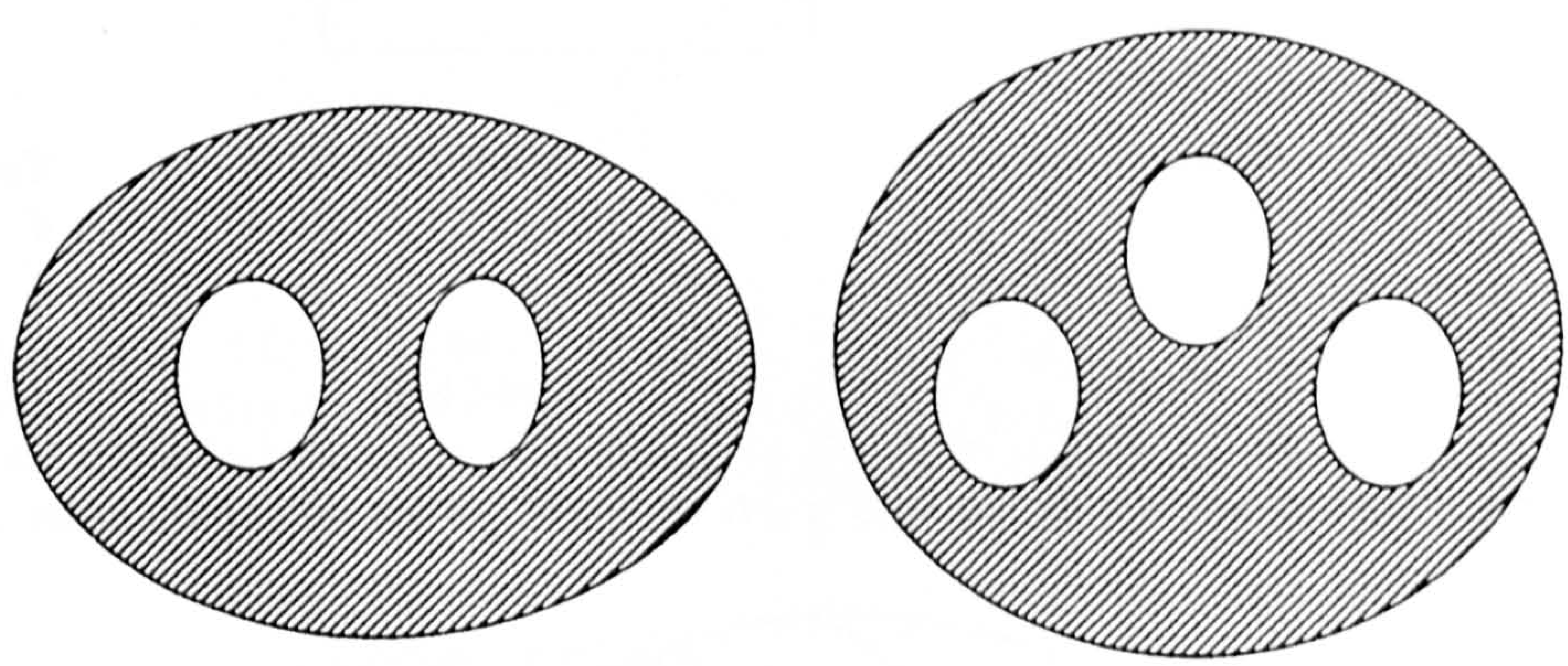
$\chi = -1$

Generally we can state that the following survey is valid: (figure 9)



$\chi = 1$ $\chi = 0$

figure 9

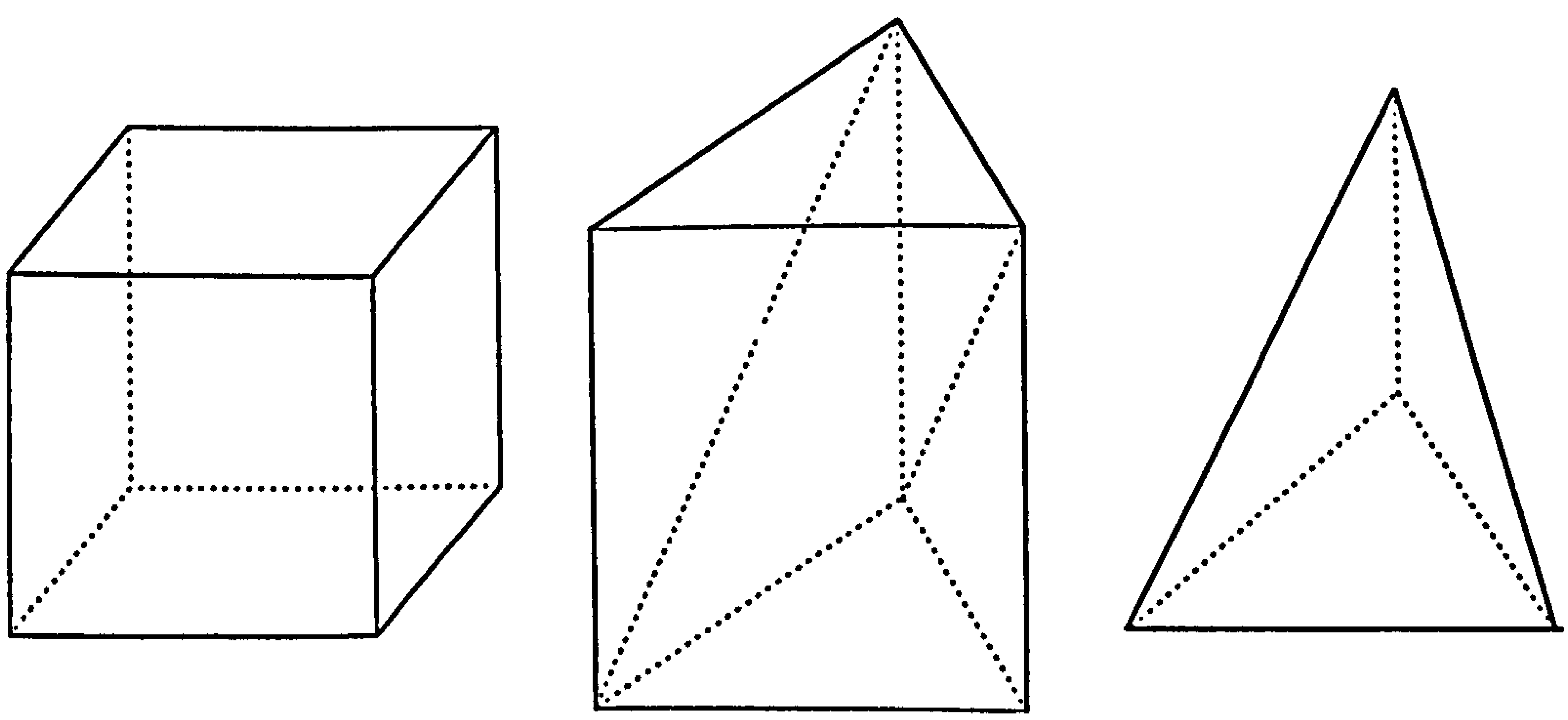


$\chi = -1$ $\chi = -2$

Thus the following formula will be valid: The Euler Characteristic of the inner area of a polygon, which has n holes is: 1- n.

The Euler Characteristic provides a close connection between the shape of a configuration and numbers. Now let us look at spatial figures. In figure 10 three spatial figures without holes are shown, which all have the Euler Characteristic 2.

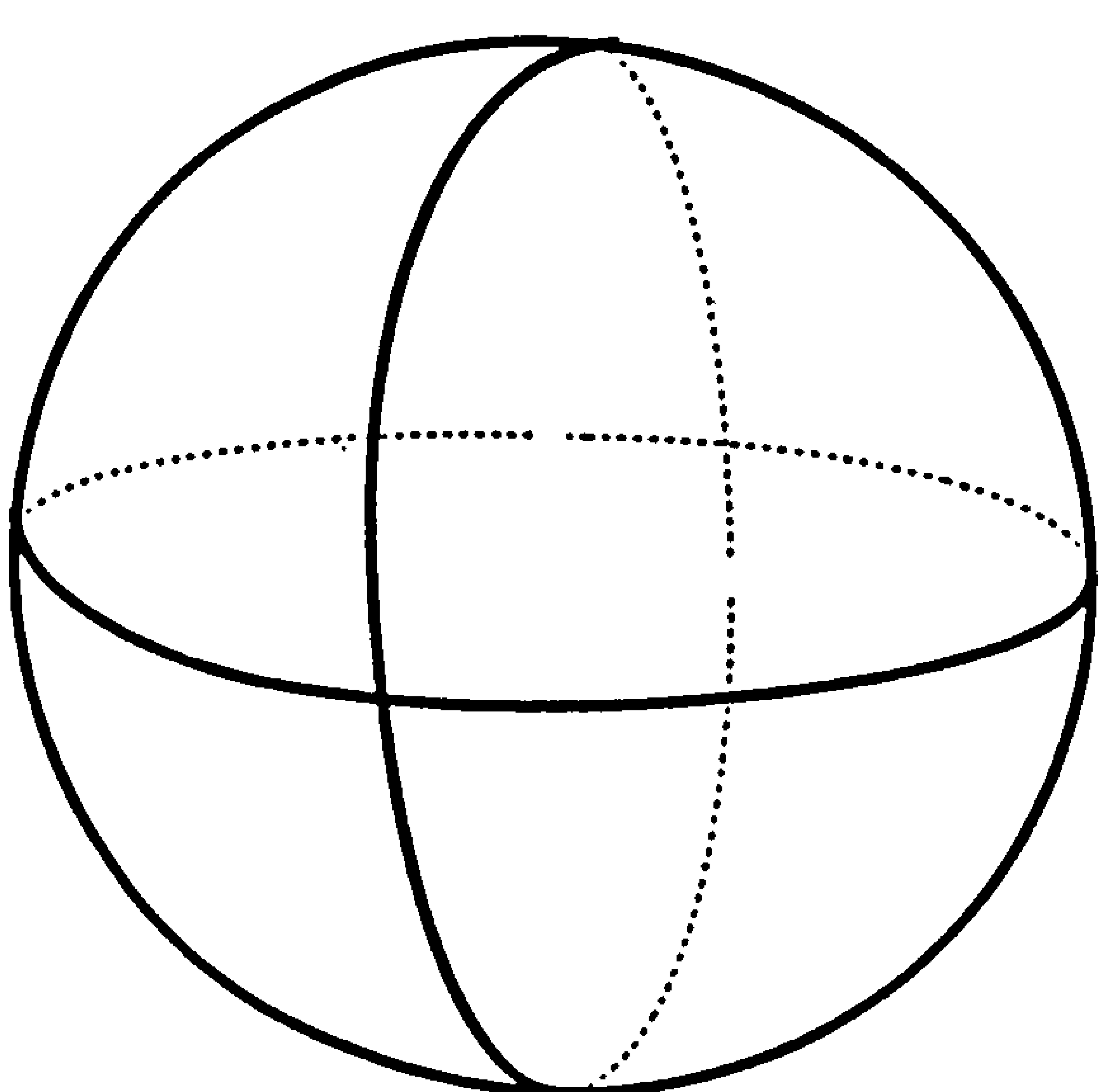
figure 10



Solid	Cube	Prism	Tetrahedron
vertices	8	6	4
faces	6	7	4
edges	12	11	6
χ = V+F-E	8+6-12=2	6+7-11=2	4+4-6=2

A general representation of Solids in figure 10 is a sphere, which is assumed to have Euler Characteristic 2 (see figure 11).

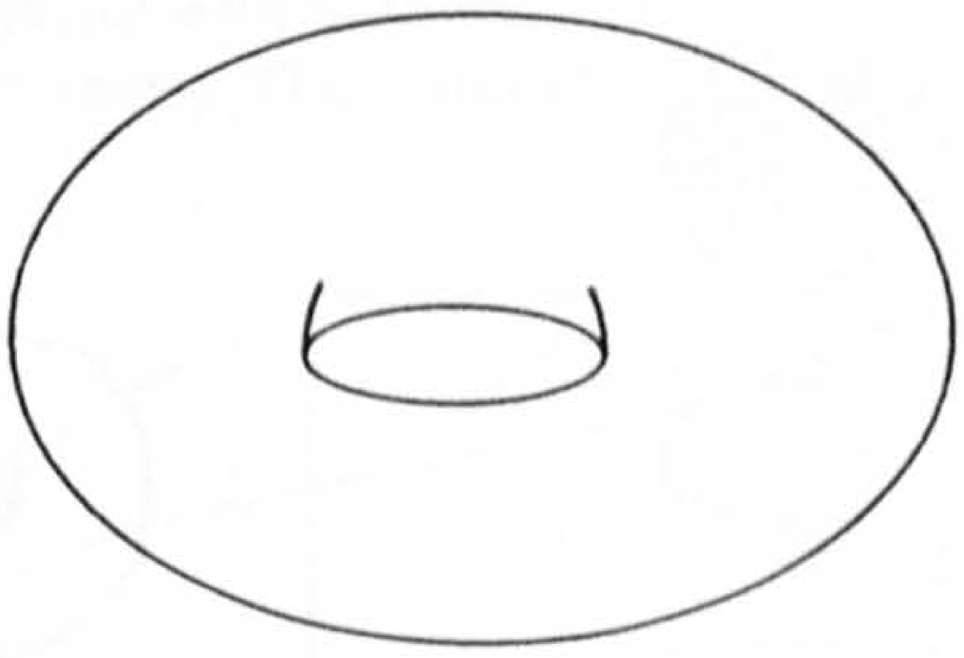
figure 11



$\chi = 2$

There is also a solid with a hole, which is called torus. It can be compared to the tyre of a bicycle (see figure 12). The Euler Characteristic of a torus will turn out to be zero.

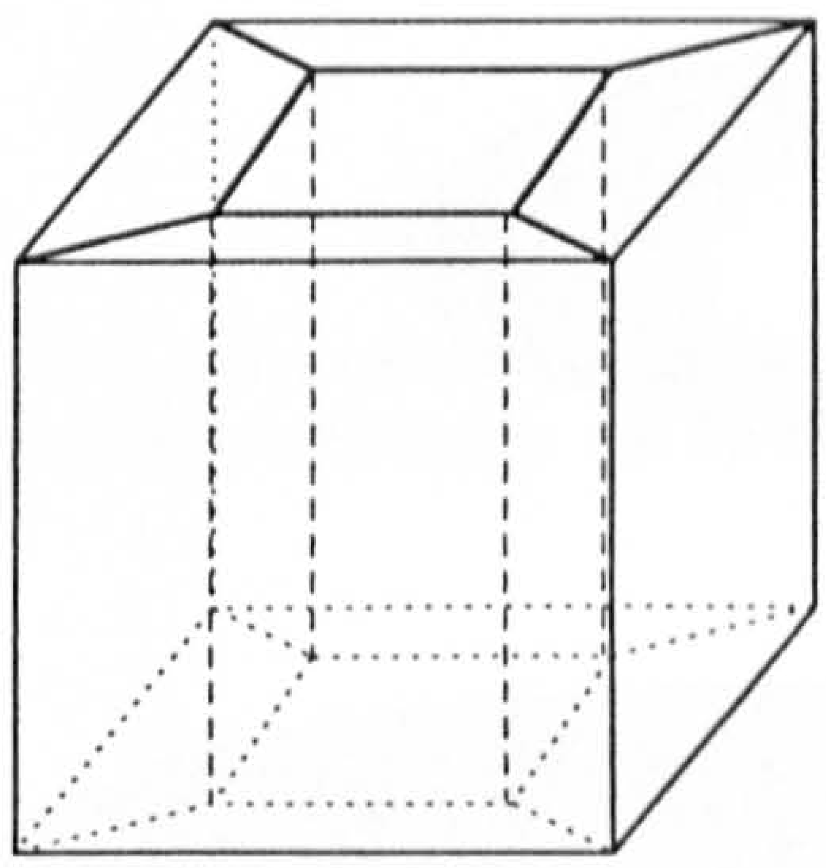
figure 12



$\chi = 0$

To compute χ, let us observe the cube in figure 13, which has a hole; the hole does not belong to the rest of the solid.

figure 13



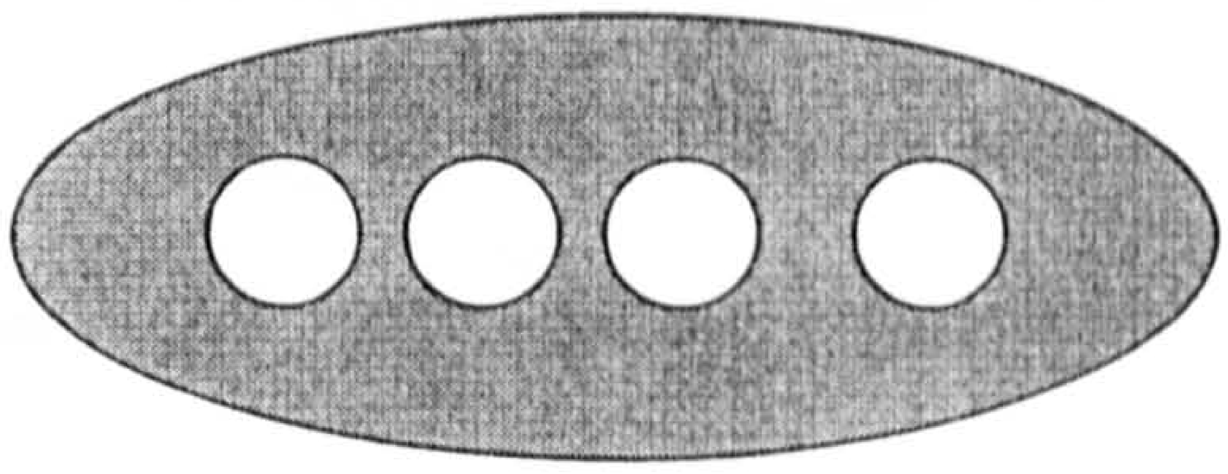
$\chi = 0$

figure 14

$\chi = -2$

The solid (with hole) of figure 13 has 16 vertices, 16 faces and 32 edges, so that the Euler Characteristic $\chi = 16+16-32=0$. Making a second hole in a solid, means that χ decreases by 2. We then have 2 connected tori (figure 14) with $\chi = -2$. So the following survey can be provided.

Flat area :

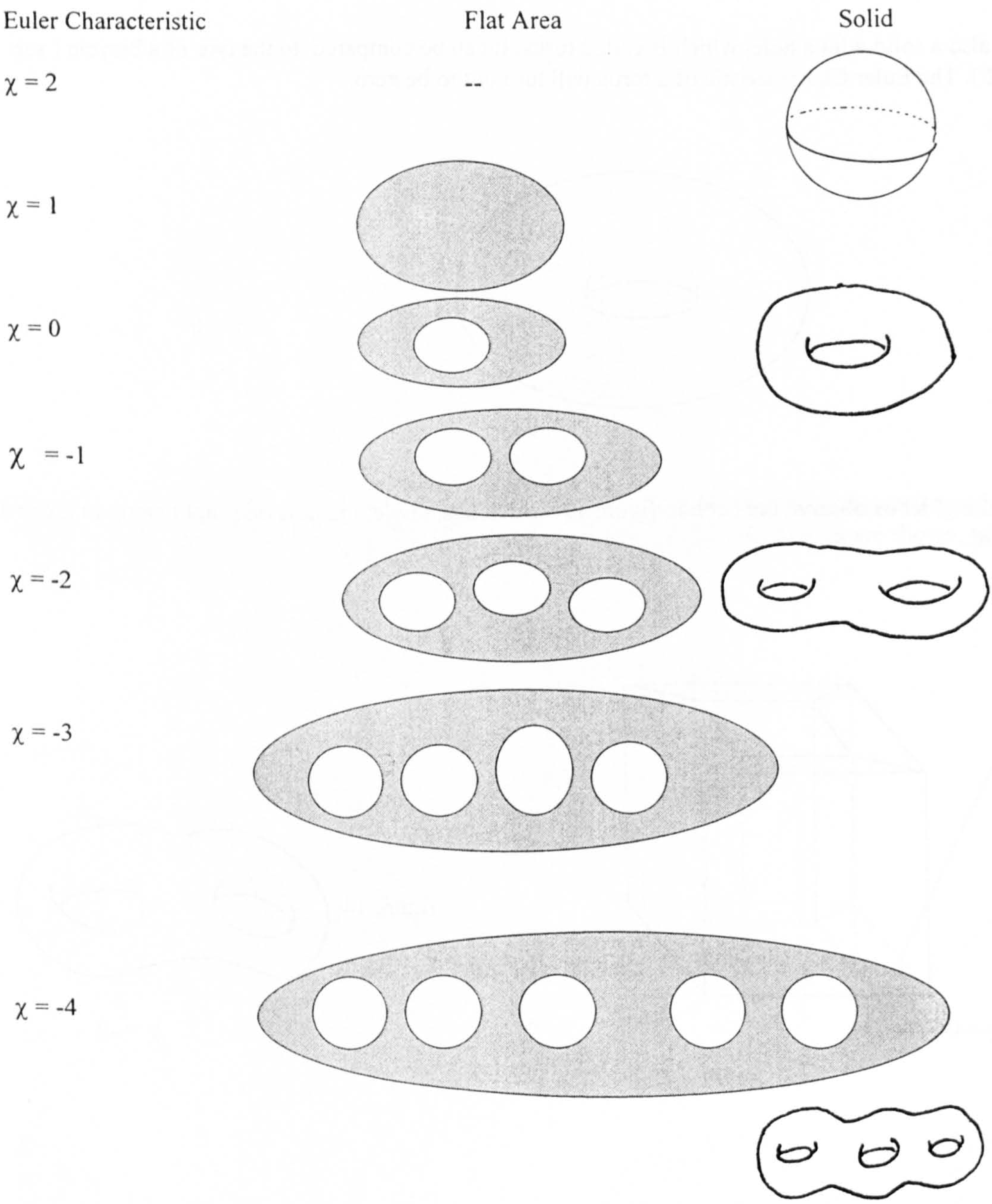


$\chi = -3$; with n holes $\chi = 1-n$

Solids :

$\chi = -4$; with n holes $\chi = 2- 2n$
n connected tori

From our survey we get the following list (1).



The list (1) provides a model, in which numbers (found by the Euler Characteristic) appear to determine the kind of configuration. The configurations are classified. There is a relationship between numbers and categories of solids and flat areas. Taking for instance $\chi = 2$, we will surely find a solid, which looks like an egg or like a prism, but without a hole in it (see figure 15).

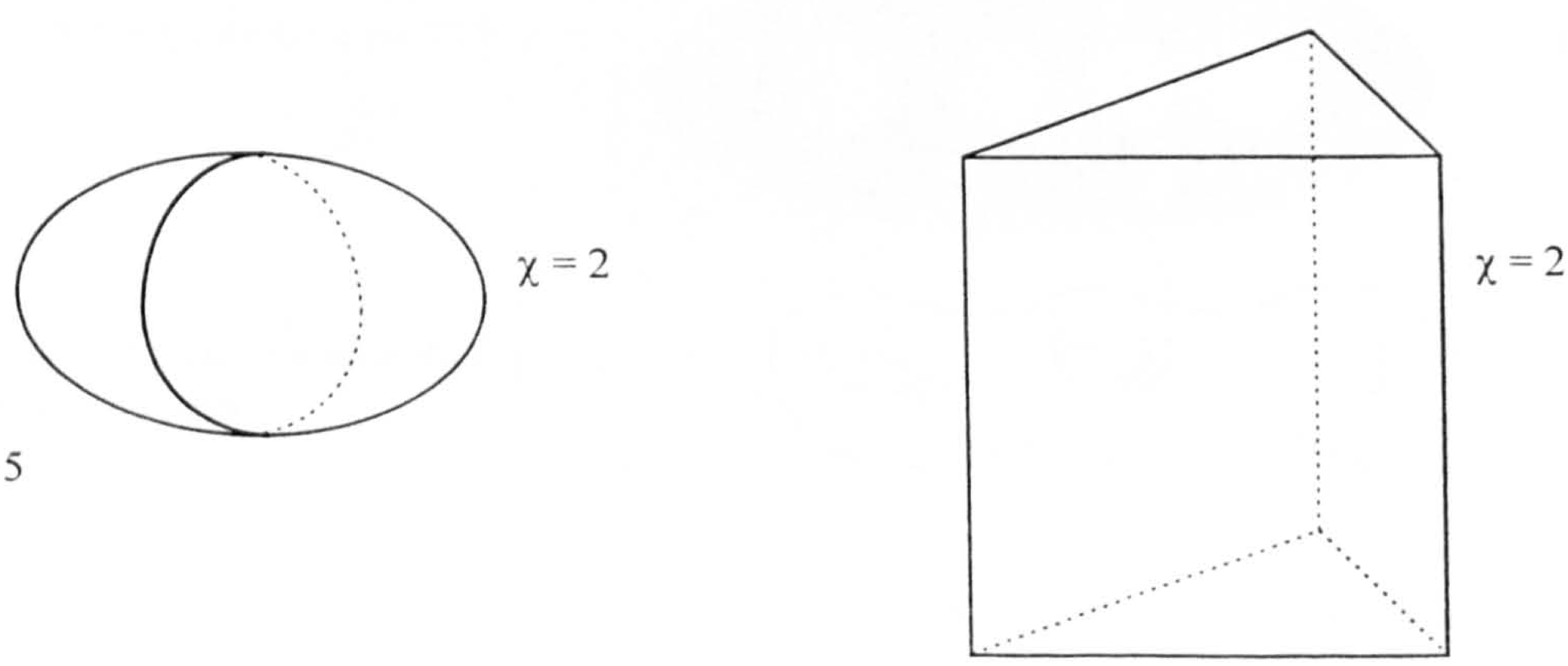


figure 15

So the Euler Characteristic provides a classification of flat areas and solids. The solids in figure 15 differ from each other, but with respect to the Euler Characteristic they are equal. ($\chi = 2$ in both cases). For that reason, the Euler Characteristic is called an invariant of this category of solids. To give another example, in figure 16 the prism with hole and the torus are different solids, but with respect to χ they are not different ($\chi = 0$ in both cases). The Euler Characteristic, again, is an invariant for this type of solid, represented by the torus.

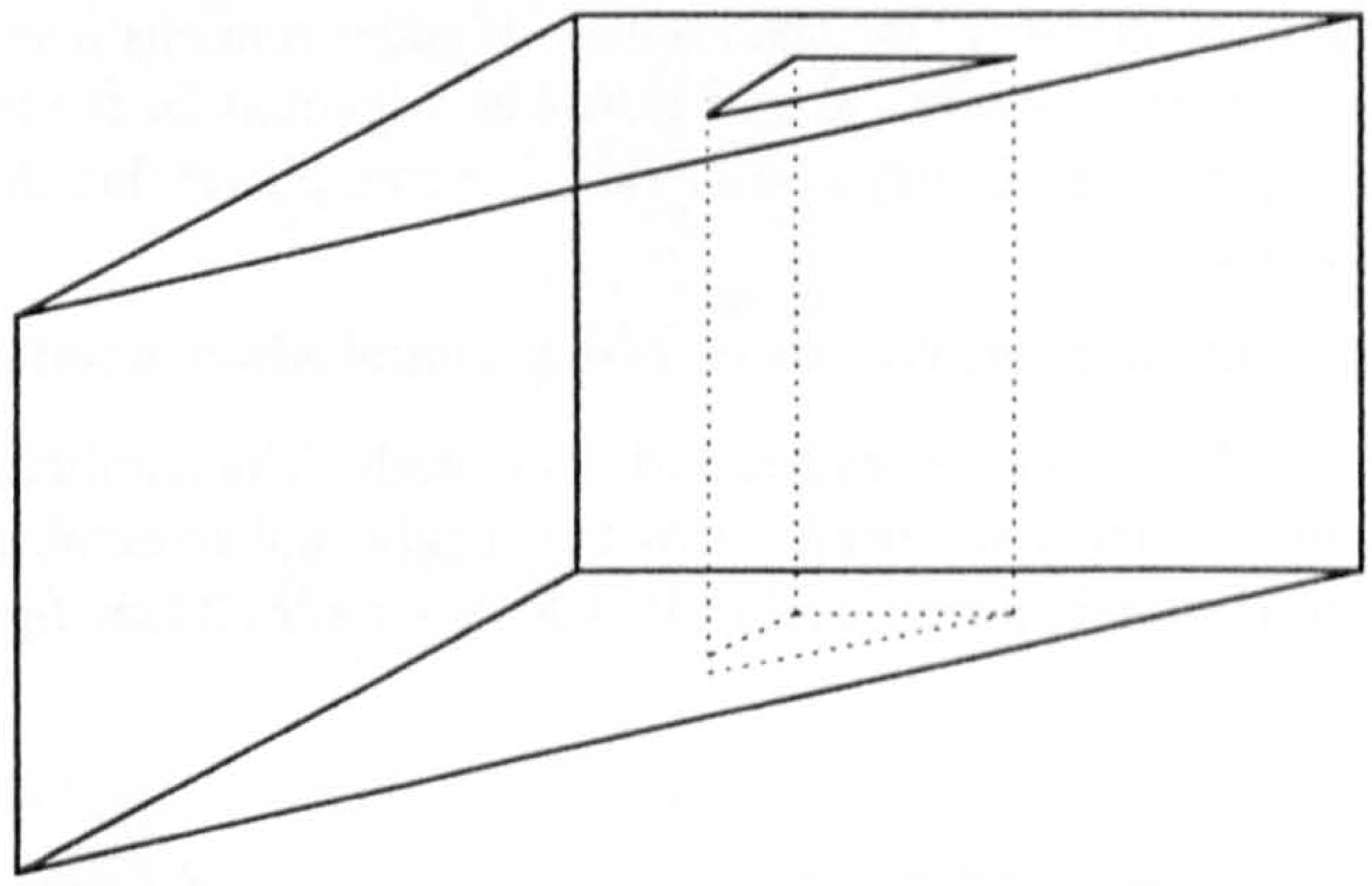
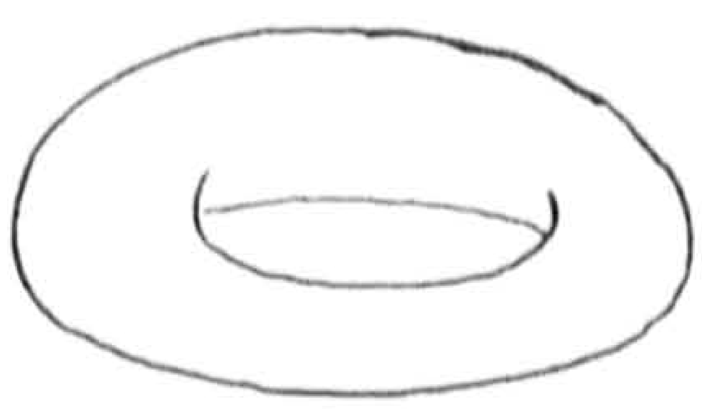


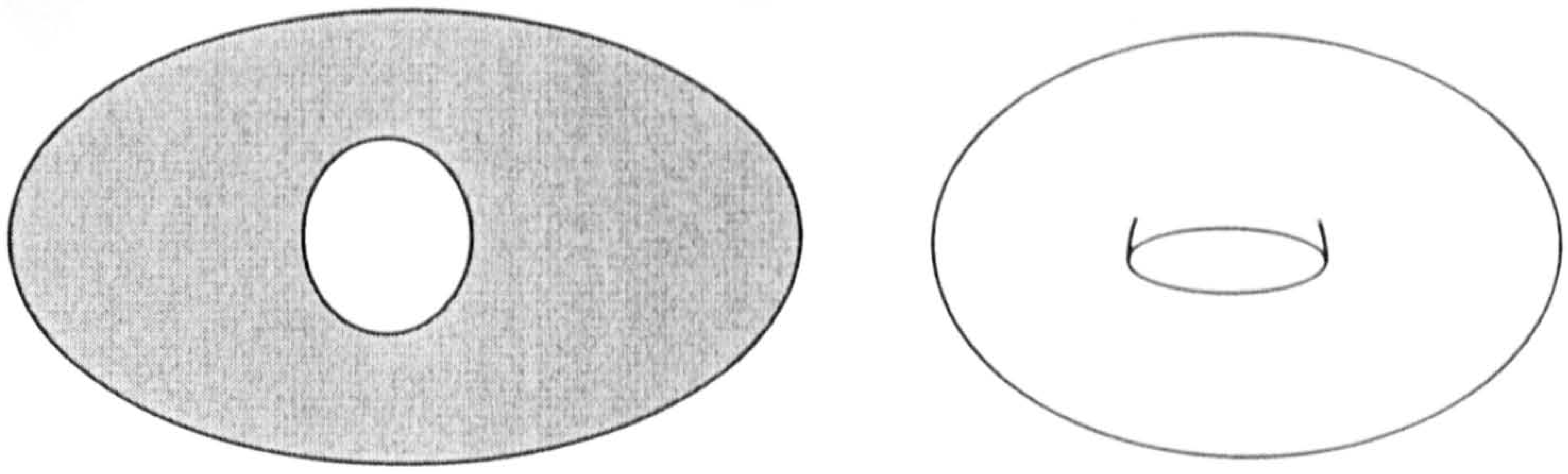
figure 16

$\chi = 0$



However, a flat area and a solid area may not be considered as belonging to the same category; for instance a flat area with a hole may not be compared to a torus, although both have $\chi = 0$ (see figure 17).

figure 17



$\chi = 0$ $\chi = 0$

This is because a torus exists in a 3-dimensional. world.

Polyhedra

In figure 1 a tetrahedron is depicted. Its faces are equilateral triangles.

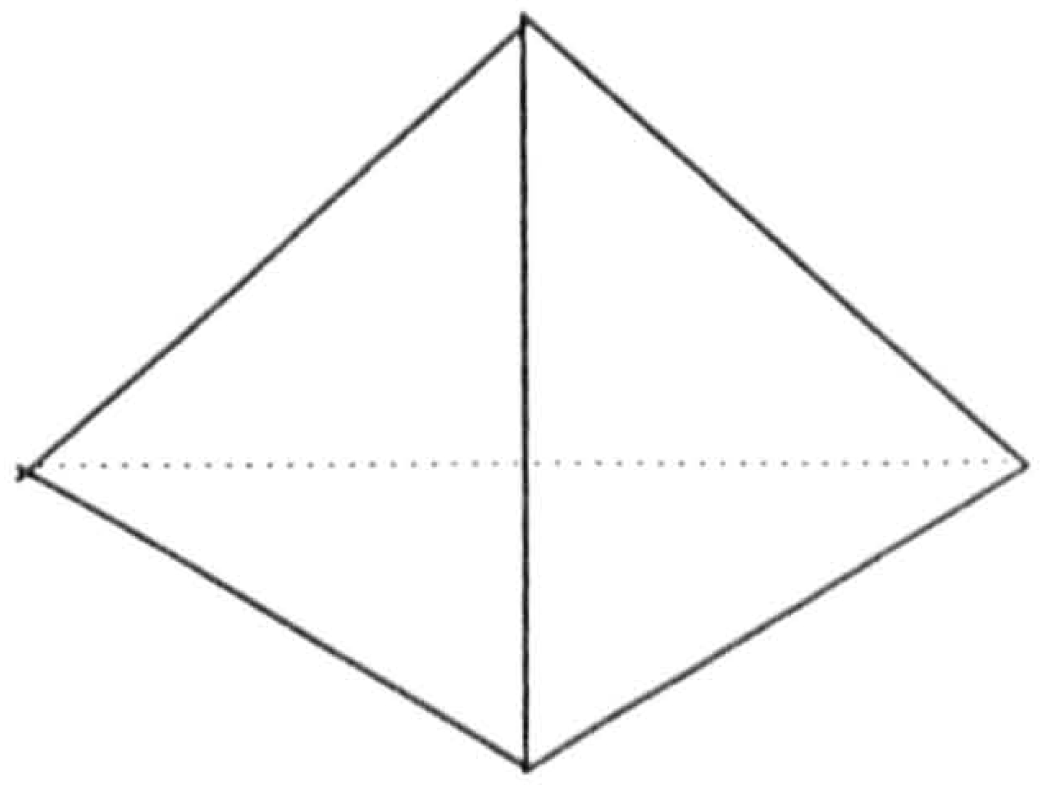


figure 1

We can easily check that the tetrahedron has 4 vertices, 4 faces and 6 edges. Consequently, the Euler Characteristic is $\chi = 4+4-6=2$, as expected. Now we try to find the dual of the tetrahedron.

In space R^3 the dual of a point is a plane. So the number of vertices has to be interchanged with the number of faces. However, these numbers are equally 4. (R^3 = 3-dimensional space). So the dual of a tetrahedron is a solid with again 4 vertices, and 4 faces. The number of edges remains invariant, because the dual of an edge is again an edge. That is understandable, because an edge may be the line of connection of two vertices. After taking the dual figures the vertices become planes, but these planes intersect along edges, which means lines.

So, the dual of a tetrahedron is again a tetrahedron. We conclude that a tetrahedron is self-dual.

A solid, such that its faces are equilateral polygons, is called a Platonic body. The tetrahedron is the only self-dual Platonic body. In our text about duality we already saw that a cube and an octahedron are dual solids; both are Platonic bodies. Both of course, have equal Euler Characteristics 2 (see figure 2).

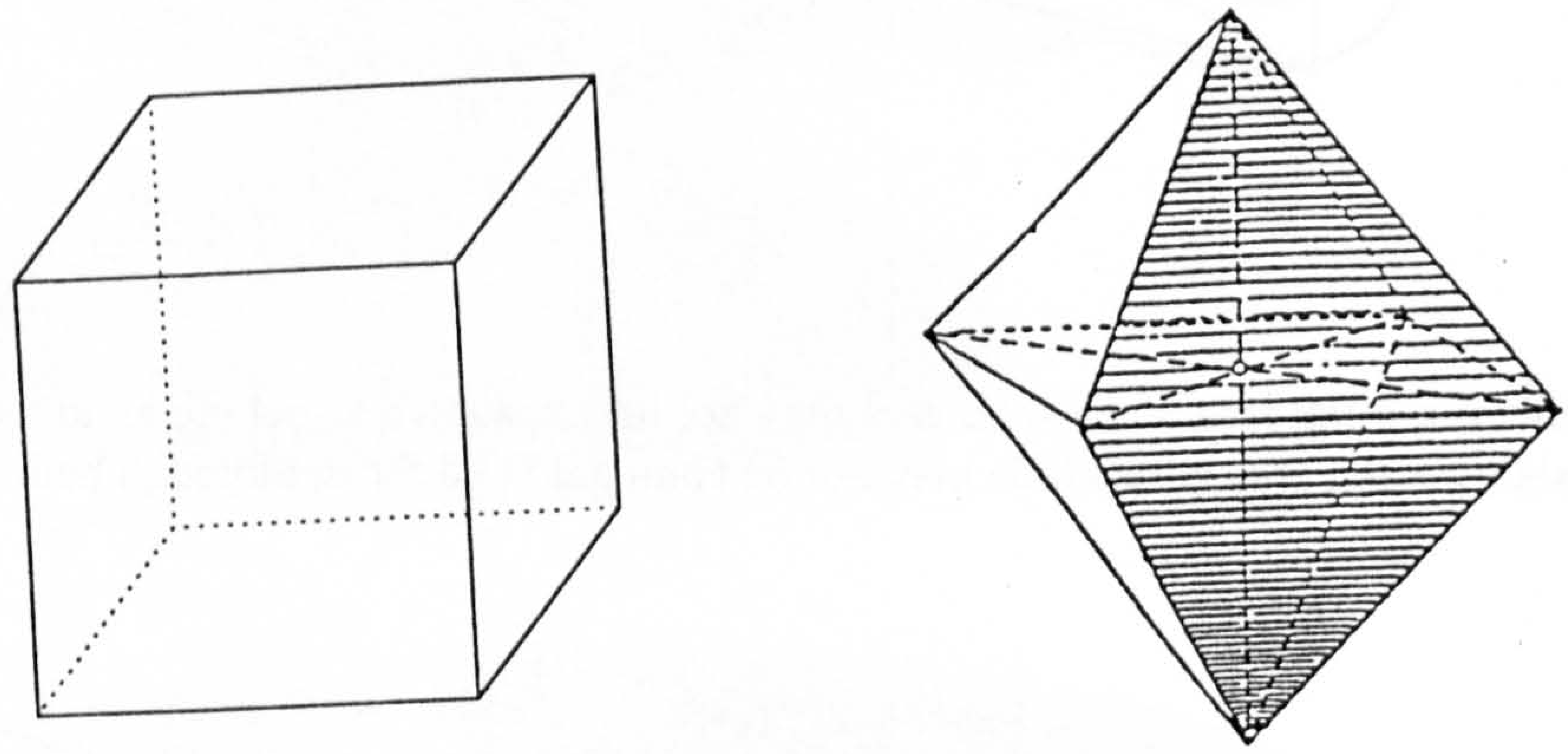


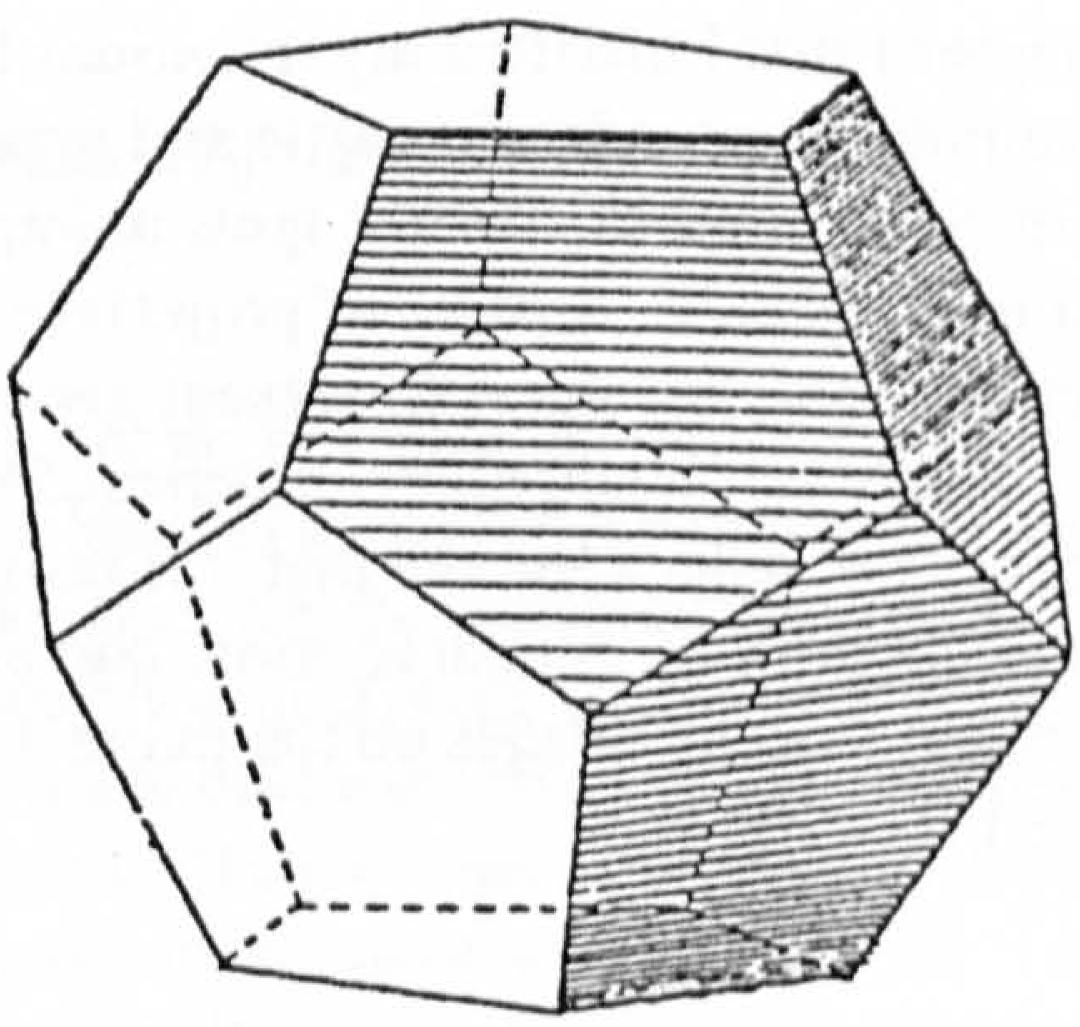
figure 2

For the cube one has : V=8, F=6, E=12, so that $\chi = 8+6-12=2$

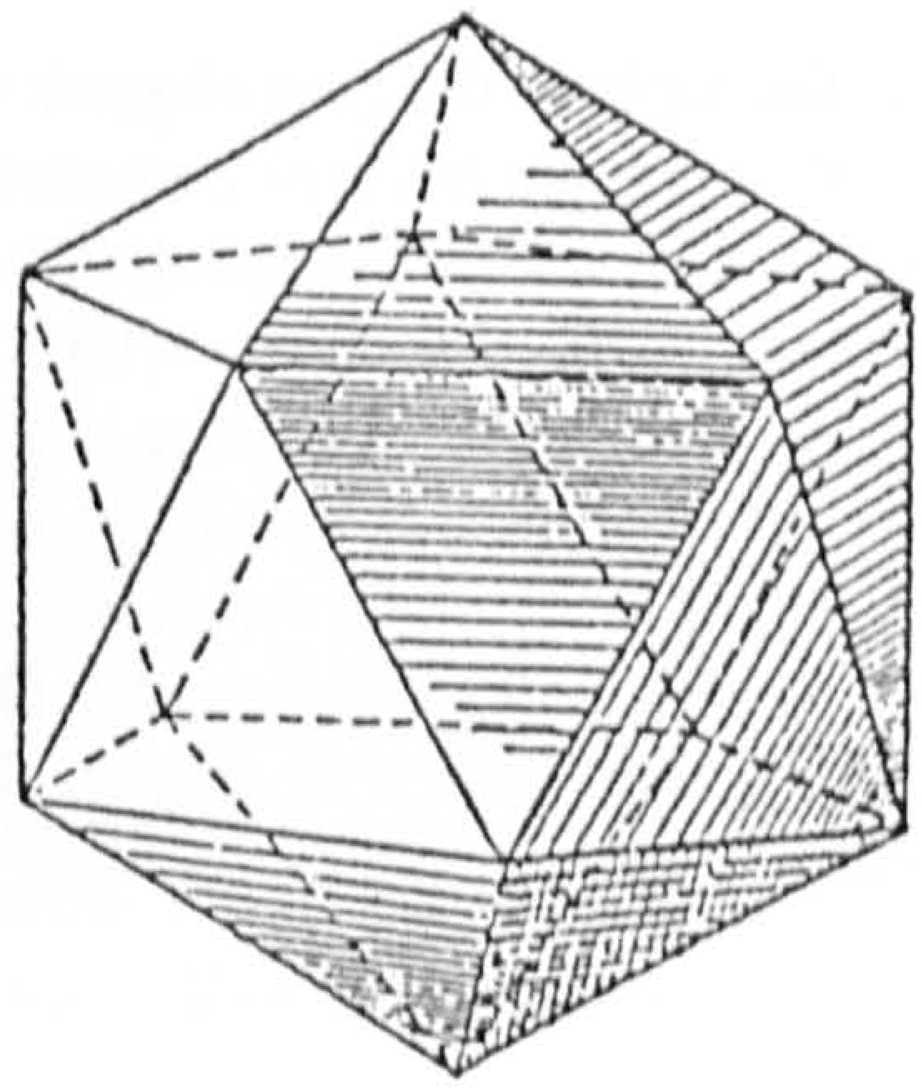
For the octahedron : V=6, F=8, E=12, so that $\chi = 6+8-12=2$

To get the dual body, V and F have to be interchanged; it yields the same Euler Characteristic. The faces of the cube are squares; the faces of the octahedron are equilateral triangles. There are 5 Platonic bodies and no more; so 2 still have to follow (see figure 3). They are the dual bodies dodecahedron and icosahedron.

figure 3



dodecahedron



icosahedron

For the dodecahedron one has: V=20, F=12, E=30, so that $\chi = 20+12-30=2$

For the icosahedron it yields : V=12, F=20, E=30 so that $\chi = 12+20-30=2$

V and F can be interchanged, which means that the bodies are each other's dual. The faces of the dodecahedron are equilateral pentagons; the faces of the icosahedron are equilateral triangles. Applying the Euler Characteristic, one has a large degree of freedom to present geometrical forms. It is not important whether a cube is depicted like figure 4 or like figure 5. Its Euler Characteristic remains the same.

figure 4

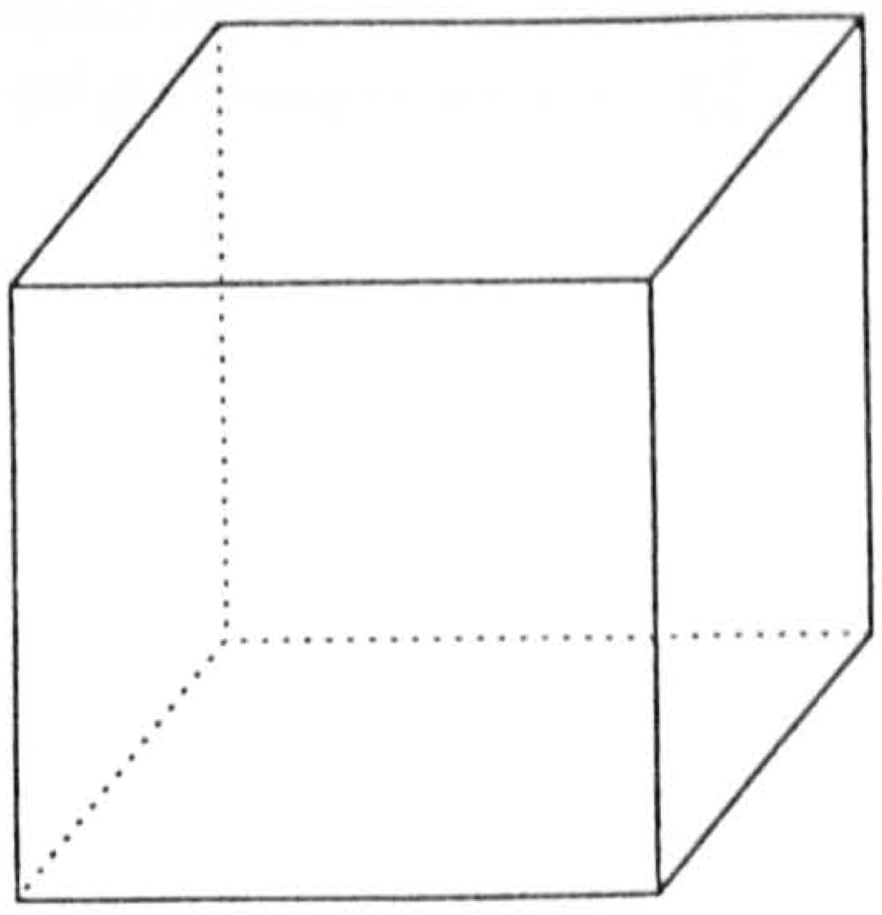
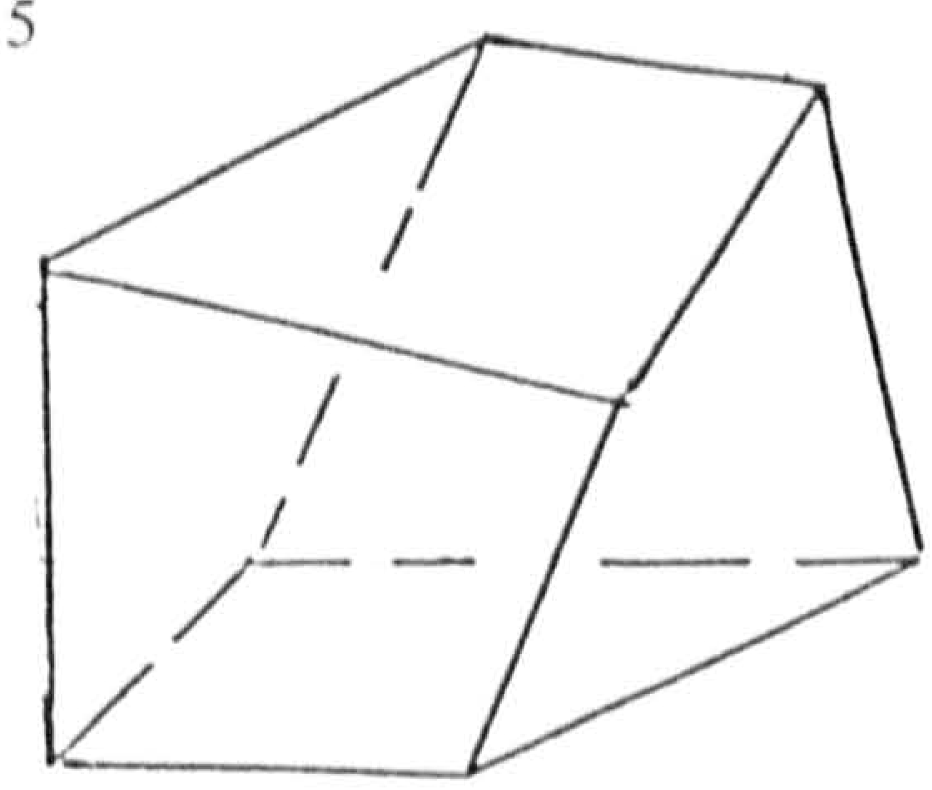


figure 5



The torus has now been demonstrated. It is such an important notion that it can not be dispensed with when visual geometry is discussed. The torus sometimes is denoted as 'a hole in a hole', which is a very visual approach.

As I said before, the Platonic polyhedra are of such outstanding beauty that they should be shown in the way we called 'educationally invalid'. Figure 4 is an example of this. The cube demonstrated in figure 4 is a specimen of 'Magic Realism' and useful for the computation of the Euler Characteristic. The distortions do not create any hindrance.

Beauty is the topic of the next chapter. It was in Chapter I that I clarified my intentions by means of analogy. Visual geometry was compared to the surrounding grounds of a castle and on these grounds there were beauty spots. I stated that it was my intention to make the beauty spots unimpeded, accessible to the general public; and that is what I intend to do in Chapter V. A piece of projective geometry was chosen as a beauty spot. The piece is called "Projections" and the aim is to present genuine Projective Geometry which may also be considered visual art . Moreover, it should be understandable to everyone. In the course of fifteen interviews with arbitrarily chosen people, I learned that it is possible to present geometry as visual art to everybody. The underlying geometry is, of course, more complicated but that is not disturbing; and almost all of those interviewed stated that the images on the pages 7 & 8 of 'Projections' were considered by them to be visual art.

Chapter V

5.1. The beauty-spot

It is extraordinary when a science produces pieces of art; but in the case of geometry this is what happens. We saw that the distortions necessary to create practically useful configurations produce bizarre drawings like the cube, which looks like 'Magic Realism' and therefore can be seen as an artistic creation. The projection of a plane on itself will appear to generate a distorted reality, yielding a fine construction, which is balanced between the 2-dimensional world and the 3-dimensional. This is shown on the first 5 pages of 'Projections'. Further, I composed a projection of two cones on the same circle, and that delivers an artistic creation on pages 7 & 8 of 'Projections'. These configurations were considered by almost all of the interviewed to be 'art'.

What is the meaning of this ? The above-mentioned configurations belong to the realm of projective geometry. By showing that projective geometry has an aesthetic value in itself, an aspect of the science of geometry is highlighted, which up to now has not been clearly demonstrated. The beauty of geometry is found in the distortions, produced by the demands of the constructions, such as the folding of two planes into one, or the configuration of two projected cones. These constructions are part of Projective Geometry, but at the same time have an artistic value. The demonstration of these products of art provides a view of the character of projective geometry, and therefore has an educational value. One of those interviewed said, after examining the pictures of 'Projections', "*I am watching a new and fascinating geometrical world.*" That is what art can do and therefore art may be considered as a means to highlight the essence of geometry.

Science in action

There is one aspect of 'Projections', not explicitly mentioned. It is the aspect of action.

In the case of Escher's Pond we saw the action of the fishes, demonstrating to us the character of the new straight lines. In 'Projections' we are travelling through the borderland of 2- and 3-dimensional geometry. The position on that border creates a scene of action, which must be presented in an aesthetic picture. The artistic aspect makes the borderland visible.

The pictures of 'Projections' which follow could be denoted as what I call 'geometrical postcards'. The tourist travelling through the geometrical landscape on the border of different dimensional worlds sends postcards home to give relatives and acquaintances a portrait of that landscape. At the same time that postcard shows how these different worlds interact, so that the spectator gets an idea of action. The beauty evoked by the contrast of 2- and 3-dimensional worlds, has to be handled carefully, because it is fragile. A minor change of the composition could ruin the artistic character.

5.2. Projections

The reader is asked to examine the next 9 pages of section 5.2. These pages contain visual material. On some of these pages it is explained how the material might be understood.

I found 15 people willing to read through these 9 pages and then be interviewed. It would probably take each of them one hour to digest the information. After a few weeks I handed them a set of questions to be answered orally. Nobody was informed beforehand which specific questions would be asked.

The questions and the discussion of these are the topic of section 5.3. These questions and their background are of course an indispensable part of the design of 'Projections'. Although the visual aspects predominate, communication to the reader is inconceivable without a thorough discussion.

From the interviews, transcribed in section 5.4, it appears that it is possible to teach visual geometry up to a good scientific level with the help of visual art. Moreover, most of the students had no more than an ordinary secondary school background and only one of them commented on the demanding level of the material.

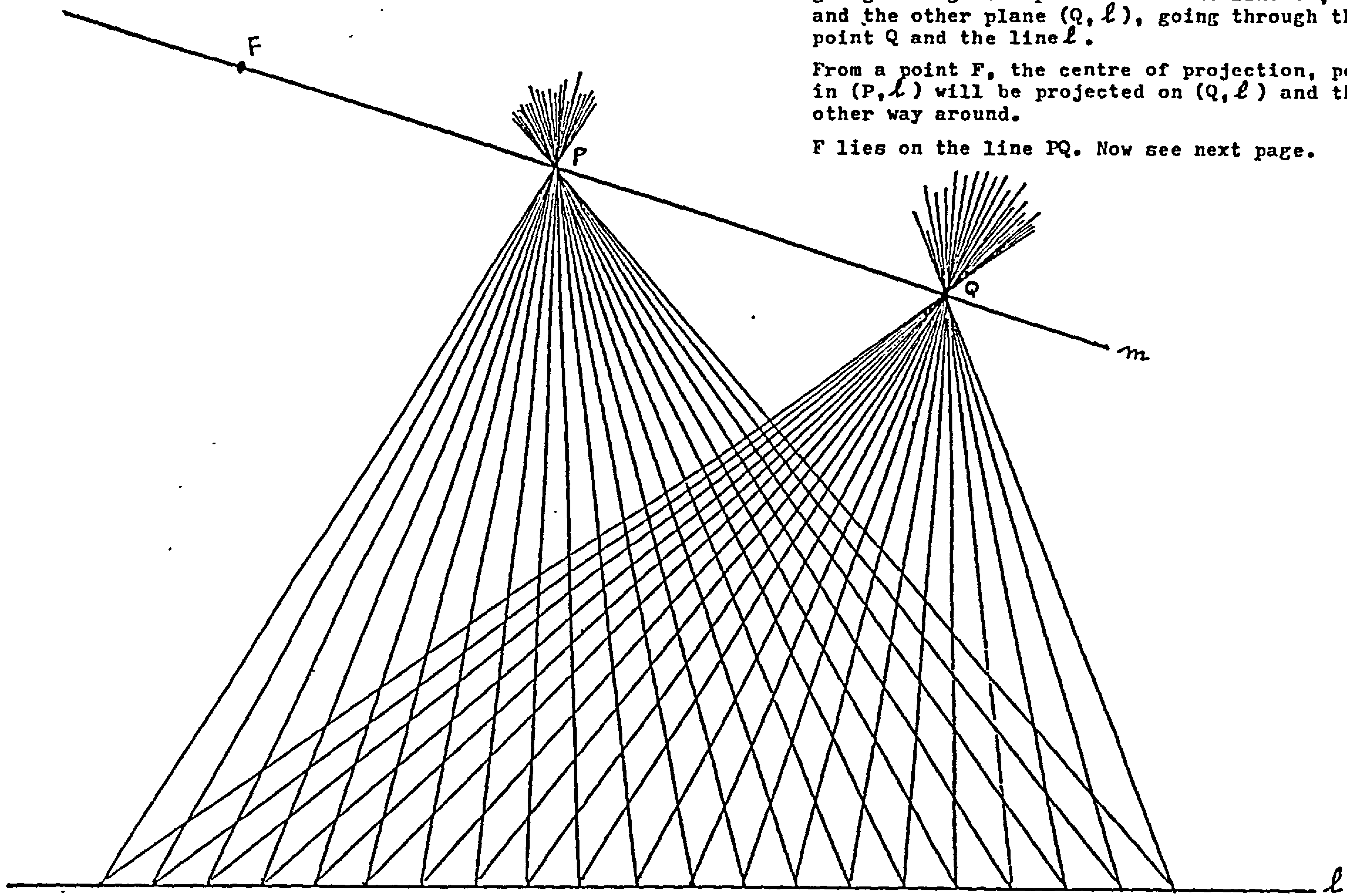
Therefore the answers of the interviewees are of paramount importance. In their reactions, the possibility of conveying scientific geometry with the help of visual art must be recognised.

At the bottom of the 9 pages the pages are numbered within 'Projections".

Imagine that there are 2 planes: One plane (P, ℓ) going through the point P and the line ℓ, and the other plane (Q, ℓ), going through the point Q and the line ℓ.

From a point F, the centre of projection, points in (P, ℓ) will be projected on (Q, ℓ) and the other way around.

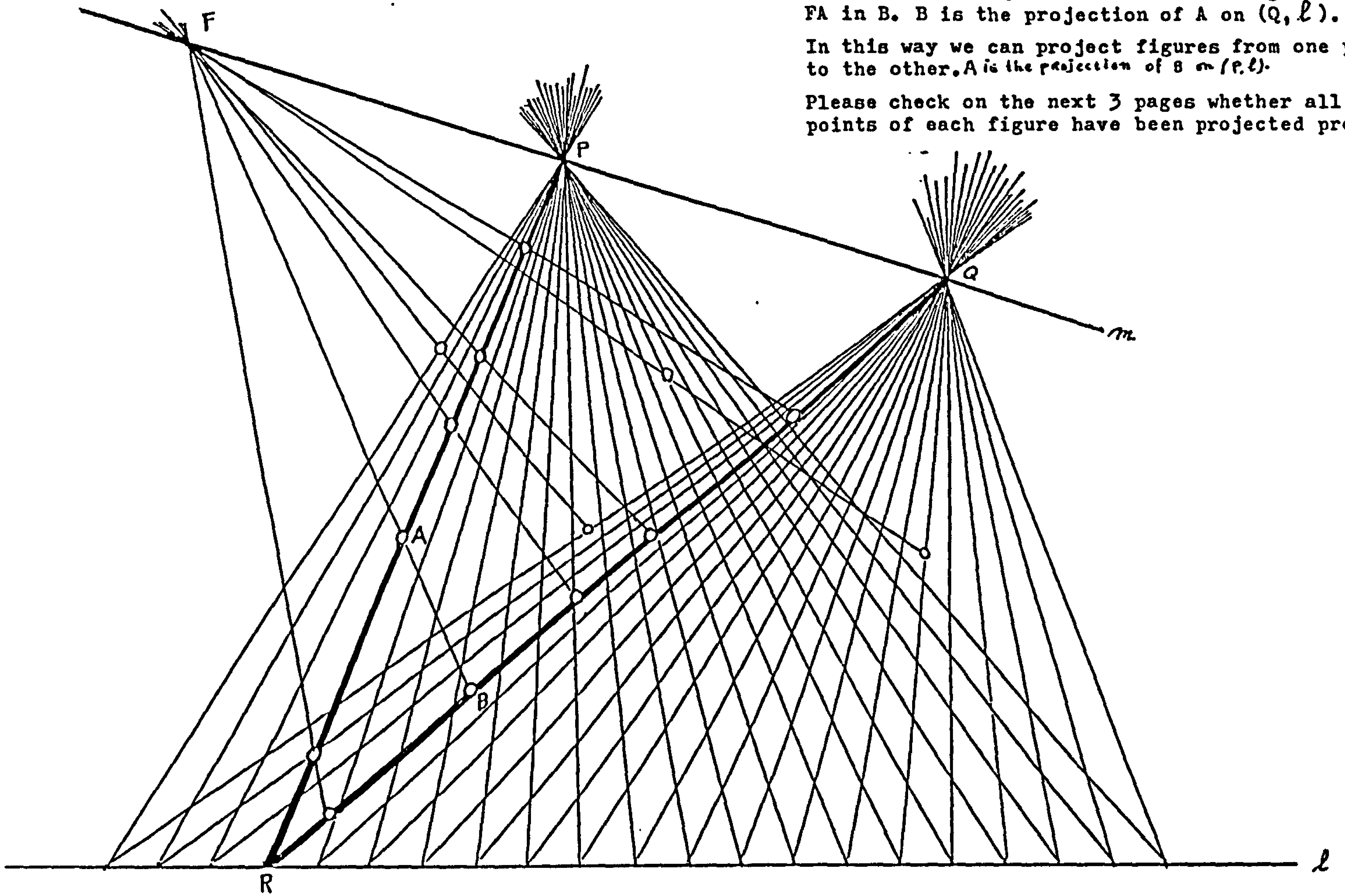
F lies on the line PQ. Now see next page.

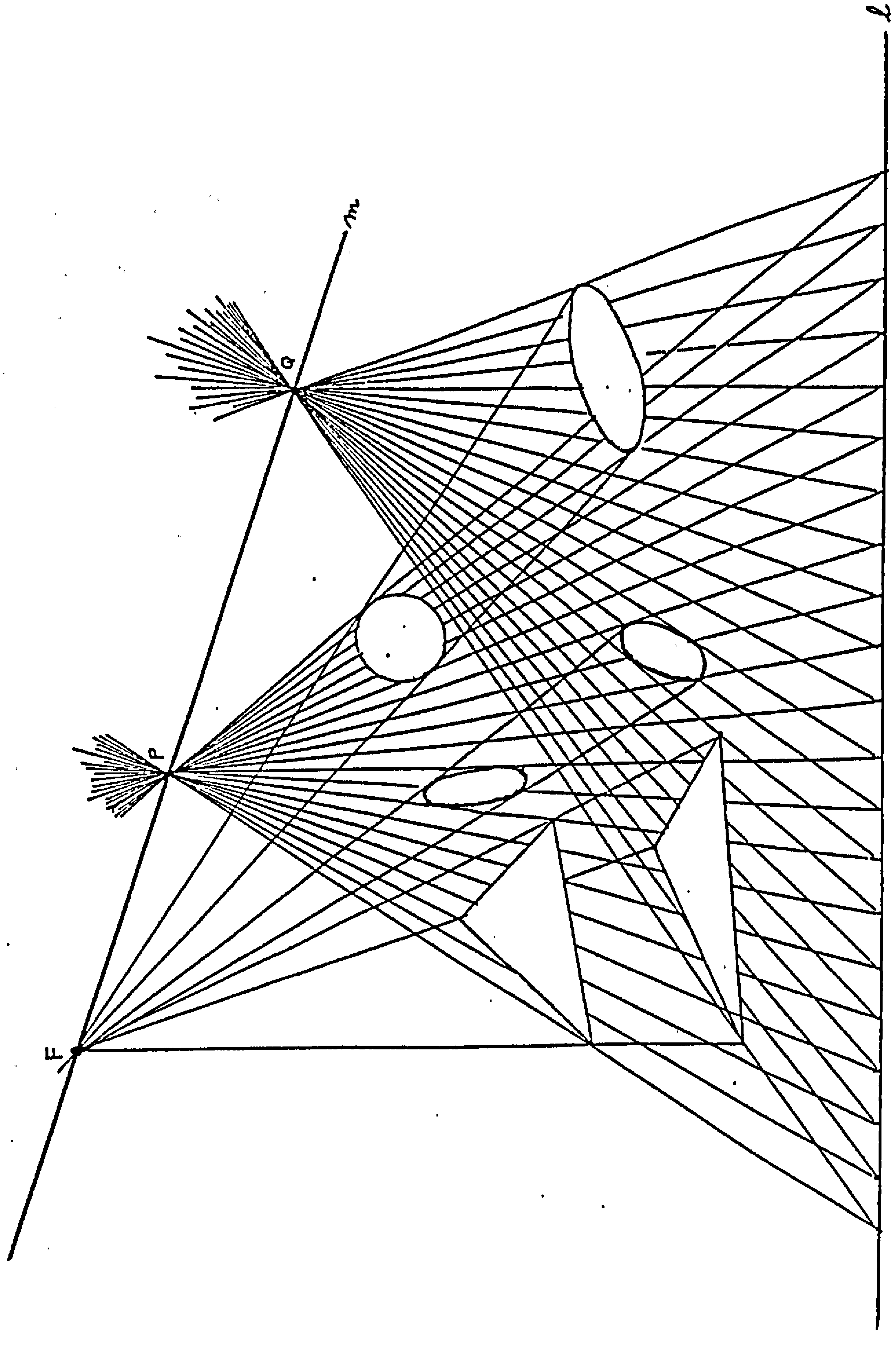


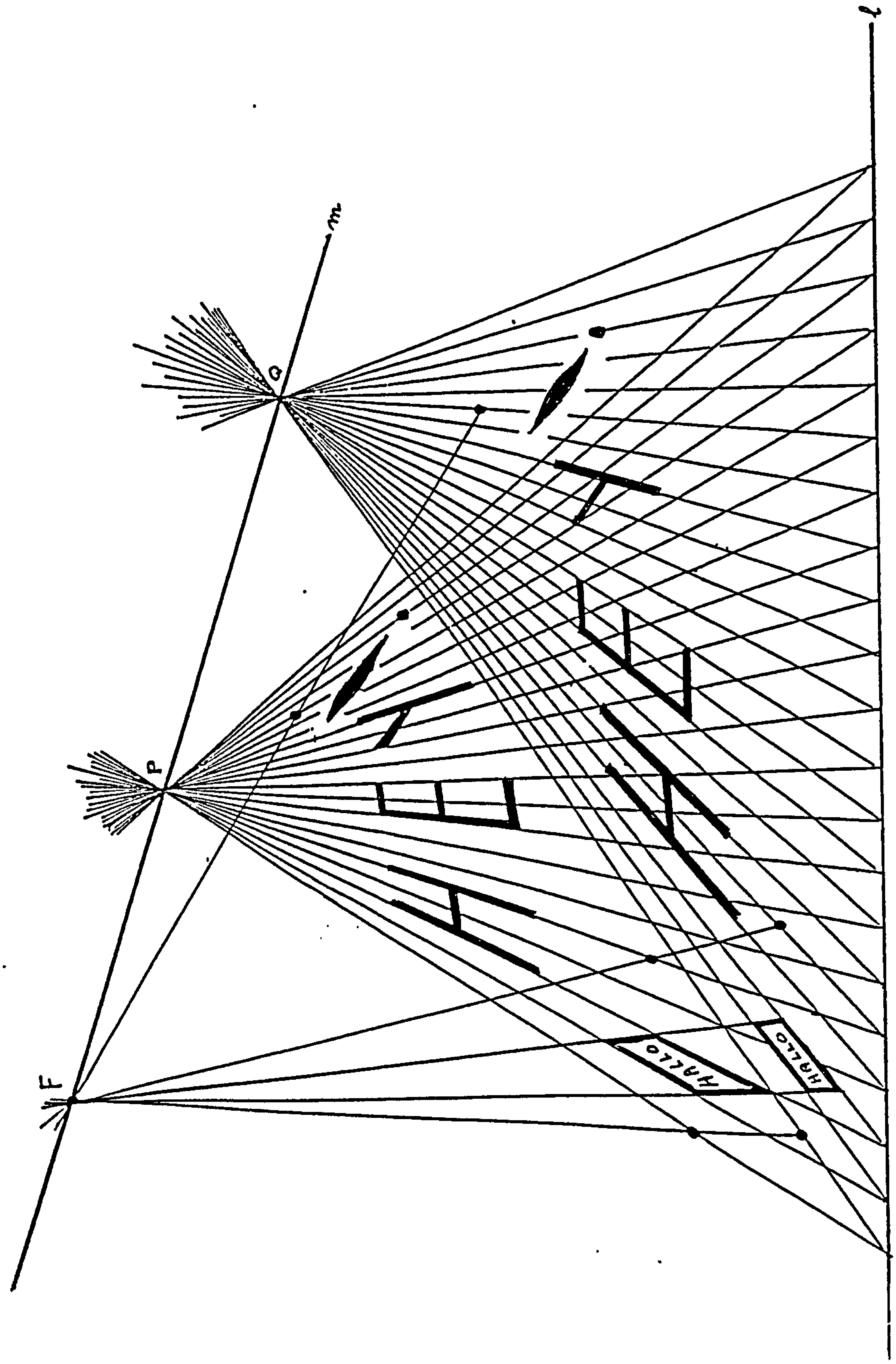
Taking an arbitrairy point A in (P, ℓ), we draw the straight line AP until it meets the line ℓ in R. The line RQ intersects the straight line FA in B. B is the projection of A on (Q, ℓ).

In this way we can project figures from one plane to the other. A is the projection of B on (P, ℓ).

Please check on the next 3 pages whether all the points of each figure have been projected properly.




F
P
Q
m
l



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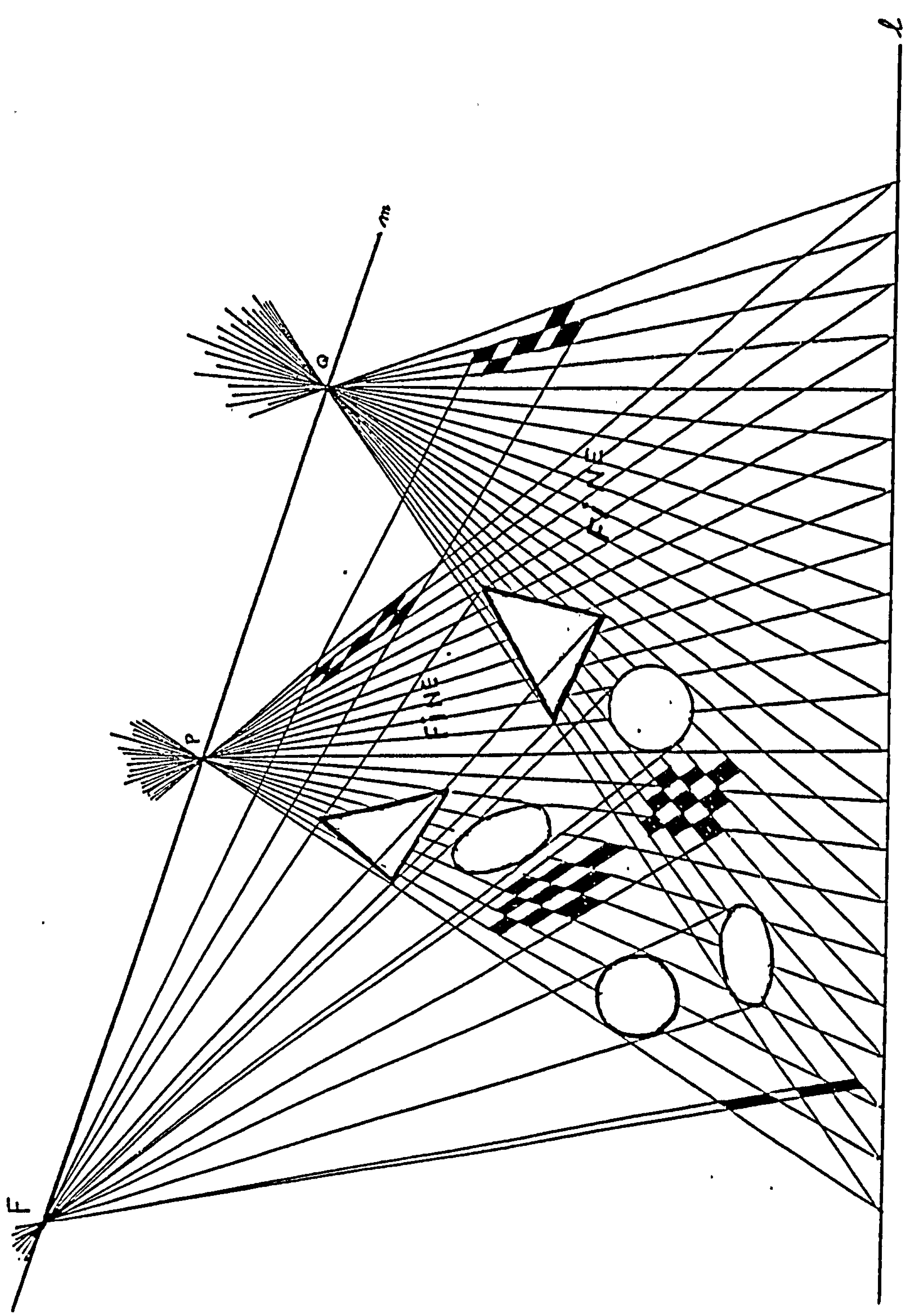

l
m
Q
P
F
FINE
FINE

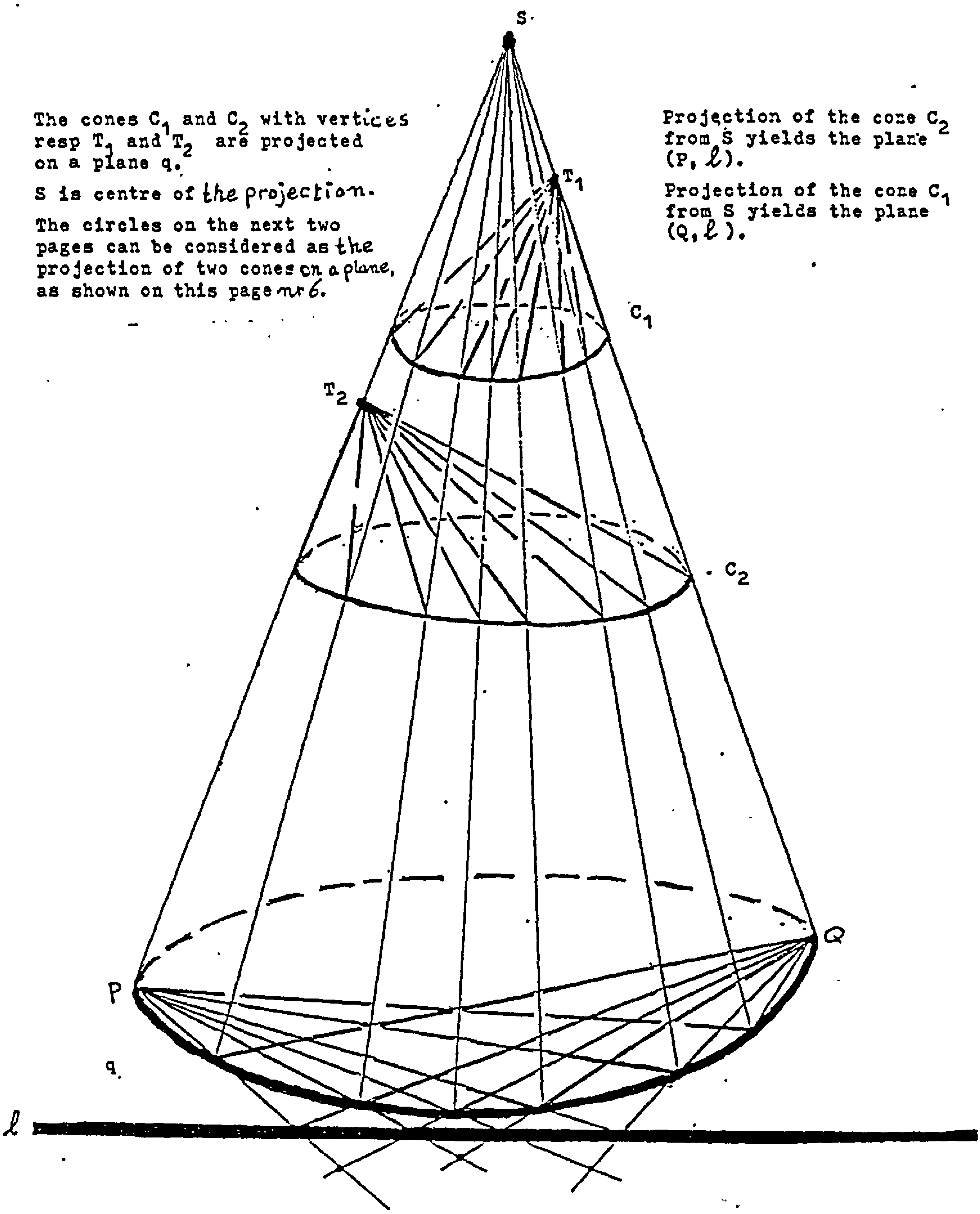
The cones C_1 and C_2 with vertices resp T_1 and T_2 are projected on a plane q.

S is centre of the projection.

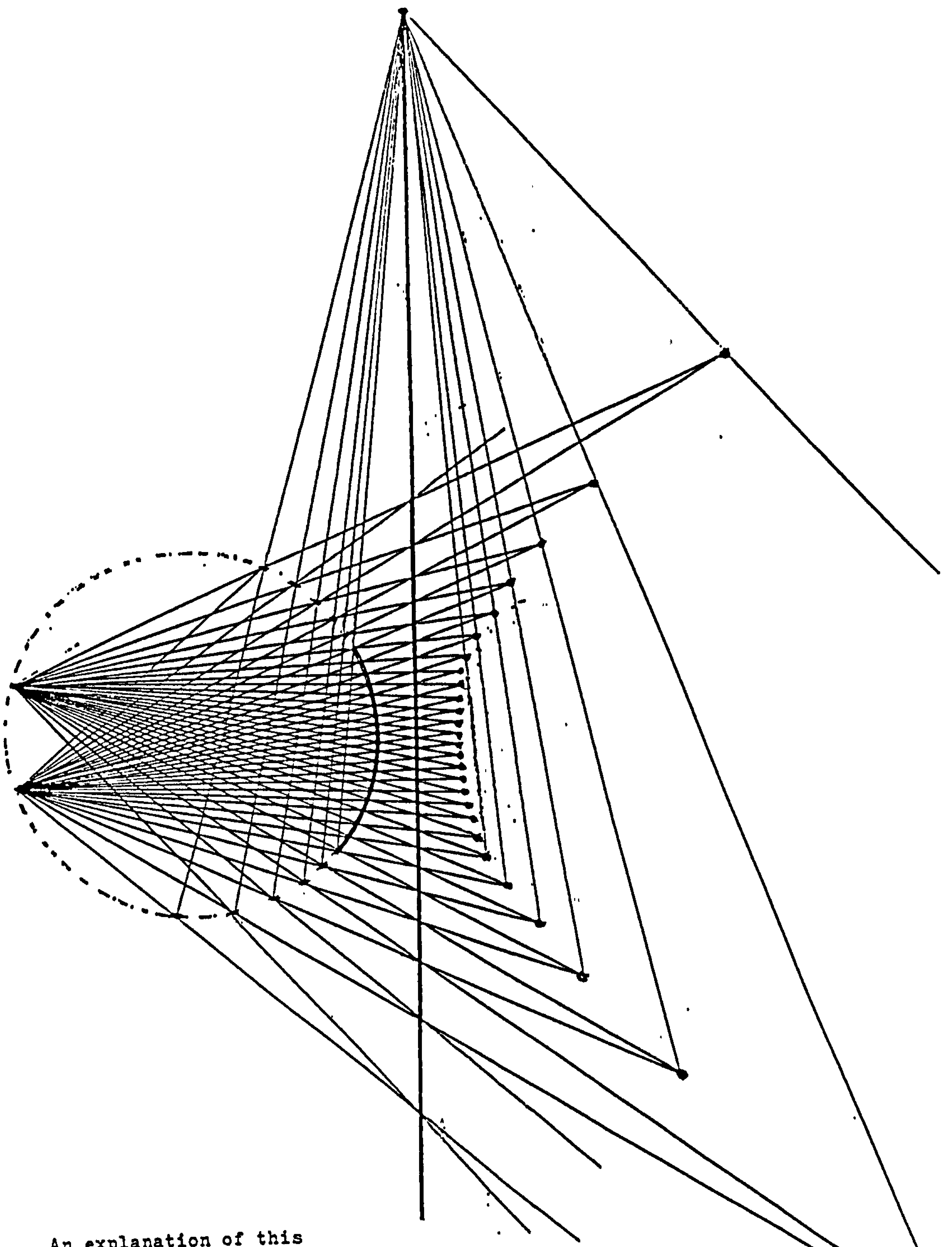
The circles on the next two pages can be considered as the projection of two cones on a plane, as shown on this page nr 6.

Projection of the cone C_2 from S yields the plane (P, ℓ).

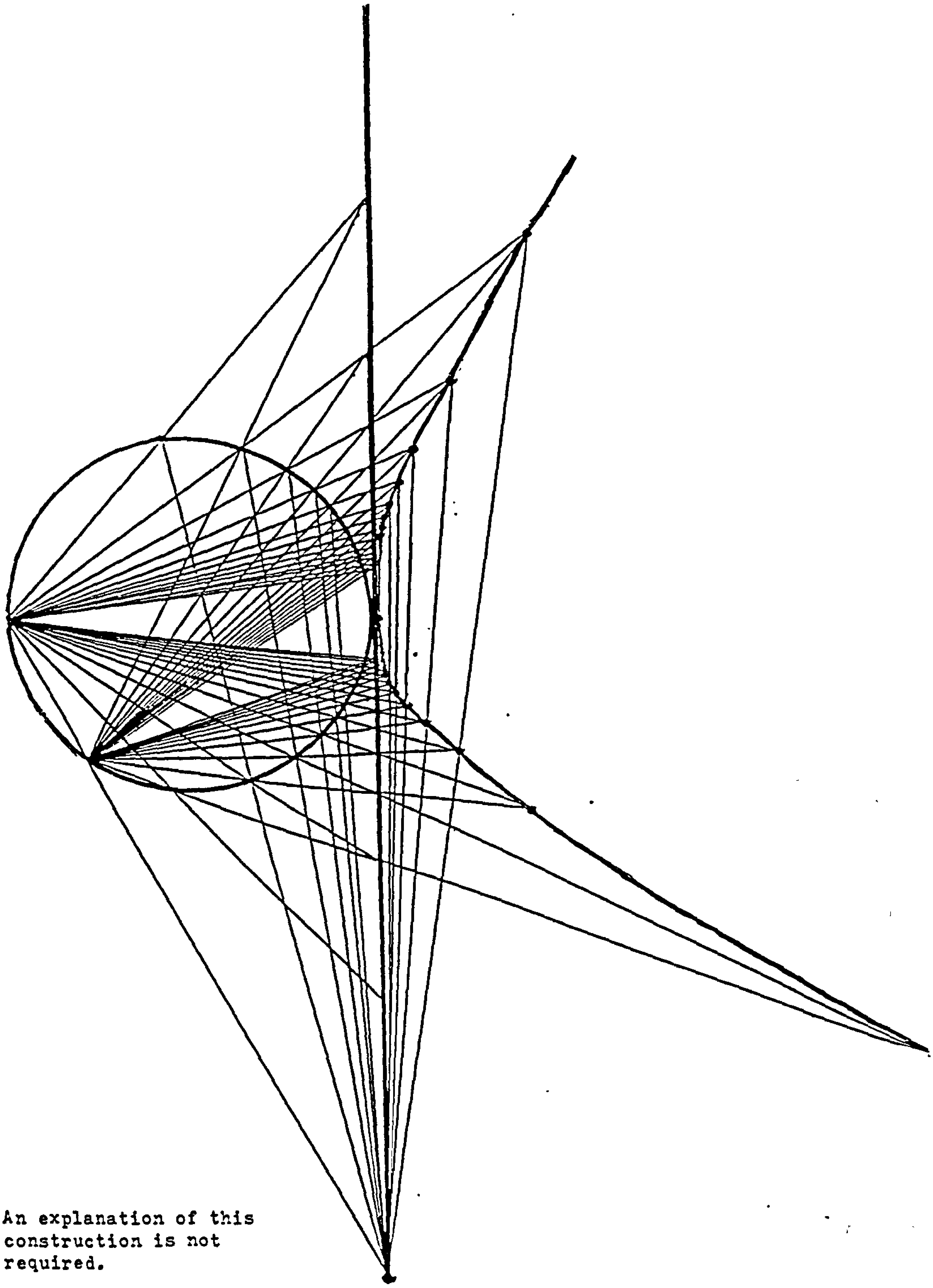
Projection of the cone C_1 from S yields the plane (Q, ℓ).



Is the construction entirely correct ?



An explanation of this construction is not required.



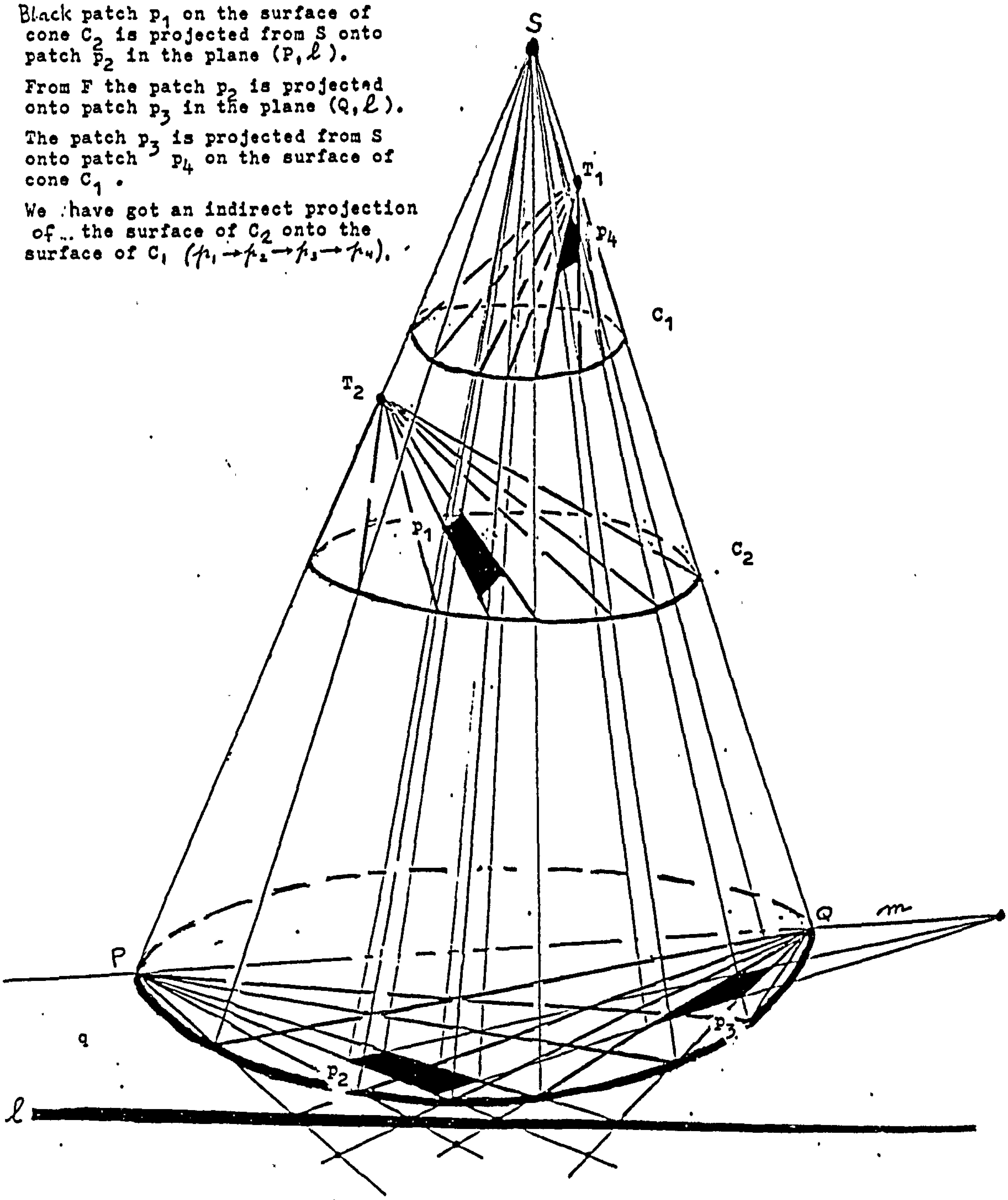
An explanation of this construction is not required.

Black patch p_1 on the surface of cone C_2 is projected from S onto patch p_2 in the plane (P, ℓ).

From F the patch p_2 is projected onto patch p_3 in the plane (Q, ℓ).

The patch p_3 is projected from S onto patch p_4 on the surface of cone C_1.

We have got an indirect projection of the surface of C_2 onto the surface of C_1 $(p_1 \to p_2 \to p_3 \to p_4)$.



<u>Summary:</u> We made a little excursion through projection-land and learned to project planes and cones.

On pages 7 and 8 we saw that the landscape, allthough beautiful, still lodges many secrets.

5.3. Questions

Now follows a list of questions and a discussion of them. My aim is to listen to the interviewees and to study their answers. Listening to one's audience is one of the most important activities in educational research.

Therefore I have included the answers of the interviewees in the main body of the text of the thesis because their reaction is decisive concerning the viability of my approach: to convey scientific geometry by means of visual art.

The interviews also determine the atmosphere in which teaching takes place and they are thus of great importance to the educational process.

Here follow the questions:

A. Give a brief account of your school/college career with regard to Math & Geometry .

<u>The numbers of the questions refer to the numbers of the pages of 'Projections'.</u>

B.

1. Can (P, ℓ) and (Q, ℓ) be considered as two different planes in space ?
2. Can (P, ℓ) and (Q, ℓ) be considered as two different planes in space if the lines m and ℓ intersect each other ?
3. Describe the deformation of the triangle under the projection from F .
4. Describe the deformation of the quadrangle "HALLO" under the projection from F.
5. Intersect the prolonged sides of both triangles with the line ℓ.
   Can you explain the result ?
6. Do you observe the error in the construction ?
7. Can a spatial model of the shown figure exist ?
8. Can you draw a similar configuration without touching the original one ?

9.1. Could the patch p_1 possibly be a circle ?

9.2. We let the patch p_2, seen from S, rotate clockwise about a point in p_2 .
Will patch p_3 consequently rotate anti-clockwise or clockwise about a point ?

C

1. Do you consider the configurations on the pages 7 & 8 to be 'Art' ?
2. Give an account of the relations between the figures on the pages 7 & 8 and the text on the other pages of 'Projections '.
3. Can 'Art' widen one's view on geometry ?

D What did you learn by studying 'Projections' ?

Discussion of the questions

We start the discussion with question B1

B1. Can (P, ℓ) and (Q,ℓ) be considered as two different planes in space?

This question is meant to stimulate the student to look at the same configuration as 2-and 3-dimensional at the same time. Particularly when the object which one is looking at, is shown as 3-dimensional, a distortion emerges, when the lines ℓ and m are thought to intersect with each other. To notice the ambiguous status of the configuration in B1 , the 2-dimensional and the 3-dimensional, is the purpose of question B1. It is a preparation for the next question, B2, which explicitly puts the question whether a 2-dimensional figure can be interpreted as 3-dimensional.

B2. Can (P, ℓ) and (Q, ℓ) be considered as two different planes in space if the lines m and ℓ intersect with each other ?

On the second page of "Projections" not only two planes are drawn , but also a projection of points of one plane on points of the other plane. It makes one think of course of two planes in space, not coinciding. However, when someone supposes that the lines ℓ and m intersect, then all constructions must take place in one plane: the plane defined by the intersecting lines ℓ and m. When, nevertheless, one admits that the two planes (P, ℓ) and (Q, ℓ) might not be coinciding, then a certain distortion emerges, which also creates a beauty in the configuration.

It is the purpose of the question B2 to evoke a sense of distortion and of beauty.

B3. Describe the deformation of the triangle under the projection from F.

It is the purpose of this question to emphasise that the projection of a figure creates a figure of the same kind, but it may be deformed under the projection. A triangle is a simple plane figure, so basic deformations may be observed immediately. The other figures are of course important too: it is shown, for instance, that the projection of a circle may be an ellipse and that the projection of an ellipse could be an ellipse which has an slightly different shape. The way figures are deformed is a very important characteristic of projections, and thus this is demonstrated visually here.

B4 Describe the deformation of the quadrangle "HALLO" under the projection from F

On page 4 we see the projection of letters and a quadrangle. A projected text can probably still be read easily; the letters are not so deformed that they are unreadable. The shape and the area of the rectangle change under projection. It is interesting how such characteristics are observed by those interviewed. Some will emphasise the change of form, and neglect the change of area. Some will talk about the differences of the letters of HALLO compared to the projected letters of HALLO. It is interesting to note which aspects of deformation are observed by one interviewee but not by another.

B5. Intersect the prolonged edges of both triangles with the line. Can you explain the result ?

Many figures are shown, We see that the projection of a circle may be an ellipse, and also that the projection of an ellipse may become an circle. The black squares demonstrate the projection of the different parts of a configuration; one part of it is less deformed by the projection than other parts. It shows that a configuration is not projected uniformly, but the projection is different for the different parts of it. What is asked in question B5 is the theorem of Desargues. Of course it can not be expected that an average student can invent a proof of that theorem. However, these questions were also answered by teachers of mathematics, who should know something of Projective Geometry. It appeared that some of the teachers had studied it long ago, but did not accurately remember the proof. Others did.

B6. Do you observe the error in the construction ?

To this question more than one answer is possible. When one is asked to find errors, the picture has to be scrutinised very thoroughly. That is its purpose; the student has to be aware that two cones are projected on one plane. It is a preliminary question for B7, and it is meant to suggest to the student that on pages 7 & 8 we look at the projection of two cones, as shown on page 6. It follows from this that the figures on pages 7 & 8 are both Art and Geometry at the same time.

B7. Can a spatial model of the shown figure exist ?

An answer to this kind of question is rather open. From one point of view one can say that it is impossible to construct any model other than a 2-dimensional one, because the straight lines have to lie in a plane. From another point of view one can maintain, that the horizontal line causes a kink in the straight vertical lines, intersecting them, so that the upper part of the drawing leans towards the spectator, but the lower part also approaches the spectator. In that case the horizontal straight line is the farthest distance from the spectator. It is also possible to look at two mountains, with their tops on the circle, based on a hyperbola shape on the ground. Here too the shape can be approached from a 2-dimensional view, or, alternatively, from a 3-dimensional view.

B8. Can you draw a similar configuration without touching the original one ?

The student is supposed to investigate how such a configuration can be drawn if you have a ruler and compasses. Actually what is asked is: what is the basic concept behind the drawing ?

Some teachers did indeed discover this concept and remarked that there is much more behind it than a superficial look will learn. That is true indeed: to create simple figures one sometimes has to go deep. However, most of the interviewees will not be aware of the deep geometrical considerations underlying the construction. Their search for the creative concept serves to convince them that it is geometry they are doing and that the artistic value is concomitant with true geometrical constructions.

B .9.1. and B 9.2

B .9.1 Could the patch p_1 possibly be a circle ?

B .9.2. We let the patch p_2, seen from S, rotate clockwise about a point in p_2. Will patch p_3 consequently rotate anti-clockwise or clockwise about a point ?

Page 9 aims to demonstrate to the interviewees that the figures on pages 7 and 8 can be seen as the projection of one cone on another, as is frequently applied in Projective Geometry. Of course this was already read by the interviewees before the interview, so that page 9 helps to explain the figures on pages 7 & 8.

C.1. Do you consider the configurations on the pages 7 & 8 to be Art ?

Perhaps the question seems to be a bit too simple, because no definition has been given about what precisely art is. One may assume that many people have their own ideas about the essence of art, while others adhere to standard definitions which can be found in books.

We might agree that Art has something to do with beauty. Sometimes people admit that drawings are aesthetic, without saying that they could be regarded as art. Others have a very personal definition of art, related to their beliefs.

For the purpose of my investigation I think it is sufficient to accept that the drawings on pages 7 & 8 are art when a reasonable majority indicates that they see them as such. When a large proportion of the interviewees state that these drawings satisfy their concept of what art really should be, I feel that this satisfies my demands. The geometrical constructions on pages 7 & 8, being in a way a distortion of higher dimensional configurations, appear to have an aesthetic value for many readers.

It has been demonstrated that within the borderland of 2- and 3-dimensional geometry lodges a beautiful landscape worth visiting. So this kind of geometry is educationally valid. For the student it refers to Projective Geometry, and thus to higher notions. The drawings provide a preliminary view of that part of geometry.

C.2. Give an account of the relations between the figures on pages 7 & 8 and the text on the other pages of "Projections".

Pages 7 & 8 form a kind of epilogue to the text on the other pages, page 9 included.

C.3. Can art widen one's view of geometry ?

An answer could be: 'Yes, geometrical action, depicted in an artistic way, may widen one's view of Geometry'.

5.4. Interviews

In this section the 15 interviews have been transcribed. The main purpose of the interviews is to get a glimpse of the teaching process that might be appropriate for 'educational geometry'.

It has not been my aim to collect statistics. 'Educational geometry' provides an unprecedented way of conveying knowledge by means of visual art so that at this stage I should restrict my investigations to the first reactions of my audience which are indeed encouraging.

Moreover, I can not think of 'educational geometry' in terms of a school curriculum. This new kind of geometry provides good opportunities for interested people to study science.

I do not consider the answers of the interviewees as data. The interviews provide a quick look into future geometry education as I see it. Therefore I have included these interviews in the main body of the text, and not in an Appendix.

Here follow the transcriptions.

A.

Female, age 26, student of musicology at the University; attended grammar school with 3 years mathematics.

B

1. Yes
2. Yes, coinciding planes. Should you look at it 2-dimensionally ?
3. Top angle widened, others narrowed; more oblong.
4. Angles tend to become right, characters become smaller.
5. Conjugated sides meet at ℓ. No explanation.
6. At the right bottom side lines are not prolonged.
7. Yes, or maybe not? It could be a statue at a warehouse.
8. Yes.

9.1.Yes.

9.2.Clockwise.

C

1. After having studied projections you look differently at the picture; It is not only Art: there is something more. 'Real' Art can be considered as a construction too.
2. There are two planes; the straight lines assemble at the hyperbola + straight lines on page 7. On page 8 there are two peaks from the circumference.
3. Reverse: Geometry can widen one's view of Art.

D

Geometry is more fun than I expected; at secondary school it was annoying and difficult - I don't know why. It was hard to understand. "Projections" is difficult too, but there are no strict rules; it looks stranger.

A

Female, age 38, housewife; attended grammar school with 3 years of geometry education; high marks for geometry and Dutch language.

B

1. Yes, if they are not glued together.
2. Yes, along each other.
3. More tilted; flatter, more towards F (3-dimensional).
4. More stretched; a trapezoid; being rectified.
5. They rotate; third dimension involved (after some explanation).
6. No.

7. The circle can be decomposed; the peaks are a bit tilted towards me; transversals are different. Very beautiful.
8. Should be possible; the hyperbola emerges of its own accord.

9.1 It will be curved; not a genuine circle.

9.2.I suspect a trick. After scrutinising thoroughly: Clockwise.

C

1. Yes, it is art. You can hang it on the wall.
2. Projections again; they represent the same idea.
3. Geometry is art too.

D You learn to look very carefully. Fine, that's OK.

A

Male economist; age 29, attended grammar school with mathematics at A-level.

B

1. It is a 2-dimensional drawing; not very spatial.
2. A plane in space is a hazy notion; it seems to be on a design; Q is a projection of P
3. Angles are different.
4. Different angles.
5. Projection is drawn well.
6. C_1 should be on the other side. F is not on PQ.
7. It seems a bow, like the Eiffel Tower in Paris. A construction with steel skeleton.
8. After some attempts it should be possible. Start off with the circle; then draw lines at the hyperbola. Tricky.

9.1.Basically; yes.

9.2.You can compare it to wheel work; anti clockwise.

C

1. Yes, it is art. It makes me think of Bauhaus. It is Art approached from the geometrical point of view. Art can be based on it. It does not blend with classic tapestry; it does not suit in every room.
2. These pages 7 & 8 are a continuation of the rest of 'Projections'.
3. It is more the other way around; you find art by studying geometry. You look differently at cathedrals, the Eiffel Tower etc.

D

That Geometry can produce fine figures; art can be based on it.

A

Male, age 38, profession: teacher of mathematics; attended grammar school with mathematics and gained higher certificate in mathematics (level comparable to B.Sc.).

B

1. Yes.
2. Yes.
3. Prolonged, stretched.
4. Compressed.
5. It is the theorem of Desargues.
6. Is $q = (P,\ell) = (Q,\ell)$?
7. If you consider it spatial, yes.
8. Difficult but possible.

9.1.No.

9.2.The same direction.

C

1. Yes, the figures are aesthetic. They puzzle you.
2. It is a circle projected on a plane.
3. It can be made attractive. By observing those figures, you learn something intuitively. You develop a feeling for Projective Geometry.

D

To observe the configurations carefully. This is stimulated by the questions. Figures can be transformed to other figures by projection.

A

Male, age 43, profession: male nurse, attended lower technical vocational school, served in the air force as a sergeant.

B

1. Yes.
2. Yes.
3. The upper one seems to be more tilted towards me in space than the lower one.
4. The lower one is more regular, more rectangular.
5. It is like the pages of a book; if you turn them, the line of the intersection remains.
6. No.
7. Seemingly, yes.
8. First you should understand the construction; it is like the elegant construction of a bridge.

9.1.No, because a circle is a plane figure; on a cone it would have to be convex in space.

9.2.Also clockwise with S as centre of rotation. (Without explanation).

C

1. Yes.
2. Basically it is the same, seen from different points of view.
3. I think so. Geometry consists of triangles.'Projections' is spatial geometry; it is 'playing with space'.

D

1. Yes, that rotating and moving in a plane is also spatial. It gives you an idea of dimension. The configuration on page 7 (of the questions) is funny and beautiful.

A

Female, age 60, profession: Senior Lecturer in Greek, Latin and Philosophy at the University, attended grammar school (with mathematics).

B

1. Very beautiful, aesthetically.
2. Less beautiful.
3. Could be an ornament in the parlour; hanging on a lamp.
4. Beautiful. Intuitively: the H in Hey is not projected rightly.
5. The small quadrangles are beautiful.
7. Magnificent: this is art.
8. Not art, but interesting. Complicated; it is spatially less balanced. It is technical, aesthetic. Nowadays the museums are exhibiting rubbish.

C

1. Yes.

A

Male, age 45, teacher of German Language. Attended grammar school with 3 years of mathematics.

B

1. Yes.
2. Yes.
3. It is perspective. The lower is more oblong. It is a deformation, caused by perspective.
4. From P it looks different compared to looking at it from Q. It is different.
5. No; it can be seen in perspective.
6. The cones are placed on the side; it is projected towards different directions.

7. Yes, under the straight line.
8. Yes.

9.1.No.

9.2.Clockwise.

C

1. Yes it is modern art; like the pillars of a bridge. You could hang the picture of page 8 on the wall.
2. Yes, there is a connection.
3. Yes, certainly. The drawings are playful, illuminated. Like flowers; one could draw geometrical lines and fields, or a DNA structure could be displayed with spiral lines.

D To discover a connection between perspectives.

A Female, age 37, teacher of mathematics, level BSc. Feminist.

B

1. Yes, a model could be seen. Girls need a context and they ask more questions.
2. Difficult; they must be coinciding.
3. Why that triangle ?
4. Area decreases; it is deformed.
5. One of the planes could be taken out of the picture.
6. We have projections from S into one plane q.
7. Wire constructions; almost not possible.
8. I think so, but it could be difficult.

9.1.No, impossible.

9.2.If they are in the same plane q; clockwise.

C

1. Beautiful and fascinating to look at. Seems like it should be finished off (by connecting the points of the hyperbola). It makes you look differently at mathematics. It could be a picture on the walls of a museum. In the figure on page 8 there is a certain tension: why do the planes break ?
2. They are projections of the cones.
3. Yes. The study of 'Projections' has changed my view of Geometry.

D

I have got a different image of the planes and space. There is something strange about the combinations space + plane. I learned something about projections.

A

Male, age 38; student of philosophy at the university; attended grammar school with 3 years of mathematics; low marks for geometry.

B

1. Yes, it might have been coloured. Compare it to the movements of the pieces in a game of chess.
2. Yes, two planes lying on each other.
3. The base becomes longer; it is squeezed. The altitude shortens, distance changes.
4. Pushed at the right hand side; tilted; distance shortened.
5. The area does not change; perspective centred on F.
6. No line m.
7. Visually it is very beautiful. I conjecture mathematical rules. A spatial model could be provided, but under the straight line the situation is different. For the whole drawing it seems impossible.
8. No, impossible.

9.1.A spatial circle. No plain circle.

9.2.Clockwise.

C

1. Yes.
2. Yes, obviously; the other pages are the materials.
3. Problems can be visualised; it is a fruitful co-operation.

D

It's fun.

A

Male, age 20, student at the Polytechnic of Textile Engineering; attended grammar school with mathematics at A-level.

B

1. Yes.
2. No.
3. The breadth is the same; count the lines. There is a rotation to the right. The size, measured in cm, changes. Area unchanged .
4. The area remains the same. Breadth unchanged (count the lines; rotation to the right). The size of the characters does not change.
5. The lines from F 'lowers' until the distance of conjugated points becomes zero.
6. A line through T_1 is lacking.
7. No, it is a projection (a 2-dimensional figure).

8. Yes, but not easy; it has to be studied thoroughly.

9.1.Circle, more an omelette.

9.2.I think clockwise, but I am not sure.

C

1. Yes.
2. Both are projected cones.
3. Yes, if you understand it a bit, you get more insight.

D. That my knowledge of geometry already has become hazy.

A

Female, age 40, lawyer; attended grammar school with 3 years of mathematics.

B

1. Could be, I can see it.
2. Why not? It need not be one plane.
3. Angles are different; the base becomes longer. It is some sort of an impossible construction with depth to it.
4. Angles are different; directions change.
5. I do not know; never studied before. The construction makes me think of astronomy; the beams of the sun.
6. I did not notice any error. Lines intersect on circle + straight line; the circle shifts a bit.
7. It is like the sun with two mountains. You could fix the spatial model; hanging on two points.
8. Compared to page 1, it can be achieved.

9.1.No, not on that bent plane.

9.2.I don't know.

C

1. Yes; page 7 is more obvious: sun on the mountains. Page 8 looks more abstract. It is more like art: you could think a long time about it. The curved line could be water.
2. Yes, seen from S on page 6.
3. Abstract art is related to geometry (Escher).

D

I have learned to understand the idea of projections; angles change when they are projected.

A Male, age 39, teacher of Mathematics (level M.Sc).

B

1. Yes.
2. Yes.
3. Projected into another plane, it remains a triangle. It becomes larger, further away.
4. It remains a quadrangle, the only invariant property.
5. It is naturally so.
6. Point F is lacking.
7. I see a spatial figure, a modern work of Art. It could be a wire bridge with a road.
8. Yes, you can consider it as a projection.

9.1.No, it would be bent.

9.2.Clockwise.

C

1. Yes, surely. It is something geometrical.
2. Yes, they are projections.
3. For pupils to study the horizon.

D

My knowledge of geometry needs to be refreshed. I now look differently at it.

A

Male. age 46, teacher of history; level MA. Attended college of Education for Primary School teaching with 5 years of mathematics.

B

1. Yes, if the directions of ℓ and m are perpendicular with respect to each other.
2. Then there are not 2 different planes.
3. Flattened; larger base, smaller altitude. The differences are minimal in the neighbourhood of ℓ .
4. Area and circumference diminish; if they follow their direction they will coincide.
5. No.
6. On the bottom side not quite symmetrical, as a cone should be.
7. No, everything is tied by straight lines. If the circle was an ellipse, you could look differently at it.
8. You should know the construction. It seems to be possible, if you know how to handle it, seen from the infinite point.

9.1.No.

9.2.Clockwise.

C

1. Yes. It is very beautiful, pretty forms; playing with lines. You look at it as an image, basic forms, artistically. Essentially it is an art approach, could be coloured. The construction on page 8 is a product of Art, based on line + point. It is created by mathematical thinking; transformations with an aesthetic frame of lines. It is an intuitive play with regularities.
2. The same rules.
3. It is methodical art, working with circle constructions.

D

It is fun to play with points and lines (dimensions; one point at infinity). It is a captivating, strange, visual world.

A

Male, age 44; teacher of economics and law (level MA) attended grammar school with 3 years of mathematics.

B

1. I can't see that.
2. Neither can I see it here.
3. The bottom side becomes longer; more sloped.
4. No major deformation.
5. It has been organised, so that the sides must coincide.
6. It looks like a crippled figure.
7. Yes, I think so. It is one of the prettiest drawings; you could dream about it.
8. No, I am not experienced enough.

9.1.At first sight and intuitively: no.

9.2.Anti clockwise.

C

1. The configuration of figure 7 could hang on the walls of a museum. How do you construct it: first circle and hyperbola? It is art; you could dream about it. Page 8 looks like a sailing ship. It is less like art, a bit strange.
2. More beautiful. Pages 7 & 8 remind me not so much of secondary school as of the rest of 'Projections'.
3. I don't believe so. The figure on page 7 is a beautiful little work of art; it does not make me think of geometry.

D

I learned something about projections; one should study it more thoroughly; first get acquainted with basic notions. The level is a bit too high, I fear.

A

Male, age 36, teacher of physics; level M.Sc. Attended grammar school with 5 years of mathematics.

B

1. At first sight it can be considered 2-dimensionally.
2. Yes, with bent planes (like a spiral surface).
3. Elastic deformation of the circumference. Another placement in space and a different orientation.
4. See question number 3.
5. It is the theorem of Desargues. Projective Geometry is something special. It takes some time to understand it. The proof should not be developed too quickly; the teacher has to grasp the concept.
6. No.
7. I think, yes.
8. Yes, it will take some time. There is a certain harmony in the construction.

9.1.No, not a flat circle.

9.2.Clockwise, connected lines shifting in the same direction.

C

1. No, art is something different. Art is an experience of the soul, while mathematics is a purely platonic world of ideas. Art and mathematics should be alongside each other; they should not be mixed up, although it is not a taboo to do so.
2. Page 6 yields a cone, which on pages 7 & 8 is flattened, projected on a plane.
3. See question number 1.

D Projective Geometry is generally considered as a model, but it is more than that. The drawing of figure 7 can be considered as a sun, radiating to the horizon, and from there to the world.

5.5. ABOUT THE INTERVIEWS

With each interviewee, the same procedure has been followed. The text of 'Projections' was handed to the student, with the request to read it. After a brief period the student was asked to choose a time for the interview. The questions were not given beforehand to the student. The purpose of this was to get spontaneous replies. That might narrow the gap between those who studied mathematics and those who did not.

For instance, question B5 is actually about the theorem of Desargues. A teacher of Mathematics will immediately recognise it, but may not be able, at that moment, to produce a proof. It is more valuable to notice that the proof has been forgotten than to have such a proof given by somebody who had the opportunity to look it up in the literature of the subject. It appeared that the study of the text of 'Projections' did generally take not more than an hour. The answering seldom took more than one hour. After the interview, the matter was discussed over coffee.

Remarkably, one of the teachers of mathematics told me that she had learned from the text. After the interview, we talked for a considerable time about projective geometry. This subject nowadays is no longer taught as much as it was some decades ago. It is a pity that a new generation of teachers in mathematics lacks the insights provided by Projective Geometry. So, I am glad to see that my material, meant for teaching students, is also valuable for teachers of mathematics.

Up to now, during the study of mathematics, it has been usual to emphasise the solving of problems by computations. The computer has emerged, with an unlimited capacity for doing computational work. However, insight and understanding cannot be produced by a computer, but have to be mastered by the student. There is almost no computational work in the material which I am offering to the students.

The answers to each question by the interviewees will now be discussed. The questions will be considered separately, and then a general conclusion follows.

Answering of B1

Most of the interviewees replied positively. One of them talked about planes which could be glued together. This is indeed the right direction: a double plane. She stated that geometry was her favourite subject at grammar school and the only subject for which she got high marks. Two of the interviewees did not recognise the two planes in one: the 3-dimensional origin of the drawing was denied. Explicitly mentioned was the possibility of the lines m and ℓ being orthogonal with respect to each other. The interviewee had a great interest in geometry and he said he had a strong visual memory. None of the interviewees had problems with the understanding of the problem.

Answering of B2.

This gave more problems than the foregoing question. Less than half of those interviewed talked about coinciding planes. Others saw it as a hazy question or simply did not see the possibility of a double plane. It was also mentioned that two bent planes were involved. One of the interviewees stated that the two planes of question B1 had disappeared into one plane . It is of course understandable that the concept of two coinciding planes gives rise to different comments; it is not in the curriculum of secondary schools.

Answering of B3.

It is notable that many of the interviewees talked about constructions in space where the constructions have remained the same. So the image has become increasingly spatial. It is interesting which parts of the triangle the interviewees were talking about: some emphasised the change of the angles, others the length of the edges;it was also mentioned that the triangle takes another position in space, so that some elements, like edges,altitude, and circumference seemingly change, but the angles have to remain invariant. It means that almost all of the interviewees considered the configuration as 3-dimensional. This 3-dimensional interpretation is different from what I expected: I was thinking of the deformation of the figures, not about their travelling in space.

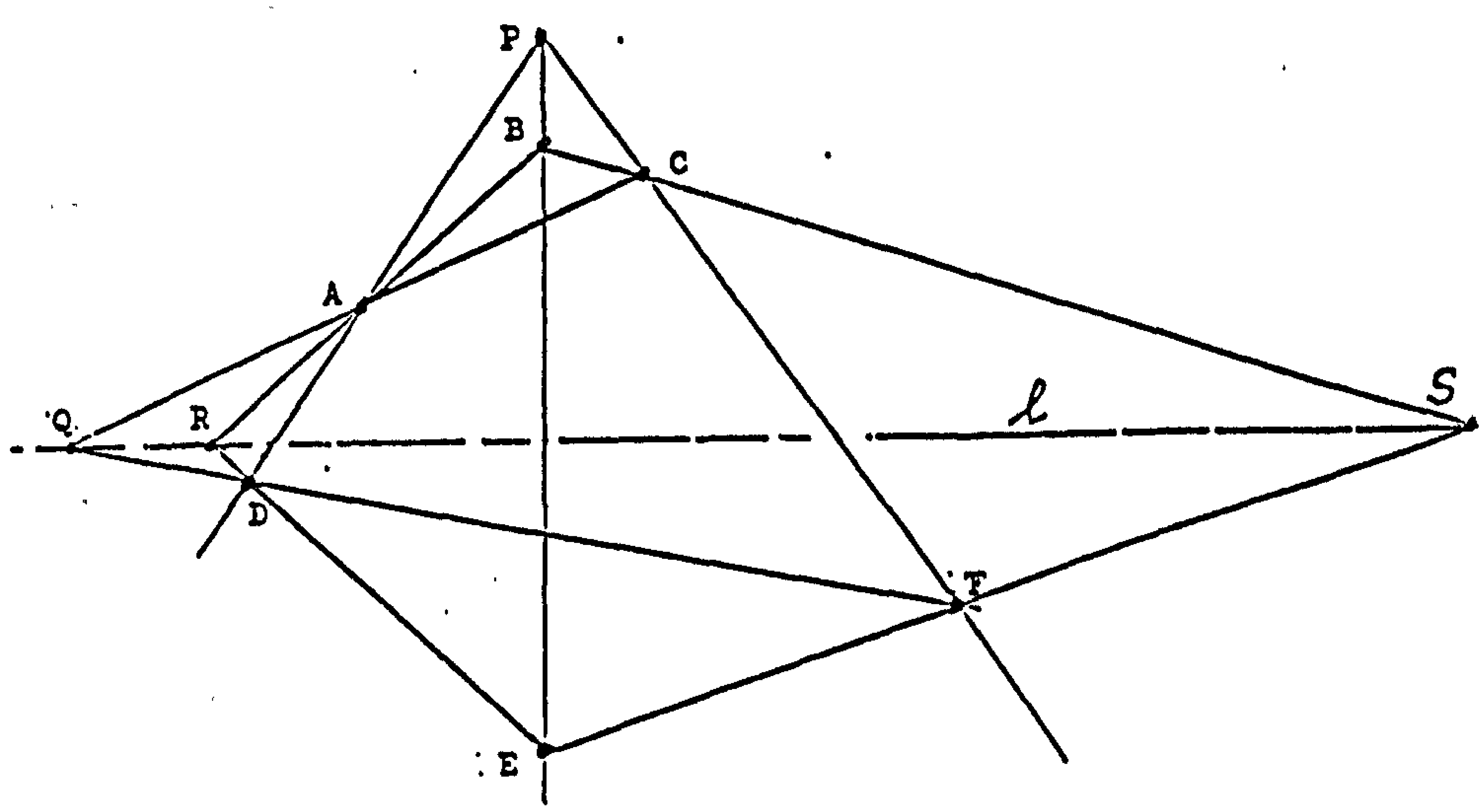
Answering of B4

More attention was paid to the deformation of the angles. The 3-dimensional status seemed no longer relevant. Both figures were compared and the differences noted. These differences were not focused on one detail: angles, compactness, the projection, the beauty of it, the quadrangle observed from different points of view, the magnitude of the area, the length of the sides of the quadrangle were all mentioned. Again, my prediction did not quite prove true: only two of the interviewees talked about the transformation of the characters HALLO. As far as I can see, the characters were considered as naturally belonging to the process of transformation, so that they undergo the same deformation as the quadrangle. The characters need not to be investigated separately. In my imagination it is quite important that a text is kept readable after it has been transformed geometrically; however, most of the interviewed did not find it special.

Answering of B5.

Only two, both teachers of mathematics, knew that the theorem of Desargues was involved. A few of the interviewees were very near to the solution. One of them answered that a third dimension was involved; and that is very true. Desargues' theorem recognises a visual proof when we involve a third dimension. The proof will be an exercise in Part II of the thesis.

figure 1



<u>Desargues' theorem reads:</u> If with two given triangles, ABC and DEF, the lines AD, BE and EF go through a point P, then the three points of intersection Q, R, and S of the edges respectively (AC,DF), (AB,DE) and (BC,EF) lie on one straight line ℓ. (see figure 1).

We translate this theorem into its dual:

<u>If with two given triangles, ABC and DEF, the points of the intersection of the edges(AC,DF), (AB,DE) and (BC,EF) lie on one straight line ℓ, then the lines AD,BE, and CF go through one point P.</u>

Because of the application of duality, a proof for the second theorem is not necessary: it follows directly from the dual theorem. In Part II we will have a visual proof in 3-dimensional space; but there is no visual proof in 2-dimensional space. Of course it can be proved analytically, so that it is evident that "Desargues" is valid in 2-dimensional space. However, the proof given with the help of figure 1 shows that is easy to jump with 'Desargues' from 3-dimensional into 2-dimensional space and the 'flat Desargues' (in R^2) follows from the 'spatial Desargues' (in R^3) when we rotate the planes ABC and DEF (in R^3) about ℓ so that they coincide and become R^2) . The construction will remain valid. In fact, the configuration in figure 1, which provides a projection of two coinciding planes on each other with help of a point P, has inspired me to design the first 5 pages of 'Projections'.

<u>Answering of B6.</u>

The incomplete parts were well scrutinised by some of the interviewees. Indeed there should be a point F on the line m through PQ, as in the projections on pages 1 to 5. Moreover, there should be more points of intersection on the right hand side of line ℓ, although one of these points fails to coincide with ℓ; nobody noticed that. Also it is questionable whether the circle PQ and the line ℓ lie in the same plane q. These shortcomings were well observed by the interviewees. The lines through P and Q intersect on the circle circumference as well as on line ℓ and that was also noticed.

So indeed the projections of the cones have been scrutinised rigorously by the interviewees, though not everyone found errors.

<u>Answering of B7.</u>

Technically, a spatial model of the given configuration cannot exist, because all the straight lines intersect with other lines; two lines lie in one plane; and all these planes have to coincide. An exception would be when the supposed straight lines have a kink at the horizontal line, so that the upper half of the drawing lies in another plane than the lower half. Most of the interviewees concluded that a spatial model is possible; a minority did not think so. However, the general idea of the interviewees about the shown figure was spatial. This emphasises that a plane figure can evoke spatial observations, and be impeccably flat at the same time, acting as a plane figure. One can stress that straight lines in most cases are drawn curved. However, in such a drawing as figure 7, the deviation from straightness is too small to give an idea of distortion. Only when the horizontal line is considered as a horizon at infinite distance, there is a distortion in the region of that horizon. This leads us to the natural idea that the horizontal line might be a horizon at finite distance and there need not be distortion when a line approaches the horizon.

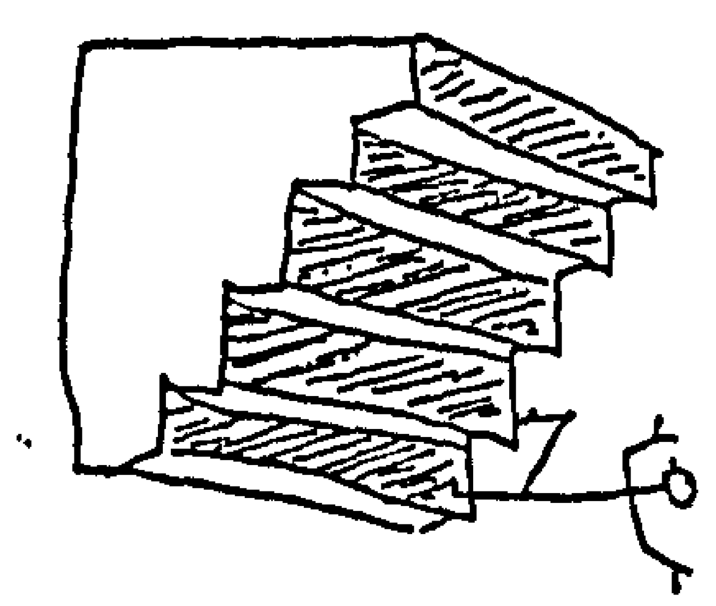


figure 1

vertical

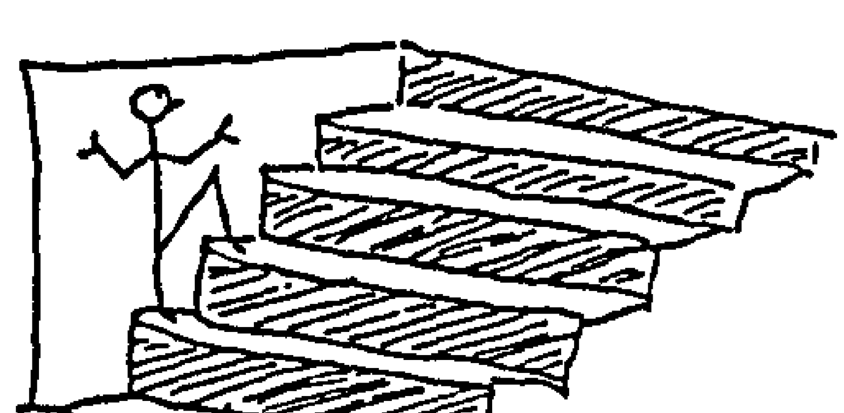


figure 2

horizontal

In the figure 1 & 2 it is shown how one can look at the same object in quite different ways. It is the 'Staircase of professor J.A. Schouten', given by him to M.C. Escher. The arrows indicate what is considered to be horizontal and vertical by the man standing on the staircase. In figure 1 the steps of the staircase are shaded; in figure 2 they are not. Both figures 1 & 2 show the same drawing, but the interpretations of the staircase are very different.

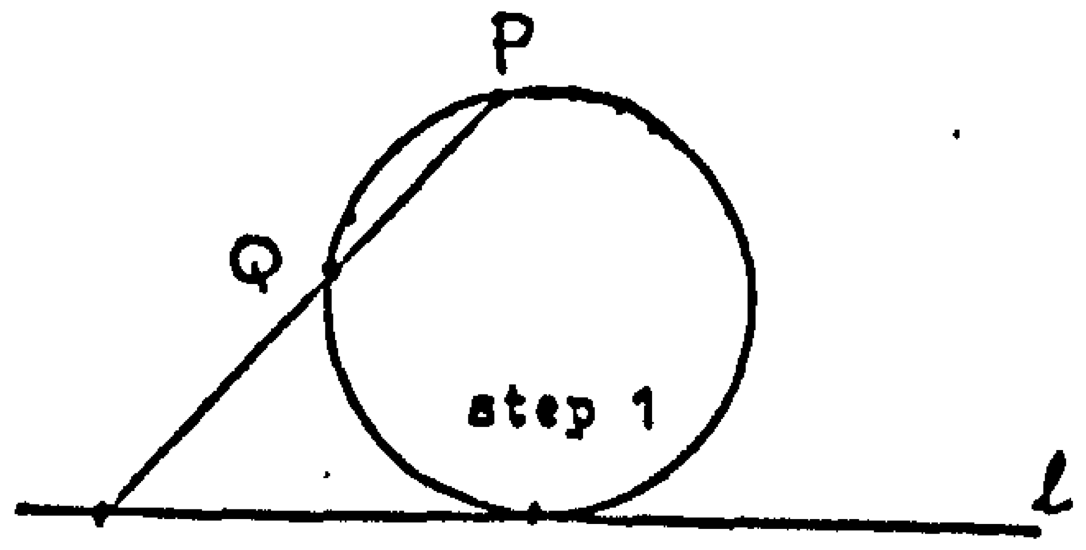
Analogously one can consider the configuration on page 7 of 'Projections' as 2- or 3-dimensional. Most of the interviewees considered it as 3-dimensional. It should be emphasised again that the configuration on page 7 of 'Projections' is completely geometrical; however, many of the interviewees assessed it as Fine Art.

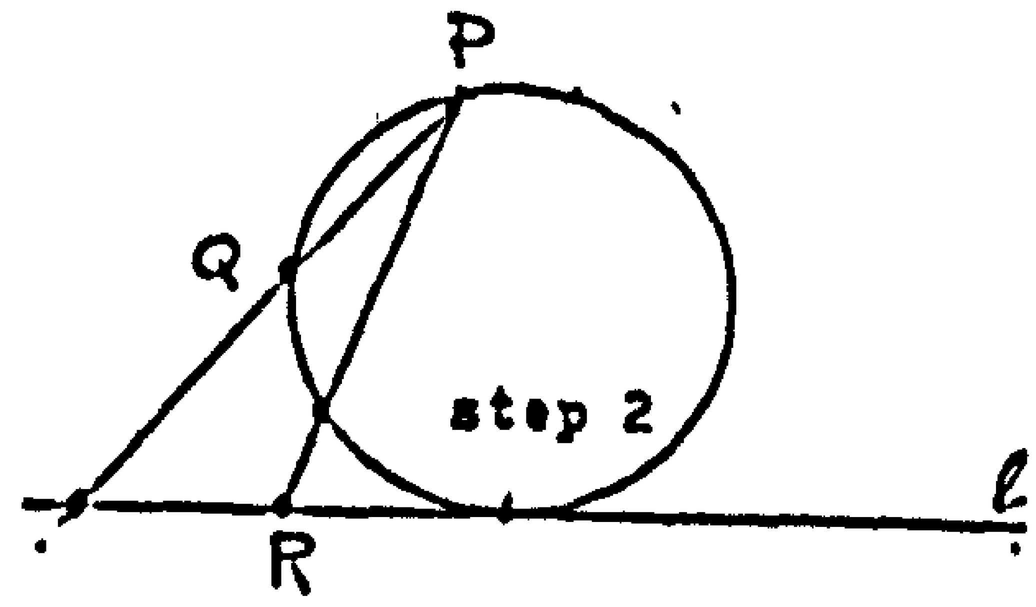
I think that the fact that it belongs to a 2-dimensional world as well as to a 3-dimensional world makes the drawing extraordinary; and the aesthetic view visualises the interaction between these worlds. The staircases depicted in figures 1 & 2 may be seen in different ways but is not evident what the geometrical importance is of these different views. Educationally it is of course important, because it shows that the most obvious conclusion is not always the only one. However, the drawing on page 7 of 'Projections' is geometrically important, because it is a geometrical construction; it also offers a dual view of it, 2- and 3-dimensional, so that it provides an instant visual concept of Projective Geometry.

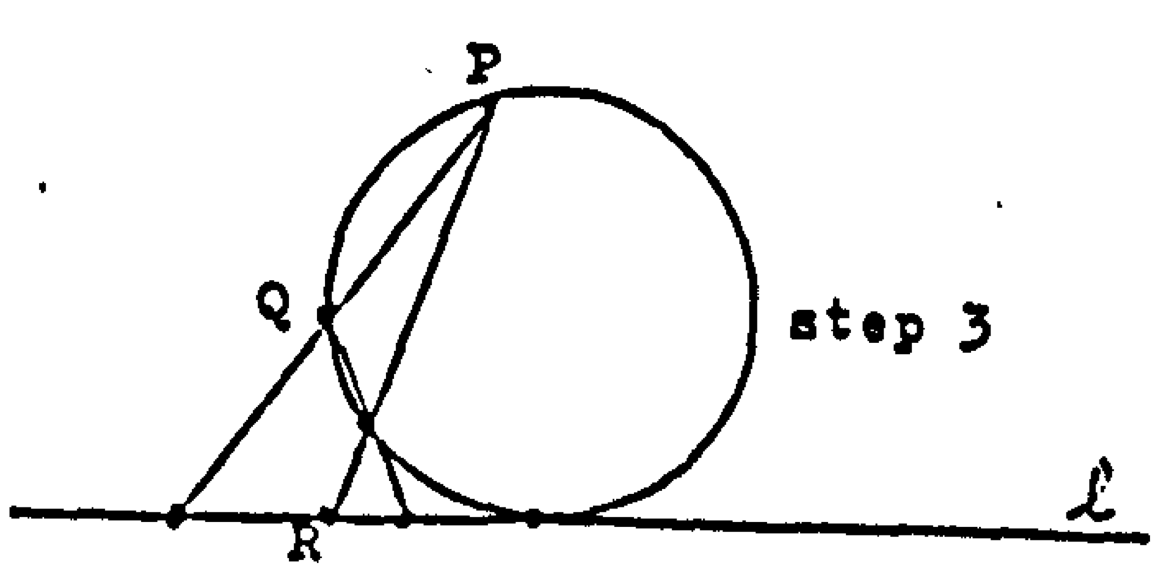
<u>Answering of B8.</u>

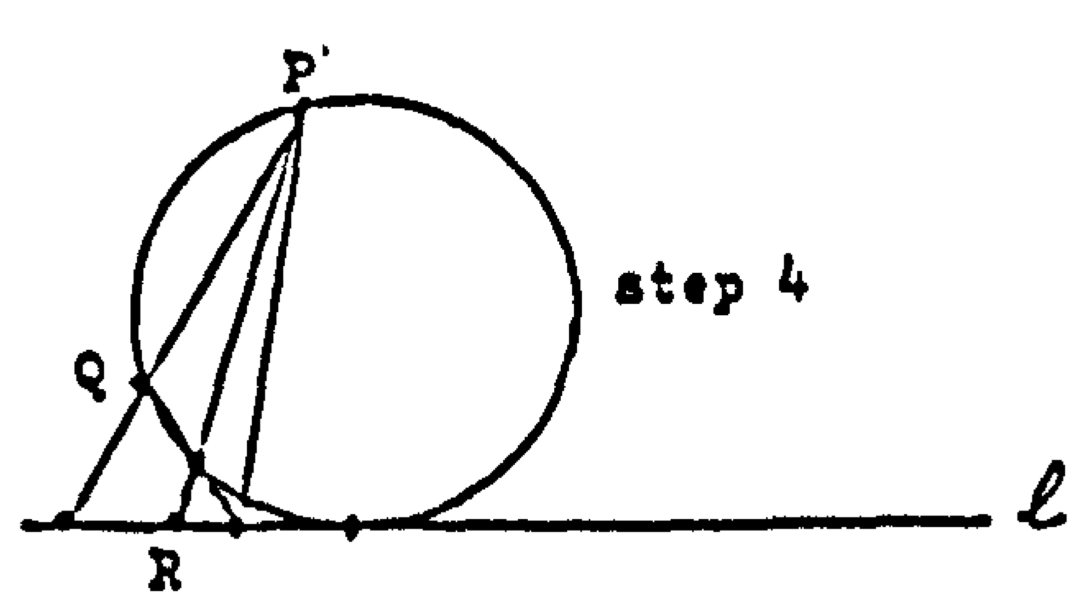
Most of the interviewees thought that it was possible to copy the drawing without touching the original. One should know the concept behind the construction. Some considered the drawing to be art; others found the previous picture more artistic. After the end of the interview, with a cup of coffee, some of the teachers of mathematics found the solution as how to start the construction.

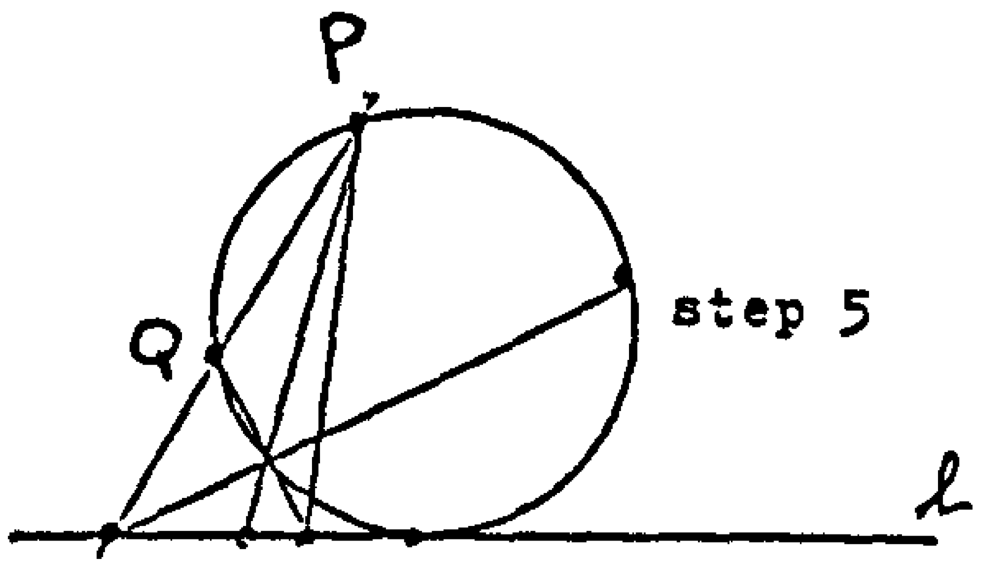
Here below the first six steps of the construction are demonstrated. Each next step gives just one additional line to draw. The line ℓ and the circle are arbitrarily chosen; they only have to be tangential to each other. The points P, Q and R are also arbitrarily chosen; the rest of the points of intersection follow compulsorily after the choice of P, Q and R. The first four steps show how to find points of the straight line left of the tangent point; the fifth and sixth steps show how to proceed on the right hand side of the tangent point

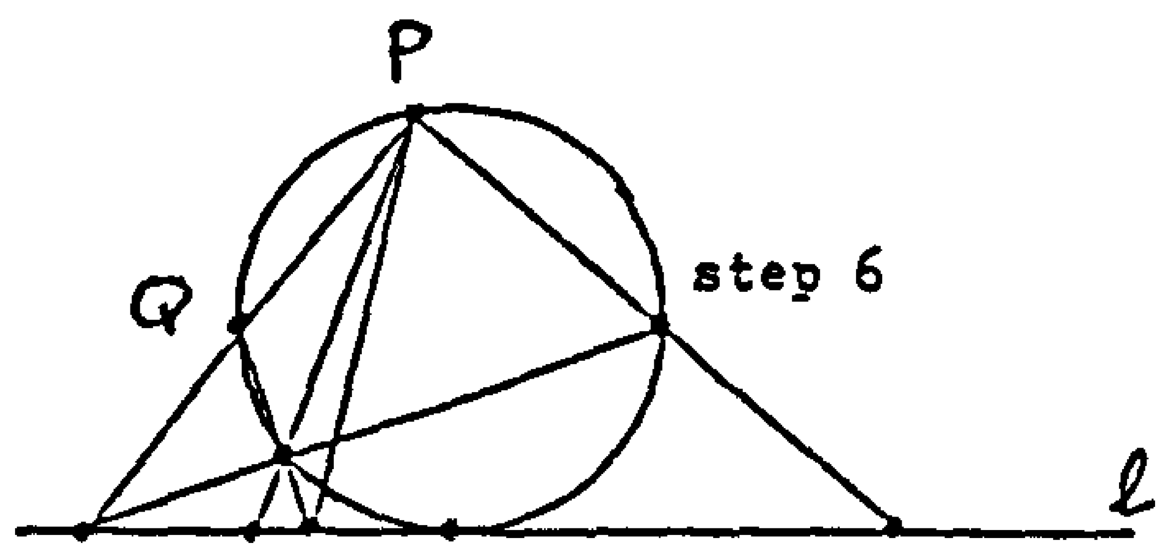












Answering of 9.1 and 9.2

The fact that 12 of the 15 interviewees concluded that a circle in the position of p_1 cannot lie on the surface of the cone indicates that the interviewees generally have a fairly good idea of spatial images. 13 of the interviewees followed higher education or university courses. 11 of the interviewees managed to indicate the correct direction of the rotation of the patch p_3. This also shows that the interviewees understood what the question was about and did not merely guess. As previously stated, the configurations on pages 6 and 9 serve to demonstrate how the constructions on pages 1 and 5 have been found and how the configurations on pages 7 and 8 can be seen to have emerged. In particular the construction on page 7 of 'Projections' can be scrutinised on page 9. It is, of course, remarkable that so many points of intersection may be found on the given circle and line on pages 7 & 8, but after the explanation of the figure on page 9 it will no longer be surprising. The construction is based on such points of intersection and is built from these points.

Generally, one can say that the questions about the pages of 'Projections' were not too difficult for the interviewees and that the answers were balanced, so that not all the wrong answers were provided by a few people and the right answers by a number of good mathematicians. Of course, some interviewees were more skilled at geometry than others; however it appeared possible to discuss all of it with those interviewees who lacked further education in mathematics. That is very important, because this type of geometry is intended for the students of 'educational geometry'. However, the teachers of mathematics and those with a more mathematical background found the material also interesting, though unconventional. The two people who did not receive higher education were very interested and answered the questions well. The 15 people involved are not, of course, representative of the rest of the population, but the results of the interviews are quite satisfactory for the purpose of this educational geometry.

Answering of C1.

It is evident that the overwhelming majority of the interviewed considered the configurations on pages 7 & 8 to be art. The picture on page 7 is generally considered more artistic than those on page 8. Many of the interviewees stated that these pictures could be exhibited in a museum and considered as Fine Art. Some of them even thought of colours. Indeed that could be an additional issue; however, not everybody has the same approach to colours. The statement that these configurations are art could be changed if the interviewee did not like the colour composition. In this case merely geometrical forms are involved. Some of the interviewees talked about the link between geometry and art; others were reminded of architecture or a landscape. It seems that the geometrical part of the drawing makes some of the interviewees look differently at mathematics. It is then considered as an insight into a new world, which is geometrical and beautiful.

Answering of C2.

All the interviewees considered pages 1, 6 and 9 to be a preparation for the pictures on pages 7 & 8; it is a more technical presentation whereas the pictures on pages 7 & 8 are more artistic. The connection between pages 7 & 8 and the rest of 'Projections' was recognised by all of the interviewed.

Answering of C3.

It seems to be important whether one looks at 'Projections' from the point of view of art or Geometry. Almost half of those interviewed looked at it as a display of art with an additional geometric aspect. About the same number considered it as a geometric project with an artistic aspect. The way people look at 'Projections' is evidently dependent on their previous education and personal preferences. It is a good thing that the different views are equally spread among the interviewees and it shows the importance of the artistic aspect being involved. Without that the material would have been less interesting for almost half of the interviewees. This is an aspect that cannot be underestimated and it shows that art is no luxury in the field of geometry but an important help to make the subject more accessible.

Answering of D.

All the interviewees answered that they had learned something and for the majority that meant that they had learned something about geometry. Generally there was a feeling that this geometry is stranger than the conventional approach, and it was welcomed. Some found it a strange and captivating world. In particular the new way of looking at the 3-dimensional universe was found interesting.

Conclusion to the interviews.

After reading the answers of the interviewees, I was satisfied with the results. One has to admit that people who agree to have such an interview must already be thinking positively about geometry, art, and such issues. Two people refused to take part in the trial and of course were not interviewed.

The people chosen can not be considered as a 'representation sample' in any valid statistical sense. Nevertheless, their reactions present a valuable contribution to the atmosphere of future geometry education, according to my ideas.

The material of 'Projections' has achieved what I hoped it would do: the interviewees were moved on pages 1 to 5 from plane geometry to spatial geometry whilst keeping the same figures. I did not expect that to happen, but it did. Also the fact that the subject was approached from different viewpoints - artistic and geometrical - in a balanced way was a satisfactory but unexpected result. So I have learned a lot from the analysis of the interviews and it has yielded unexpected results which are better than I had foreseen.

It is evident that the reading of 'Projections' and the answering of the questions has produced good results. Almost all of the interviewees had a positive opinion about 'Projections'. It was a beauty spot along the footpath we followed across the grounds of visual geometry. Remember my parable in Chapter I, section 1.2. At the gate of the gardens we met professor Barrau who warned us that things might become wild. Then we visited a rural Art-museum where Magic Realism could be admired, together with paintings of seventeenth-century artists. The museum was situated in an old coach- house in the castle's grounds. In the forest we discovered a magic pond, crowded with fish which tried in vain to reach the bank.

The footpath led us to a beauty spot and from there one had a brilliant view of the hills and valleys of the vast territory of Projective Geometry. And now we are facing the thorn hedge which professor Barrau talked about. Behind that hedge are ambushes and hazards. There is an indefinitely deep hole in the ground called 'elementary geometry'. One pitfall has the illustrated designation: 'Analytical Stop'. A gate in the hedge shows us the entrance to a labyrinth. Fortunately a sign warns us of: 'educationally invalid' ways out ! Thus far the analogy.

The elementary notions have something to do with intuition, I assume. An important author on that subject is Spinoza (1632 - 1677). Remarkably, he already warns of 'educationally invalid' knowledge. Here follows a comment on Spinoza by G.H.R. Parkinson (Spinoza, 1989, page 239 & 240):

In this important note Spinoza summarises and adds to what he has said so far about the different ways of knowing. It is helpful to compare with this the account given in the earlier TRACTATUS DE INTELLECTUS EMENDATIONE. Here what the Ethics *call the first kind of knowledge is subdivided in two, giving four kinds of knowledge in all. These four ways of perceiving which I have used so far to affirm or deny something without doubt'(G ii 10) are :*

I. The perception which we have from hearsay, or from some so-called "conventional sign"

II. The perception that we have from uncertain experience, that is, from experience which is not determined by the intellect, but is only so called because the experience happened as it did by chance, and we have no other item of experience (experimentum *) which conflicts with it, so it remains with us as unshaken.*

III. Perception in cases where the essence of a thing is inferred from another thing, but not adequately; as when we gather a cause from some effect, or else when it is inferred from some universal that some property always accompanies it.

IV. Finally, perception in cases where a thing is perceived either through its essence alone, or through the knowledge of its proximate cause.

In part III of the text we read : *"Perception in cases where the essence of a thing is inferred from another thing, but not adequately"*.

This phrase could be used to define 'educational geometry', in the sense that it concerns applying local geometry erroneously on global geometry.

Chapter VI

6.1. Intuition

There are stories about mathematicians who in their sleep were wrestling to solve some extremely difficult and complicated problem. In the morning, when they had awoken, the problem seemed utterly simple and it appeared to have reached a solution.

One is inclined to assume that intuition has played a role in that process: the mathematician was not able to solve his problem while he was rationally thinking in the daylight; but, when evening shadows fell and he went asleep, his mind remained active and worked unconsciously and was no longer hindered by the noise of everyday life. This is only a superficial sketch of what was going on but if one tries to explain the process psychologically, the matter might at least appear to be ambiguous.

The question of what precisely intuition is, is a contested matter. It could be questioned how concepts like straight lines, planes, points, and so on have come to our mind. Is that a matter of intuition? Professor Barrau, whom I have quoted several times in the thesis, testifies that elementary geometrical knowledge has reached us from experience and imagination but that a thorough treatment of this elementary material makes us run into great trouble. So he is glad to avoid the issue because he has written a book on analytical geometry in which the definition of these vexed topics is not required.

I studied Fischbein's book "Intuition in Science and Mathematics", quoted above, but I could not accept that his approach would lead to fruitful results. This is discussed in: 'Review', Chapter VI, section 6.3. In the above story of the sleeping mathematician, the process is mainly considered as a mechanical one. The mathematician is sleeping but his mind works on almost mechanically, and that is an objection I have to such assumptions. I cannot agree that the solution of the problem is a matter of some sort of automatic pilot guiding the mind. What really happens in the mind can not be presented as mechanical. Now this discussion of the working of the mind is mainly based on my conjecture that essentially intuition cannot be understood in a scientific way and I find support for this conjecture in the fact that the seemingly simple and basic notions in geometry cannot be defined scientifically but have to be produced by experience and imagination (Barrau). If the mind could continue on its intuitive path like a robot, why does it not arrive at an accurate and exact assessment of these 'simple' concepts?

Seemingly the very basic notions of geometry are not accessible to the scientific mind but can be touched by intuition. How is it then acceptable to consider intuition as a matter of scientific processes during sleep at night? I cannot answer this question. Equipped with these assumptions, I came upon the views of Descartes and Spinoza. Descartes' presentation of 'deduction' is impressive and I have tried to produce an example myself in Chapter VI, section 6.2. For the matter of intuition Spinoza's approach did solve the deadlock I felt there to be. Spinoza distinguishes three kinds of knowledge. (Spinoza, 1989, pages 69 & 70)

According to Spinoza, we proceed to an adequate knowledge of the essence of things with the help of the third kind of knowledge which he calls intuition. What I consider as the decisive step in his thinking is that the third kind of knowledge cannot be derived from the first or second kind. If that were not so, there would not be a third kind of knowledge at all, but only the first and second kind. That is of basic importance because it clearly means that intuition is something in itself and in its own right.

If intuition can be seen as a special kind of reason then the concept of intuition will be of less value. For this reason I have taken Spinoza's approach to intuition as a basis for my investigation into the phenomenon of intuition in mathematics. The application of Spinoza's 'intuition' has in my case been restricted to the question of 'straightness' and its meaning for visual geometry. So with the help of Spinoza's interpretation of 'intuition' I have come to the view that straightness is an intuitive concept which cannot be scientifically explained or understood. I presume that straightness can be found when we observe a genuine horizon that is a virtual line which cannot be reached, because it travels with you while you are moving.

So when we accept that straightness stems from the horizon, it follows that it is a notion borrowed from global geometry. In this case a global notion - straightness - is applied to local geometry to make a line segment straight. Now - returning from global geometry - the procedure is not educationally invalid but on the contrary highly useful to be applied in architecture and carpentry and all kinds of practical work.

I might now further discuss the issue of the sleeping mathematician and his mind wrestling unconsciously with the complicated scientific problem. Is it likely that the problem is solved as a result of this nightly activity?

First of all I have to explain that I have been involved in the interpretation of dreams for the last twenty years. In the mid-nineteen-seventies I began recording my own dreams and the dreams of others. I did a lot of thinking about the issue and with the help of popular books on this issue and the reading of works by C. Jung and others I tried to come to an explanation of these nightly phenomena. The issue was fashionable in the eighties and many groups of interested people met frequently to discuss their dreams and the dreams of others. Now there is less interest in dreams and their interpretation. Apparently there are no new results to provide an incentive for further investigation. There may also be other reasons for the end of regular meetings of dream-analysis groups. When I started my study on the subject, I was afraid of what dreams might have to tell me. What also struck me was that most of the dream dictionaries based their views on the works of Carl Jung so that any new dream dictionaries could not add much to those already existing. Only books derived from, say, ancient Egyptian sources generated new approaches.

It is not my intention to evince any theory concerning the explanation of dreams or relate them to the matter of visual geometry which I am investigating. The work I did on the clarification of dreams has taught me, however, that it seems to be quite normal to extract from your dreams the explanations which you are most eager to have. In other words the realm of the dream becomes a kind of wishing well, which echoes the wishes of the interpreting dreamer the next morning

The world of dreams does not always behave according to one's wishes. To return to our sleeping mathematician, I have to say that I cannot believe that his nightly activities are a continuation of his work in the daytime. That explanation is simplistic and, even worse, it is little more than an echo emerging from the wishing well.

6.2. Intuition in geometry

Professor Barrau writes:

"In some or other way, usually by appealing to experience and imagination and without arousing the sleeping dogs of a premature criticism, elementary geometry education works itself through this thorn-hedge and further behaves with exemplary punctuality, as to wipe off the memory of its somewhat wild debut."(Barrau, 1918, page 11).

Barrau says in other words that when you are talking about elementary geometry, your statements are likely to be wild. I will remember that!
The following quotation by R. Descartes (1596 - 1650) serves to illuminate his concept of deduction:

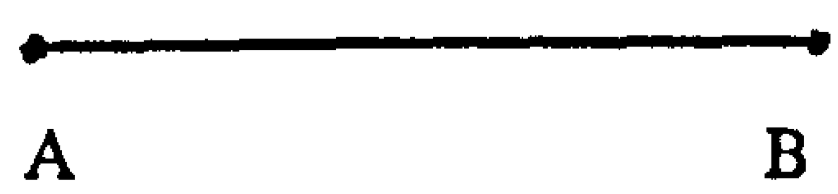
"Hence now we are in a position to raise the question as to why we have, besides intuition, given this supplementary method of knowing, viz. Knowing by deduction, by which we understand all necessary inference from other facts that are known with certainty. This, however, we could not avoid, because many things are known with certainty, though not by themselves evident, but only deduced from true and known principles by the continuous and uninterrupted action of the mind that has a clear vision of each step in the process. It is in a similar way that we know that the last link in a long chain is connected with the first, even though we do not take in by means of one and the same act of vision all the intermediate links on which that connection depends, but only remember that we have taken them successively under review and that each single one is united to its neighbour, from the first even to the last. " (Descartes, 1967, page 8)

In figure 1 the chain which Descartes mentioned is portrayed.

figure 1

Now we take a visual straight line AB (figure 2). See Def. (1).

figure 2

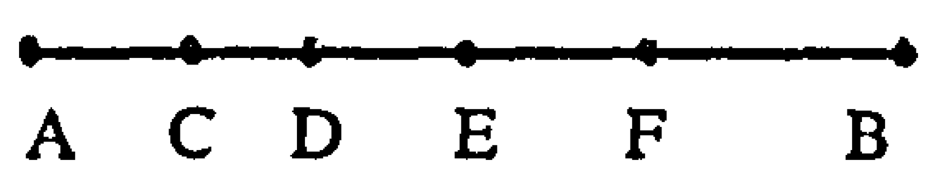


On that line AB we find more points: C, D, E and so on (figure 3).

figure 3

How are the points A, C, D, E and B connected? Not in the way Descartes' chain is organised. No point is linked to its neighbour by some chain-like connection. In fact there are no neighbouring points. If in figure 3 the points E and B could be considered to be neighbours, then a point F can be found between E and B so that E and B are no longer neighbouring points (see figure 4).

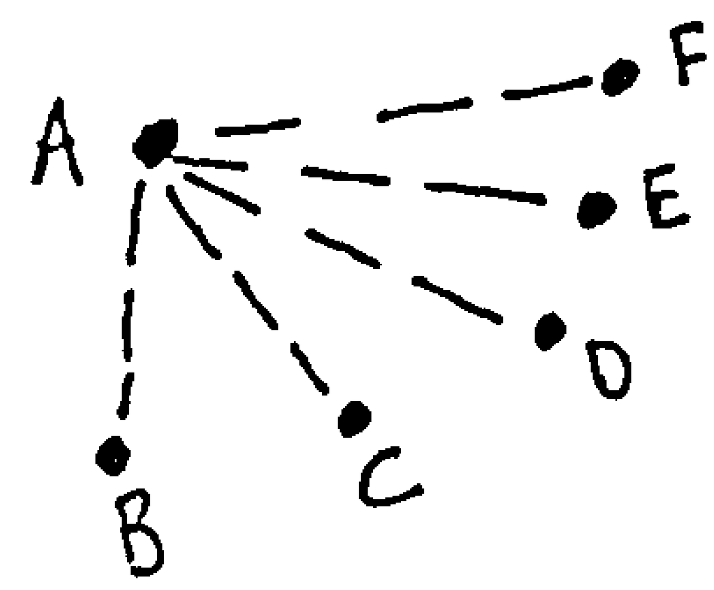
figure 4



One would say that these points are connected by the straightness of the visual straight line AB (figure 4). However, we have seen that visual straight lines can be perceived as curved.

There is a different way to look at the visual straight line AB in figure 4. We saw in Chapter III that global straight lines can be represented by just two points and nothing more. Thus we consider the couples AC, AD, AE, AF, AB and so on as global straight lines (figure 4). All these global straight lines (figure 4), consisting of just two points, are connected by their common point A (figure 5).

figure 5



Now there is a chain-like connection between the points of a visual straight line AB (of figure 4). Is it comparable to Descartes' chain?

In fact it seems to be simpler. Each step (for instance the link between AC and AD) has only to be repeated for any other two points on the visual straight line AB (figure 5).

Further, Descartes explains the relationship between deduction and intuition in the following quotation:

"Hence we distinguish this mental intuition from deduction by the fact that into the conception of the latter there enters a certain movement or succession, into that of the former there does not. Further deduction does not require an immediately presented evidence such as intuition possesses; its certitude is rather conferred upon it in some way by memory. The upshot of the matter is that it is possible to say that those propositions indeed which are immediately deduced from first principles are known now by intuition, now by deduction, i.e. in a way that differs according to our point of view. But the first principles themselves are given by intuition alone, while, on the contrary, the remote conclusions are furnished only by deduction"
(Descartes, 1967, page 8).

I consider the drawing of figure 5, following Descartes' concept, as a deduction from the image of figure 4. The way back from figure 5 to figure 4 is not a matter of deduction because it requires the notion of straightness which has disappeared in figure 5. It was supposed in figure 5 that the lines given by just two of their points are straight but it is no longer the visual straightness of figure 4.

To return from the image of figure 5 to the image of figure 4, I consulted Spinoza's view on intuition which is described by him in the following quotation:

"Third, from the fact that we have common notions and adequate ideas of the properties of things (Coroll., Prop. 38, Coroll. And Prop. 39, and Prop. 40, Part II). And I shall call this reason (ratio) and knowledge of the second kind (cognitio secundi generis). Besides these two kinds of knowledge there is a third, as I shall show in what follows, which we shall call intuitive knowledge (scientia intuitiva). Now this kind of knowledge proceeds from an adequate idea of the formal essence of certain attributes of God to the adequate knowledge of the essence of things. I shall illustrate these three by one example. Three numbers being given, we are to find a fourth, which is in the same proportion to the third as the second is to the first. Tradesmen without hesitation multiply the second by the third and divide the product by the first: either because they have not forgotten the rule which they received from the schoolmaster without any proof, or because they have often tried it with very small numbers, or by virtue of the proof of Prop. 19, Book VII., of Euclid's Elements, namely, from a common property of proportionals. But in very small numbers there is no need of this, for when the numbers 1, 2, 3 are given, who is there who does not see that the fourth proportional is 6? This is much clearer because we infer the fourth number from the very ratio which, with one intuition, we see the first bears to the second."
(Spinoza, 1989, pages 69 & 70).

As far as I can see, Spinoza's third kind of knowledge is not derived from the first and second kinds of knowledge and so intuition is an independent source of knowledge.

In my opinion 'straightness' is an intuitive notion, belonging to the third kind of knowledge described by Spinoza. Considered in this way, 'straightness' can not be found by using, for instance, a ruler. The notion of straightness is just there as an intuitive concept.

Straightness is not the same as a straight line, of course. A straight line might be denoted as a curve which has straightness in each of its points but this 'definition' does not work at all. How do we check straightness of a line? A slight deviation, not measurable but present, defies every means to determine the straightness of the line.

The above considerations have important consequences for the notion of symmetry. Up to now I have not used this notion. One could expect, however, to find in a thesis on visual geometry that the notion of symmetry would be very important. The problem is, however, that there is no adequate way to handle parallel lines.

Let us take, for example, a square. It is an example of what symmetry would be. Opposite edges are assumed to be parallel. But this parallelism is connected with the educational invalidity of the square. Visual parallel lines can not be mathematically parallel, as we saw in the example of straight lines running to the horizon. See figure 6.

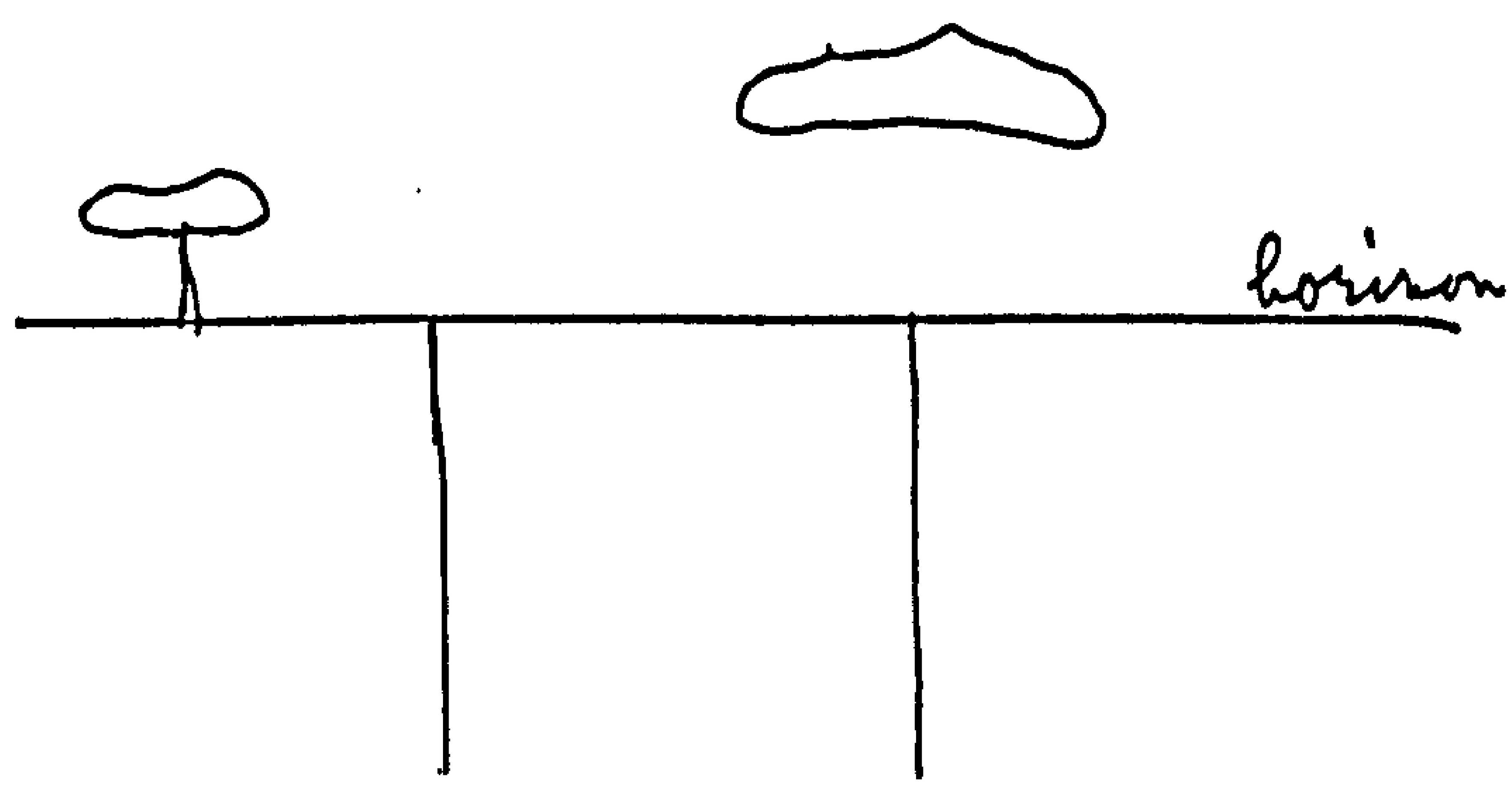
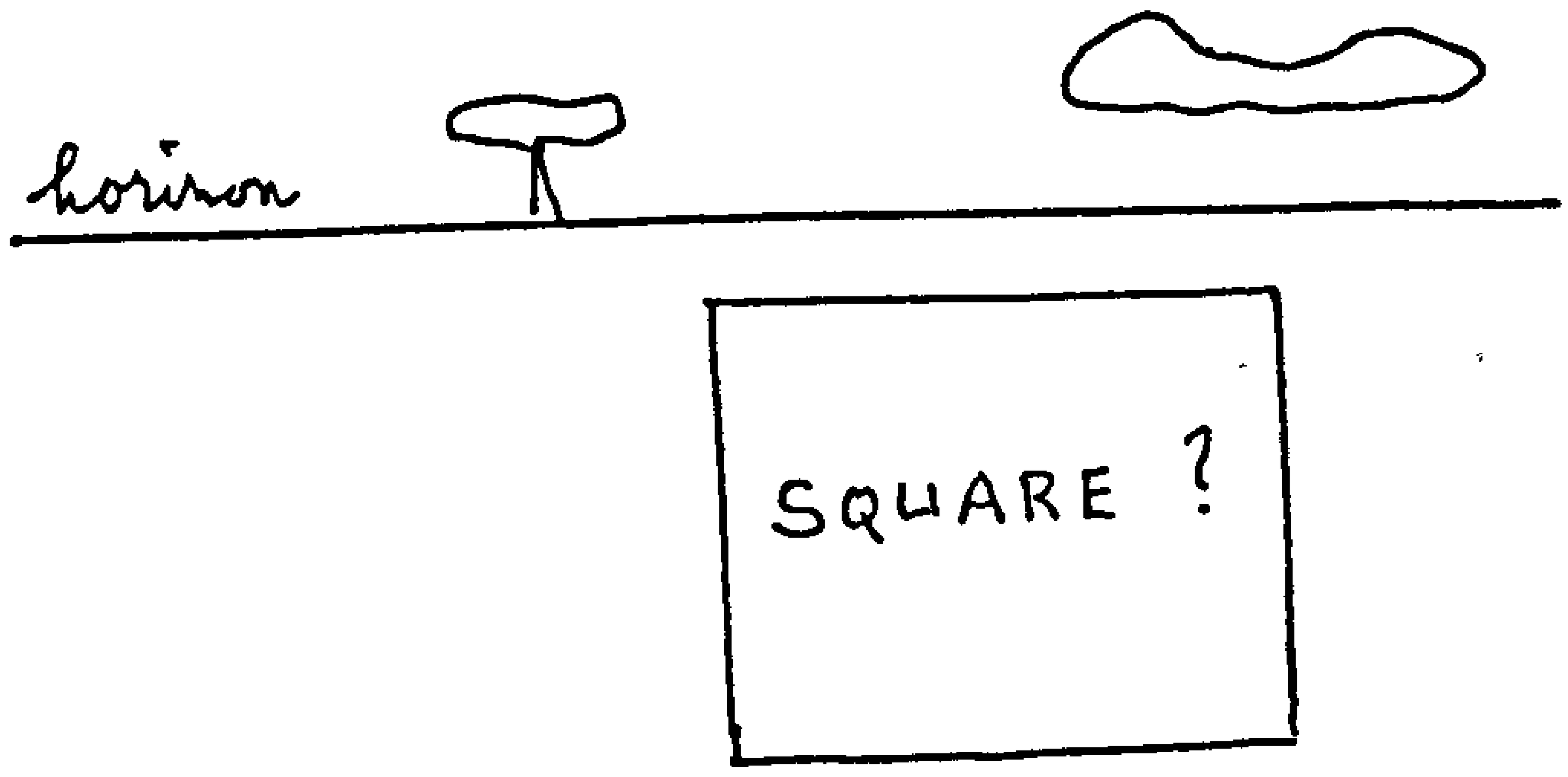


figure 6

In figure 6 we observe visually parallel straight lines running to different points of the horizon. The lines cannot be parallel but nevertheless they do not intersect. Can these two lines be edges of a square, or, rather, can a square exist as shown in figure 7? (see figure 7).

figure 7



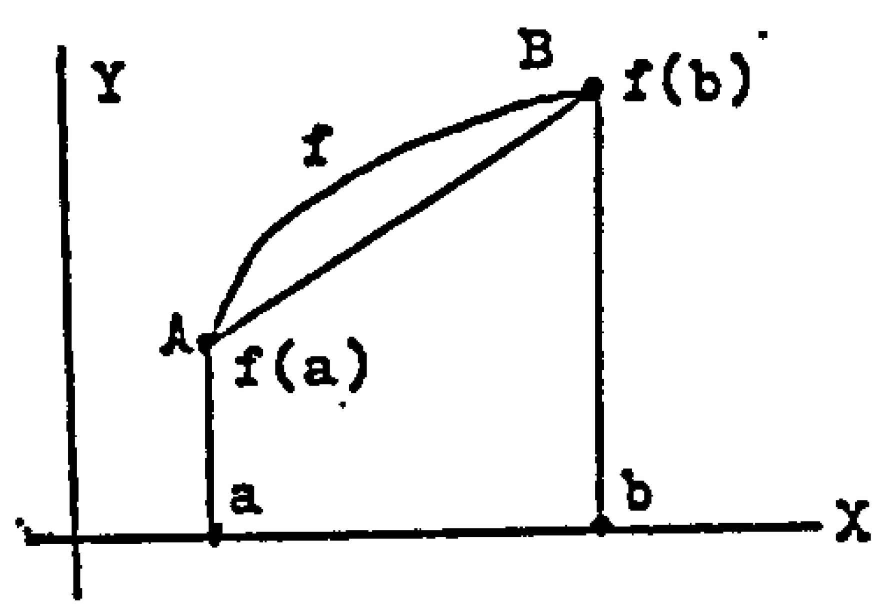
Two lines running visually parallel to the horizon and two lines, perpendicular to the first pair, parallel to the horizon (figure 7). Such a square can not exist but the symmetry of it is at stake too. So, for these reasons, the notion of symmetry is not involved in the establishment of a visual geometry. I conjecture that symmetry is a purely artistic concept. One also finds symmetry in numbers. For instance the number 7 lies between 5 and 9 and one could state that 7 is the midpoint of the pair (5,9). However, such an interpretation of symmetry is arbitrary. One has to consider that the distance between 5 and 7 is equal to the distance between 7 and 9. That way of looking at symmetry is a result of the definition that the distances between two consecutive numbers are all equal. But why are these distances all equal? By convention, I would assume.

Apparently, visual geometry (or Educational Geometry) is evading all the difficulties attached to notions like symmetry, straight lines, axioms and so on. It seems comparable to what Professor Barrau writes about Analytical Geometry:

"It enables us, as gradually will appear, to sail round the difficulties of a purely geometrical theory of axioms."
(Barrau, 1918, page 11).

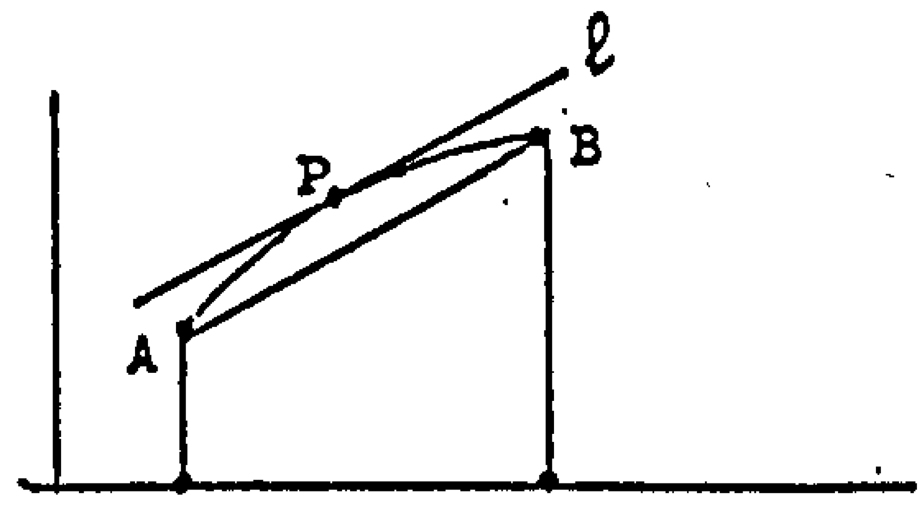
An example will now demonstrate that intuition can also be found in ordinary mathematics textbooks: Given a smooth function f, visualised in figure 8.

figure 8



It will intuitively be certain that there exists a point P on the curve such that the tangent to the curve going through P will be parallel to the straight line AB (figure 9).

figure 9



This is a good example, which was provided by Professor Dr A.W. Grootendorst.

6.3.Review

A discussion of Fischbein's "Intuition in Science and Mathematics" now follows. It provides the opportunity to quote other authors who have written on the subject of Intuition. A review of Fischbein's book "Intuition in Science and Mathematics" appeared in 1988 in "Educational Studies in Mathematics (19) 499-502 ", written by Brian Greer of the Department of Psychology of Queen's University, Belfast. The review starts with concise quotations from the views of 2 other authors, Reber and Gregory, on the subject of Intuition:

"Intuition. A mode of understanding or knowing characterised as direct and immediate and occurring without conscious thought or judgement. There are two distinct connotations which often accompany this term: (a) that the process is unmediated and somehow mystical; (b) that it is a response to subtle cues and relationships apprehended implicitly, unconsciously. The former borders on the unscientific and it is not recommended, though it is certainly common enough in non-technical literature; the latter hints at a number of difficult but fascinating problems in the study of human behaviour in the presence of complex situations (Reber, 1985)."

"Intuition. This is, essentially, arriving at decisions or conclusions without the explicit or conscious process of reasoned thinking. It is sometimes thought that intuitions are reliable; and indeed we do act most of the time without knowing why or what our reasoning may be. It is certainly rare to set out an argument in formal terms, and go through the steps as prescribed by logicians. In this sense, almost all judgements and behaviour are 'intuitive'.

The term is used in philosophy to denote the alleged power of the mind to perceive or 'see' certain self-evident truth (Latin intueor, to see). The status of intuition has, however, declined over the last century, perhaps with the increasing emphasis on formal logic and explicit data and assumptions of science.

'Woman's intuition' is, perhaps, largely the subtle use of almost subliminal cues in social situations from gestures, casual remarks, and knowledge of behaviour patterns and motives. Psychologists find these important matters for living almost impossible to formulate (Gregory, 1987)."

After these quotations Greer characterises Fischbein's book in a clear way as follows:

"As he points out, intuition has a long history in philosophy, with a variety of often conflicting interpretations. In particular, there are the incompatible views of intuition as the source of the only really reliable knowledge, and as a fallible route to knowledge not to be trusted. Fischbein is not concerned with the mystical aspect mentioned by Reber above; his treatment is based on the belief that intuition is open to psychological investigation, like any other form of cognition, through the interplay of theory construction and experimentation. He bypasses many of the philosophical debates by shifting the focus from truth to usefulness - what adaptive role does intuition play, regardless of whether or not it is correct ? Indeed, he treats correct and incorrect intuitions as behaviourally indistinguishable. (Note that, at one point, he carefully defines "correct" as "accepted by the contemporary scientific community"(p.37.)"

With reference to the above quotation I would like to state the following. If I was forced to make a choice between the distinct connotations: (a) intuition is mystical; and (b) intuition is open to psychological investigation; then I would certainly choose (a).

Intuition, as I see it, can best be described in the way Spinoza did, saying that it is a third kind of knowledge, independent of reasoning.

Further, Greer describes the 'educational dilemma', put forward by Fischbein:

"The educational implications of the existence of the two types of thinking - intuitive and formal - are clear.

The central issue is what Fischbein characterises as an 'educational dilemma'. In introducing any concept, it is natural to begin with simple, concrete, instances of that concept. The intuitive, primitive models which result may continue to exert a tacit influence when the concept has been made more general, abstract and formal. Experience makes it clear that it is not possible to avoid this by going straight to the formal, abstract version of the concept. So how can we deal with the dilemma?"

Commenting on this 'educational dilemma', I would remark that from my point of view such a dilemma does not exist. I cannot accept that intuitive notions should be considered 'primitive' or 'simple'. Spinoza's third kind of knowledge, described by him as knowledge of the essence of things, seems far from primitive. Rather, the above- mentioned, so-called 'formal, abstract version of the concept' is a primitive attempt to explain the inexplicable.

I would say that the concept of 'intuition' is so difficult to handle that a quantitative approach seems senseless. Let us look at Professor Grootendorst's example, demonstrated in figures 8 & 9 of Chapter VI, section 6.2. If we ask 100 people whether the assumption is correct, then probably 100 will (intuitively ?) state that it is correct. But what if 1 of those 100 says that it is not correct? Call this person: Q. It is useless to persuade Q that he is wrong, because then Q's intuition is 'helped', and thus distorted, in the right (?) direction. Referring to the overwhelming majority of yes-voters means a defeat, because then the intuition of a single person does not count anymore. Such a person Q, who will be considered as too stupid, or too stubborn to understand what the question was, will not be asked a second time to give a view. So far, so good. In most cases this will be the right approach. However, what happens when Q is indeed right ? He might be too intelligent to say yes to a faulty assertion.It is conceivable that Professor Grootendorst's statement is wrong. In Mathematics there are many examples of theorems, which were accepted as true for some time in the past but which finally turned out to be wrong. For this reason I think that it will not be feasible to accept 'intuition' as a member of the scientific family.

CHAPTER VII

TOWARDS A CONCLUSION

It may be worthwhile to consider my motives in designing an art-based education in higher geometry. Decades of mathematical and geometrical research show that there are enormous challenges. Therefore, it

might be useful to look at related subjects. Visual art is likely to provide a good approach to visual geometry since these two subjects have so much in common. Actually, this method of comparing visual geometry to a different subject is not unusual. The most traditional approach to visual geometry is the analytical one. Numbers are assigned to line segments to denote length and angles are determined by means of a similar process. Points are defined by means of a system of co-ordinates and are usually attached to a couple of co-ordinates.

The visual aspects of geometry are given less emphasis by the analytical system and are largely replaced by computational work.

However, geometry is not purely analytical and the visual aspects should not be ignored. They have, as I have demonstrated, important educational values. It is, for instance, regrettable that in order to apply analysis and computations to geometry, visual aspects have had to be distorted and deformed so that visual geometry is ignored. This happens, for instance, in the standard representation of a cube.

Visual art does not carry the burden of the relevance of formulae and computational work. So, contrary to analysis, visual art yields a much more free approach to visual geometry and pictures are allowed to be more natural. Moreover, the application of visual art provides a way for students to learn geometry, even for those who might never be competent at analysis but who nevertheless have a good spatial imagination and a talent for the study of geometrical topics.

It has to be mentioned that analysis and its application to geometry have already produced a kind of visual art. The standard representation, for instance, of a cube can be considered to have been largely deformed by the impact of analysis, but at the same time the distorted image belongs to a branch of visual art named 'Magic Realism'.

Because of the beauty of such images I do not wish to remove them completely, although I would characterise them as educationally invalid since their visual presentation might be an obstacle to the student in pursuing further studies. Beauty, however, is an inevitable companion of visual geometry and beautiful images should be handled carefully and not callously discarded. Thus, in some cases a distorted cube may be maintained for its outstanding beauty.

Furthermore, I have shown examples of the application of art to visual geometry.

There was Escher's Pond, which not only demonstrates the picture of a global straight line, but also forgoes the educational invalidity of the standard presentation, which is a line segment on a sheet of paper.

Summarising, my aim has been to design an art-based visual geometry which avoids educationally invalid geometrical representations. One additional aspect of geometrical visual art is that it makes geometry much more attractive. This will stimulate students to visualise geometrical concepts. It comprises, as I call them, the beauty spots of geometry. I provided an example, entitled 'Projections', which is an artistic view of a piece of Projective Geometry.

I have called the ignoring of global items in geometry "educationally invalid" because it prevents students from deepening their understanding of the subject. Thus a local geometry is needed in which global items are used properly. In Part II I will provide examples of this through a lesson on the topic of Duality with suggestions for teaching Group theory and Topology. Furthermore, there is a need to provide the student with more information on global geometry. A model of non-analytical visual geometry emerges in which local and global properties are strictly distinguished. This model should be accessible to students who are interested in geometry and who wish to drop the obligation to pursue analysis and computations. The model would be equally valuable for somebody following a specialist mathematics course. The model may be seen as educationally valid. The existence of such a model is in the interests of geometry education because it paves the way to a deeper knowledge of the subject.

It was at the beginning of Chapter I in Part I that I stated that it has become fashionable to look around you to detect geometrical objects rather than to consult textbooks on the subject. However, looking around you, you will see global geometry which is ignored by the textbooks in which mainly local geometry is treated. One of the aims of my thesis is to bridge the gap between these two kinds of geometry and point out the relationships between local and global geometry so that students may understand both branches better.

CONCLUSION OF PART I

At the end of Chapter I the 'Basic Paper' opened with the intention of designing an Art-based education in higher geometry. Visual art has indeed played a dominant role in my research, since it makes geometry so much more attractive. However, that is not the only reason. The power of art conveys the essence of geometry in a very short time.

The use of analysis has been dispensed with because for many students it brings an end to their geometrical studies. I can not agree with scientists who state that analysis has the final say in geometry. Nothing can replace visualisation.

This does not mean that I reject the use of analysis in the science of geometry, but it should not negate the experience of space which visual geometry can provide. The application of visual art as a means to convey scientific knowledge will not generally be welcomed. It may be felt as a transgression of subject borders. However, I have to emphasise that the design of an Art-based education in higher geometry is intended to provide a different view of geometry and its teaching. To achieve this, the influence of traditional mathematics has to be eliminated because otherwise the old, ossified patterns will easily reclaim their rights.

Current geometrical pictures are so distorted by the demands of analysis and computations that one can no longer speak of visual geometry. I have tried to introduce a new visual geometry which is artistically based rather than being an extension of arithmetic.

Analysis should be a gardener in the landscape of visual geometry but no longer the landscape-architect. For a good gardener is much better than a bad landscape-architect.

Here ends Part I of the Thesis.

PART II
of the Thesis

CHAPTER VIII

INTRODUCTION TO PART II

The second part of the thesis is dedicated to background and practical work. The first chapter is a continuation of the treatment of duality in Chapter IV. The concept of duality will be deepened and a new notion, the important notion of 'ideal' points, is discussed, which leads to a useful concept. The notion of ideal points plays an important role in the way we look at duality. Duality will mainly be considered as a local matter. The ideal point represents the global part of the theory of duality.

The theorem of Desargues can be proved by merely visual means so this is a moment to display a truly visual proof. Desargues' theorem contains much of the theory of duality; and the reverse theorem is the dual case. Pascal's theorem and its reverse, Brianchon's theorem, are also demonstrated to the student. These theorems are basic to the development of Projective Geometry.

In Projective Geometry there is the so-called Cross Ratio. It is a basic notion but it requires so many computations that it would disturb the progress of visual geometry if it was used.

The lesson on duality provides a model of how to teach geometry without the use of computation and formulae.

There is also a chapter on Group Theory in which some suggestions are made for teaching that subject. One can move (rotate or translate) a geometrical figure (for instance, a square or a tiling) so that the result of the move leaves the initial figure invariant. This is a limitation of the possible moves of the figure but it is visual geometry in the sense that the configurations are not distorted.

Further, there is Chapter X treating triangulations where the dissections of basic geometrical objects, like torus, projective plane and cylinder, are considered. The treatment is brief, but aimed at raising interest in the subject. These three subjects, Duality, Group Theory, and Topology (triangulations), demonstrate the level at which the activities of the students are assessed.

Next, in Chapter XI, there are short historical notes about the history of visual geometry at Primary Schools in The Netherlands.

Chapter XII shows a view of contemporary visual geometry as introduced in secondary schools in The Netherlands. The emphasis on the visual aspect of geometry has strengthened since the nineteen-sixties . The notion of 'lines of sight' is new and it is promising. This is also a result of the work of the Wiskobas team, which was active in primary schools in the nineteen-seventies in The Netherlands. In parts of secondary school education in the Netherlands (at the HAVO-schools) the role of computations and formulae has been reduced in favour of a more visual approach to geometry. (HAVO = Higher Comprehensive Secondary Education). This is a disadvantage for those students who want to pursue a technical course after their secondary school career.

There is a chapter on the Van Hiele Model in secondary school geometry. It provides a complete outline for the teaching of geometry to pupils aged 12 - 16 and it has proved to be successful in practice. The question is whether the Van Hiele Model may be seen as relevant for our Educational Geometry.

In Chapters XIV, XV and XVI ideas from ancient Greece play an important role. Antiquity appears to have a certain relevance nowadays. I have considered the question whether one may involve ancient philosophers in contemporary research. If we make an unqualified use of ancient ideas, it is possible that we may change the ideas of the ancients beyond recognition; and it may not be possible to borrow ideas from the ancients without being an expert on ancient history and dead languages.

I am aware that I have applied Platonic and Euclidean notions in a way of which the ancients might not approve. Nevertheless I feel that ancient Greek concepts are still so fruitful that it would be a waste not to use them.

At the end of Part II there is a final conclusion.

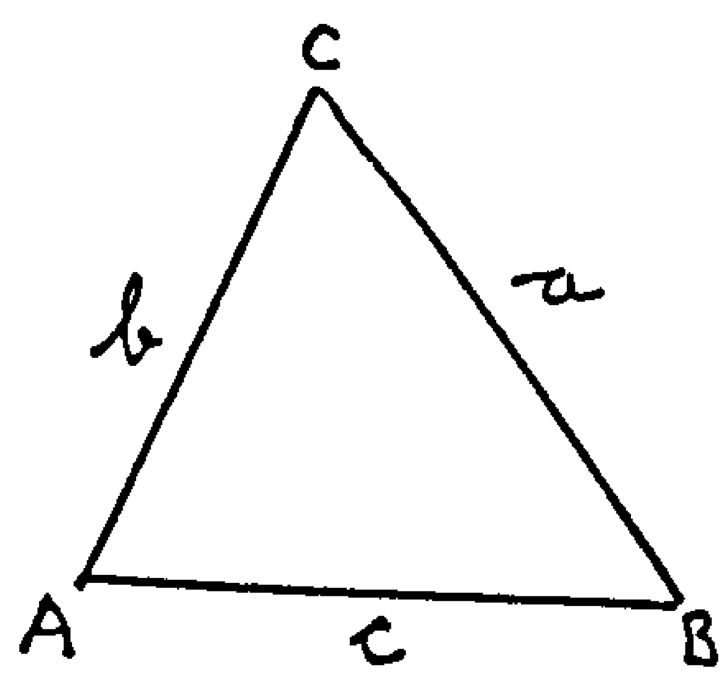
8.1 DUALITY

Duality is a subject which is very suitable for enhancing the understanding of geometry without carrying out computations or applying analysis. One of the items of duality is the study of a relationship between points and straight lines. A point is then considered as an intersection of two straight lines. Straight lines will be denoted simply by 'lines' in the following.

A problem arises when two lines must be seen as parallel because they apparently do not intersect. The mutual equivalence between lines and points seems to be invalid in such cases. The assertion: 'two lines intersect at one point' ceases to be true. There is also a statement: 'two points are connected by a straight line'. Is that always true?

We will start generally by raising the question in what way points and lines are assigned to each other by duality. Take, for instance, a triangle ABC with edges a, b and c (figure 1).

figure 1



One may conceive that vertex A is dual to edge a and analogously B is dual to b and C to c.

Let us now suppose that edges a and b (AC and BC) happen to be parallel. Then point C disappears. Does any duality remain between a line (AB) and a non-existent point C? (see figure 2)

figure 2

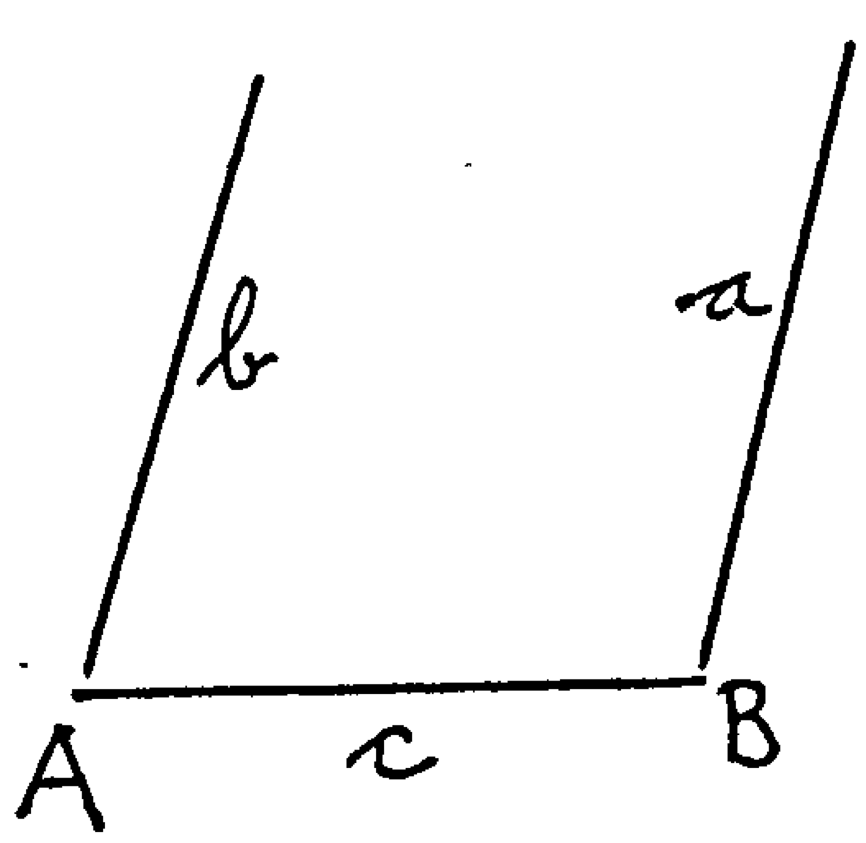
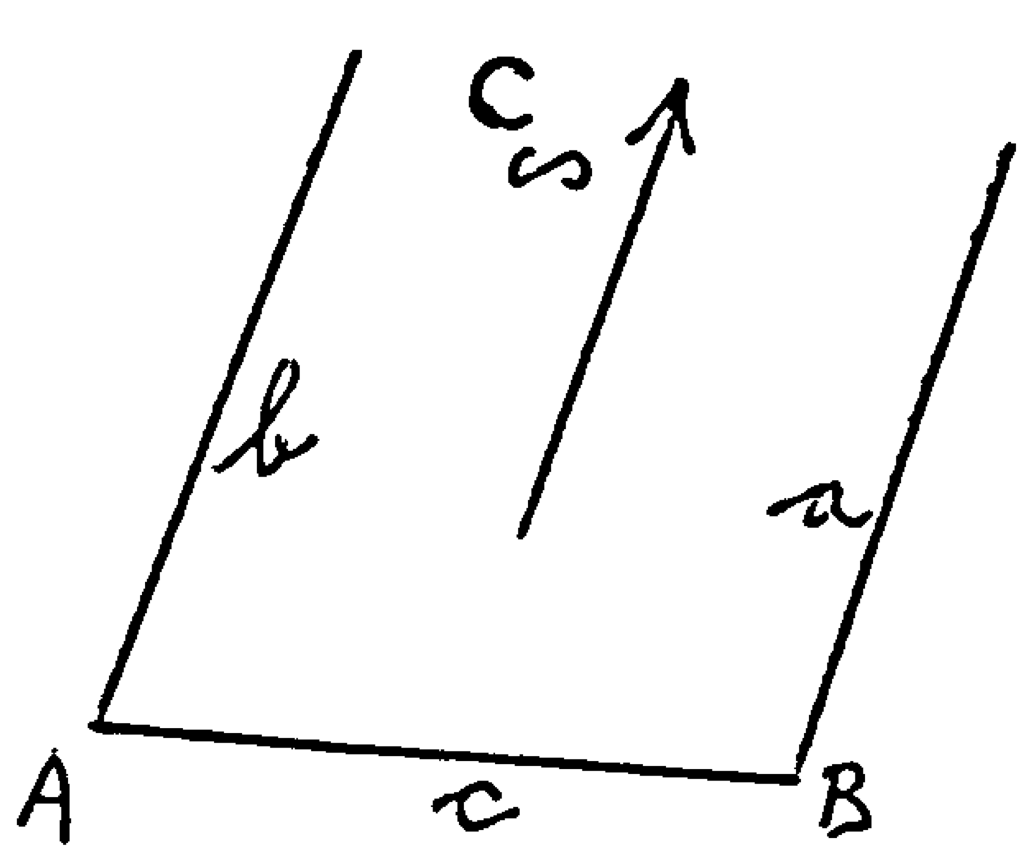


figure 3



In Projective Geometry, it is assumed that parallel lines such as a and b in figure 2 meet at a so-called ideal point C_∞ ; and that ideal point C_∞ is considered as dual to the line c (= AB)

It is of great importance to answer the question whether the assumption of such 'ideal' points is educationally invalid or not. In the following we will discuss that issue.

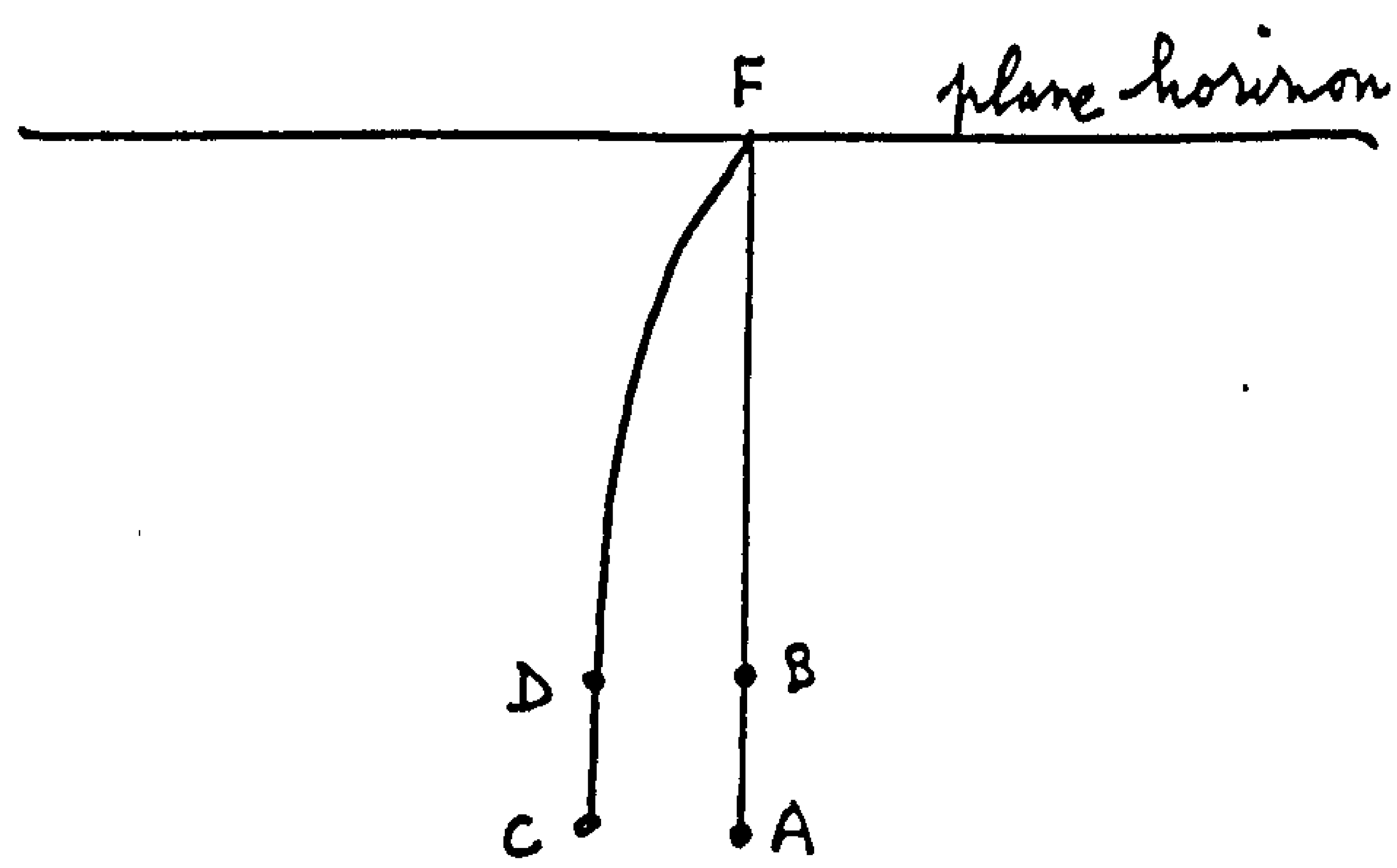
Let us consider the two dual statements:

(1) Two straight lines intersect at one point.

(2) Two points are connected by one straight line.

Let's recall the image of two parallel straight lines in a plane (figure 4).

figure 4



We remember that in the situation of figure 4 we are standing on a plane of infinite width, looking towards the horizon. The plane seemed to have the shape of a saucer; and when we followed two parallel straight lines with our eye, running to the horizon, the picture of figure 4 would emerge.

The problem is now: do the lines AB and CD intersect at a real point? By a 'real point' we mean a point which we can pinpoint somewhere, which could literally be touched with a finger.

Visually point F is the point of intersection of the lines AB and CD. But no point of the lines AB and CD is represented by F. A possible point of intersection of the lines AB and CD cannot be point F. The point, indicated by F, is only a visual point and it cannot be located; not on AB; not on CD and not on the horizon. The horizon itself is only visual and it is built up of non-existent visual points. So point F appears to be quite 'untouchable'.

This kind of non-existent, untouchable points are required to make assertion (1) true. In Projective Geometry it is assumed that the lines AB and CD of figure 4 do intersect at a point G, which is called the 'ideal point' of the line AB and of the line CD. This ideal point G is not located anywhere and can not be pinpointed. We may not even assume that it is infinitely far away. It simply has no place in space.

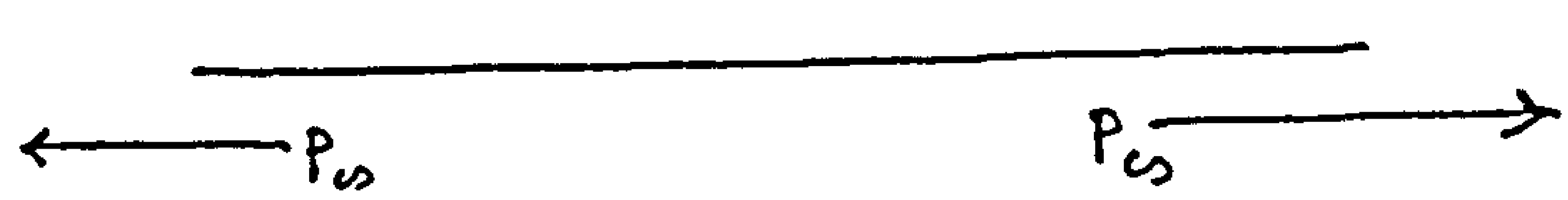
From the above it follows that every set of mutual parallel lines shares just one ideal point. Such an ideal point is often thought of as the direction of a certain set of parallel lines. Ideal points are used in Projective Geometry as equivalent 'really existing' touchable points. We have to acknowledge here that ideal points are not accepted in every branch of geometry but they are used in Projective Geometry.

When two points of a line are given, we can always draw a straight line connecting them. If only one point is given but we know the direction of the line, then we are able to draw the (straight) line through that point with the given direction. So the direction of the line is considered as the second point on it to make it possible to draw the whole line.

In our new terminology this means that we can draw a straight line when a real point is given plus the ideal point of the line. This is a generalisation of statement (2): 'Two points are connected by one straight line'. One of the given points may be an ideal point. The generalisation goes even further. Given two ideal points, then there is a straight line supposed to connect these two ideal points. That straight line is called the 'line at infinity'. Every plane is equipped with a 'line at infinity' and all the 'lines at infinity' in R^3 form the 'plane at infinity' of the three-dimensional space R^3.

In a plane (or in R^2) the line at infinity intersects every 'real' line in an ideal point. The ideal point, as we noted, cannot be located but sometimes it is thought to be infinitely far away to the left as well as to the right (figure 5). The left infinity is identified with the right infinity. This is necessary to fulfil the requirements of statements (1) and (2) above.

figure 5



Is the assumption of the existence of ideal points educationally valid? Without these ideal points, duality would be no more than a local phenomenon. Ideal points remind us of the existence of global straight lines without actually portraying such global figures.

It is my opinion that ideal points may not be considered educationally invalid. In figure 4 we have indeed observed two parallel straight lines, running towards the same 'dot', so that it seems natural to accept a point of intersection of two parallel lines.

Let us now recall the presentation of a global straight line by just two points (figure 6). See Chapter III, section 3.3,

figure 6

The global line of figure 6 is well represented too when we suppose that one of the points, say B, is an ideal point. Then it can be compared to the 'untouchable' point F of figure 4.

In a way the geometry of ideal points and the line at infinity form a shadow-geometry of, for instance, the geometry of the plane. A rotation of the plane corresponds to a translation of the infinite line. A reflection of the plane in an arbitrary straight line on that plane corresponds to the reflection of the line at infinity with respect to the ideal point of the line of reflection. These new 'ideal' points and 'lines at infinity' are also tools to handle visual geometry.

The application of Duality has a clear advantage. Geometry is largely simplified because theorems which are valid keep their validity when they are dualised, if they can be dualised.

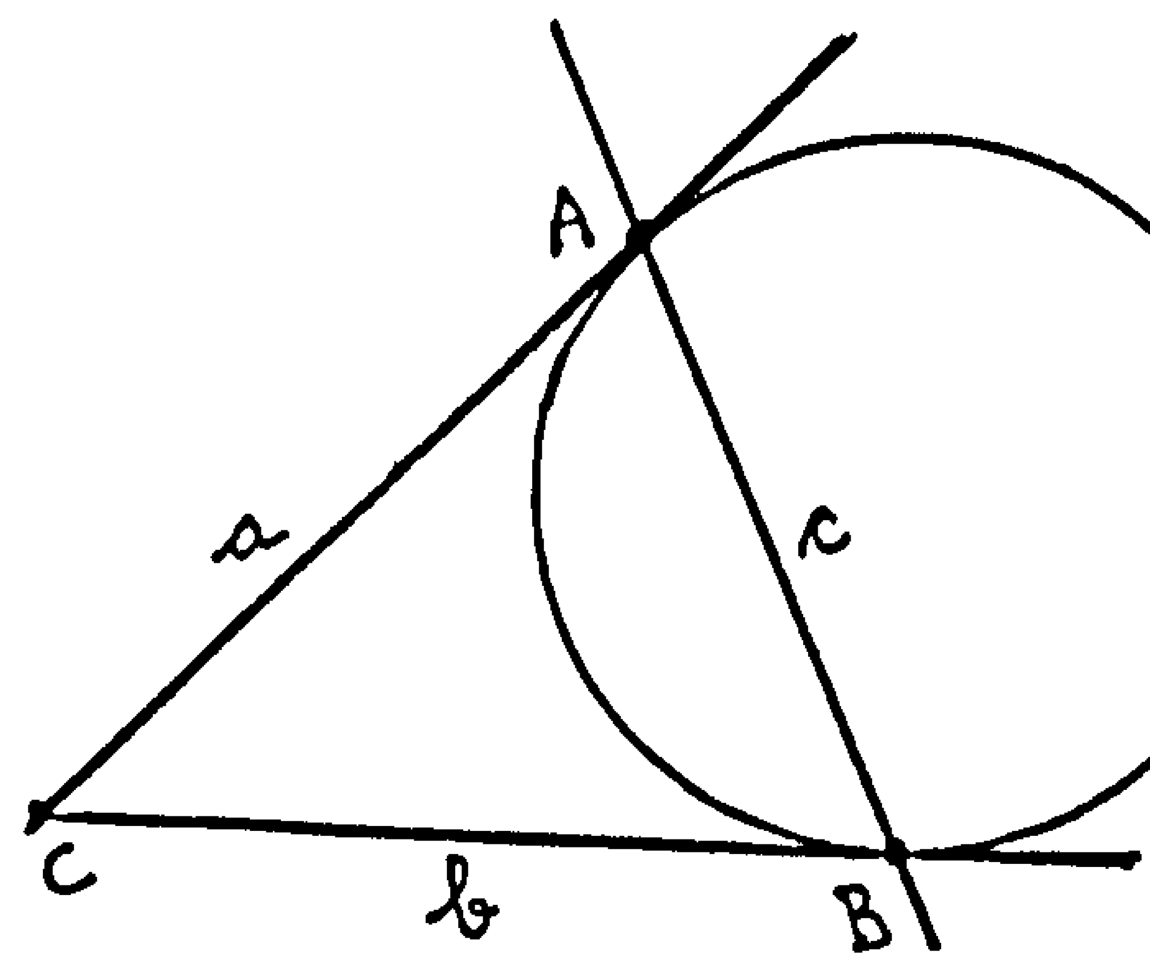
Examples of this are given in the theorems of Desargues, Pascal, Brianchon and others and will be demonstrated.

Duality is not a simple matter. From history we learn that two excellent French mathematicians, Gergonne and Poncelet, had a long and bitter disagreement about the accurate and correct meaning of the concept of Duality. It is described in the Historical Notes (Chapter VIII, section 8.3).

An example of duality in a plane is the assignment of points to lines so that a point on the circumference of a given circle is assigned to the line tangential to the circle at that point.

This is shown in figure 7.

figure 7



A is assigned to a, B to b, and C to c (figure 7).

On the following pages more exercises will be given - including applications of the theorem of Pappus (300 AD.), Pascal (1623 - 1662), Brianchon (1783 - 1864) and Desargues (1593 - 1662).

The reader is not required to do the exercises 1-10 in the sections 8.2 and 8.4 of Chapter VIII before reading on. The exercises are no more than suggestions for a curriculum and no essential information will be missed by skipping them.

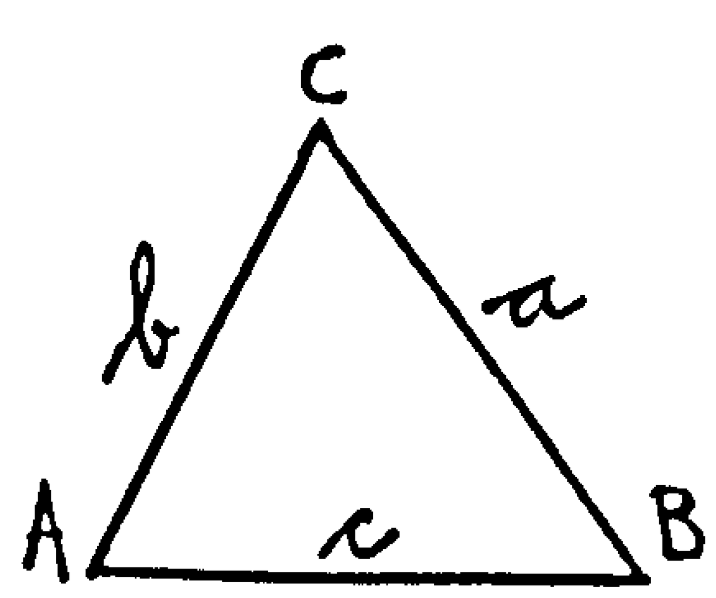
8.2. Lesson on Duality.

Some geometrical objects are self-dual. A triangle has 3 vertices and 3 edges so the dual of a triangle is a triangle again. To each vertex the opposite edge is assigned.

Have a look at figure 1. The vertices A, B and C and the edges are a, b and c. A is opposite a, B opposite b and C opposite c.

Is this the way duality is organised or is it possible to assign points to lines in another arrangement?

figure 1



In figure 1 the self-duality of the triangle has been applied to itself instead of to a second triangle. We will now assign points to lines in the case of two different triangles. These two may be contracted to one and then a case of self-duality arises. Look at the figure 2 and 3.

figure 2

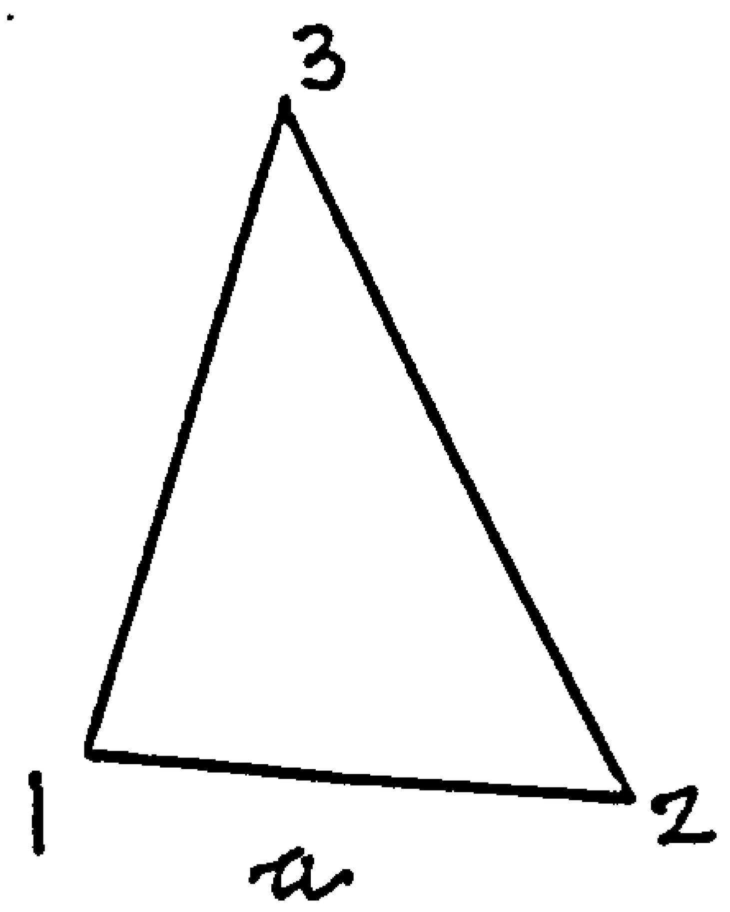
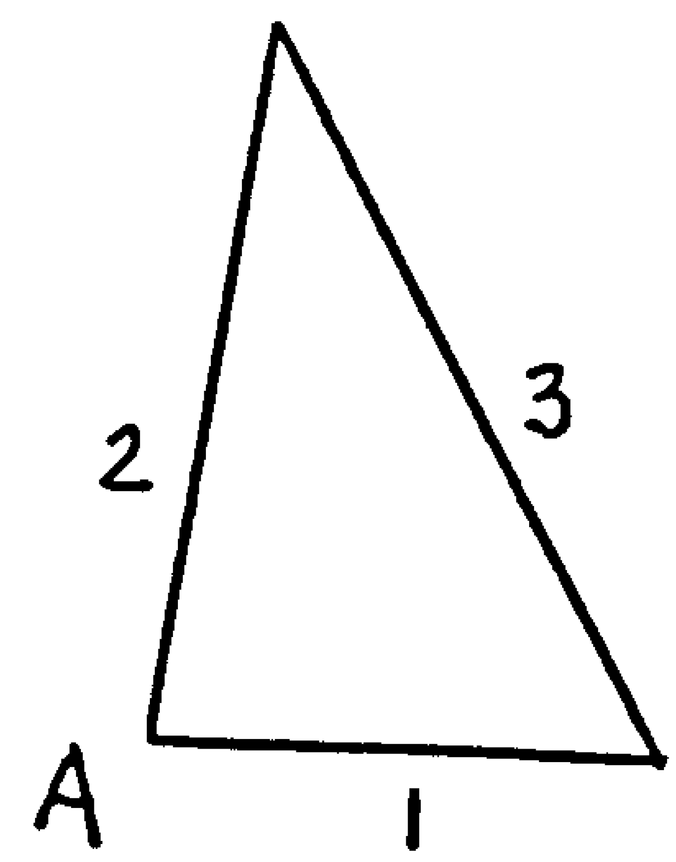


figure 3



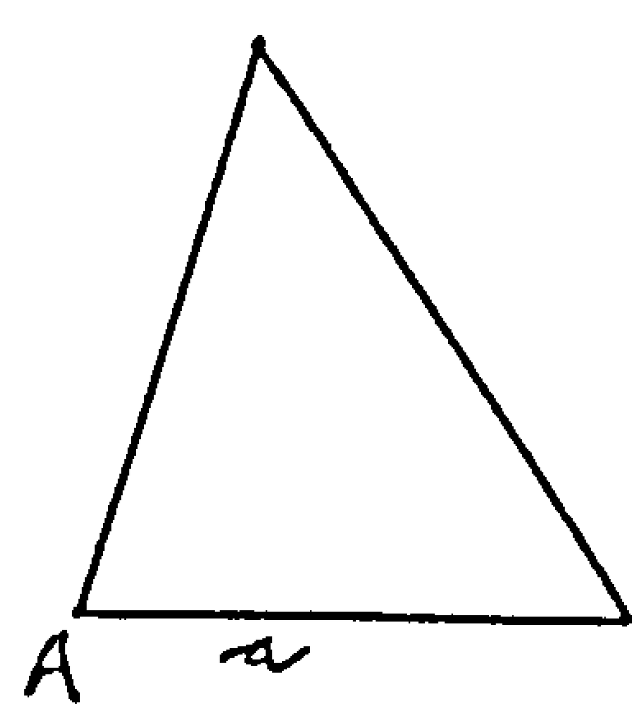
In figure 2 the vertices are numbered 1, 2, and 3. In figure 3 the edges are numbered 1, 2, and 3.

We now apply duality in the sense that (1,2) in figure 1 denotes the edge connecting the vertices 1 and 2 while (1,2) in figure 3 denotes the point of intersection of edges 1 and 2.

In figure 2 we call (1,2) edge a and in figure 3 we call (1,2) vertex A.

Contracting figure 2 and figure 3 to one triangle in figure 4, we get:

figure 4



In figure 4 vertex A is not opposite edge a.

EXERCISE 1. Assign appropriate characters to the remaining vertices and edges of the triangle in figure 4.

EXERCISE 2. Assign numbers to the vertices of figure 5 and to the edges of figure 6 so that figure 7 is the result.

figure 5

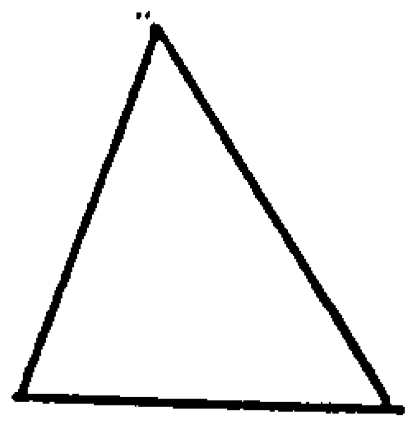


figure 6

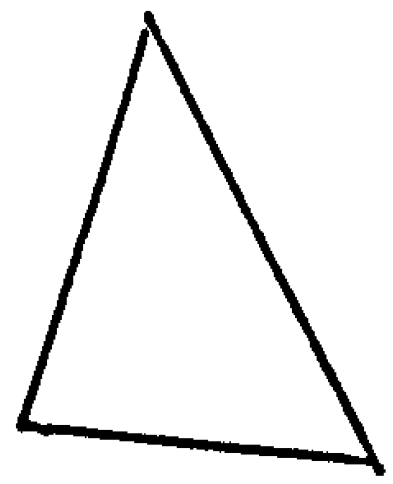
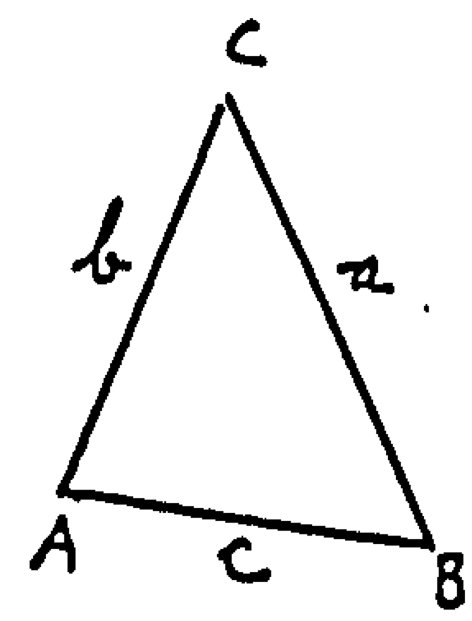


figure 7



So the assignment of points and lines is dependent on the numbers placed at vertices and along edges. This is also true for the tetrahedron which is self-dual.

EXERCISE 3. Place numbers along the edges in figure 9 so that vertices of figure 8 are assigned to opposite faces.

figure 8

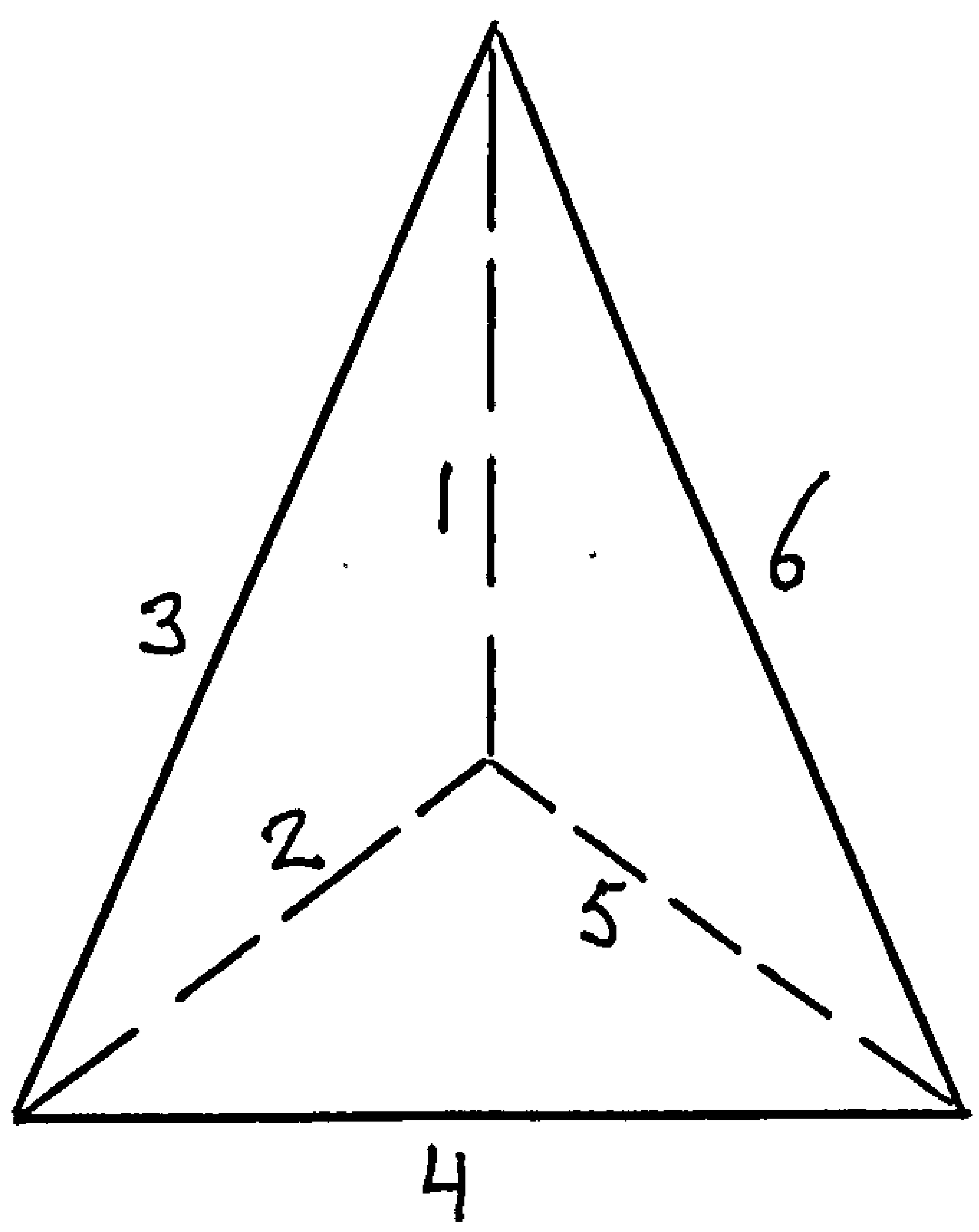


figure 9

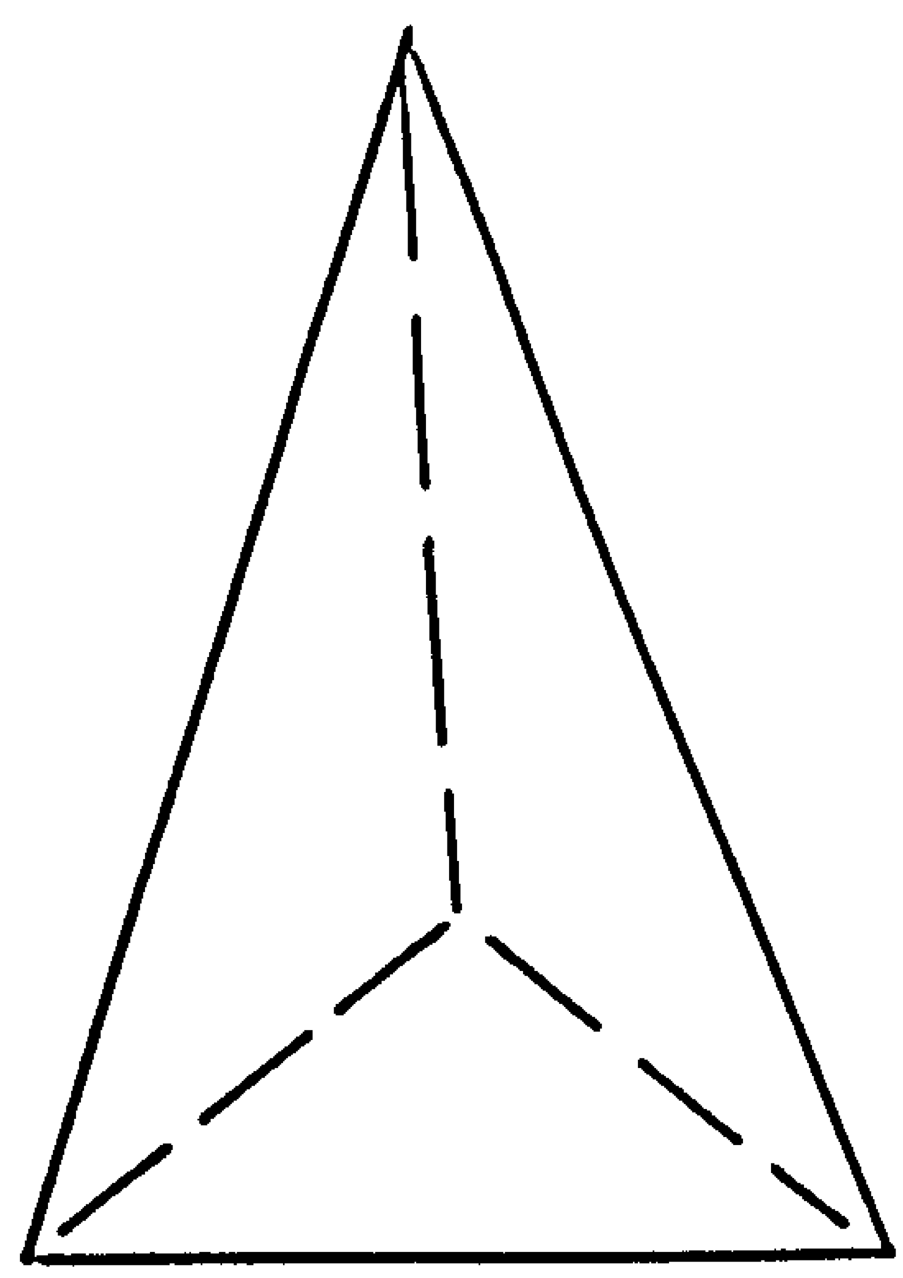


figure 10

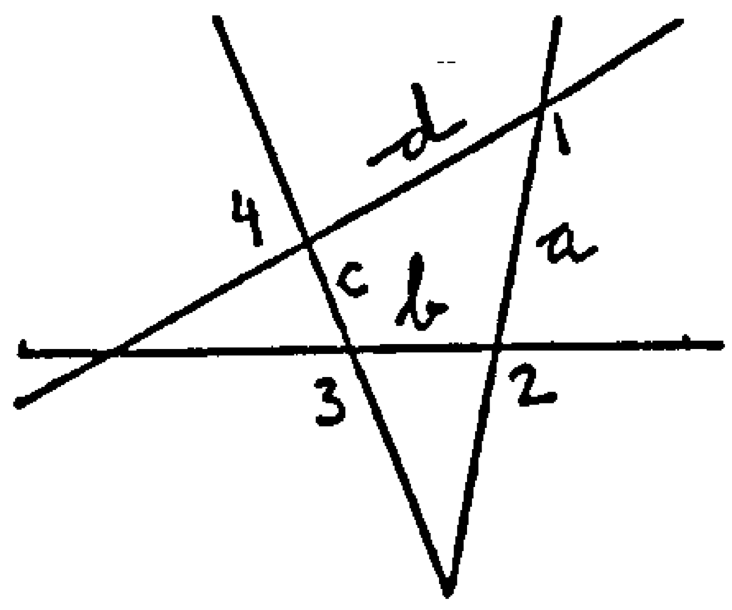


figure 11

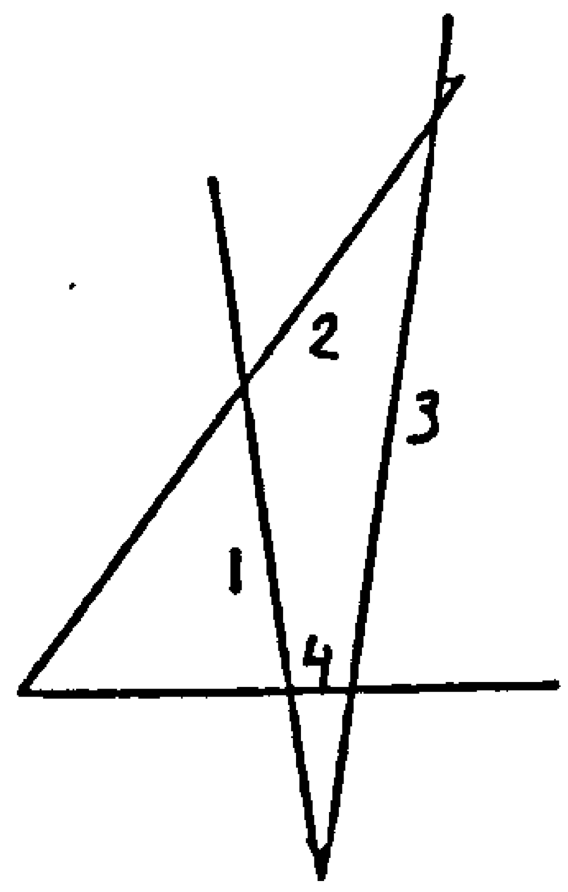
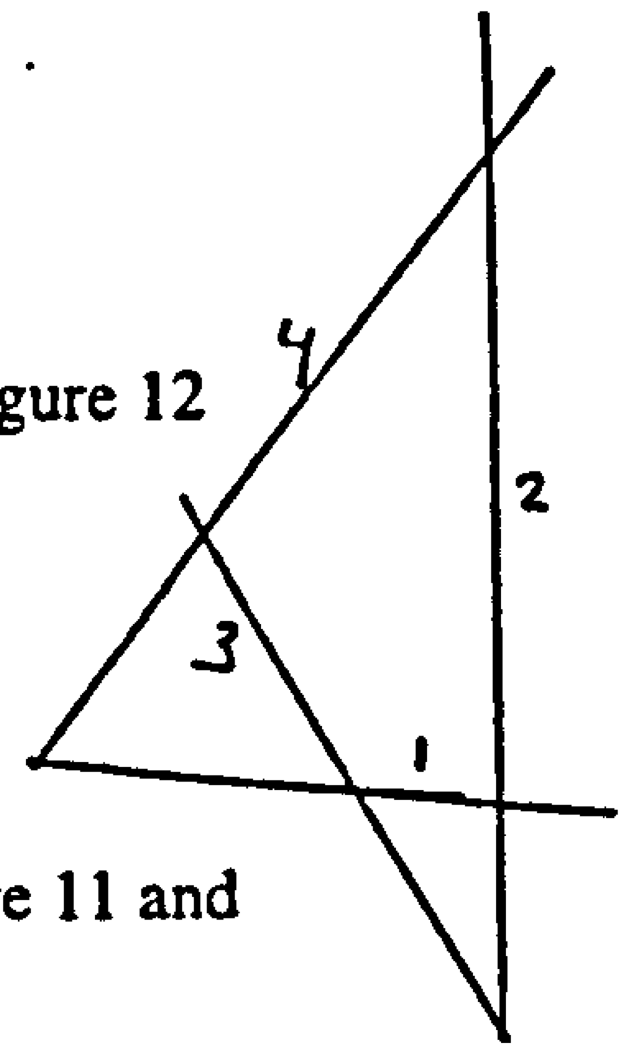


figure 12



EXERCISE 4. Figure 10 is dual to figure 11.

Figure 10 is dual to figure 12.

Place appropriate characters at the vertices of the quadrangles of figure 11 and figure 12.

figure 13

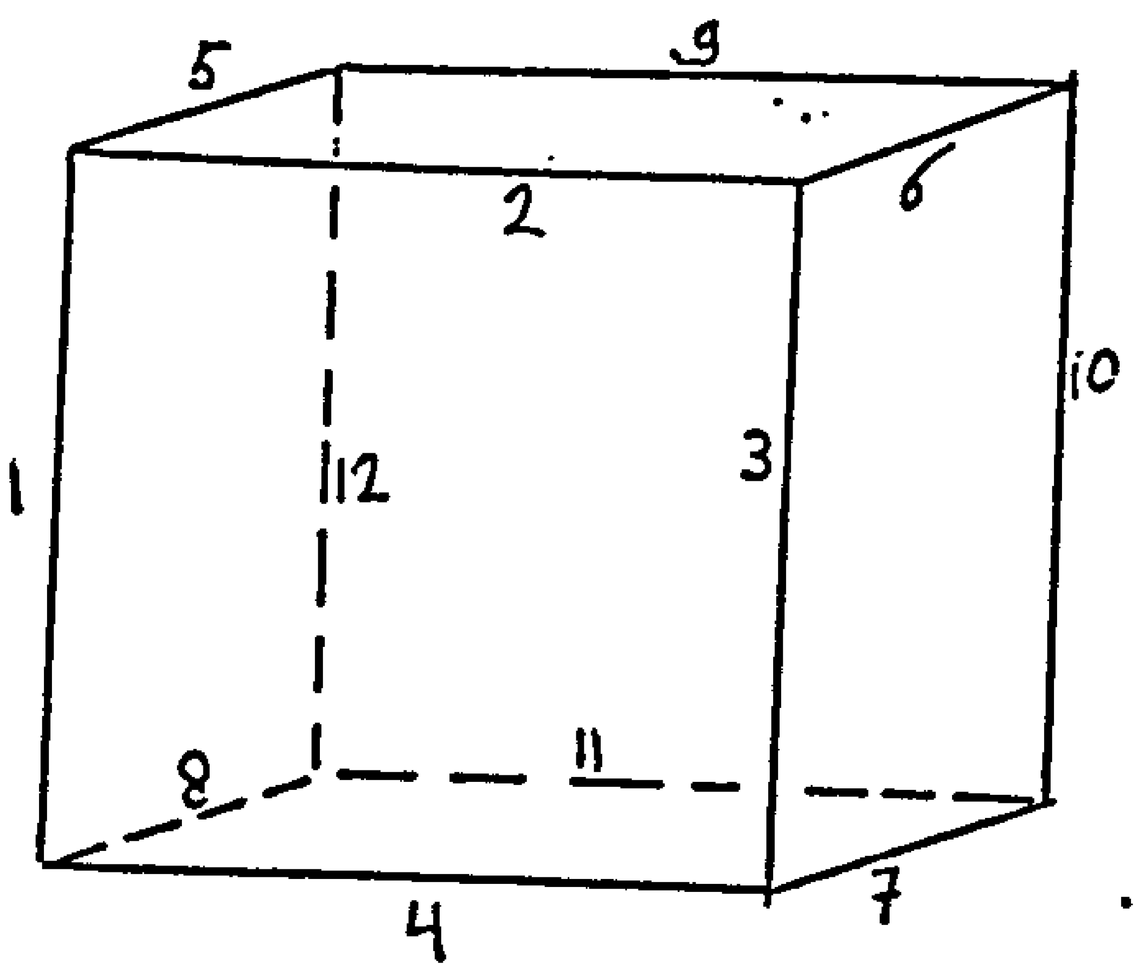


figure 14

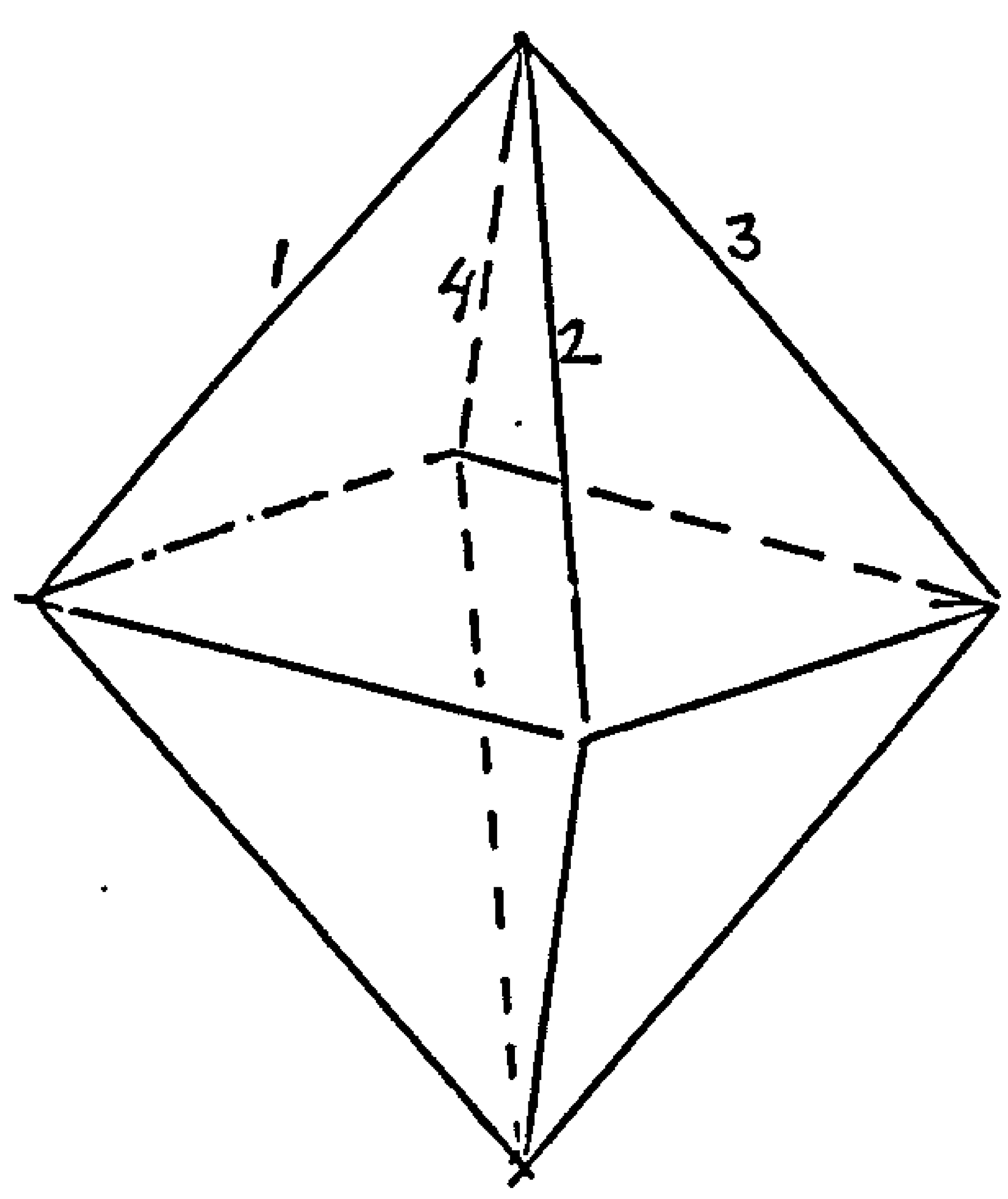


Figure 13 is dual to figure 14; edges are dual to edges. Vertices are dual to faces.

EXERCISE 5. Complete the numbering of the edges of figure 14.

8.3. Historical Notes

Duality was developed much further in the first half of the nineteenth century after a fight between two French mathematicians. Jean Victor Poncelet (1788 - 1868) received his mathematical education at the Ecole Polytechnique in Paris where he was a student of Gaspard Monge (1746 - 1818). He became an officer in the army of Napoleon and was made a prisoner of war at Krasnoi in Russia. From there he had to walk to the stronghold of Saratov. He describes the journey as follows. "Dressed in the remainders of a French uniform, eating the black bread of the Russian farmers, one had to walk the long stages which separated Krasnoi from Saratov; icy and silent planes in that fatal and exceptional winter of 1812 where temperatures were felt which made the mercury in the thermometer curdle."

Poncelet had to keep his spirits high to escape from mental destruction. He tried to reconstruct the lessons he had received from Monge in Paris but he had forgotten almost everything. Trying to recollect issues from his memory, he developed new ideas about duality and Projective Geometry.

Monge was an authority on Descriptive Geometry. This branch of geometry is related to the architecture of fortresses. Monge joined Napoleon's army to Egypt and he became Earl of Peluse and a senator but after Napoleon's defeat all his titles were abolished.

Joseph Diaz Gergonne (1771 - 1859) began his career as an artillery-officer but became a lecturer in mathematics. He was a professor at Montpellier in France and editor of the "Annales mathematiques" which was established by him and named after him. Gergonne had also discovered new items about Projective Geometry and duality but slightly differently from Poncelet's concept. Between Poncelet and Gergonne a bitter struggle emerged concerning the right interpretation.

After Poncelet's return to Paris in 1814, he delivered a treatise about his new ideas and concepts at the French Academy of Science. Three members of the Academy, Arago, Cauchy and Poisson, were appointed to report on Poncelet's treatise and they ended up doubting the validity of his principles. Poncelet was highly offended by this verdict.

Duality and Projective Geometry were developed further by Jacob Steiner (1796 - 1863).

8.4. Theorem of Desargues

Long before the nineteenth century certain theorems already existed which could be dualised. One of those theorems was developed by Desargues (1593 - 1662). He was an army-officer under Richelieu. He wrote about perspective and he is considered to be the founder of Projective Geometry.

In figure 15 his theorem is depicted. The two triangles ABC and DEF are called perspective because seen from P they cover each other visually and completely. P is called the centre of projection. ABC is a projection of DEF but one can also say that DEF is a projection of ABC from the centre P.

We can see something very remarkable here: the two planes in which ABC and DEF intersect along a line λ. We note that AB and DE intersect on λ ; also AC and DF intersect on λ and finally BC and EF intersect on λ too.

This is Desargues' theorem: <u>If two triangles are perspective, then the corresponding edges intersect at three points which lie on one straight line.</u>

(corresponding means: being projections of each other from P; perspective means: corresponding vertices or edges cover each other seen from P).

EXERCISE 6. Prove the theorem of Desargues with the help of figure 15.

There is also a dual version of Desargues' theorem. If one of the versions has been proved, then it follows automatically that the dual version holds good as well.

EXERCISE 7. Dualise Desargues' theorem.

Note: The dual of Desargues' theorem is its reverse. No proof is needed for this reverse theorem because the original theorem has been proved.

It is worthwhile to give an elaboration of the concept of Duality.

In figure 16 a circle is drawn with points numbered 1, 2, 3, 4, 5 and 6 on the circumference.

On a circle a point is dual to the tangent line to the circle through that point. So the tangent lines can be numbered 1, 2, 3, 4, 5, and 6 as well.

The point of intersection of two tangent lines (for instance 1 and 2) is denoted by (1,2).In this way we can denote the points of intersection in figure 15 by (1,2), (2,3), (3,4),(4,5), and (6,1). 'Opposite' intersection points are:

(1,2)	and	(4,5)	connected by straight line a
(2,3)	and	(5,6)	connected by straight line b
(3,4)	and	(6,1)	connected by straight line c

The remarkable fact is that the straight lines a, b and c go through one point. This is always true although the points 1, 2, 3, 4, 5, and 6 may be chosen arbitrarily on the circumference of the circle.

Theorem of Brianchon: If six points are arbitrarily chosen on the circumference of a circle, then the straight lines, which connect opposite points where tangents through those six points intersect, will go through one point.

Note: the above theorem of Brianchon is a simplified version of a more comprehensive theorem.

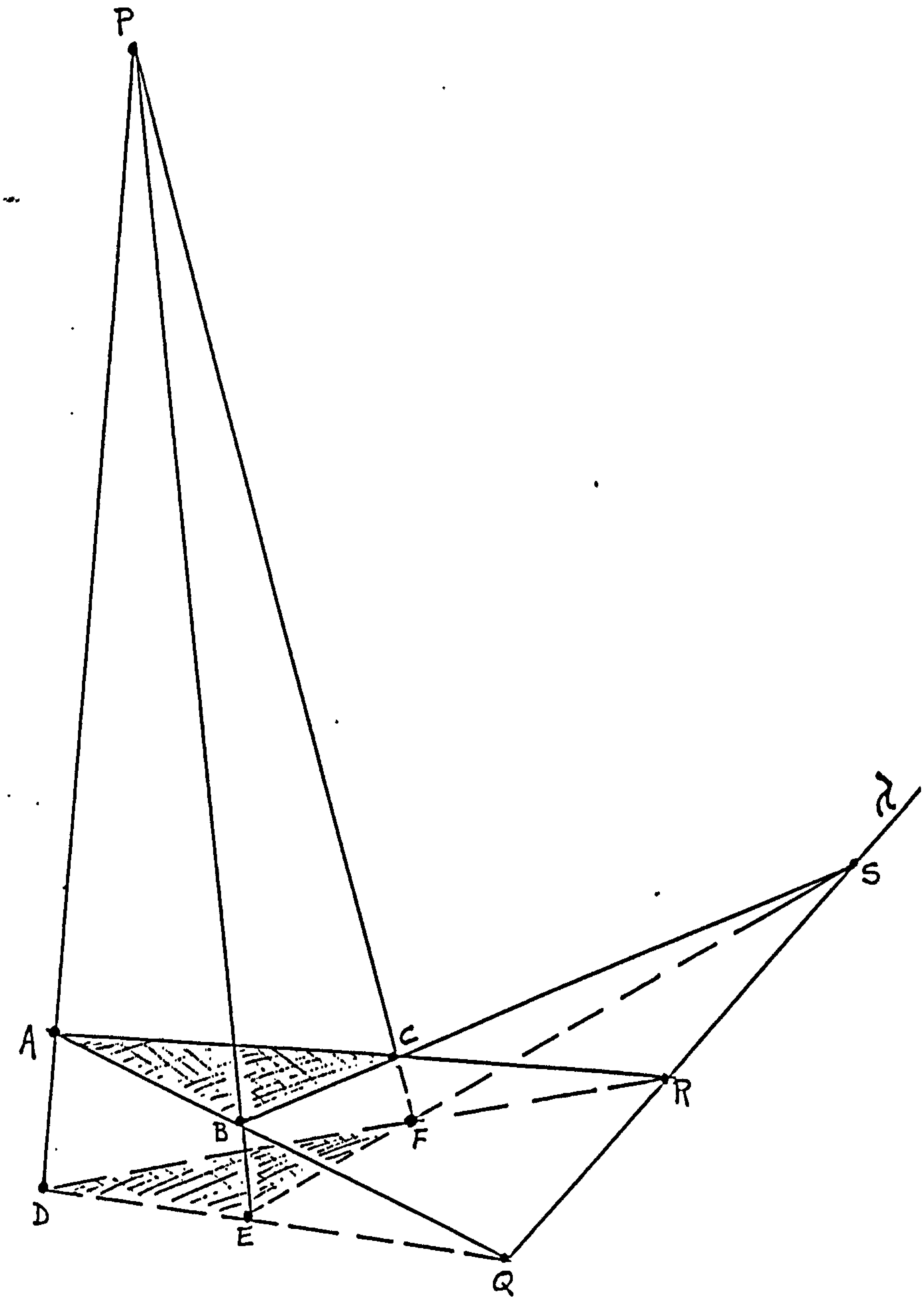
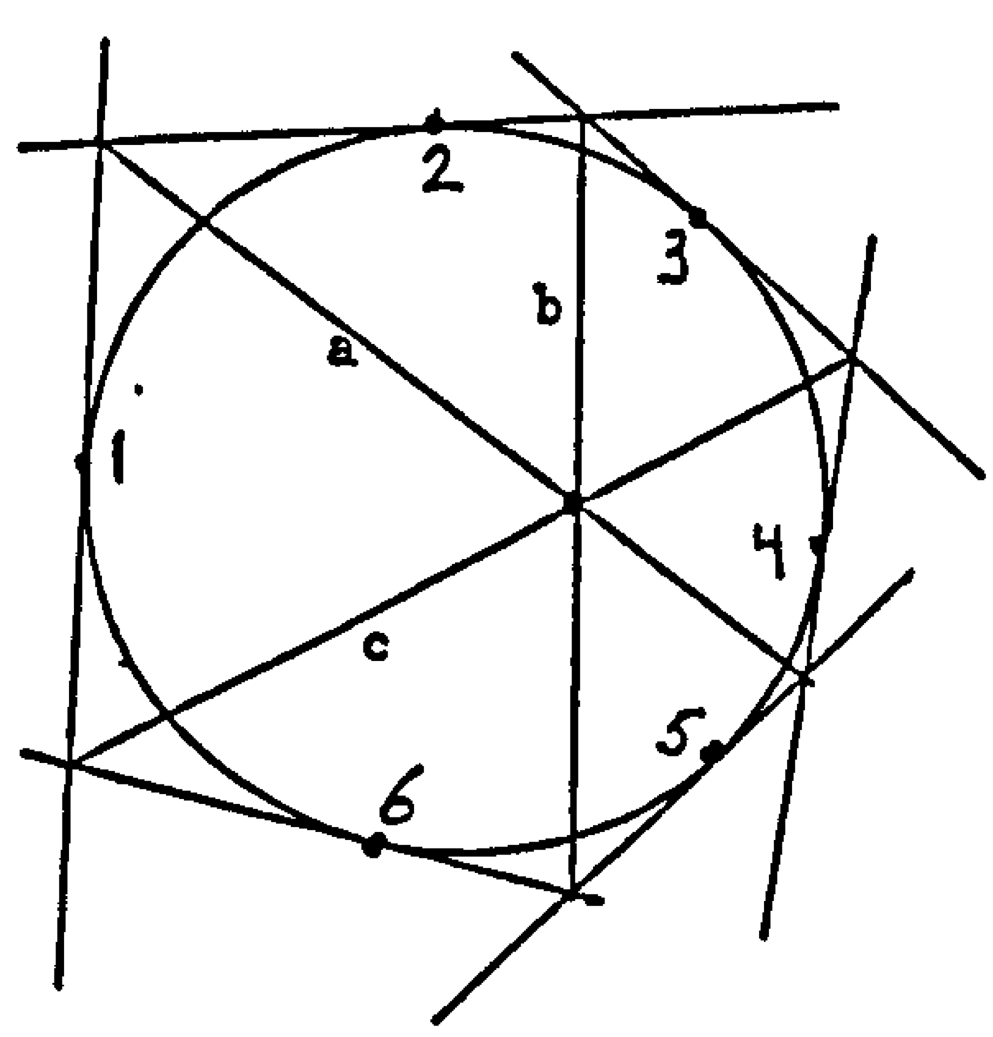


figure 15

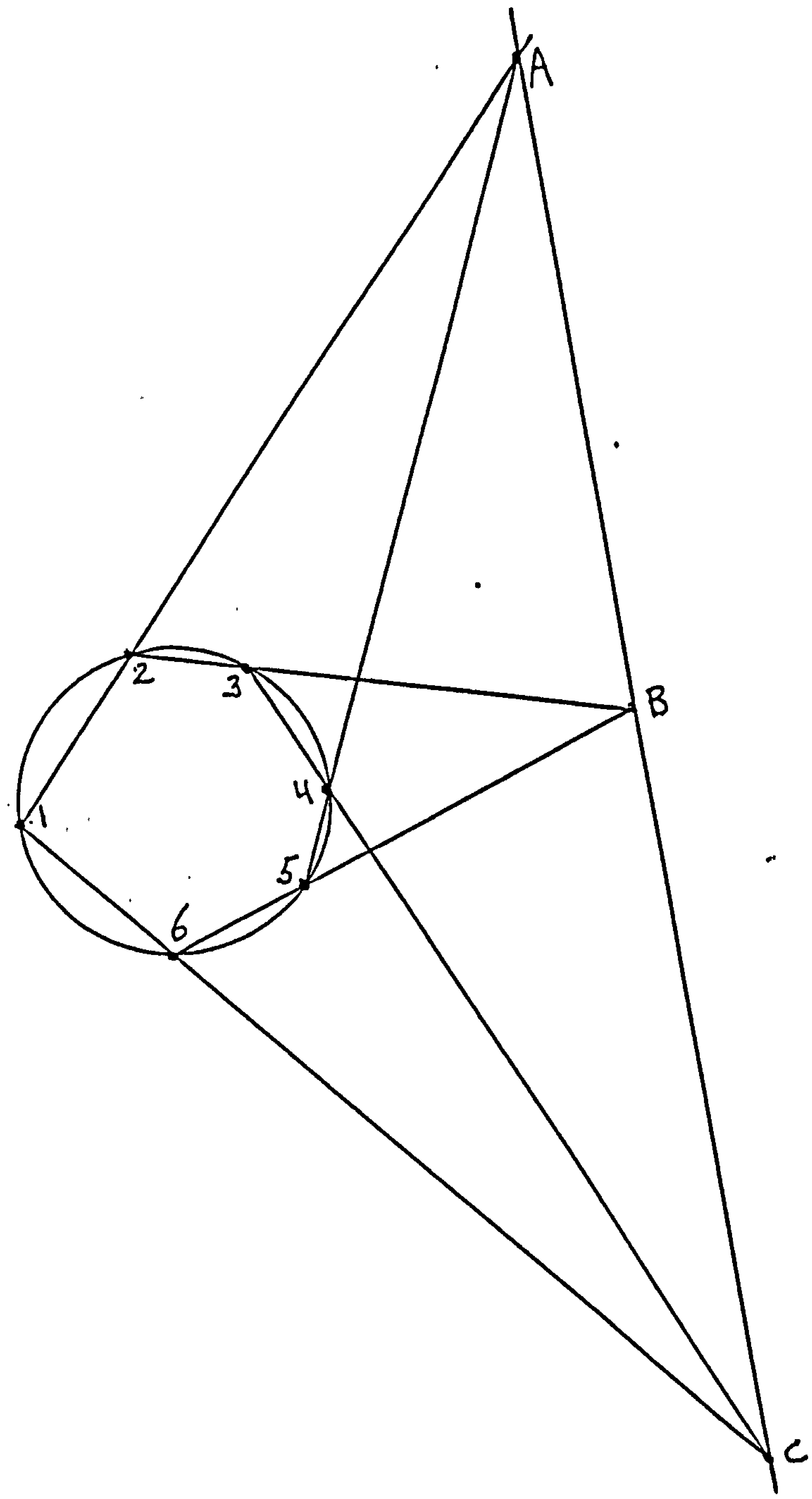
figure 16



EXERCISE 8. State the dual theorem of 'Brianchon', using the circle of figure 17.

Remember: <u>Points on the circumference of a circle are dual to the tangent lines to the circle through those points.</u>

figure 17



This exercise is an application of Pascal's theorem.

We continue the lesson on Duality. In figures 16 and 17 we saw a hexagon. The hexagons looked circular which means that the edges intersect outside the circle or on the circumference.

We start a new numbering of the figures.

For instance, in figure 1 no two edges intersect in the interior of the hexagon. In figure 2 one cannot distinguish an interior or an exterior of the hexagon; nevertheless we have also six vertices 1, 2, 3, 4, 5, and 6 in figure 2. The point of intersection of the lines (1,2) and (5,6) is not considered as a vertex of the hexagon in figure 2.

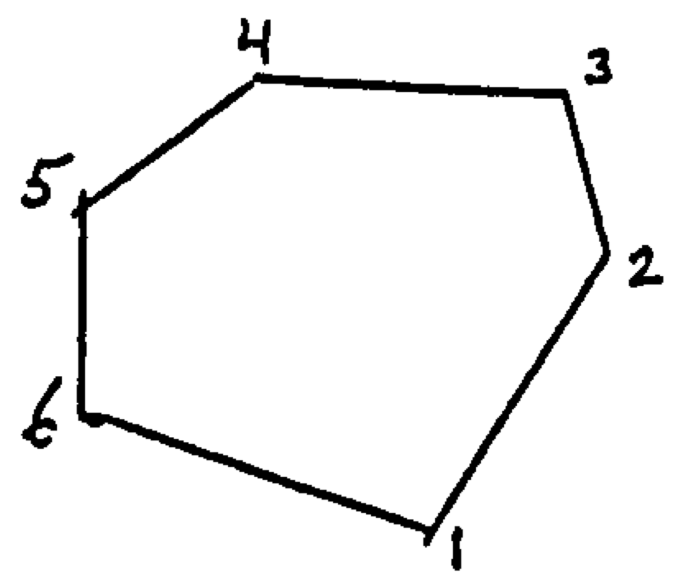


figure 1

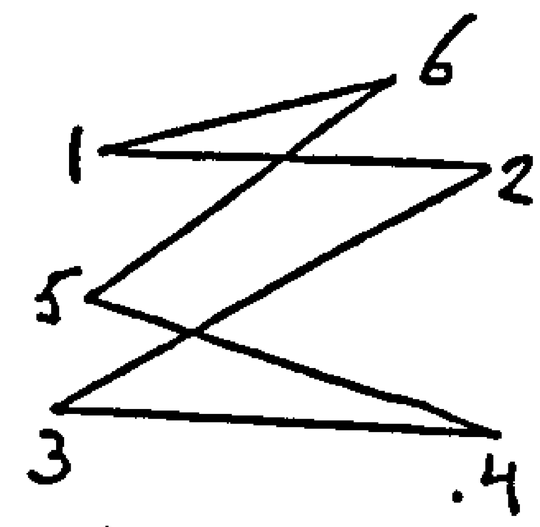


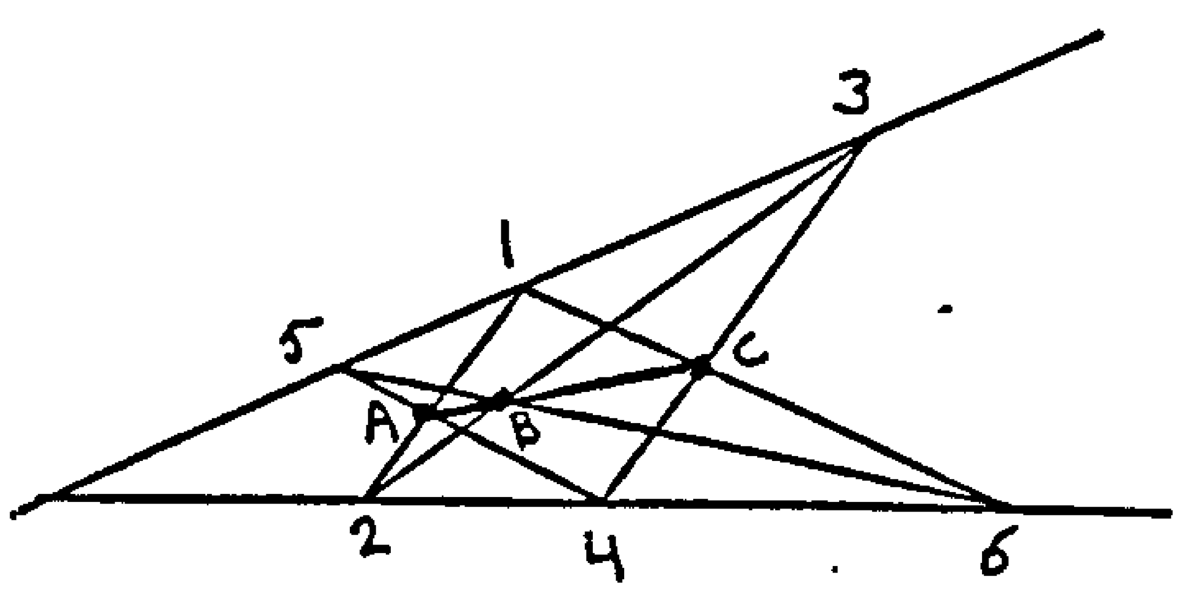
figure 2

The edges of the hexagons in the figures 1 and 2 are denoted by (1,2), (2,3), (3,4), (4,5), (5,6), and (6,1). Lines like (3,6) are considered as diagonals.

We might conceive that the vertices of the hexagon in figure 1 lie on a circle. We will picture the vertices of the hexagon in figure 2 on two intersecting straight lines (figure 3). The remarkable fact is that the points A, B and C, where opposite edges of the hexagon in figure 3 meet, lie on one straight line.

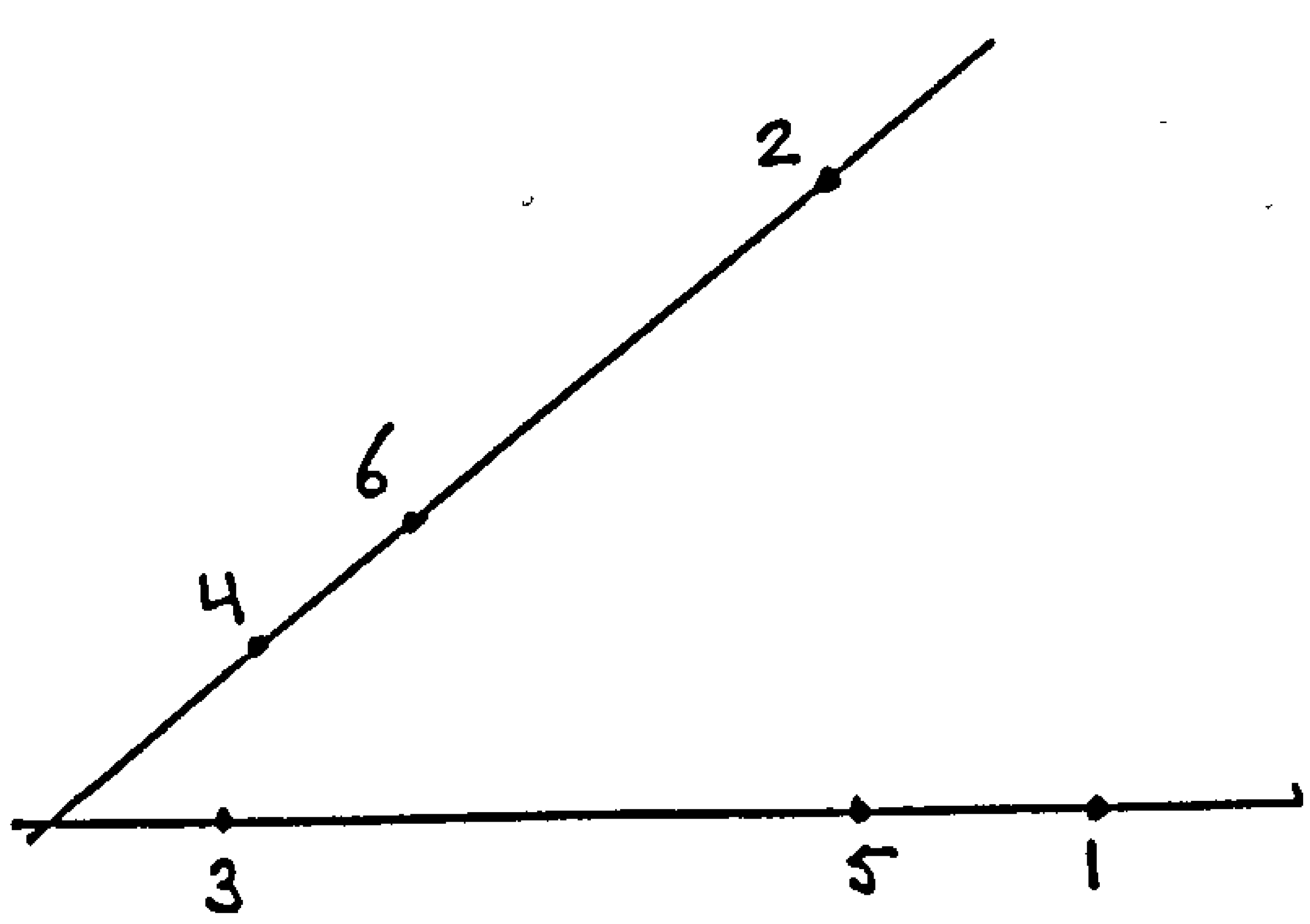
This is the theorem of Pappus, an ancient Greek geometer. Sometimes that theorem is referred to as: 'Pappus-Pascal'.

figure 3



EXERCISE 9. Construct the accurate positions of A, B and C, which are points of intersection of opposite edges of the hexagon (figure 4).

figure 4

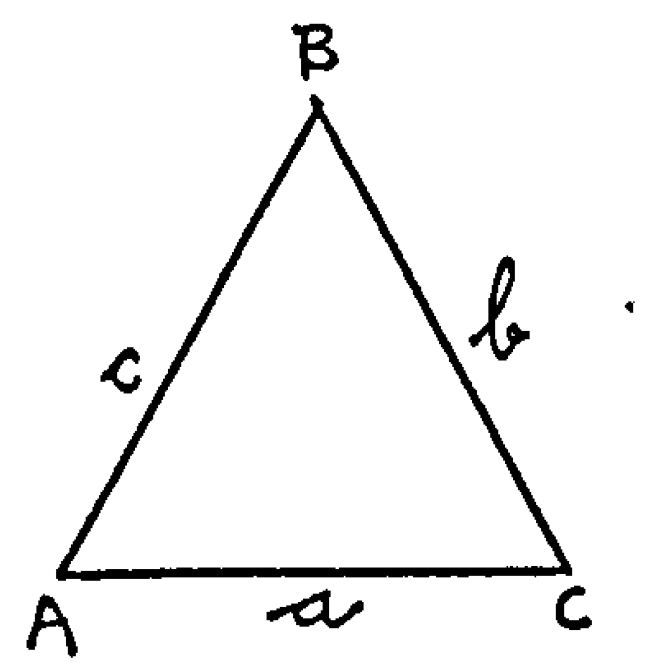


Pappus' theorem reads: if the vertices of a hexagon lie on two straight lines such that three lie on one and the other three on the second straight line, and these two straight lines do not contain an edge of the hexagon, then the points of intersection of opposite edges lie on one straight line.

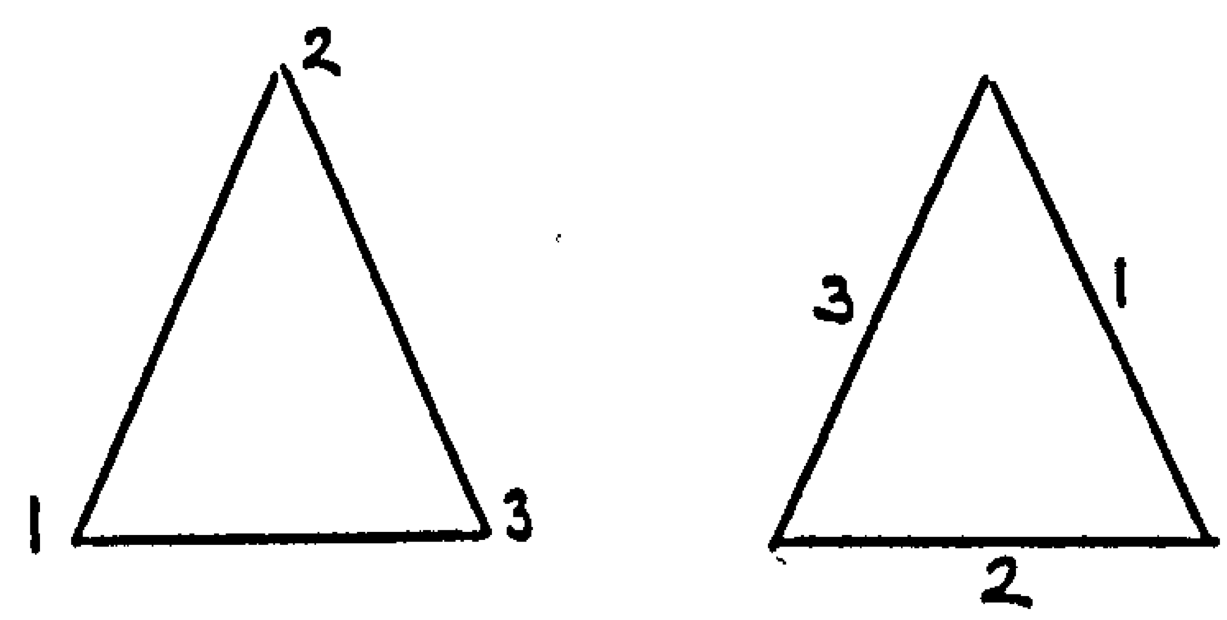
EXERCISE 10. Dualise Pappus' theorem and demonstrate it with the help of a configuration which you should draw.

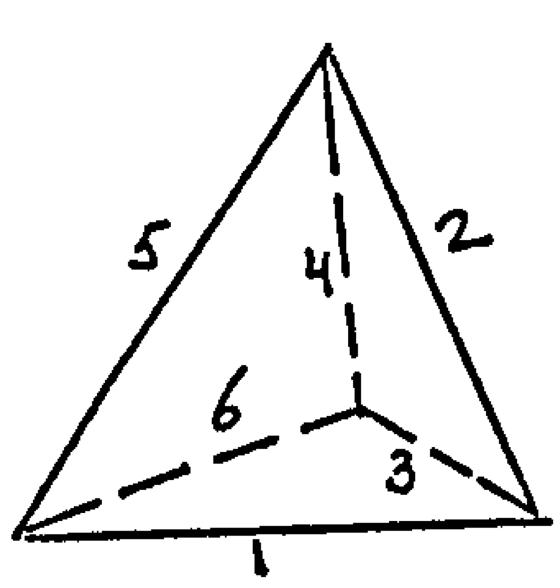
ANSWERING OF THE EXERCISES

EX. 1

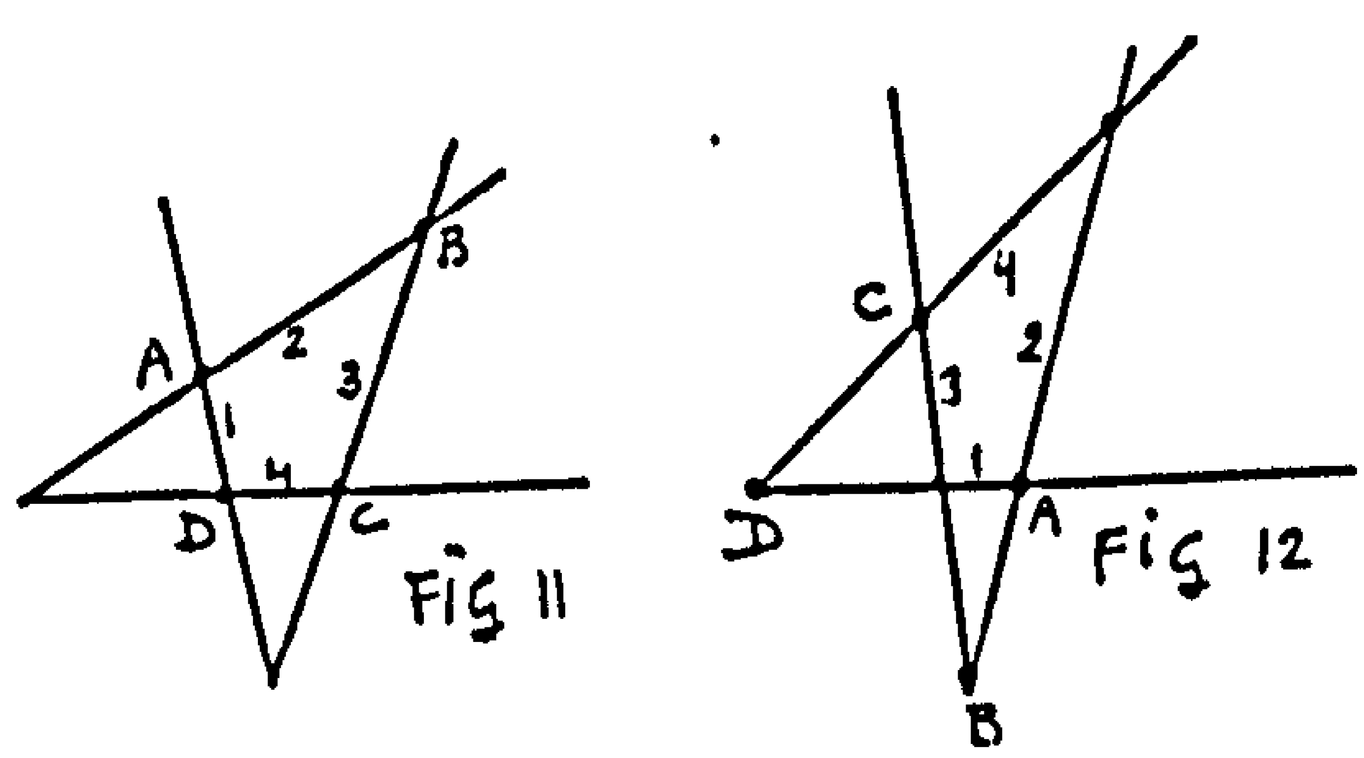


EX. 2



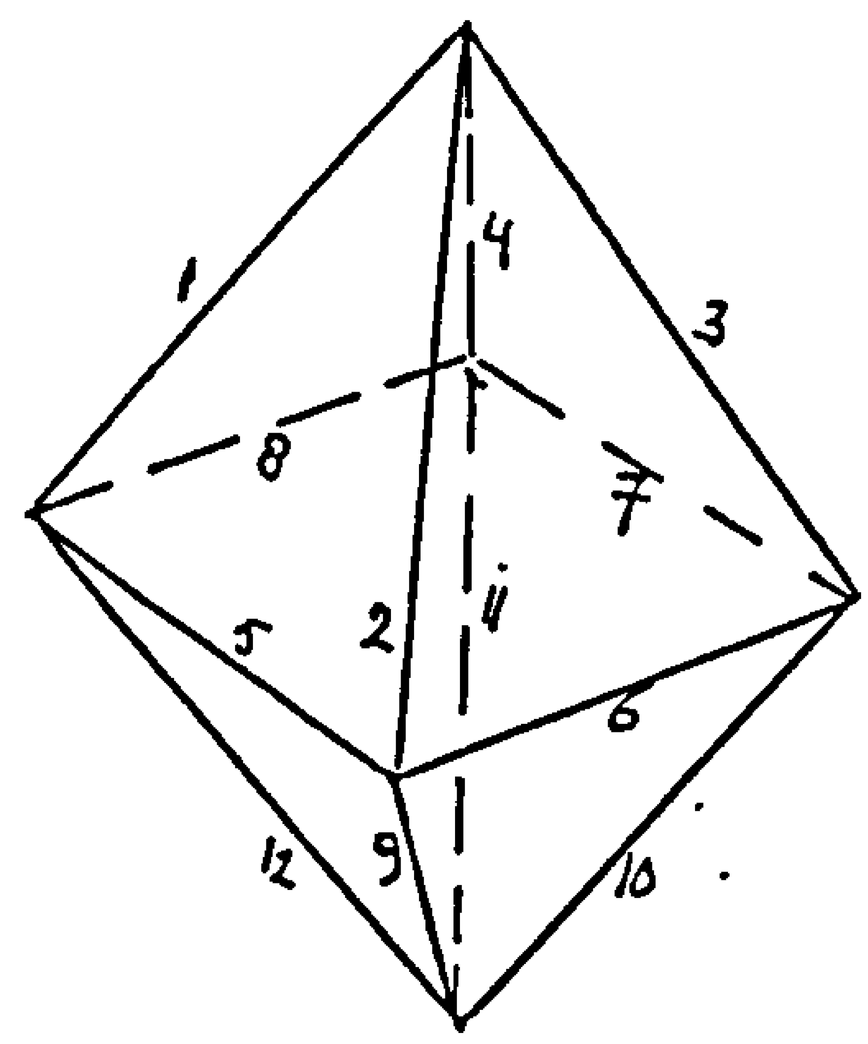


EX. 3



EX. 4

EX. 5



EX. 6

The planes ABC and DEF intersect along λ.

The plane PBEFC intersects λ in S.

The plane PADBE intersects λ in Q.

The plane PADFC intersects λ in R.

So S, R and Q lie on λ.

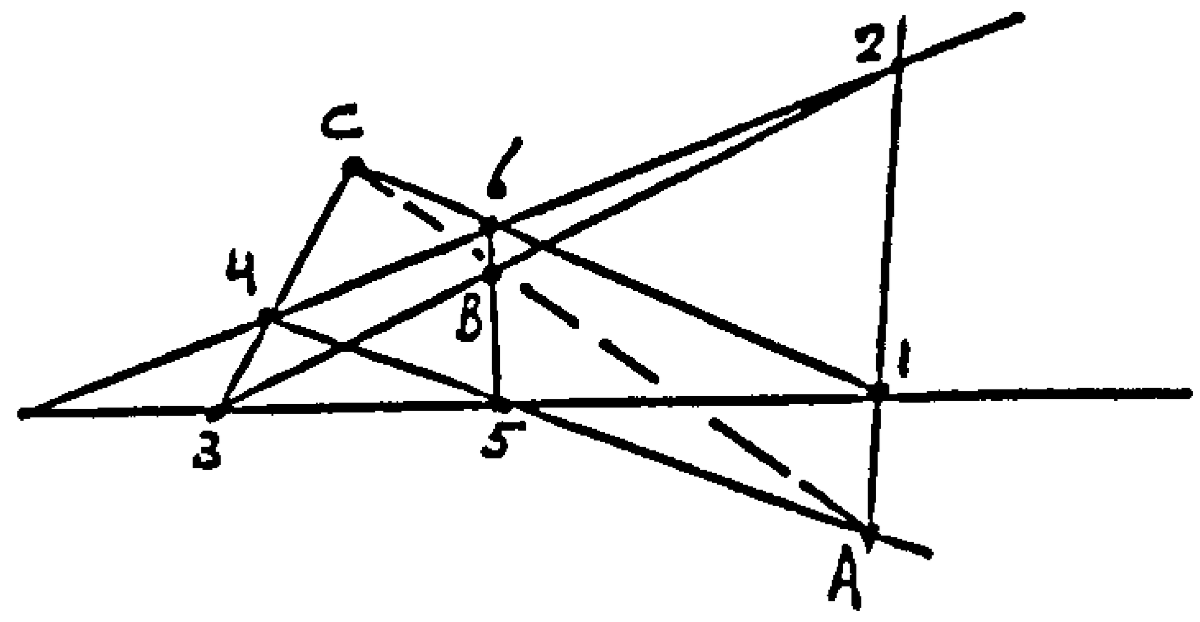
EX. 7

If the corresponding edges of two triangles ABC and DEF meet at points Q, R and S on a straight line λ then the triangles are perspective (with a centre P).

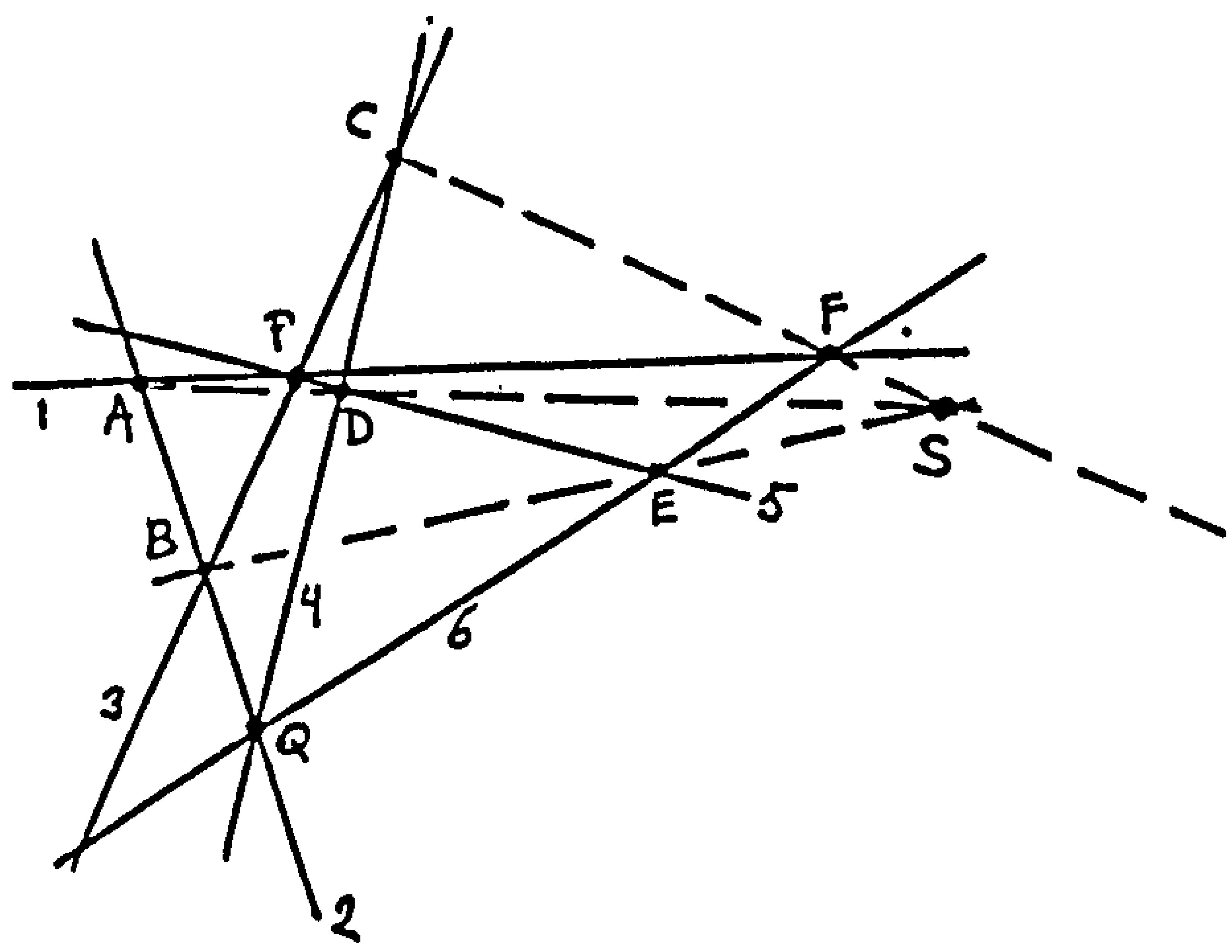
EX. 8

If the points of a sexagon lie on a circle, then the opposite edges intersect at three points which lie on one straight line. The edges (1,2) and (4,5) are opposite edges (figure 17)

EX. 9



EX 10



If three edges of a hexagon go through a point P and the other three edges go through another point Q, so that P and Q are not vertices of the hexagon, then the lines connecting opposite vertices go through one point S.

8.5. Projections

The pages on Projections in Part I, Chapter V, are called 'The beauty spot'. A lot of Projective Geometry is shown there but disguised as artistic drawings. The carrying out of visual constructions may provide that touch of magic which is found in an artistic drawing but it can also emerge from an ordinary geometric construction.

In the following part I have displayed a set of questions which ask the student to walk once again through the beauty spot of 'Projections'. The answers are also given so that the student is able to check his/her proficiency on the subject. Referring to the cone, which is demonstrated in 'Projections', (Chapter V, section 5.2, pages 6 & 9 of 'Projections'), the student might take up the study of conic sections which is highly interesting; duality is also applied in it.

The questions and the answers in the following refer to pages of 'Projections', Part I, Chapter V.

QUESTIONS.

1. Do the straight lines ℓ and m lie necessarily in one plane?

2 If you suppose that the straight lines ℓ and m lie in one plane, then which lines of the plane will remain unchanged under the projection from F ?

3. Describe the deformation of the circle under the projection from F.

4. Which elements are wrongly projected?

5. Describe the deformation of the 'tiles' under the projection from F.

6. Do you observe an error in the construction?

7. Describe the emergence of ellipses in the interior of the circle.

8. Is a spatial model of the shown figure possible?

9.1. If the patch p_1 rotates clockwise about T_2 seen from T_2, will the patch p_4 rotate clockwise about T_1, seen from T_1 ?

9.2. Could patch p_2 possibly be an ellipse?

ANSWERS

1. No, there is no need that ℓ and m intersect.
2. Each line going through F will remain invariant as a line, even the line PQF. However, of the line m every point will remain invariant.
3. It becomes an ellipse.
4. The 'H' of HEY and the dot near P.
5. They become longer and smaller.
6. The line emanating from Q at the right hand side does not intersect ℓ.
7. Subsequently intersecting lines intersect on an ellipse
8. No, not if all the lines which are drawn straight are straight in reality. Yes, if the horizontal lines have a kink.

9. 1. Yes

9. 2. Yes

CHAPTER IX

9.1. Group theory

After 'duality', another subject may be found which can be part of the curriculum of 'Educational Geometry'. Such a subject should meet the expectations of 'visual geometry'.
This means that the geometry involved should not be distorted by 'mathematical' demands. So we are looking for areas of mathematics in which geometrical images are used in such a way that no 'educationally invalid' operations can occur.
In the text below it is demonstrated that certain parts of Group Theory do not contradict the basic assumptions of 'Educational Geometry' so that these areas of mathematics may be seen as eligible to be part of the curriculum of 'Educational Geometry'.
Let us start with an exercise. Try to think of a flat plane in which no visual figures can be perceived: no straight lines, no points, no triangles, no circles in the plane, nothing else but the plane itself. Can a horizon be observed? Presumably the plane becomes invisible when no figures can be seen on its surface.

Are we going to deny the visual image of a plane because it turns out to be essentially invisible? No, we reject an invisible plane and assume that it is possible to draw figures on the plane's surface. In order to observe these figures, we suppose that the plane can be perceived by the eye. Space, even a 2-dimensional one like a flat plane, is apparently invisible but we can mark the boundaries by drawing lines on the surface. These lines limit the areas on the surface in order that the surface is divided into parts and we are able to perceive the boundaries.

Further, the exercise requires you to conceive that the plane moves in itself. To understand this, consider the following example. Water in the sea moves, but the shores are fixed. The coast does not move; it remains unaffected by the movement of the water. One might say that the sea only moves 'in itself'. Applied to the case of a geometric plane this means that the plane and its points move only in the plane itself. When they move, they remain in the plane in which they lie. The points do not leave the initial plane as a result of the move. For instance, let a plane rotate about some fixed axis in space which is perpendicular to the plane (see figure 1).Then almost all the points of the plane move but the plane as a whole maintains its initial position in the space R^3. The rotated points are still in the initial plane. It is also possible to translate a plane so that the position of the plane as a whole does not change in R^3. Again the translated points remain in the initial plane. Has the plane changed visually after the move? No, of course not. So here is a move of a plane which we know has taken place but we do not observe any change. The position of the plane in space has not changed; yet we assume that the move has taken place. Now suppose we denote our move by M_1. After this we carry out a new move M_2; a translation or a rotation or both. Denote the new move by $M_2.M_1$. So from the right to the left we read which moves have been consecutively carried out.

A series of moves may emerge so that the final situation can be denoted by:

$M_j=....M_i....M_3.M_2.M_1$. These moves M_j are called the elements of a group and the multiplication $M_j.M_i$ is called a group operation. So a group has a group operation and it has elements which are the single moves. Multiplication of the elements occurs according to the prescribed group operation which in this case is the consecutive carrying out of the moves. Let us now think of a firmly marked plane covered with triangles, rectangles, points, circles and so on and let it rotate or translate. All the figures will then change position and they will be seen to change visually by an observer. Such moves cannot be accepted as elements of the group of moves which should leave the figures visually invariant. So this will be our criterion for an element of our group. The moves of the plane in itself have to leave the configuration of the plane visually invariant. The allowed moves of the plane will be depend on the chosen configuration in the plane. The demand that the configuration does not visually change as a result of the group operation could of course be dropped but then we are no longer talking about 'Educational Geometry'.

Let us for instance rotate a circle about its centre. Visually the circle does not change at all. Such a move is allowed as an element of our group. We also need a so-called neutral element, denoted by M_0. That implies: no move at all. So for instance $M_2.M_0.M_1=M_2.M_1$.

Now consider the rotations of a plane in itself about a fixed point, say P. A circle which has its centre in P will remain invariant under every rotation of the plane about P. We can denote the rotation by the number of degrees of its angle of rotation. So one could say that $180°+180° = 0°$, because after two U-turns the initial position has been reached again. We call the group an addition-group, because a rotation of 30° followed by a rotation of 60°, for instance, will yield a total rotation of $60°+30° = 90°$. The elements of the group can be added and thus produce a new element of the group. One may say that the group operation is addition.

Of course we can define negative angles like - 45 ° to denote that the direction of rotation has been inverted. Next we could consider the move of an equilateral triangle whose centre is the point P. Such a triangle allows a rotation of the plane of 120° or 240° about P so that after the move no visual change of the triangle may be observed. Such moves can be accepted as elements of our group. In section Chapter IX, section 9.2 examples will be provided of these kinds of groups.

Now imagine that the plane is covered by a grid which has no visual limit except for the horizon. It is possible to translate the grid so that it covers itself or rotate it about a point so that it covers itself. Such a point is not necessarily a point on the grid but could for instance be the centre of a square which is part of the grid. However, such a move of the grid (and hence of the plane) is visually allowed only when no other figures change visually as a result of the move of the grid (and thus a move of the plane). There are also reflections: if we reflect a square using a diagonal as an axis of reflection, then the figure will not have changed visually after the reflection. Thus reflections are also eligible to join our group.

In conclusion, a special configuration, which could be a triangle, rectangle, circle and so on, might give rise to a group of moves of the plane which leaves the visual appearance of the figure invariant. These moves are acceptable in a visual geometry which does not for instance allow mathematical squares. These are, as we have determined, too distorted to serve in visual geometry. Now a contradiction seems to emerge. In the case of the square, I allowed moves which will leave the square visually invariant; but the square as such is rejected as a visual geometrical object. Is that right?

The answer is yes. In the case of the plane, we allowed moves which leave mathematical (rejected) squares invariant, for instance the rotation 90 °, 180 °, or 270°. Although the mathematical square is rejected as a visual figure, it may very well be used as a source of rotations and reflections which then can be legitimately applied to the moving plane. In visual geometry there is nothing wrong with that. Remember Professor Barrau's statement about the somewhat wild debut of elementary concepts in geometry!

Let us look at a mathematical square and erect a line through the square's centre, perpendicularly on the plane in which the square lies (see figure 1).

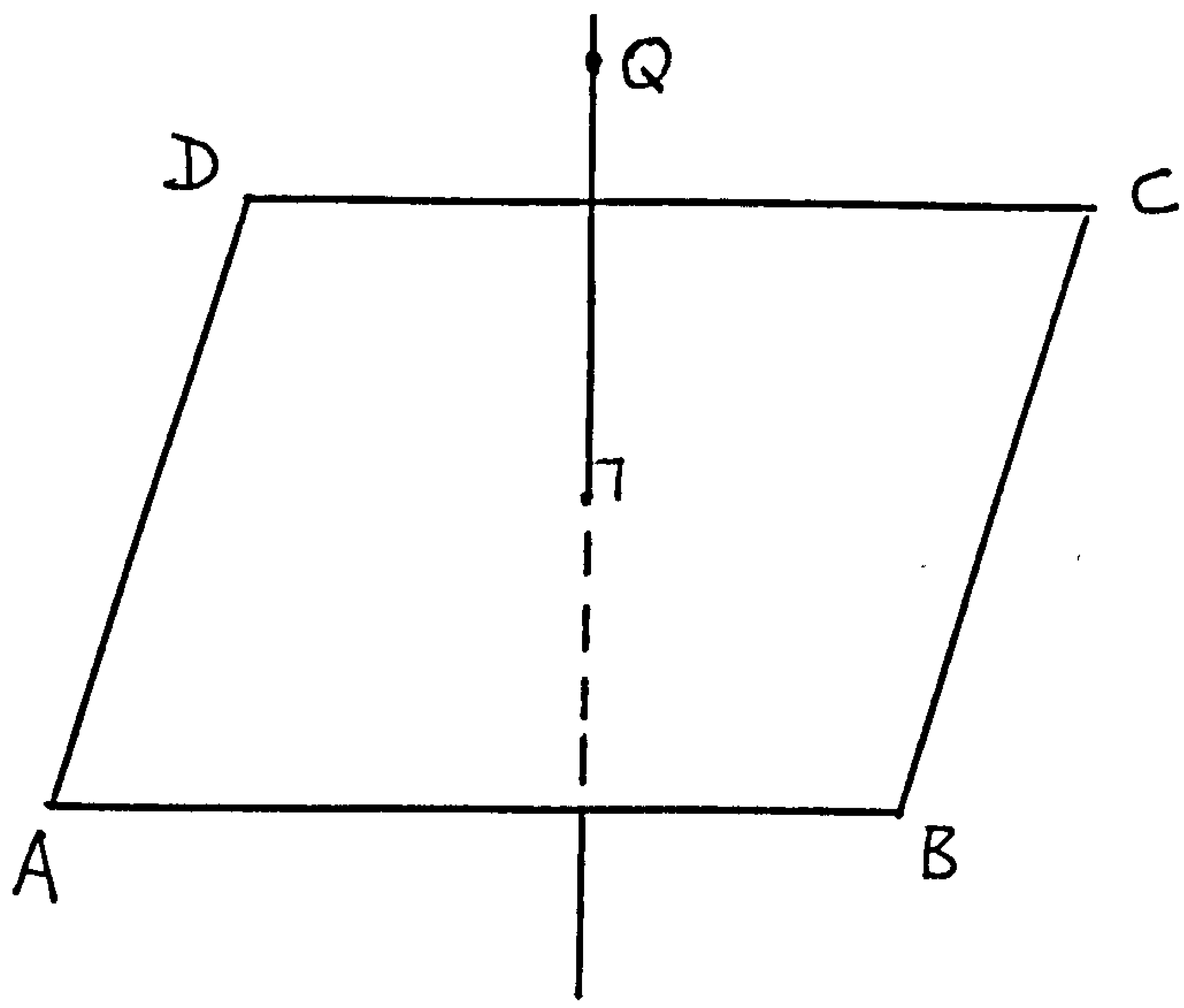


figure 1

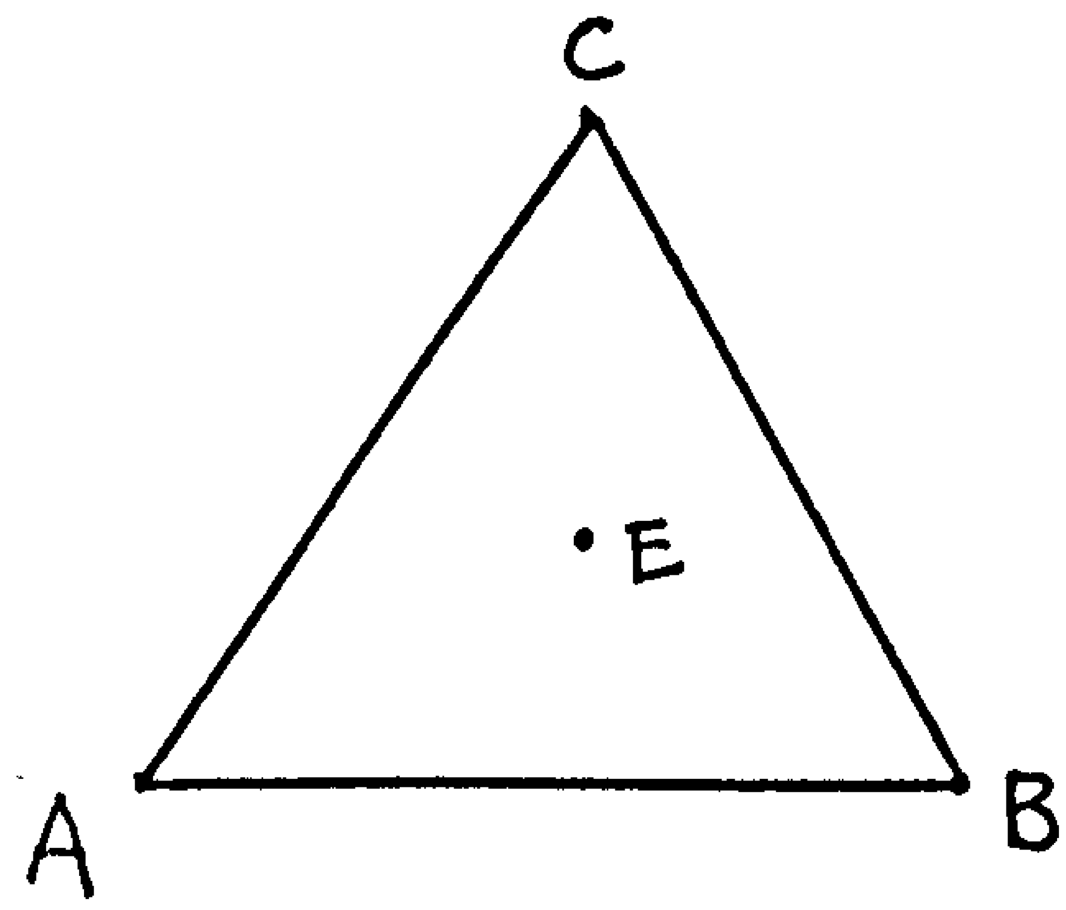
On that perpendicular line we choose a point Q. Now we assume that, seen from the point Q, the square looks perfectly symmetrical. As we stated in Chapter VI, section 6.2, the acceptance that a line is visually straight, can be seen as a result of intuition. Intuition is the main means of checking the straightness of a line, the flatness of a plane. And the 1-dimensional straight line is a visible space, but only seen from a position in a 3-dimensional space and then only for a small part. Remember that the visual image of a global straight line consists of no more than 2 single points. So in figure 1 we may only intuit that the lines AB and CD are visually straight and thus it is by intuition that we suppose that a rotation of 180 °, which interchanges AB and CD, will leave the square visually invariant. Next, in section 9.2, some visualised examples of groups will be demonstrated which can be practically applied in visual geometry.

9.2. Theory of Groups. Exercises.

After the more theoretical discussions in section 9.1., we will now demonstrate practical exercises in which group theory is involved according to the assumptions of 'Educational Geometry'.

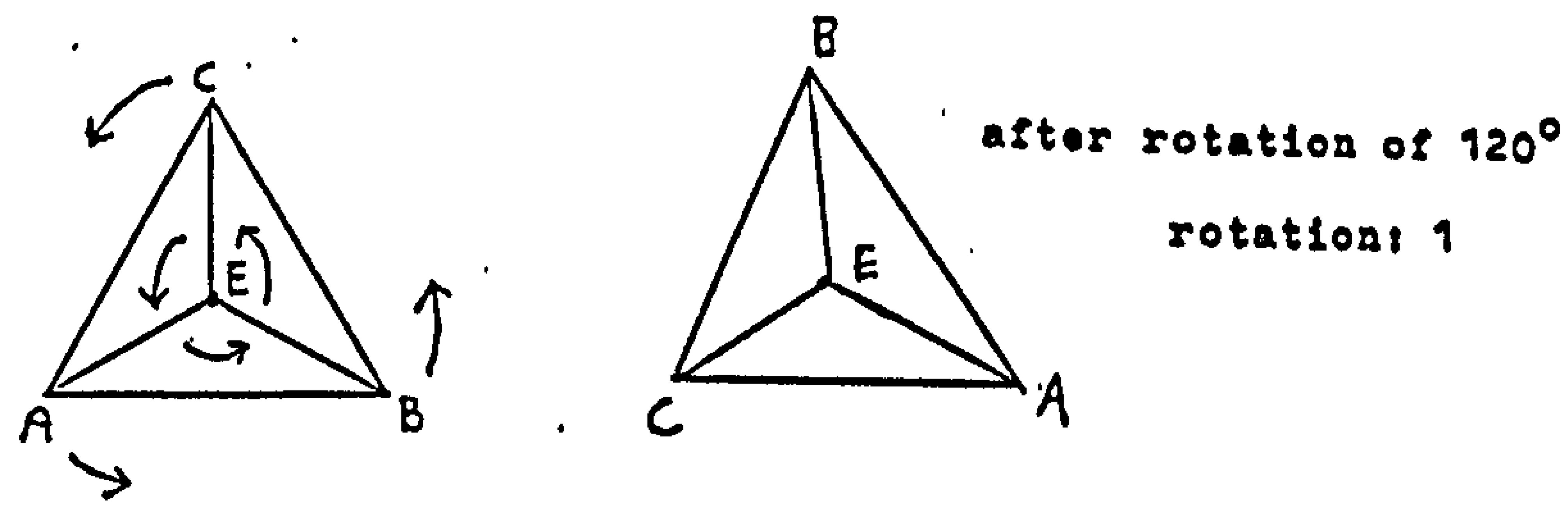
Consider the equilateral triangle ABC in figure 1: we observe it has a centre E, about which the triangle may be rotated.

figure 1



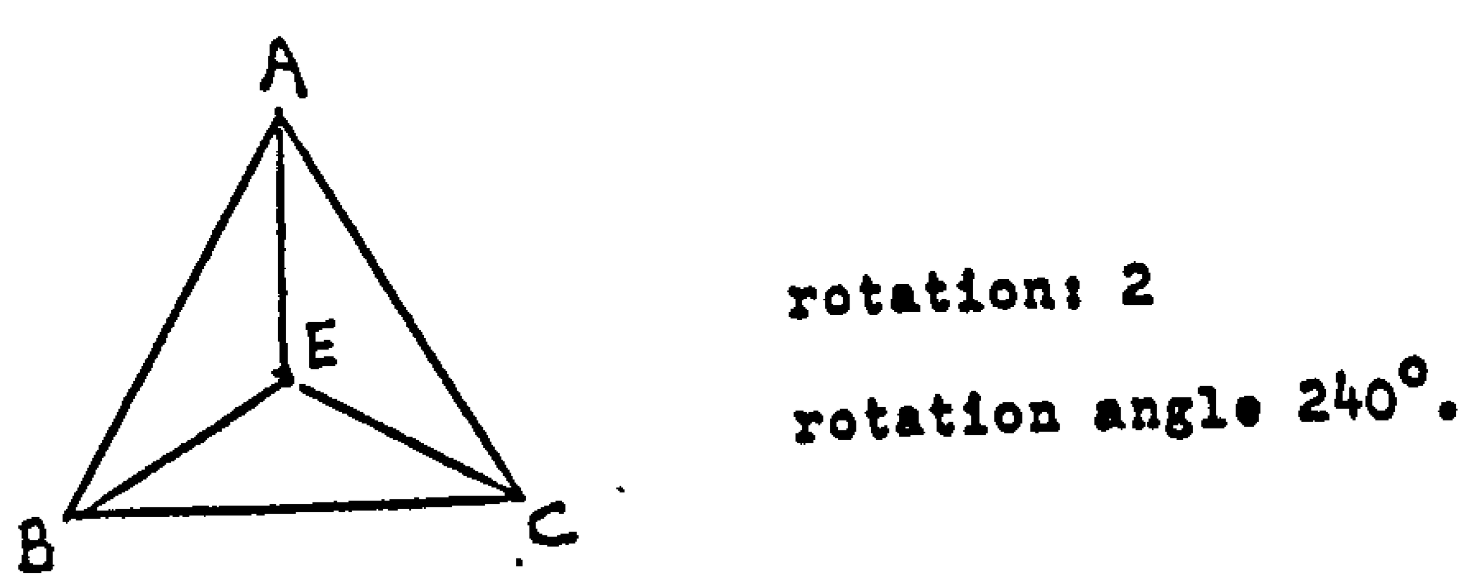
Drawing the lines AE, BE, and CE, we get three congruent triangles AEB, BEC and CEA; subsequently the triangle is rotated about E by an angle 120° (figure 2).

figure 2



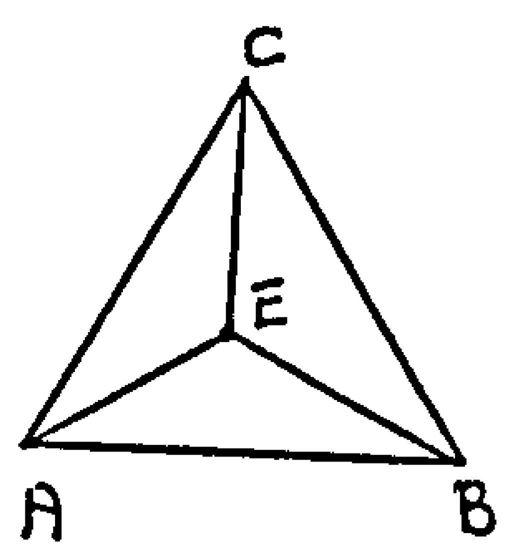
We call that rotation :1, because the rotation angle 120 ° has been taken once. Further, the rotated triangle rotates again, again by an angle 120 °; this rotation is called: 2, because from the initial position the rotation angle is twice 120 ° (figure 3).

figure 3



Rotating further from the position of figure 3 by an angle 120 °, means that we are back in the initial position; this will be called: O (figure 4)

figure 4



initial position:

rotation: 0

Relating the angle of rotation to numbers we get :

Rotation angle	associated number
120°	1
240°	2
360°	0

We may add the angles of rotation: 120° + 120° means a rotation angle of 240 °. However it is simpler just to add the associated numbers: 1+1=2. So, with the help of the numbers 1,2, and 0 we may add: 1+1=2; 0+1=1; 1+2=0.

That last addition is special; it means that when you rotate by an angle of 120° and subsequently by an angle of 240°, you are back in your initial position, which is denoted by: 0. For instance, if you rotate by an angle of 600 °, this is 360 ° + 240 °, which for the associated numbers means : 3 + 2 = 2 and we have taken 3 = 0, because rotating three times 120 ° is 360 ° and you are back in the initial position.

Consequently, the associated numbers are 0,1,2 and 3 = 0, so that 1+1=2; 1+2=0;2+1=0 ; 2+2+1 =2 and so on. We say that the numbers 0,1,2 are computed modulo 3; so that 3 = 0 . Such a set of numbers 0,1,2 with 3 = 0 is called an addition group . If, for instance, a square is considered, then the associated numbers of the rotations (90°) form an addition group modulo 4; so that 4 = 0. (see figure 5).

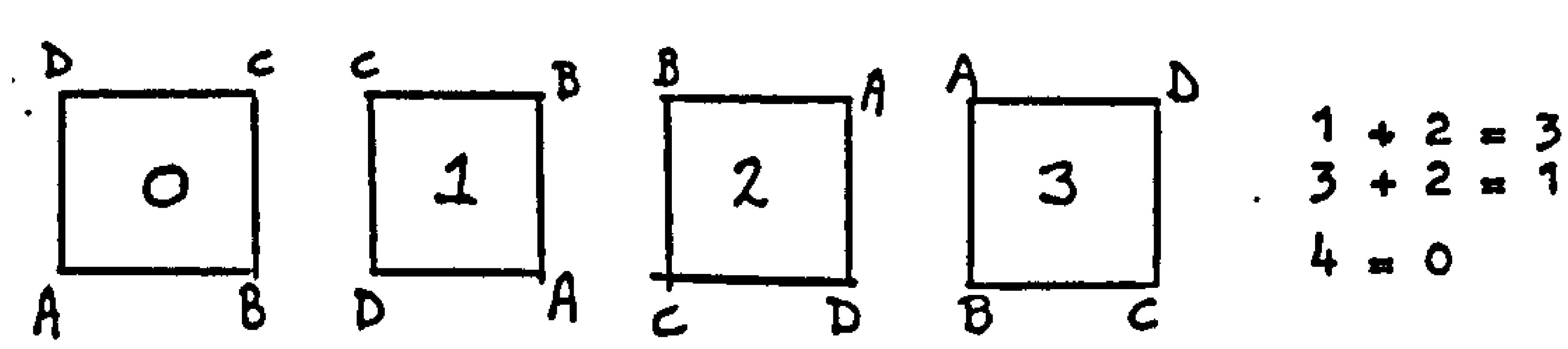
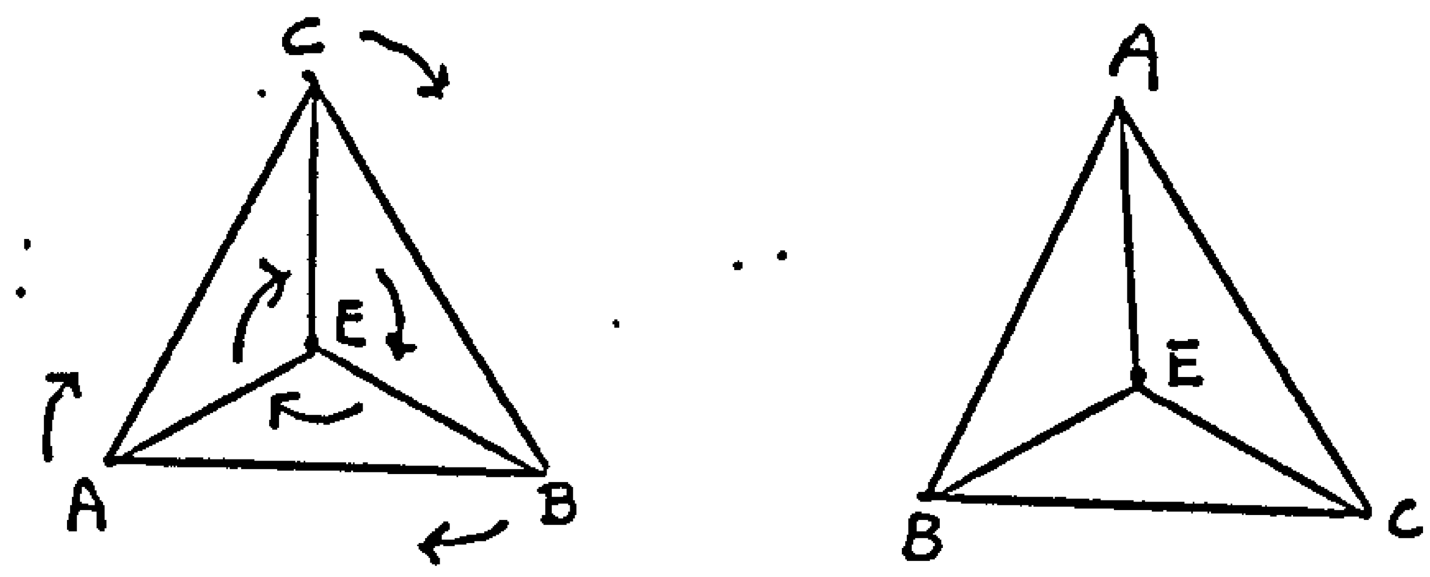


figure 5

In figure 2 we have rotated anti clockwise; for a clockwise rotation the associated numbers will be negative. So an associated number of -1 means:

figure 6



rotation -1 = 2

But in figure 6 we are in the same position as rotation 2. So, one may say: -1 = 2 modulo 3. In other words: the negative numbers are equal to the ones we already had:

-1 = 2

-2 = 1

-3 = 0

The addition group with three elements will be denoted by C_3. So C_3 contains the elements 0,1,2 and 3 = 0. This is the group we studied in figures 1,2,3 and 4. The set of numbers 0,1,2,3 will be denoted by C_4; in that group one has 4 = 0; it is the rotation group of rotations by angle 90°. Consequently the group C_5 will comprise the elements 0,1,2,3,4 and 5=0 ; the rotation angles are 72°, 144°, 216° , 288° and 360°. These groups are visualised in figure 7

figure 7

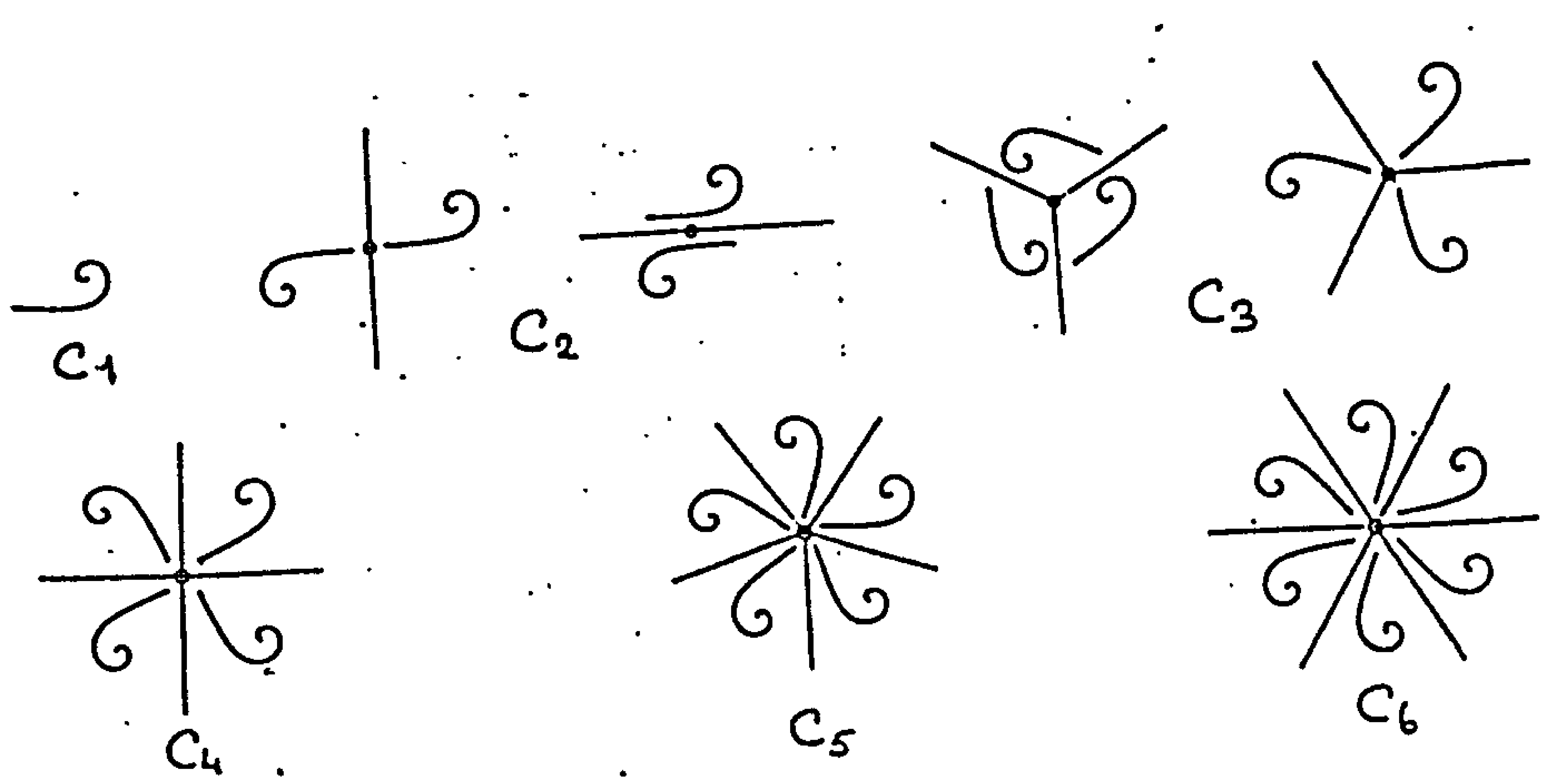


Figure 7 is a photocopy, taken from Struys, 1986, page 38.

Besides rotations there are also reflections. In triangle ABC (figure 8) the line CD can serve as an axis of reflection. The points A and B are interchanged so that the triangle is reflected in the axis, but point C remains invariant, because it is on the axis.

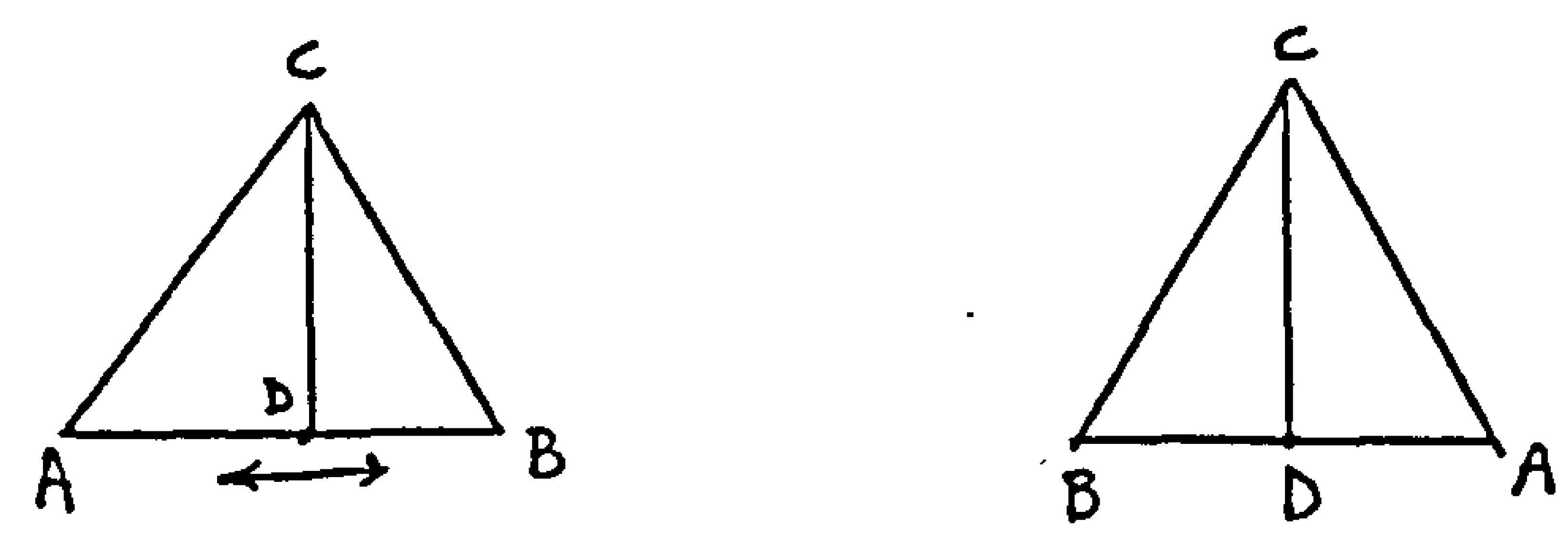
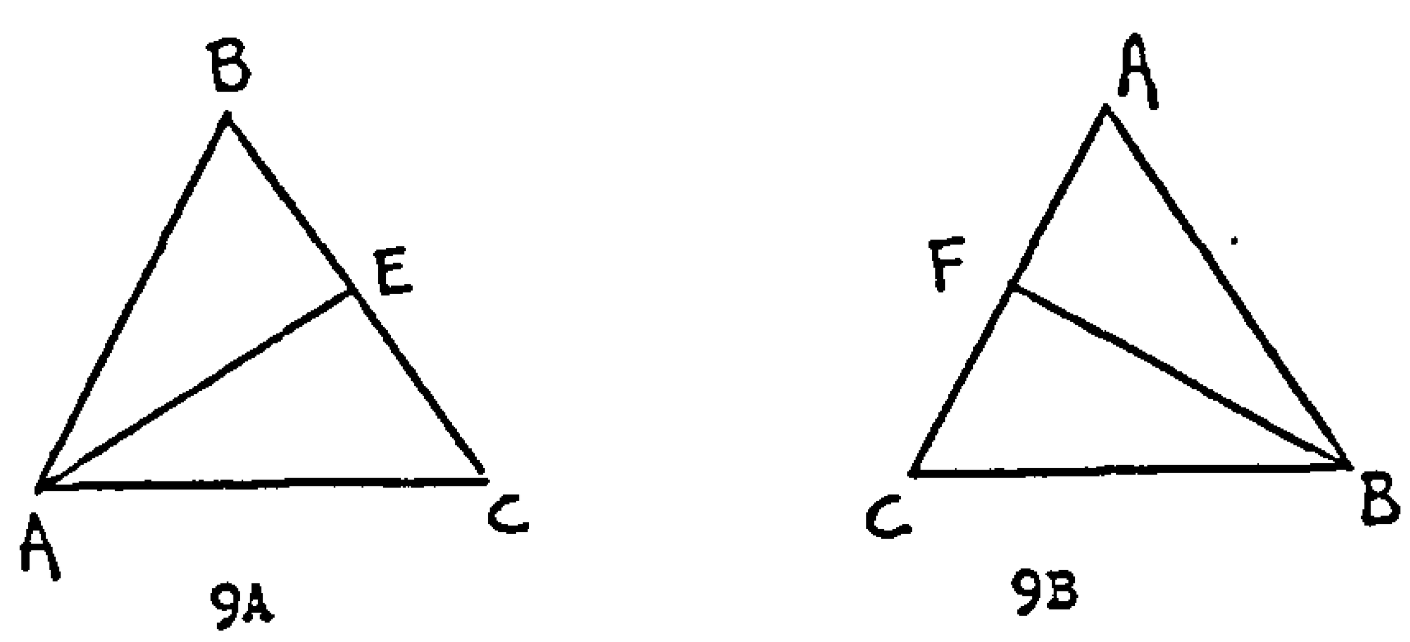


figure 8

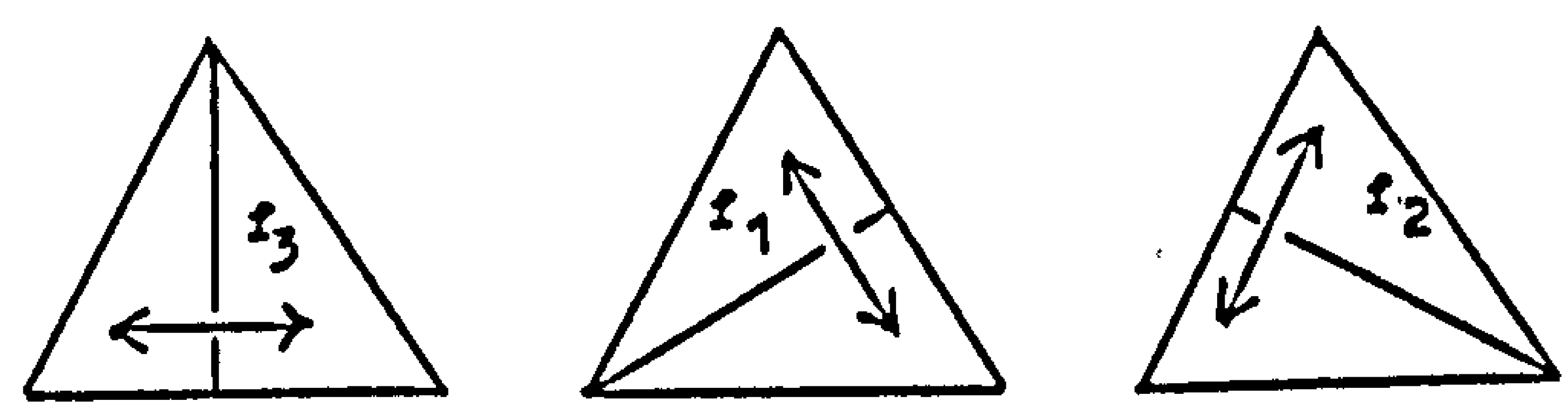
Analogously one can reflect in the line AE and in the line BF (figure 9).

figure 9



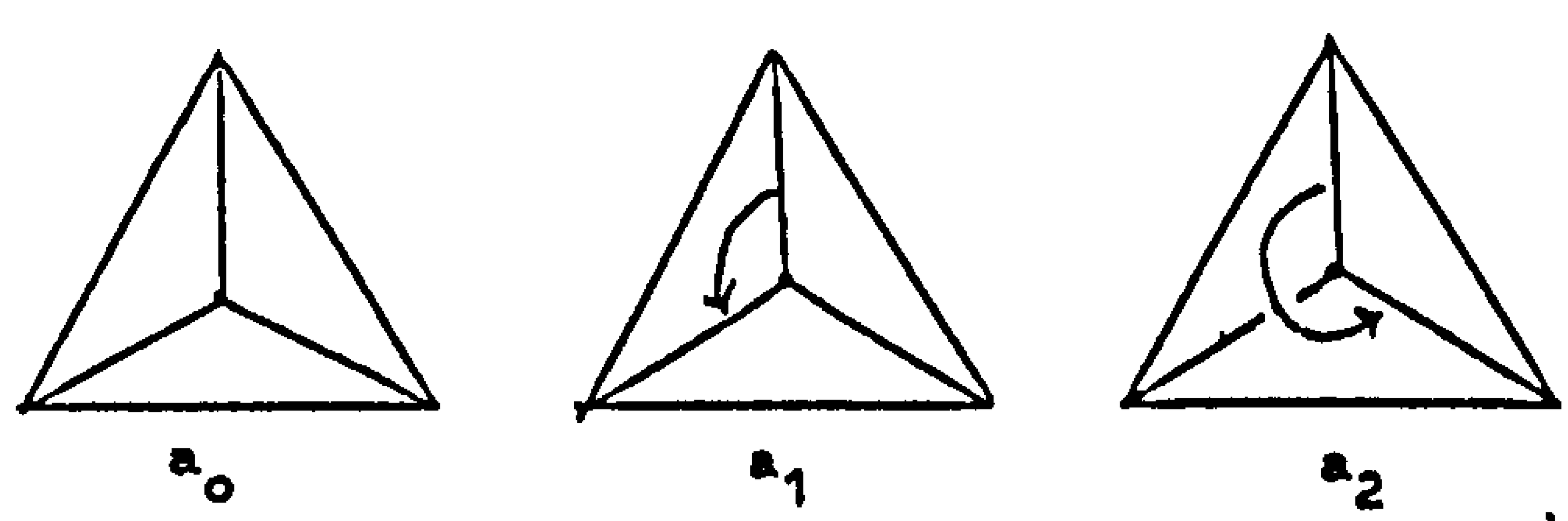
The rotations of the figures 1,2,3 and 4 will be denoted by a_1, a_2, a_0, so that a_2 means: an anti-clockwise rotation with angle 240 °. The reflection in figure 8 in the vertical axis will be denoted by f_3. The reflection in figure 9A in the axis with positive slope will be denoted by f_1 The other reflection, of figure 9B, is f_2. In figure 10 we have a survey of the reflections.

figure 10



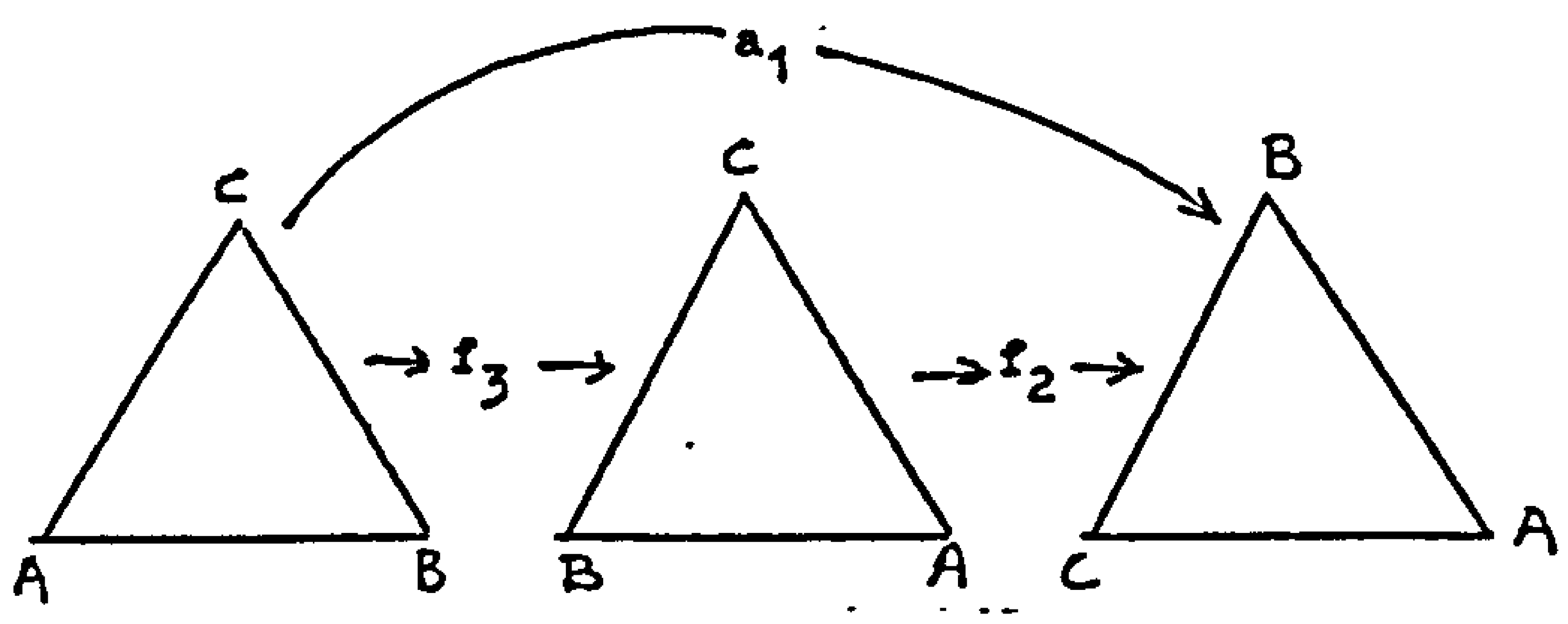
The rotations are also shown (figure 11).

figure 11



It is possible to find a product of reflections. The product f_2*f_3 means : first apply f_3 and after that f_2. This will be shown in figure 12.

figure 12



The reflections can also be applied when there are more axes of reflection: for instance if we have a square. Then 4 different axes are demonstrated. See D_4 in figure 13, which shows the reflections in one, two, three axes and so on.

Following from figure 12 we have: $a_1 = f_2 * f_3$

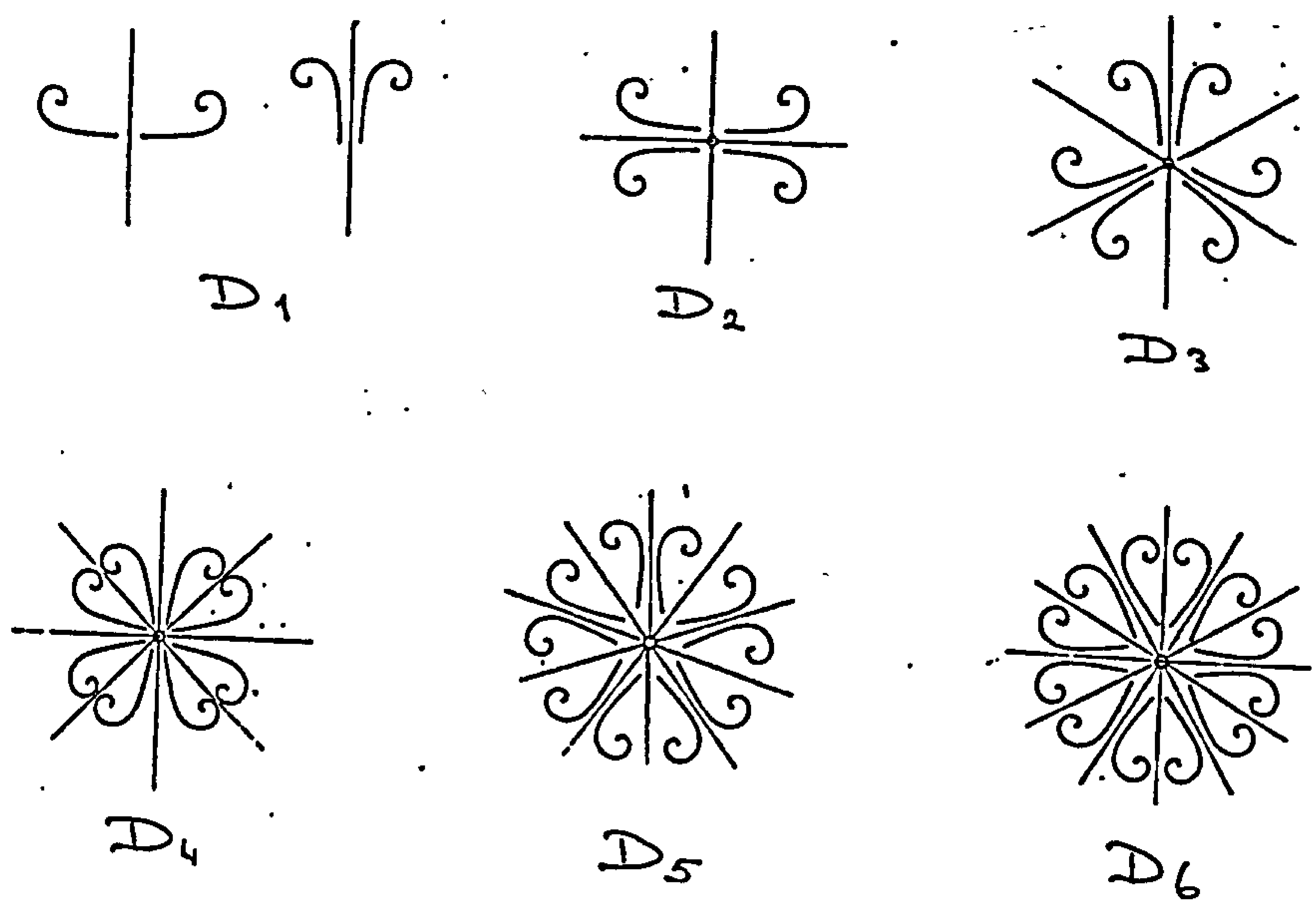


figure 13

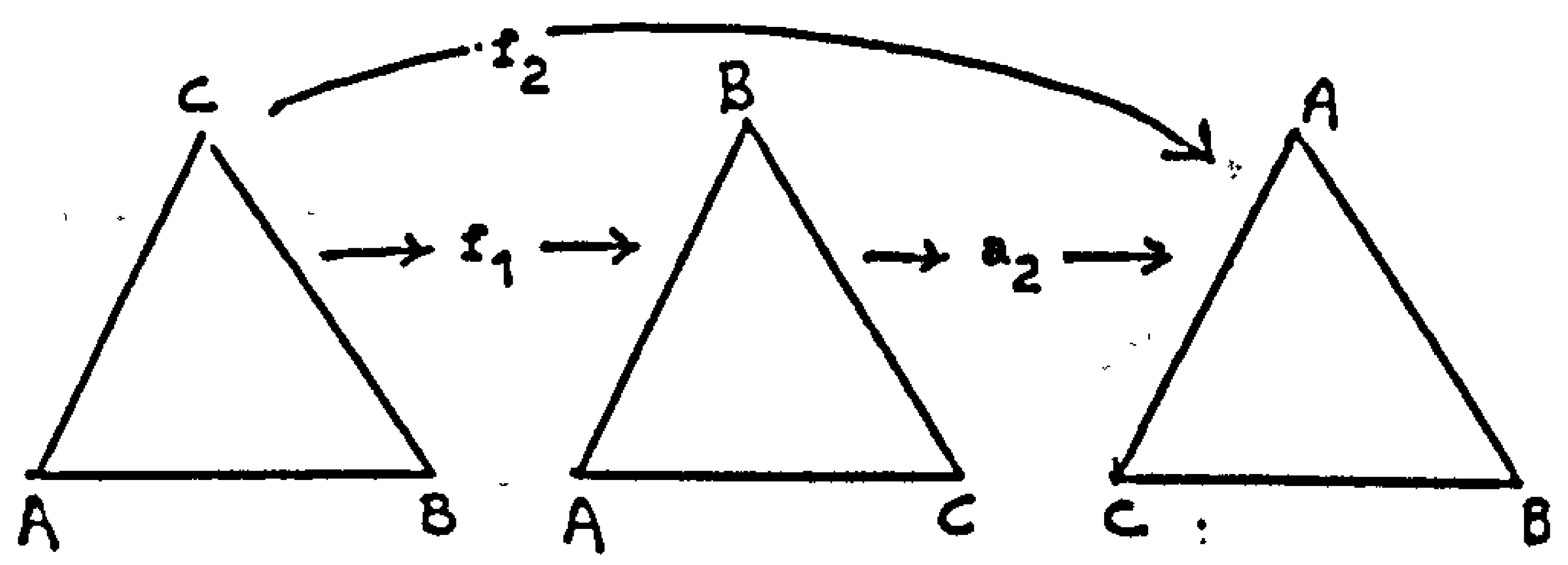
Figure 13 is a photocopy, taken from Struys, 1986, page 39.

We now form a group of the rotations and reflections of the equilateral triangle. The rotations and reflections are called the elements of the group and the group itself is denoted by S_3.

In figure 12 we saw that a product of reflections is possible; the product f_2*f_3 means: first f_3 and subsequently f_2. The result is the triangle at the right side of figure 12; this triangle however is also the result of the rotation a_1, applied to the initial triangle at the left side. Our conclusion is $f_2*f_3 = a_1$, so that the product of two elements of the group S_3 yields another element of S_3.

A new example : a_2*f_1 means : apply first f_1 and after that a_2 (figure 14).

figure 14



We see that the result is f_2, so that $a_2*f_1 = f_2$.

Proceeding this way, one can design a scheme which demonstrates all the possible products of rotations and reflections in S_3. The element of the vertical column on the left is written first, and to the right of it an element of the horizontal top row is placed, and this provides the products of elements.(figure 15)

S_3	a_0	a_1	a_2	f_1	f_2	f_3
a_0	a_0	a_1	a_2	f_1	f_2	f_3
a_1	a_1	a_2	a_0	f_3	f_1	f_2
a_2	a_2	a_0	a_1	f_2	f_3	f_1
f_1	f_1	f_2	f_3	a_0	a_1	a_2
f_2	f_2	f_3	f_1	a_2	a_0	a_1
f_3	f_3	f_1	f_2	a_1	a_2	a_0

figure 15

	a_0	a_1	f_1	f_2
a_0	a_0	a_1	f_1	f_2
a_1	a_1	a_0	f_2	f_1
f_1			a_0	a_1
f_2				

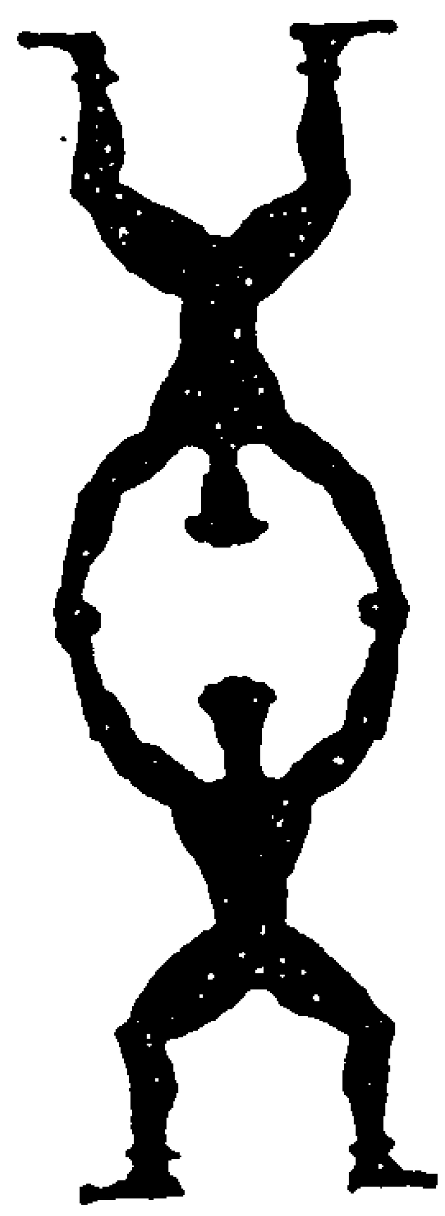


figure 16

ANSWER:

	a_0	a_1	f_1	f_2
a_0	a_0	a_1	f_1	f_2
a_1	a_1	a_0	f_2	f_1
f_1	f_1	f_2	a_0	a_1
f_2	f_2	f_1	a_1	a_0

Finally a question is given:

Question :

The image of figure 16 allows one rotation of 180° and two reflections. A group scheme is shown which gives the product of these movements. For example the product a_1*f_2 means : first apply f_2 and then a_1 .

The task is to complete the diagram.

Here the treatment of the theory of groups ends. More material may, of course, be found in textbooks on the subject. In the next Chapter, the subject 'Triangulations' will be discussed. Triangulations are frequently applied in surveying. Studying the subject promotes the understanding of geometry as a science. 'Triangulations' is a continuation of the material on "The Euler Characteristic", studied in Chapter IV, section 4.4.

CHAPTER X. Triangulations

The subject 'Triangulations' offers the opportunity to apply the assumptions of 'Educational Geometry'. The Euler Characteristic refers to properties of solids and plain configurations which are independent of the particular way of visualisation. This has already been explained in Chapter IV, section 4.4. So this type of 'visual' geometry (or rather topology), connected to the Euler Characteristic, is clearly eligible to be part of the curriculum of 'Educational Geometry'. It will be worked out below.

From part I, Chapter IV, we remember Euler's Characteristic:

$\chi = V - E + F$

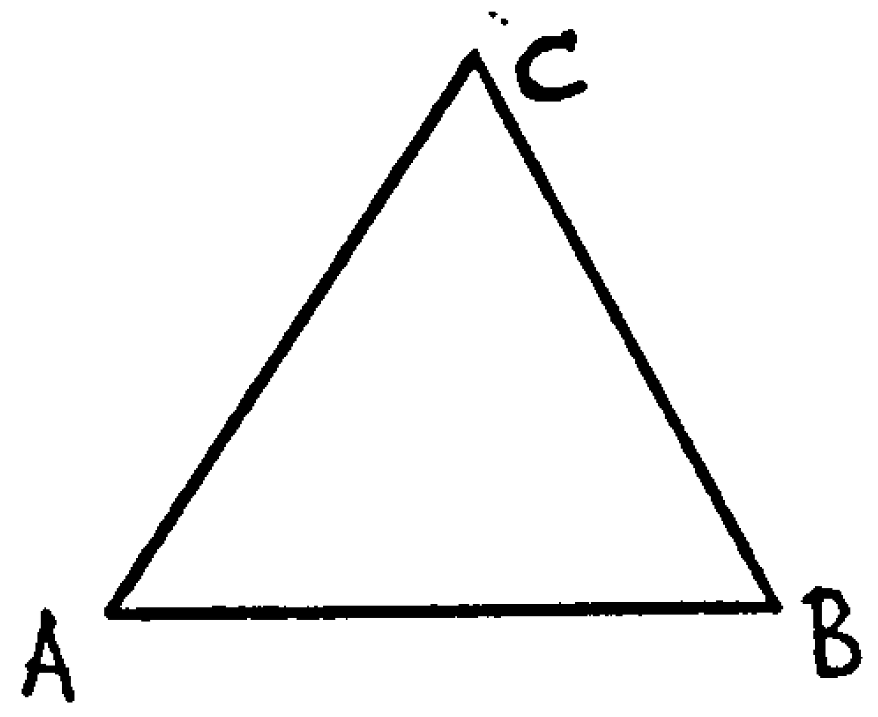
in which V= number of vertices

E = Number of edges

F = Number of faces

Applying this formula to the triangle of figure 1, we get 3 vertices A,B,C; 3 edges AB,BC and AC and 1 triangular face ABC.

figure 1

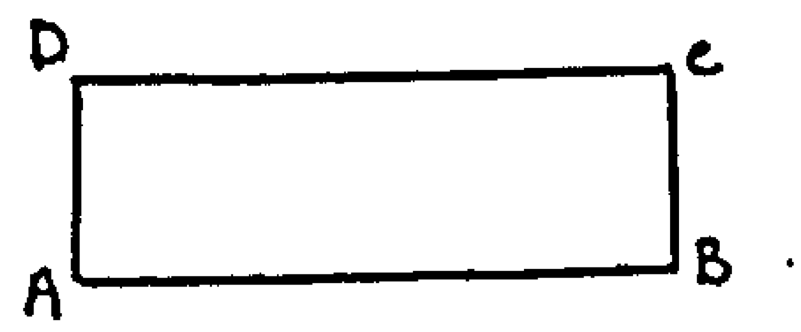


V = 3; E = 3 and F = 1 so that $\chi = 3-3+1$

We can also compute χ for the quadrangle ABCD of figure 2, which then yields :

$\chi = 4-4+1=1$

figure 2



A new step is, that we triangulate ABCD in figure 2; by this we mean that ABCD is dissected into triangles. That might be done as shown in figures 3, 4, or 4a.

figure 3

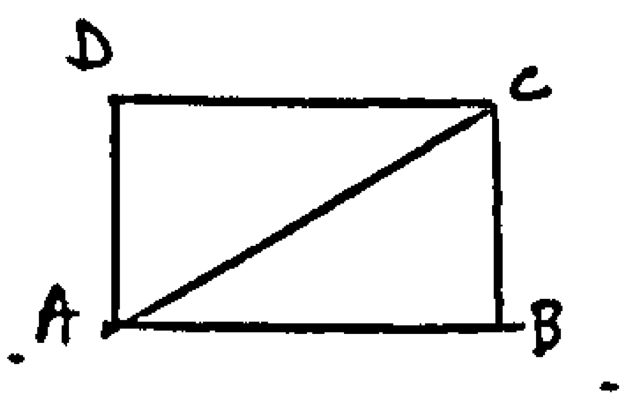


figure 4

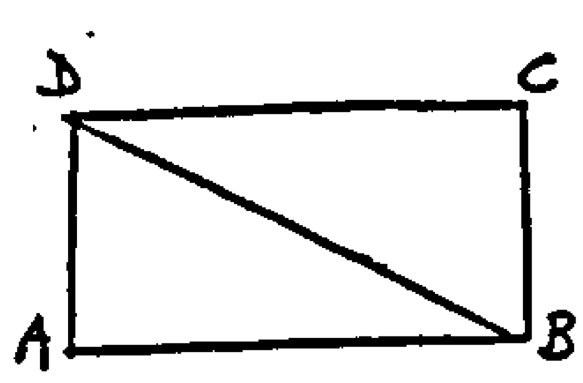
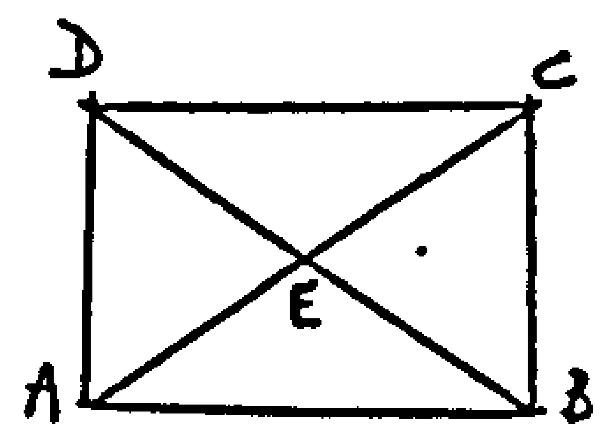


figure 4a

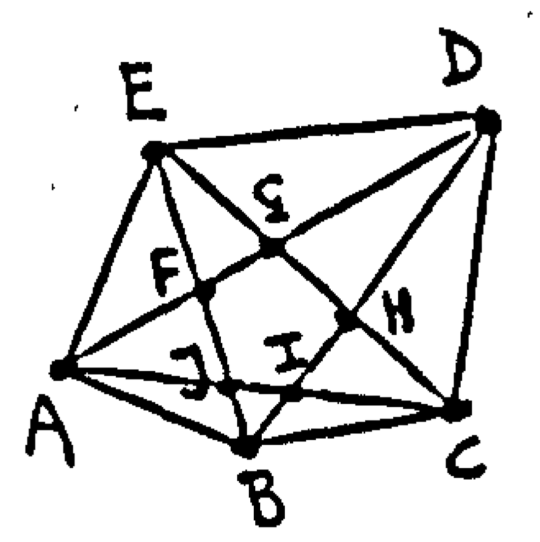


$\chi = 4-5+2=1$ $\chi = 4-5+2=1$ $\chi = 5-8+4=1$

The Euler Characteristic is not dependent on the way the quadrangle ABCD is dissected, but any dissection will yield: $\chi = 1$. In figure 5 we take for example a pentagon ABCDE and dissect it.

The Euler Characteristic yields: $\chi = 10 - 20 + 11 = 1$, so the Euler Characteristic appears to be invariant for polygons like those shown in figures 1 - 5 .

figure 5



The Euler Characteristic can also be computed when spatial figures are involved. In figure 6 a cylinder is drawn.

figure 6

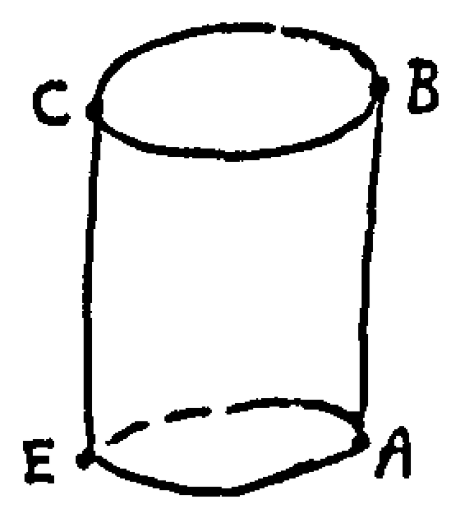
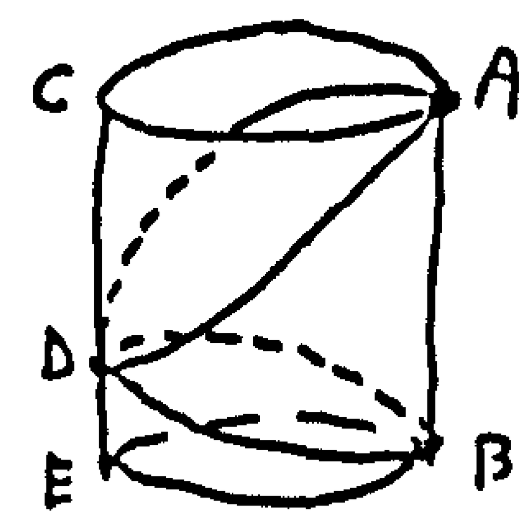


figure 7



In figure 7 an attempt is made to dissect the cylinder into triangles. Such a dissection of a spatial figure may become complicated when we have to compute χ . So we cut the cylinder along line AB and it can be placed as a flat plane on the desk (figure 8).

figure 8

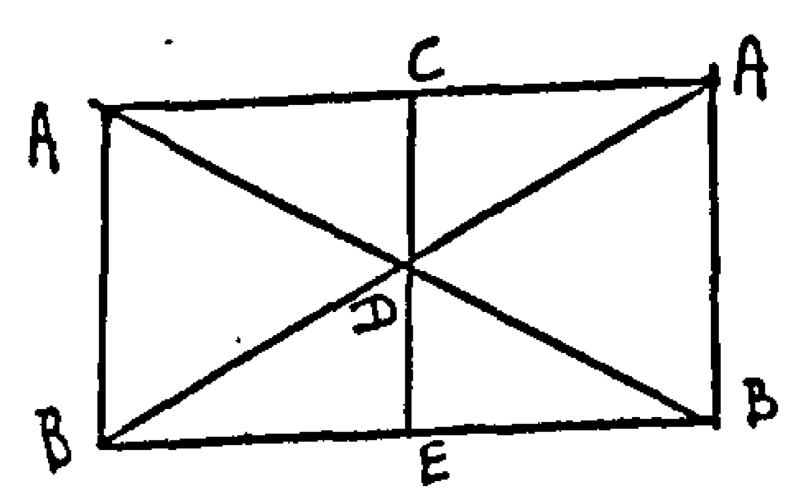
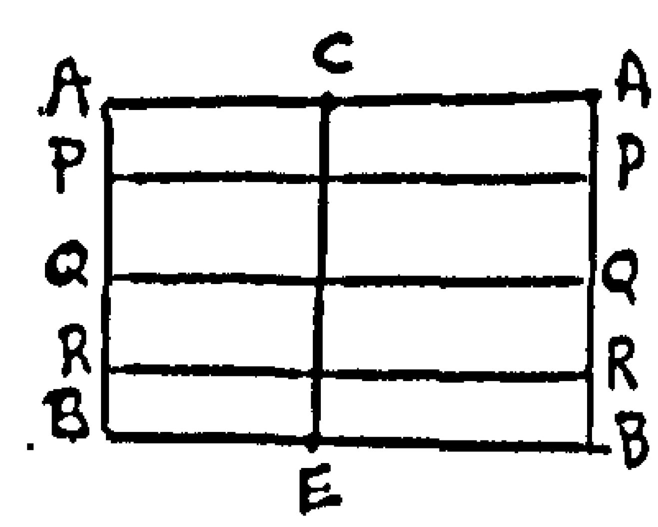


figure 9



In figure 8 there is an ambiguity. When we talk about triangle ABD, it is not clear which of the two triangles ABD in figure 8 is denoted. This follows from the fact that line AB, along which the cylinder has been cut, is drawn twice in figure 8. We say that the line AB on the left side and the line AB on the right side of the image of figure 8 are identified, which means that they are one and the same line. The points are also identified, as is shown in figure 9. To get a correct dissection of the cylinder of figure 6, an additional line FG has to be drawn (figure 10 and figure 11). No ambiguities as in figure 8 are allowed from now on.

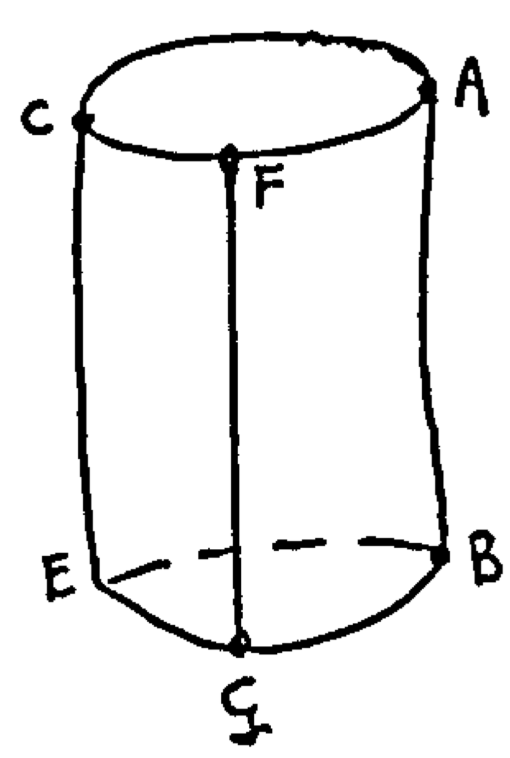


figure 10

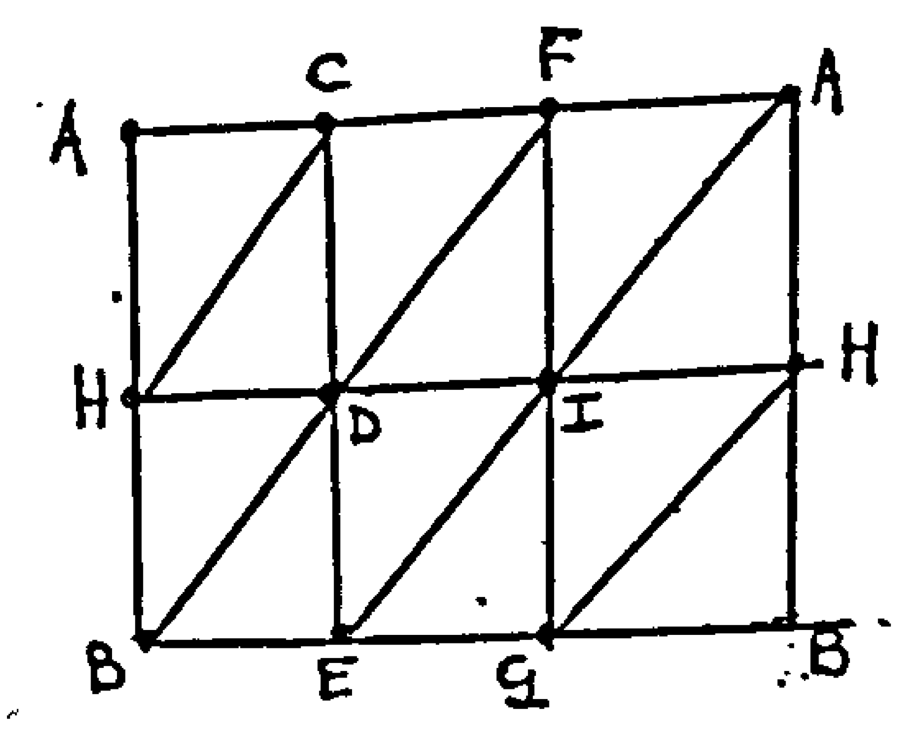


figure 11

In figure 11 we have V = 9; there are 12 depicted, but the points A,B and H count double. F = 12 because we count 12 different triangles; E = 21: we count 23 but AH and BH count double.

So, the cylinder of figure 11 has $\chi = 9-21+12 = 0$

EXERCISE 1: There is a minimal dissection of the cylinder with V = 6, E = 12 and F = 6

Note: Exercises 1, 2 and 3 are meant as an example of what the contents of a textbook of 'Educational Geometry' could be. The reader is not expected to do these exercises because no essential information is lost when the exercises are skipped. The exercises are no more than examples of practical work of 'Educational Geometry'. The answers will be provided below.

Our next object will be the Möbius strip. It is a rectangular strip with the dotted edges glued 'upside down' (see figure 12)

figure 12

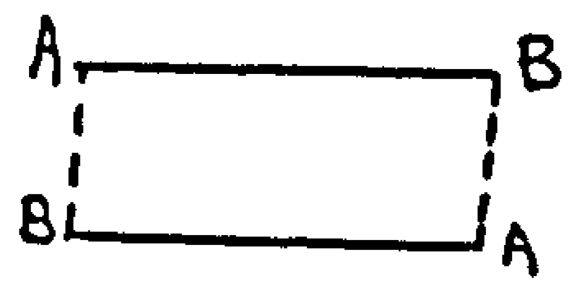
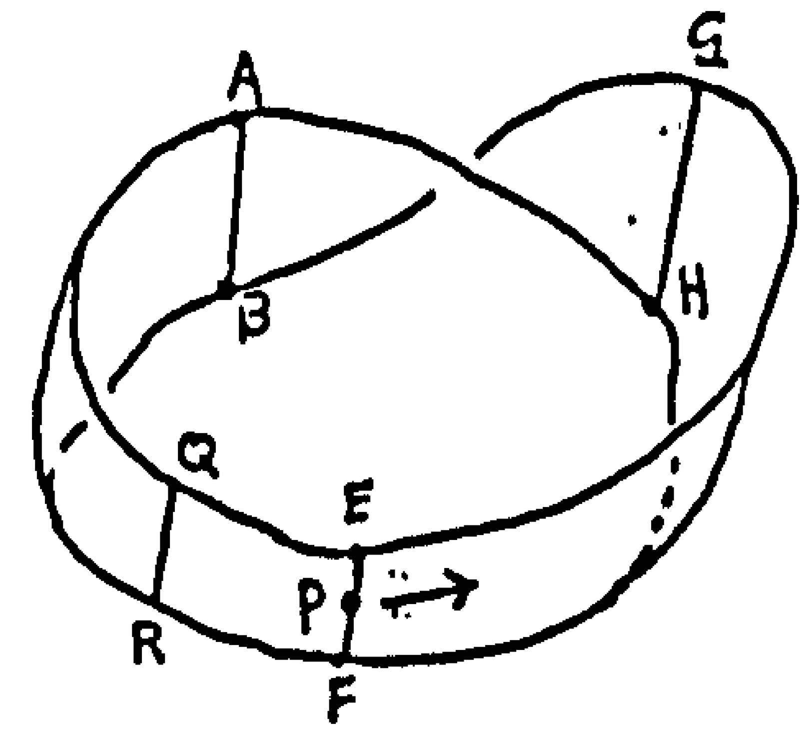


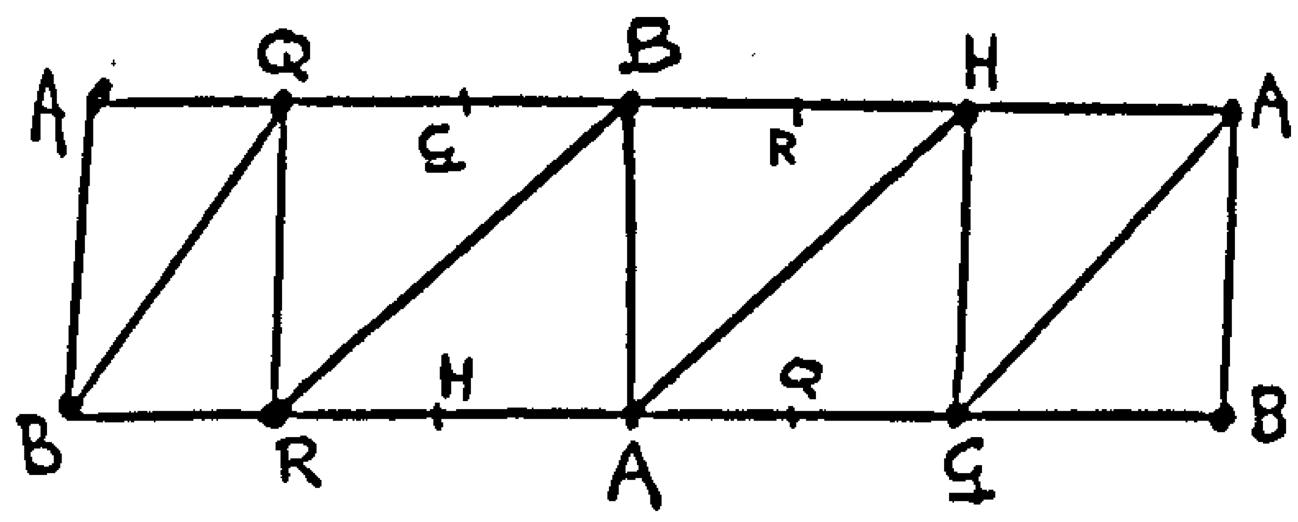
figure 13



The surface of a Möbius strip has no front and no back. When P moves in the direction of the arrow along the surface (figure 13), it will pass GH, BA, RQ and finally arrive at the initial point of departure but E and F will have interchanged positions and P is invisible at the back of the surface. Such a surface is said to be non orientable. For the Euler Characteristic we consult figure 14 and we compute:

$\chi = 6 - 15 + 8 = -1$ because $V = 6$; $E = 15$; $F = 8$

figure 14



Further, there is the projective plane. It can be represented by a circle of which antipodal points with respect to the centre are identified (figure 15). It is very difficult to find a spatial model for the projective plane. However, a dissection in triangles can be found when we observe figure 16 and compute $\chi = 1$.

figure 15

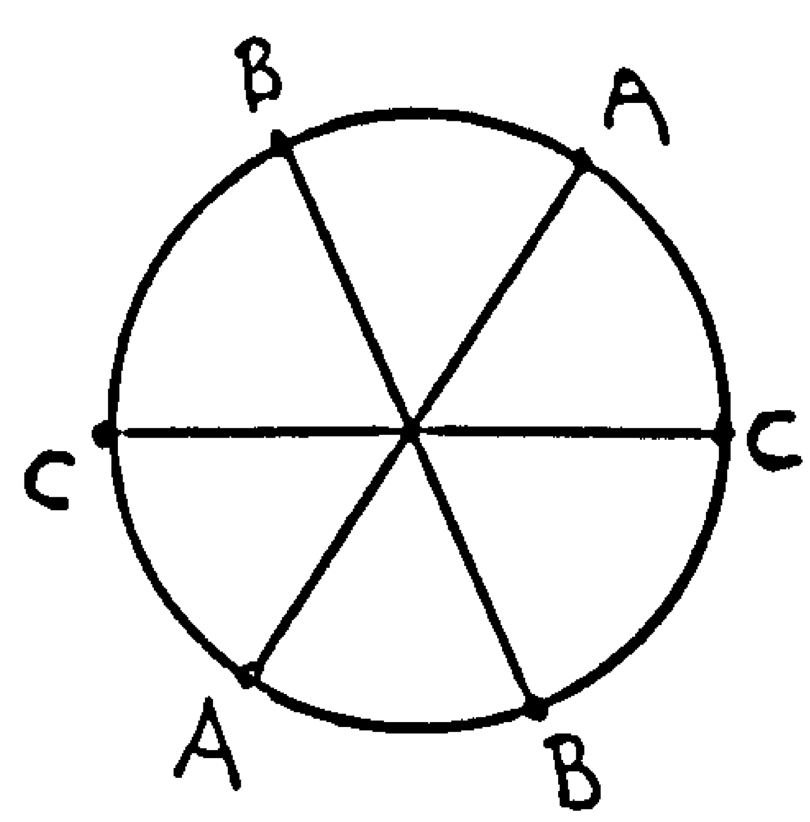
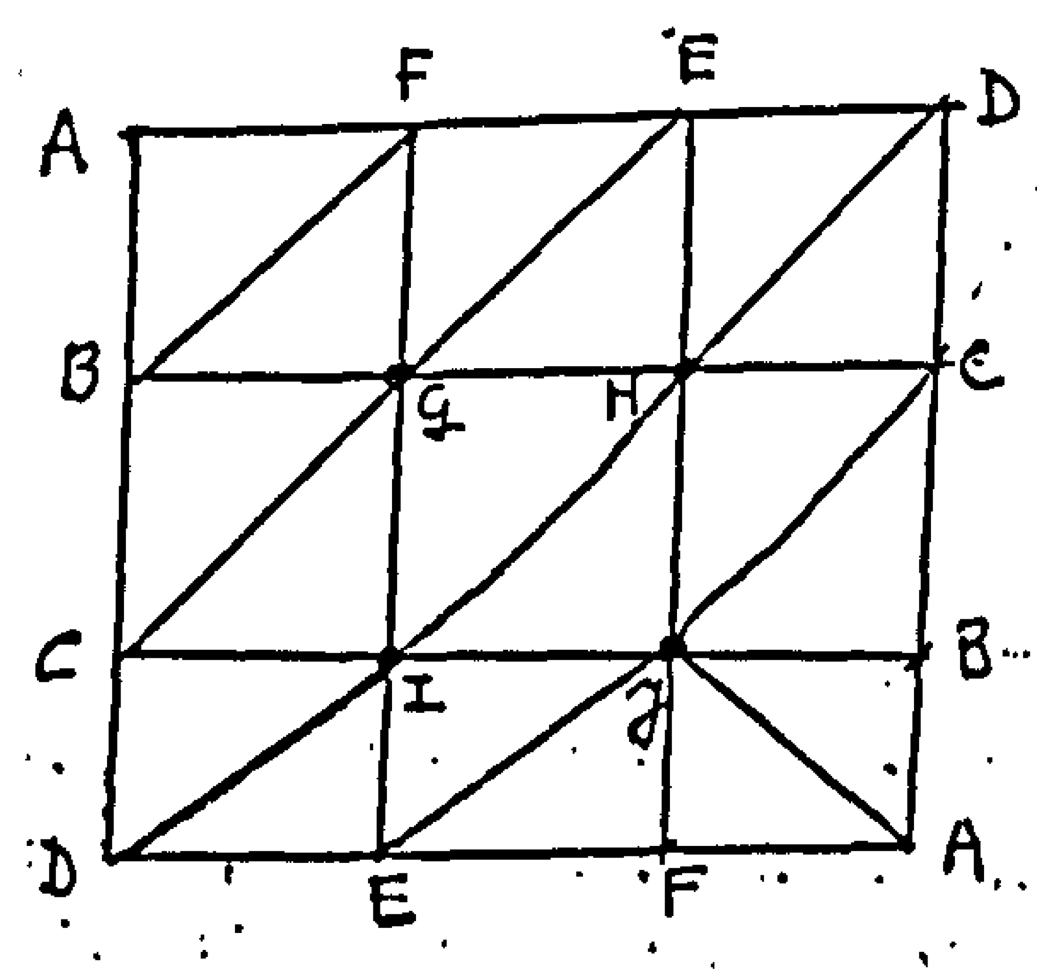


figure 16



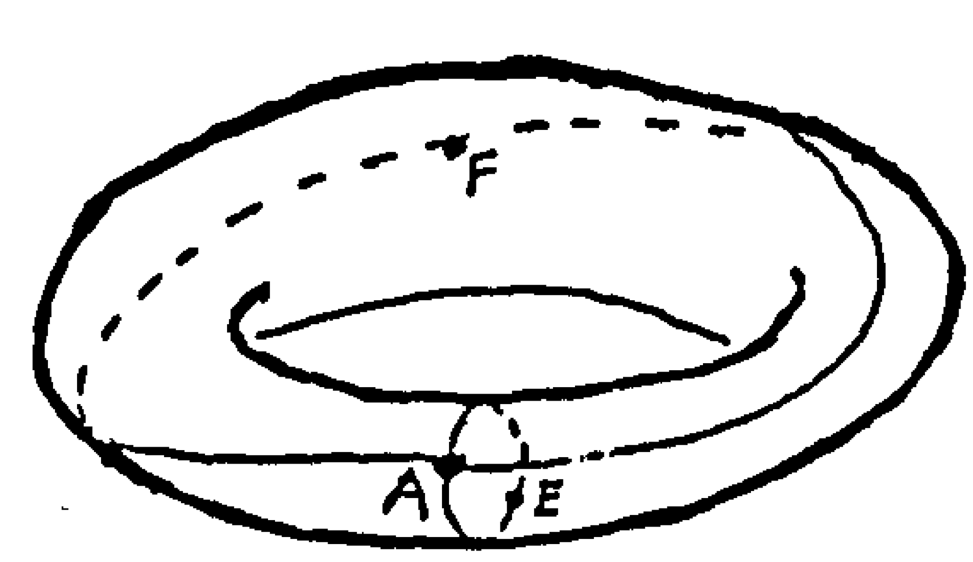
$\chi = 10-27+18=1$

$V = 10; E = 27; F = 18$

EXERCISE 2: There is minimal dissection of the projective plane with $V = 6; E = 15;$ and $F = 10$

Finally we will compute the Euler Characteristic of a torus with the help of triangulation. From Part I, Chapter IV we already know that $\chi = 0$ for the torus. A torus is a spatial body that can be compared to the tyre of a bicycle (figure 17).

figure 17



Alternatively we can consider that torus as a rectangle whose opposite sides are identified (figure 18).

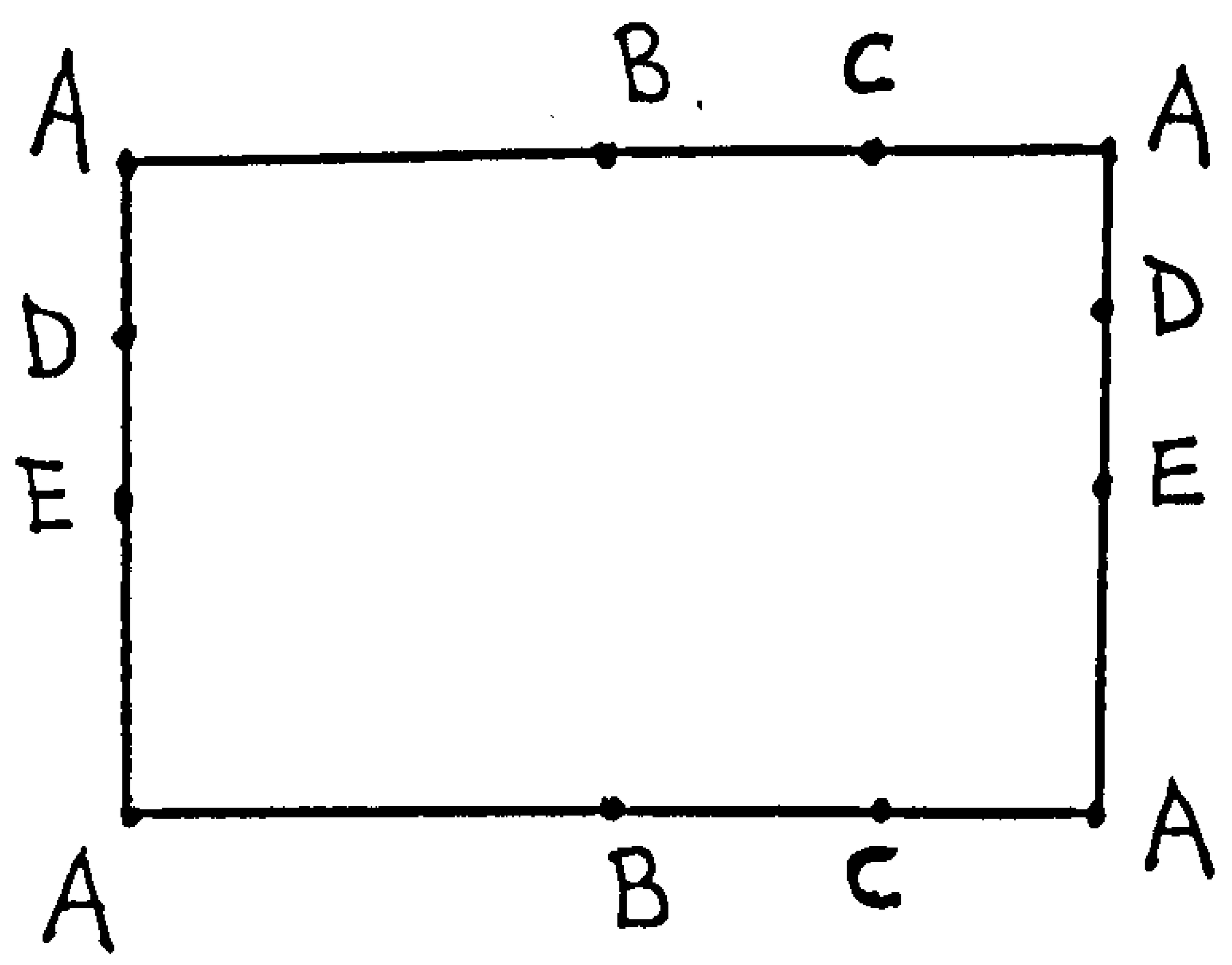


figure 18

The torus of figure 17 has been cut along the circles AF and AE. This leads to the following dissection:

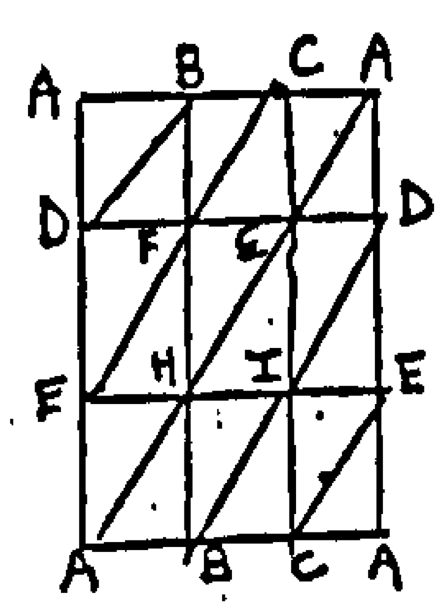


figure 19

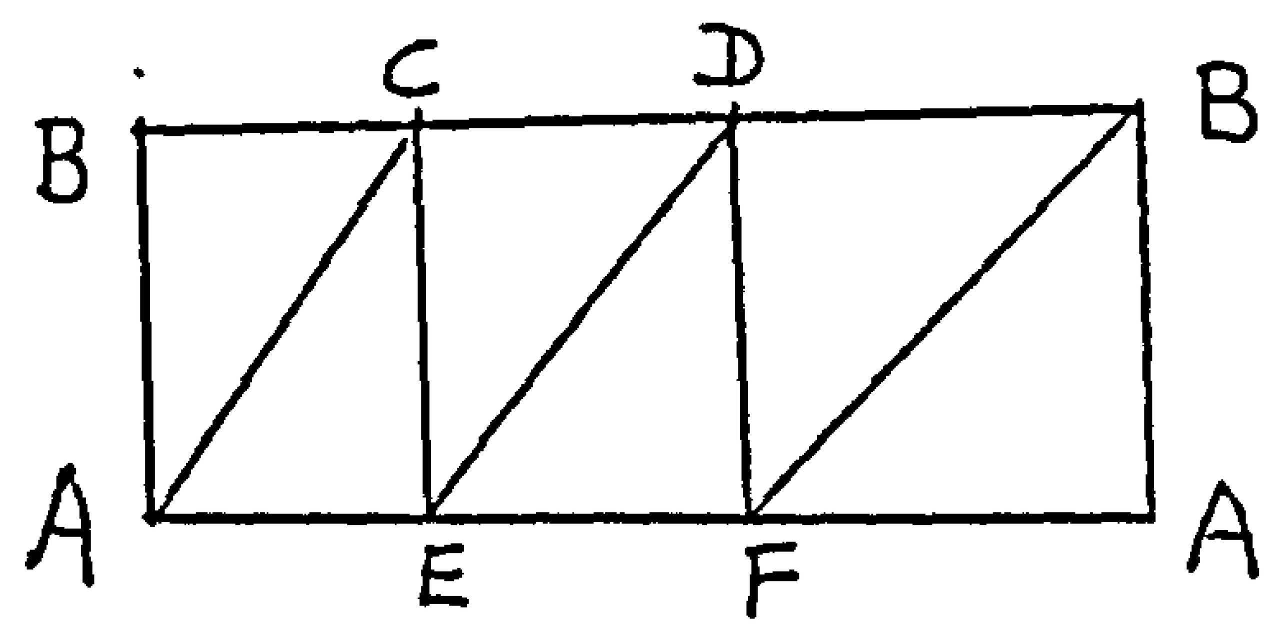
in which $\chi = 9 - 27 + 18 = 0$ because V = 9, E = 27 and F= 18.

<u>EXERCISE 3</u>: There is a minimal dissection of the torus with V = 7, E = 21 and F = 14

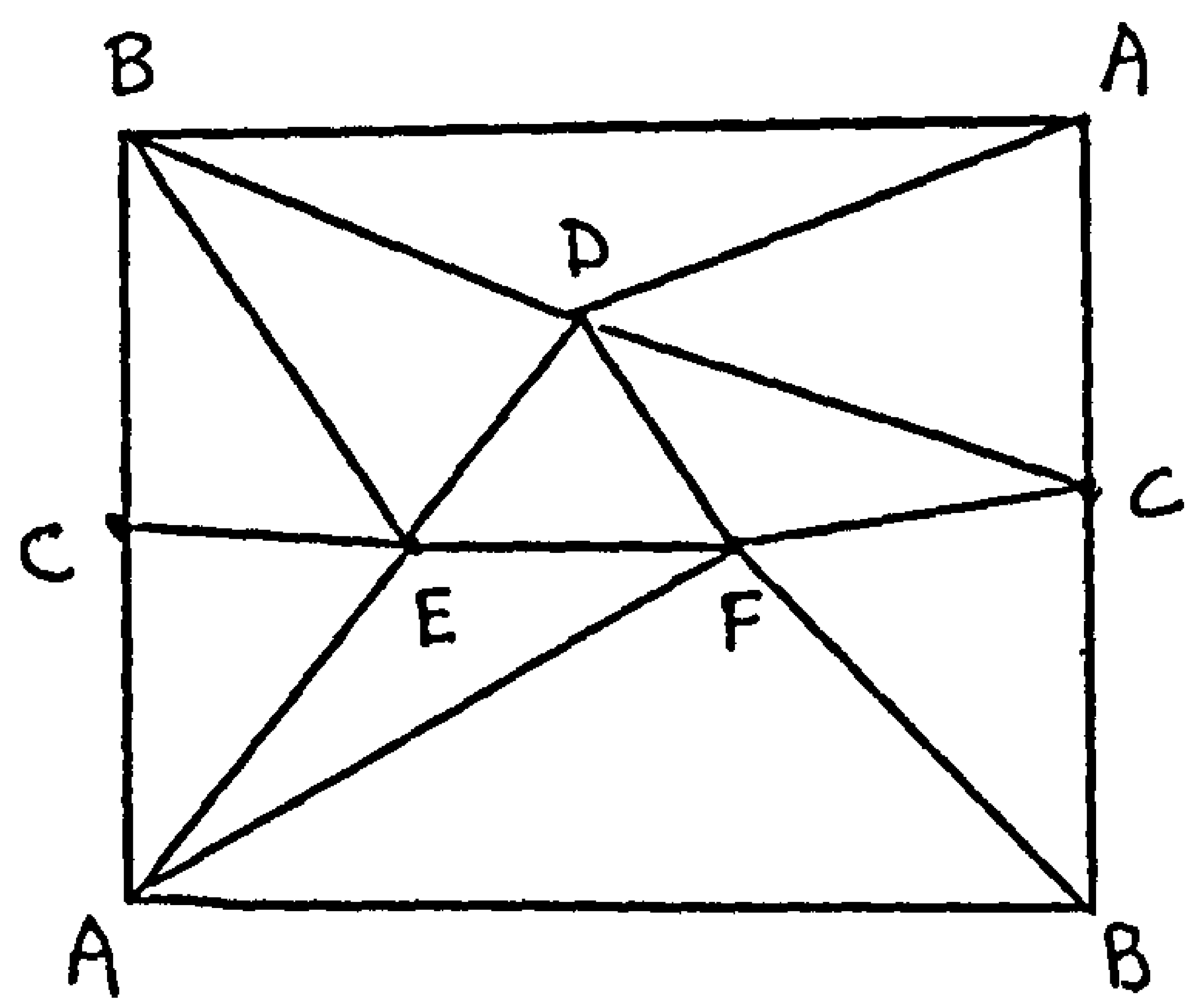
Answer to the exercises

EXERCISE 1

Minimal dissection of a cylinder



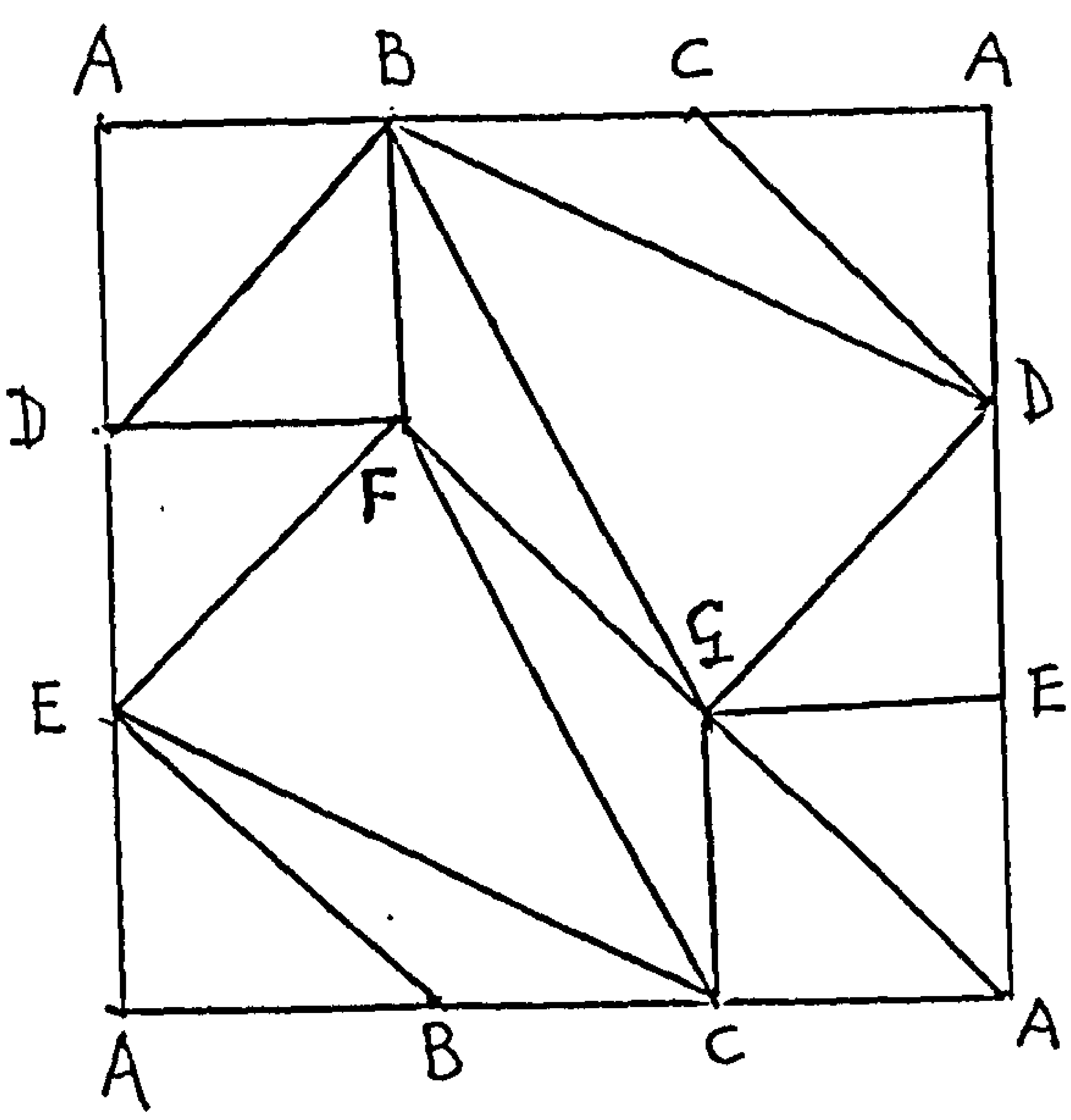
EXERCISE 2



Minimal dissection of a projective plane

EXERCISE 3

Minimal dissection of a torus



CHAPTER XI

Visual Geometry in Primary School Education

For children attending Primary Schools, space has a special meaning because these children are growing. When they are small, at about the age of 3, the rooms of their parental home seem to be castle-like. Some years later these children will notice that the enormous spaces of their early years have shrunk to more 'normal' dimensions, meaning more standard according to grown-up sizes. Therefore I don't think that young children should be bothered with questions like: 'Try to choose a unit distance'. The real unit distance of a young child changes rapidly. Measured by its own height, to a child the world shrinks every day and such a transition cannot be translated into a choice of standard distances determined or chosen by the child itself.

On the contrary: at the age of say 7 or 8 years the child needs a stable unit determined not by itself, but by the grown-up world. Then it can measure its present size and estimate its future size.

One has to remain aware of the relativity of distances. In the classroom a distance of 10 metres often covers the full width of the room, but outside the school a distance of 10 metres seems to be much smaller, compared to other, visually larger distances in the open air. For children, some inverse growth/size ratio seems to be valid: the bigger you grow, the smaller the surroundings become. According to Jean Piaget, spatial images have a privileged character.

"In the case of spatial images, on the other hand, the contents to be presented are spatial like the imaged forms which represent them and spatial operations (displacements, projections, etc.) are again figurative transformations and so, in a sense, figures in space. There is then more or less complete homogeneity between form and content and this is enough to account for the privileged character of spatial images." (Piaget, 1982, page 681).

The growth of a child can be measured in centimetres so the path on the way to maturity can be expressed in the number of centimetres indicating the child's height. In this case a spatial operation (measuring the child's height) does not only reflect its physical growth but also its mental growth.

In the above quotation, there is a reference only to the mathematical aspect of spatial figures but not to the psychological impact of a shrinking space to a growing child.

Before me on my desk I have a copy of a geometry textbook written in 1887 in The Netherlands. It is called "Aanschouwelijke Meetkunde", which means 'Visual Geometry'. The visual character of the book is emphasised in the introduction where we read: "In the treatment of the content, that means the geometrical bodies, we have taken as a base the perception of figures as much as possible". (Koenen, 1887, page 3). The same introduction says: "Above all it was our aim to represent the knowledge of general properties and practical validity in the most simple way and not through the dense maze of a series of reasoning."

"Aanschouwelijke Meetkunde" is a textbook for those preparing to sit an examination to become a primary teacher. In the early nineteenth century there were no Colleges of Education in the Netherlands and the aspiring teacher was educated by the head of the primary school where he had been a pupil. The principal had had the same kind of education. One was allowed to start studying to become a primary teacher at 14 so that hardly any secondary school knowledge was required. In 1878 an Education Bill was passed by the Dutch parliament to enhance the level of the Primary School teachers' education. The education of young children was not made compulsory until 1901 (and then with a parliamentary majority of only one, because one of the 'no' voters had caught a cold and stayed at home).

"Aanschouwelijke Meetkunde" comprised 80 pages and 70 figures, of which five have been copied below. Practical constructions are demonstrated.

figure 1

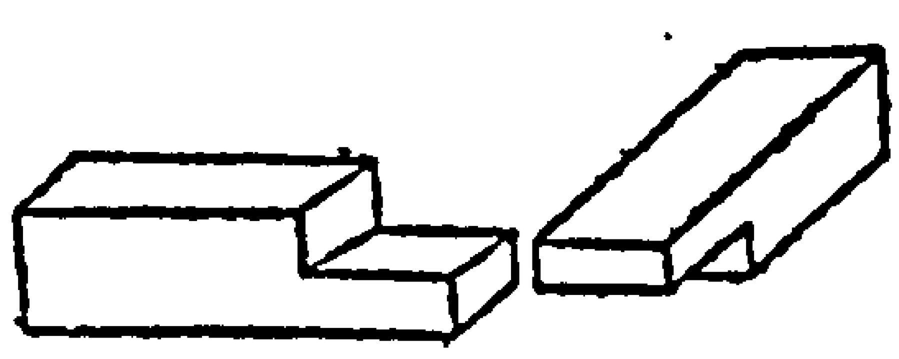


figure 4

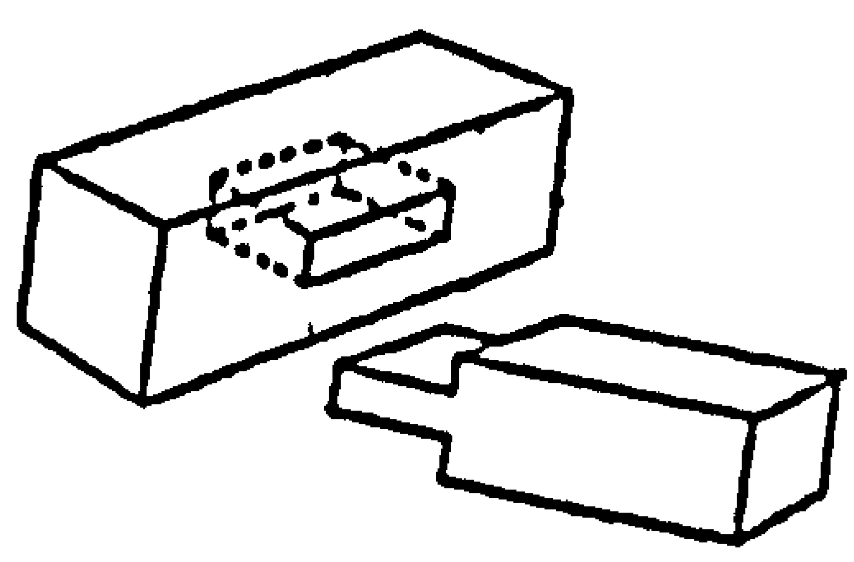


figure 2

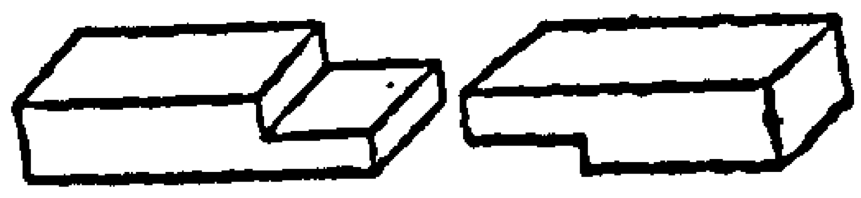


figure 5

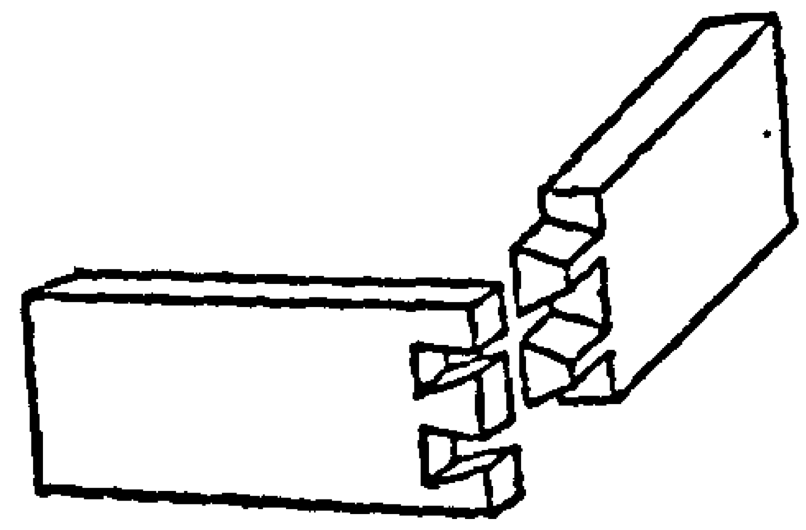
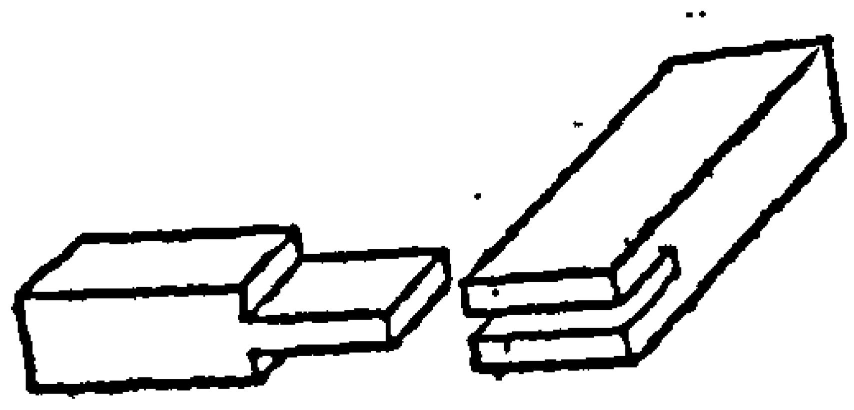


figure 3



Among the users of the textbook were 14-year olds who had hardly had anymore than primary school education. It is understandable that the authors could not bother such students with complicated reasoning. Such reasoning, as Professor Barrau (Barrau, 1918,page 11) had testified, formed a thorn-hedge which could only be handled in a cursory and superficial way. From the figures 1 to 7 we see that on the one hand geometry is practical but also that conic sections are portrayed, which represent a more theoretical mathematics.

figure 6

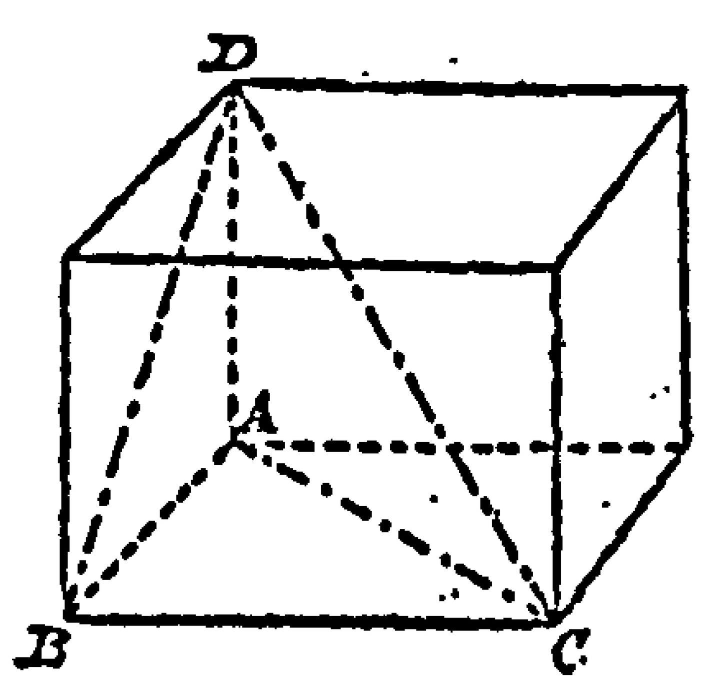
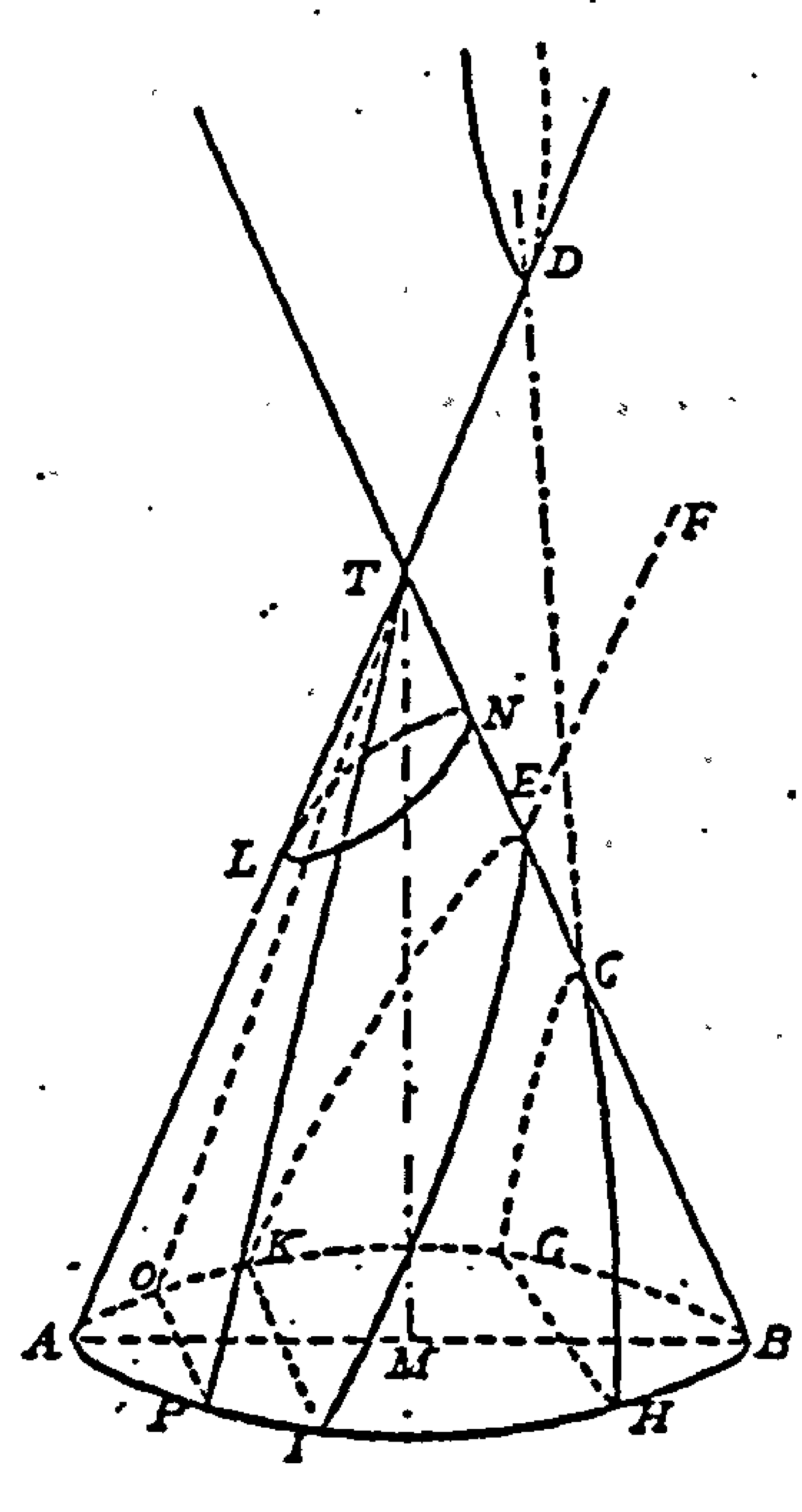


figure 7



Figures 1 to 7 are photocopies taken from "Aanschouwelijke Meetkunde".

It is precisely because the education of future primary teachers in the 19th century was little more than basic that knowledge had to be conveyed visually. For that reason a visual geometry was developed in textbooks like "Aanschouwelijk Meetkunde" which emphasised practical usage and simple explanations. From figure 6 we see that it is educationally valid because a horizon is used, although it is not visible.

Many practical exercises are found in the book so that a reasonable skill at practical geometry can be obtained by working through it. This kind of geometry education lasted until the Second World War. During that period (1887 - 1940) Primary School teachers qualified to teach at the Colleges of Education, where their future colleagues were educated. This changed in the course of the nineteen-fifties, when it became education policy to boost the level of students at Colleges of Education and teachers at such Colleges were required to have a university education or a comparable background.

Now we will consider a textbook "Rekendidactiek", which in English means "Education in Arithmetic". It was edited in the Netherlands in 1965 and written by P. Woestenenk."Education in Arithmetic"was written for use by students of Colleges of Education, where future primary teachers were educated. Geometry had become a part of arithmetic and the subject comprised 9 pages. Spatial figures were removed completely. The geometry section of the textbook did not have more than three pictures which are copied below (figures 8,9, and 10).

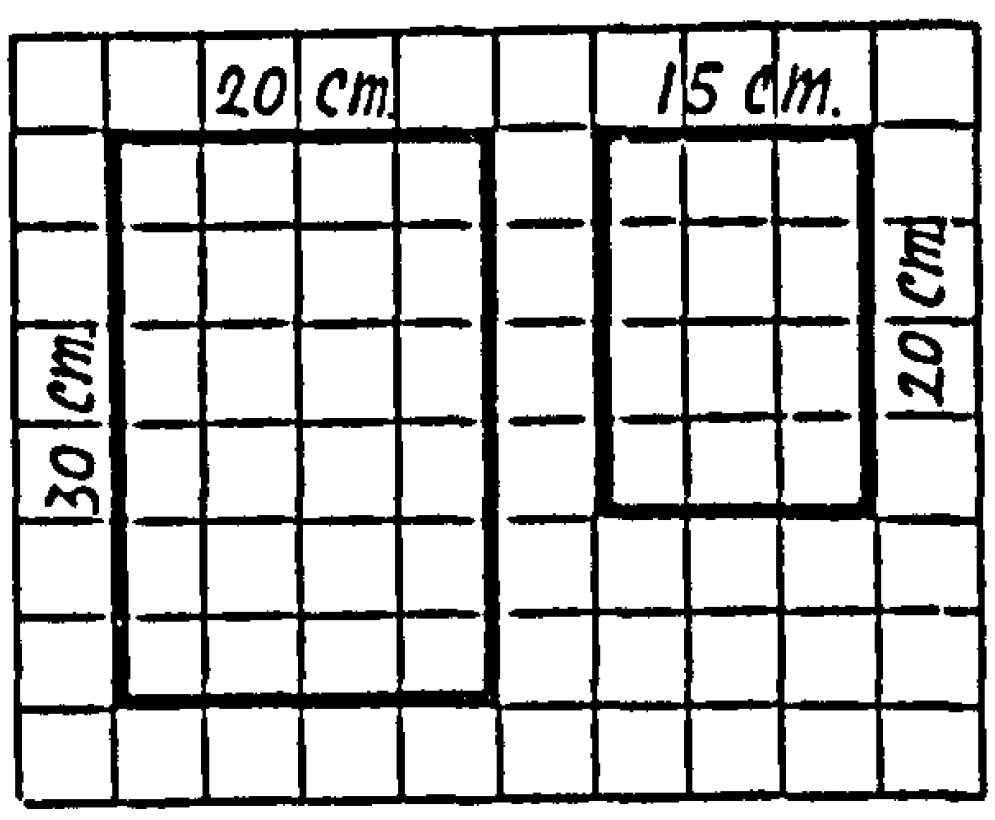


figure 8

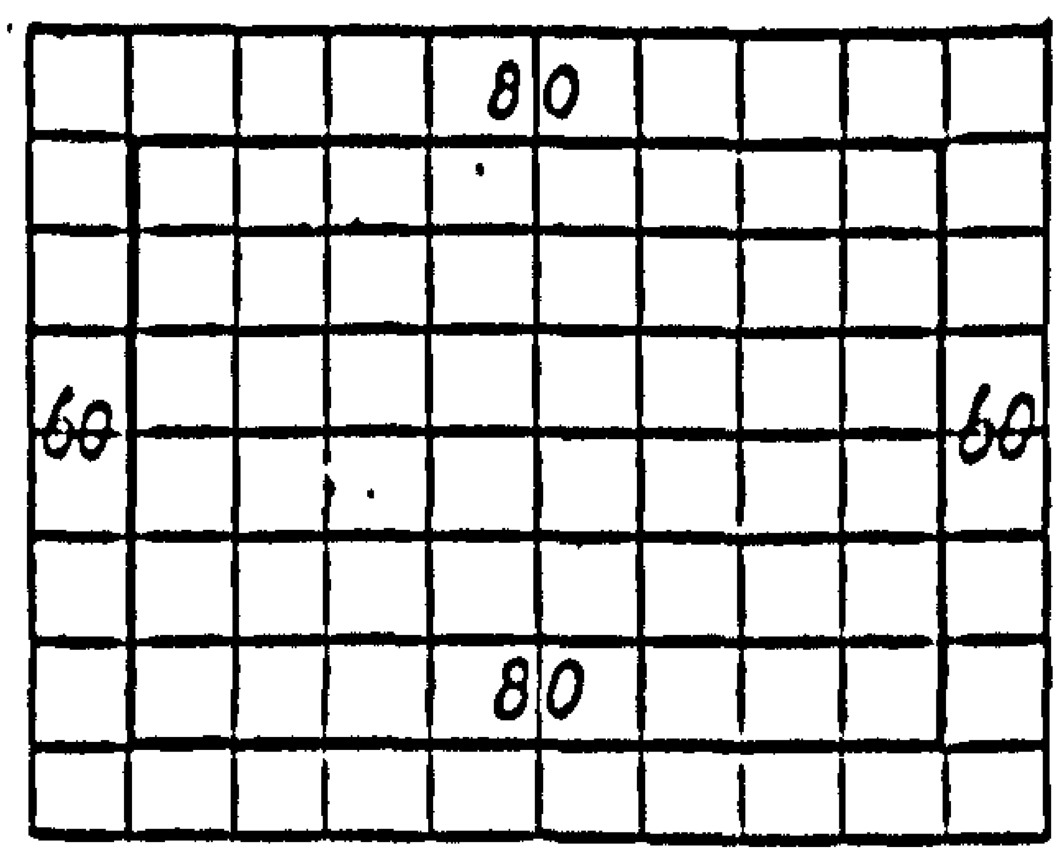


figure 9

Figures 8 & 9 are photocopies from Woestenenk, 1965.

Figures 11 to 16 are photocopies, taken from Nauta, 1946. It is an exercise book on arithmetic, in use at primary schools in 1965, entitled: Fundamenteel Rekenen (which means: Basic Arithmetic).

figure 10

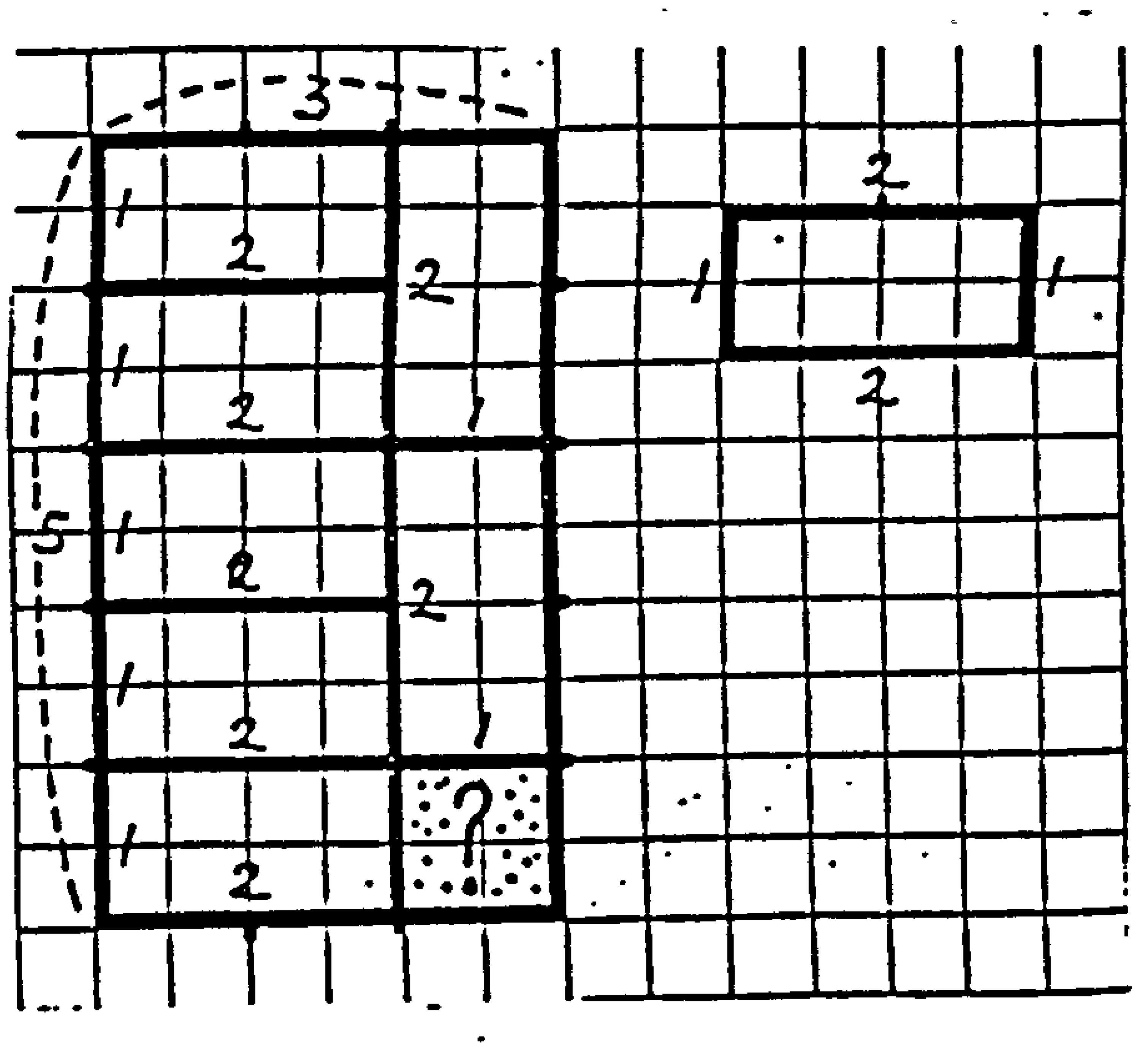


Figure 10 is a photocopy taken from Woestenenk, 1965.

figure 11

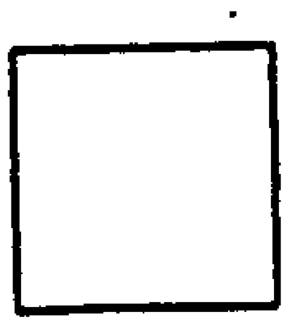


figure 12

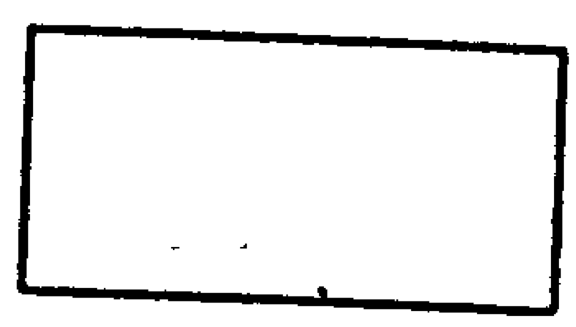


figure 13

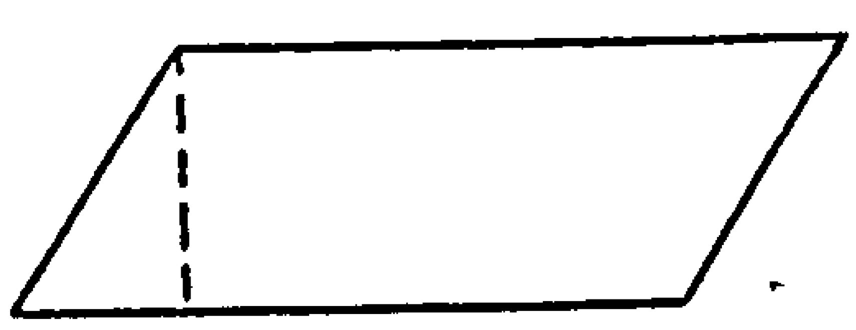


figure 14

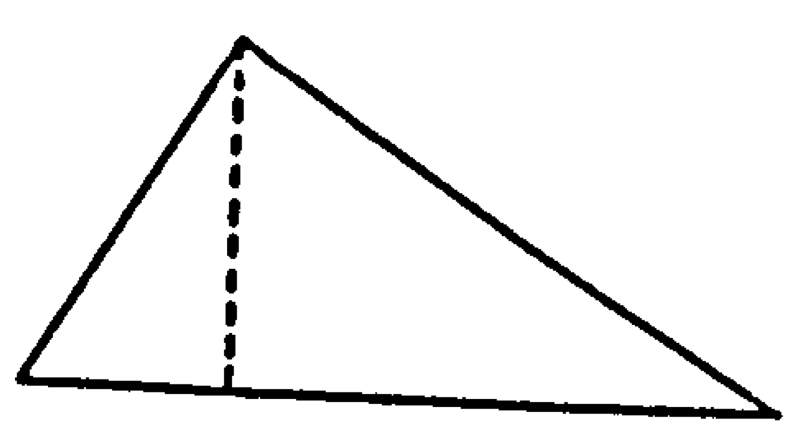


figure 15

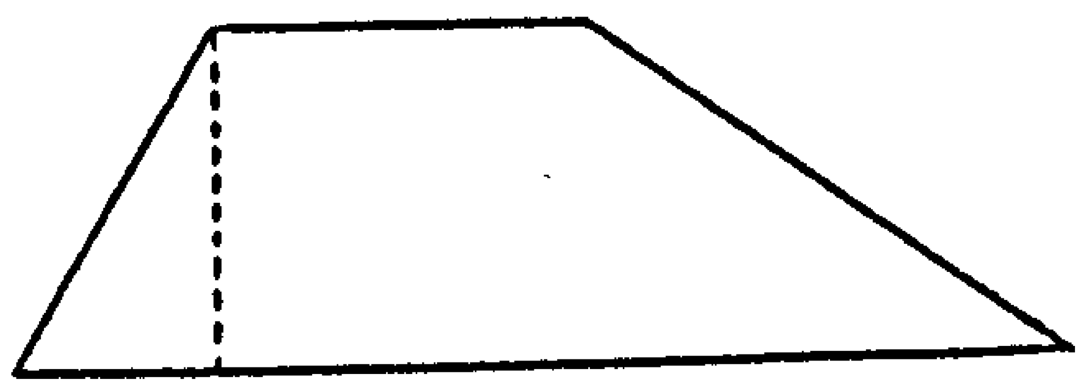
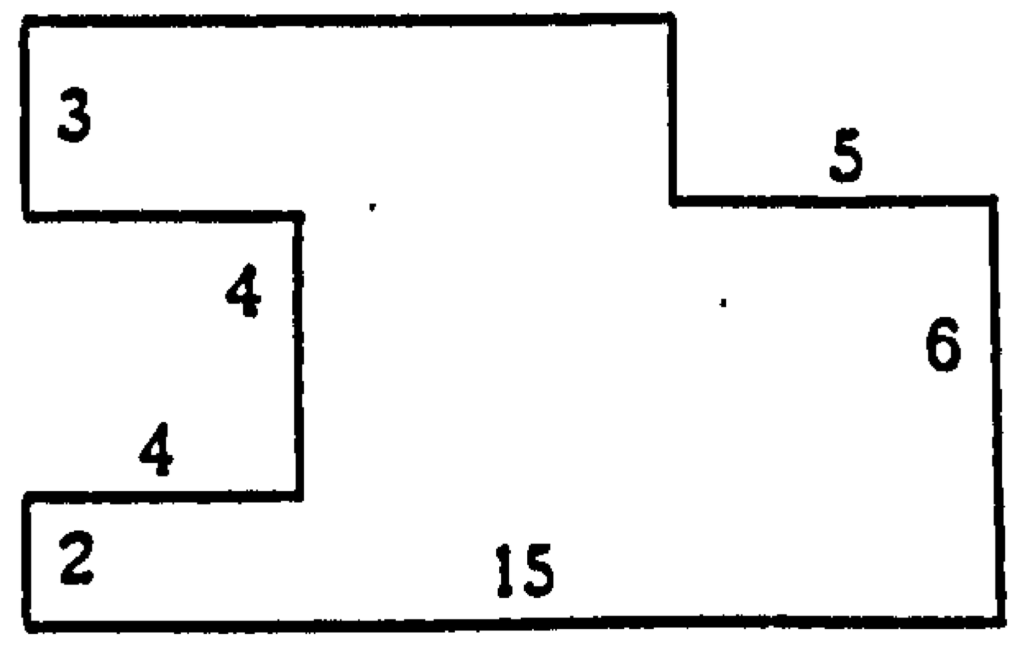


figure 16



Let us consider for a moment how these examples from 1887 and 1965 relate to the 'educational geometry', developed in Part I. The visual geometry of 1887 not only refers to applications in carpentry; it also demonstrates conic sections and it even handles perspective. So essentially it provides a visual picture of some topics of the science of geometry.

These portrayals however are not much more than illustrations, with respect to the scientific content of geometry. The pictures are cartoon-like. No proofs are given; Pythagoras is not even mentioned. There are formulae to compute the volume of spheres and their parts. We should notice that "Aanschouwelijke Meetkunde" was written for students of what we would now call Higher Education. In those days (1887) the education of primary school teachers was considered as primary education. So the material used for the teachers' education was not too different from what the primary school pupils had to learn, although the teacher, of course, was expected to have a better command of it.

This is very different from the 'educational geometry', introduced in my thesis. There the Theory of Groups, Duality and Topology and the differences between Local and Global Geometry are highlighted. The notion of a horizon appears.

In some respects the 'Educational Geometry' is unprecedented, but there are of course many links to existing curricula of geometry. Considering Woestenenk's textbook (1965) it appears that he sticks to the abstract approach to geometry, such as: 'area = length x breadth'. In my view this is not a very fruitful approach for primary school purposes. On the next pages some examples will be demonstrated of a more modern and better explanation of geometry at primary school level. These examples are produced by 'Wiskobas', a renewal movement which was active in The Netherlands from the late nineteen-sixties until about 1980.

So my conclusion is that these snapshots which I have shown from the history of Primary School Geometry make clear that 'educational geometry' has roots in the primary school curriculum as far as the visualisation of geometry is concerned, but that the treatment of geometrical knowledge in old-fashioned education and 'educational geometry' is essentially different and they cannot be compared with each other.

On the next two pages more modern developments and approaches, a result of the Wiskobas movement will be demonstrated. It yielded a new presentation which also meant that the visual aspect of geometry was emphasised more. The aim of Wiskobas was not only to improve primary school mathematics but also to enforce changes at secondary school level. Their approach to mathematics can be summarised in the word 'Realistic' which means that it should be related to the world around us and not exclusively to material in textbooks. With new ideas implemented at primary school level, secondary schools would have no choice but to continue that process. The Wiskobas movement was terminated in 1981 by the Ministry of Education as a result of cuts in financial aid. Nevertheless the concepts are still here and they may be implemented further. The realistic approach to mathematics has become a new area for teachers who dislike endless exercises of a similar kind. Instead they may produce their own materials according to their own tastes.

Some examples of realistic pictures, used for educational purposes, are displayed.

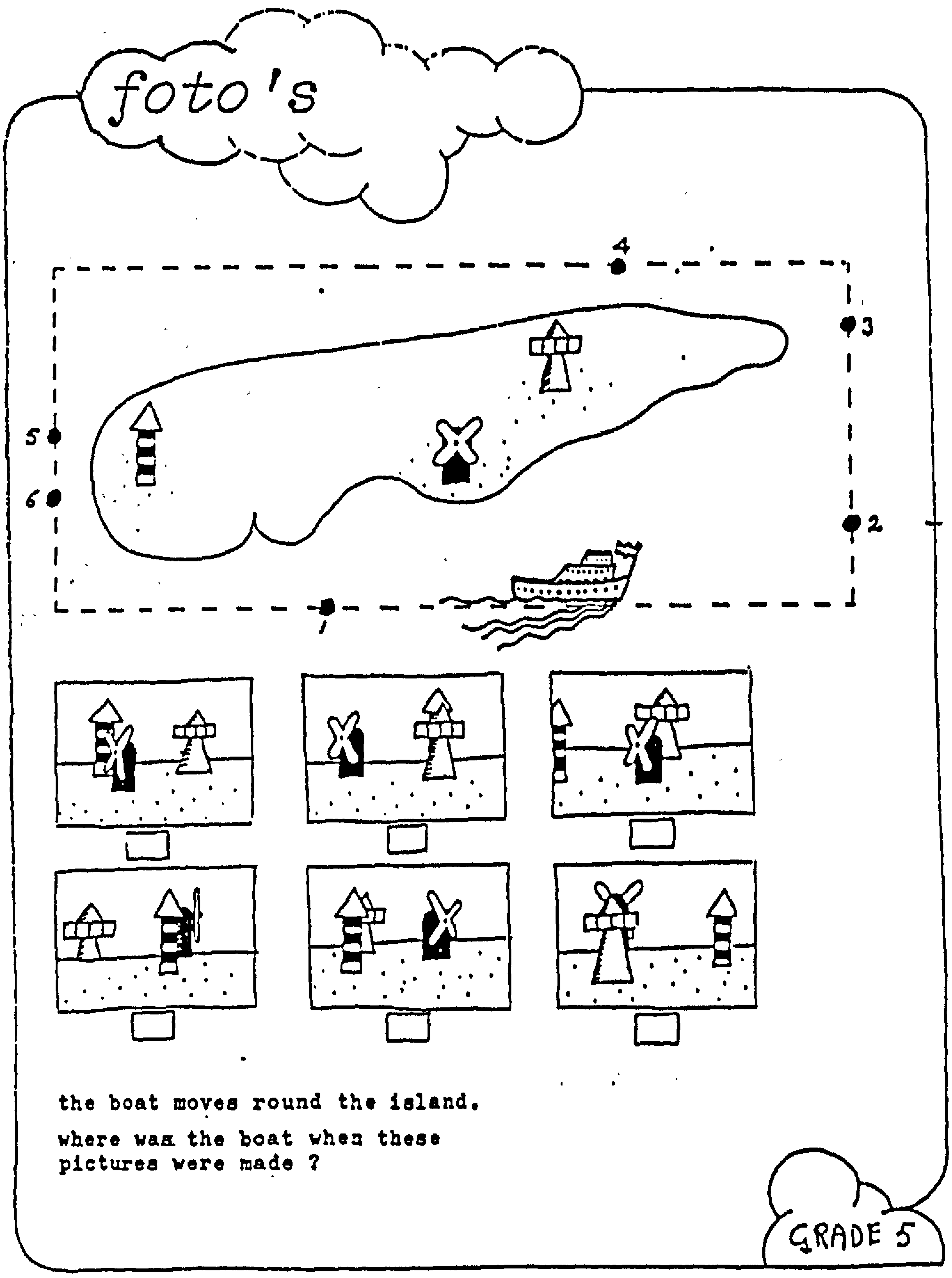
Example 1 was produced in 1977 by the Institute of Development of Mathematics Education in Utrecht; example 2 has the same origin. In example 1 a global notion is involved: the reader is looking towards the horizon. The line along which the observer watches is called 'line of sight' which nowadays has become an important idea in secondary education of geometry. It will be discussed in Chapter XII.

Example 2 is really educationally valid: it has perspective and so it uses a horizon which is not visible but can be conjectured. Notice that if we assume that the source of light is left of the mountain in example 2, then one gets a different view, as often occurs in spatial images. Nowadays at Colleges of Education for primary school teachers a textbook in 3 volumes written by Fred Goffree is generally in use, entitled "Wiskunde & Didactiek" (Mathematics and Didactics). Much more attention is paid to geometry than in 1965, especially its spatial aspect. From this book we show 2 examples: 3 and 4 (Goffree, 1986, Vol. II, pages 138 & 142). The question is whether the verger, living in the black house indicated by an arrow in Example 3, is able to see the church bells (Example 4).

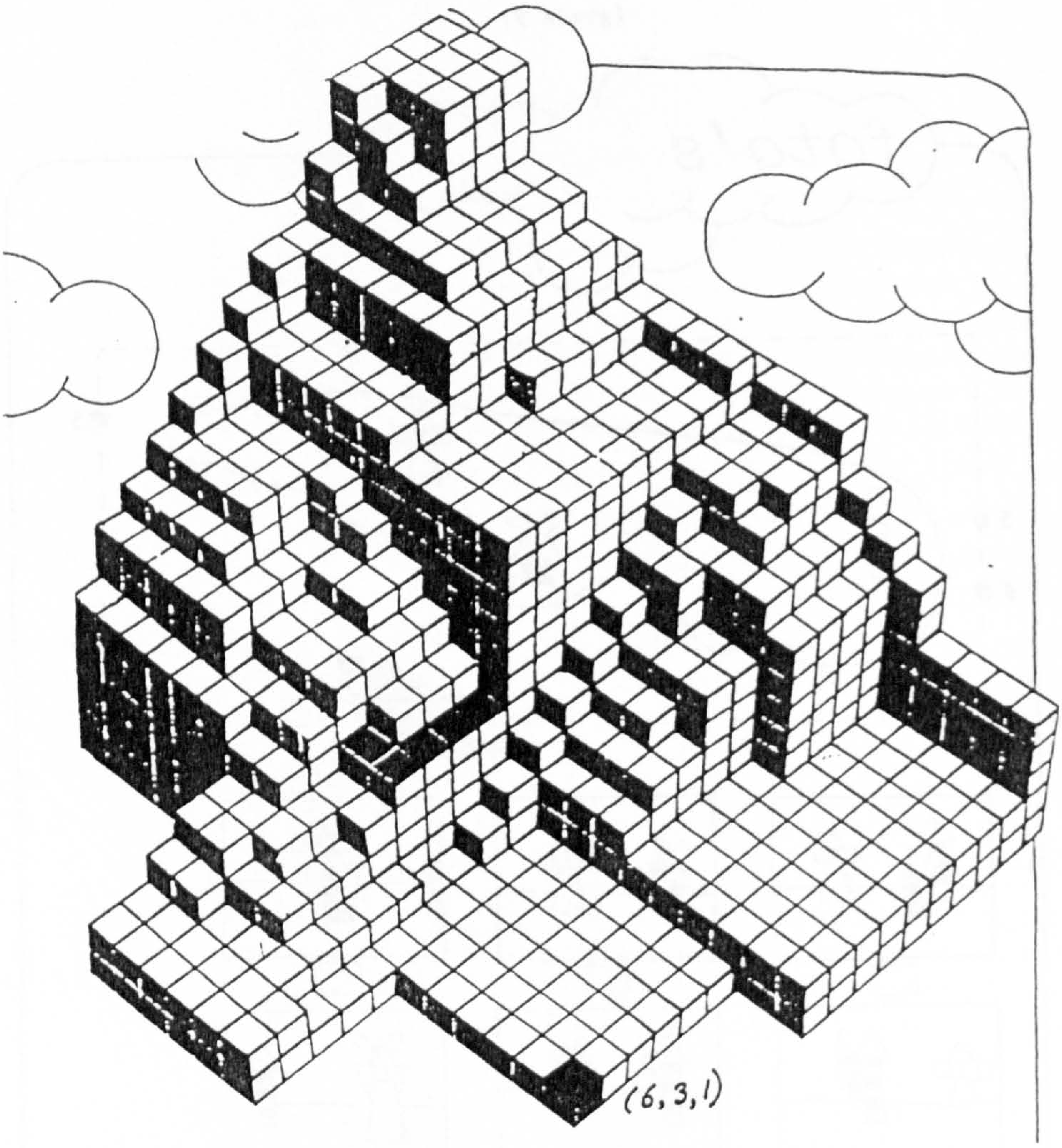
In the next two chapters (XII and XIII) attention will be paid to the visual aspects of geometry education at secondary school level. Issues from the secondary school curriculum will be investigated for useful applications of 'Educational Geometry'.

Example 1 was produced in 1977 by the Institute of Development of Mathematics Education (IOWO) in Utrecht, The Netherlands.

Example 1 of Wiskobas Geometry
(grade 5)



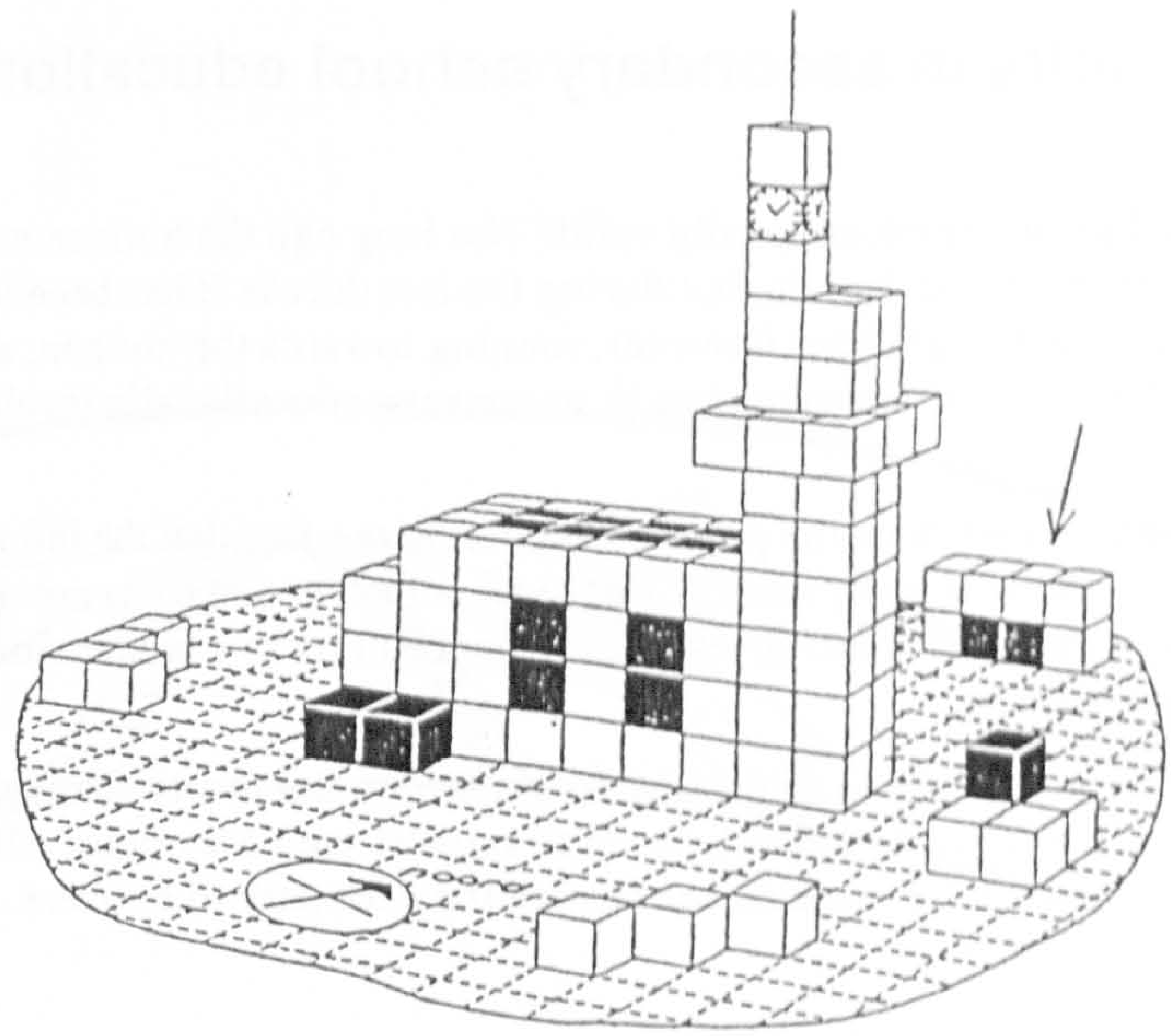
Example 2 of Wiskobas Geometry
Grade 8



Where is the location (12,11,11) ?

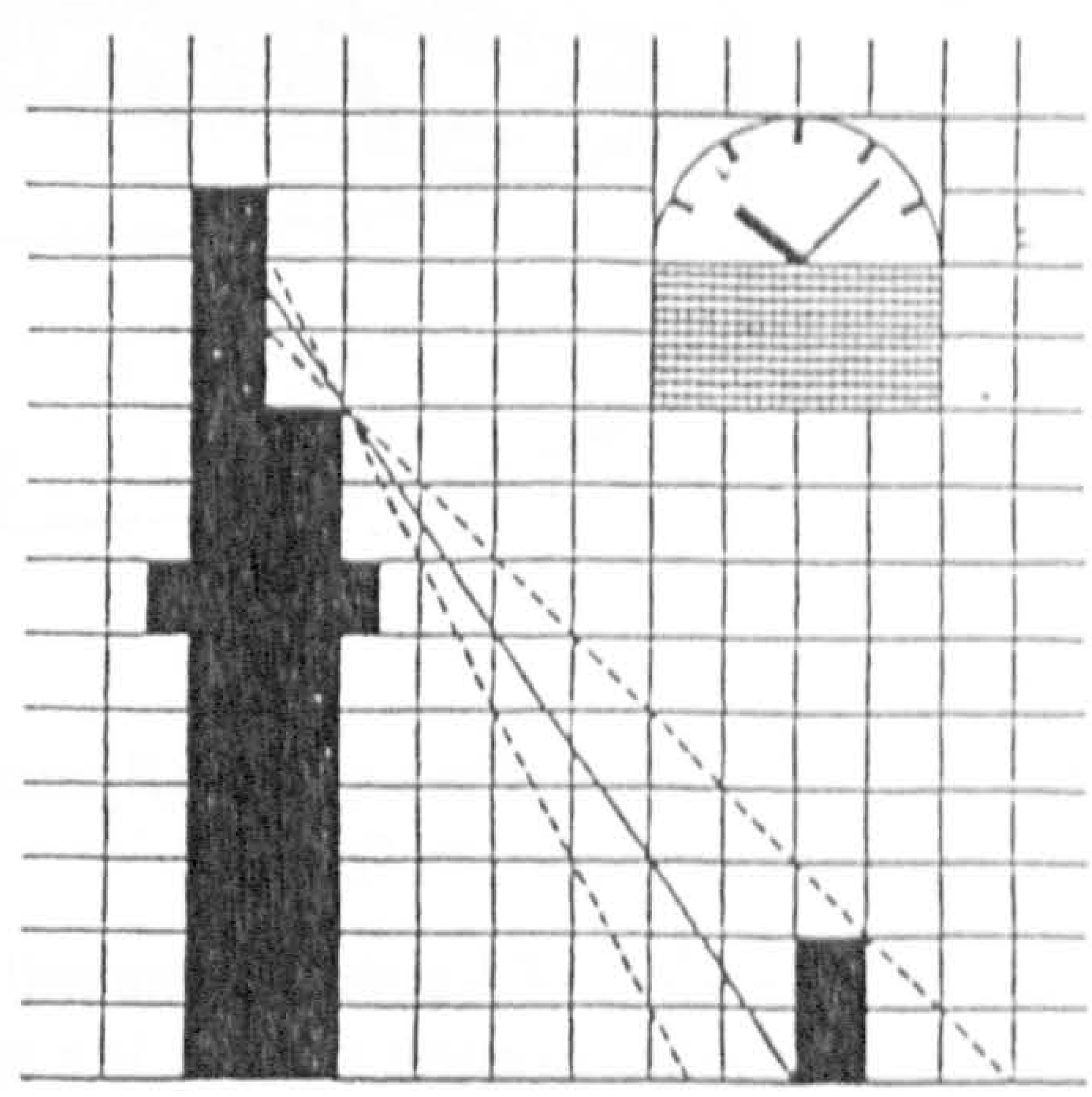
Example 2 was produced in 1977 by the Institute of Development of Mathematics Education (IOWO) in Utrecht, The Netherlands.

EXAMPLE 3



Example 3 is a photocopy taken from Goffree, 1986, Vol II, page 138

EXAMPLE 4



Example 4 is a photocopy, taken from Goffree, 1986, Vol II, page 142

CHAPTER XII

Visual geometry in secondary school education

Is secondary school geometry educationally valid? Not long ago the horizon was not dealt with in the secondary school geometry curriculum, but during the last decade it has been introduced in some schools in The Netherlands. The straight lines however, running towards the horizon, are not bowed but presented as 'visual straight lines'. Such a presentation is, as we saw, educationally invalid.

Does this matter? Should we draw the pupils' attention to the fact that the information in the pictures in their geometry textbooks is actually wrong? For most pupils, geometry is not a subject which they will need in their future careers. That fact gives one the opportunity to pay attention to unconventional subjects, such as perspective.

It would certainly be worthwhile to discuss in the classroom the fact that straight lines are often visually curved. Examples of photographs might be used to support such assertions. Allusions to non-Euclidean geometry may be presented; for instance concerning the geometry of a sphere.

In The Netherlands a study about geometry education entitled "Achtergronden" appeared in 1992 in which the mathematics secondary school curriculum designed for the coming decade was treated. The picture of figure 1 is a photocopy, taken from Abels, 1992, page 53, figure 24.

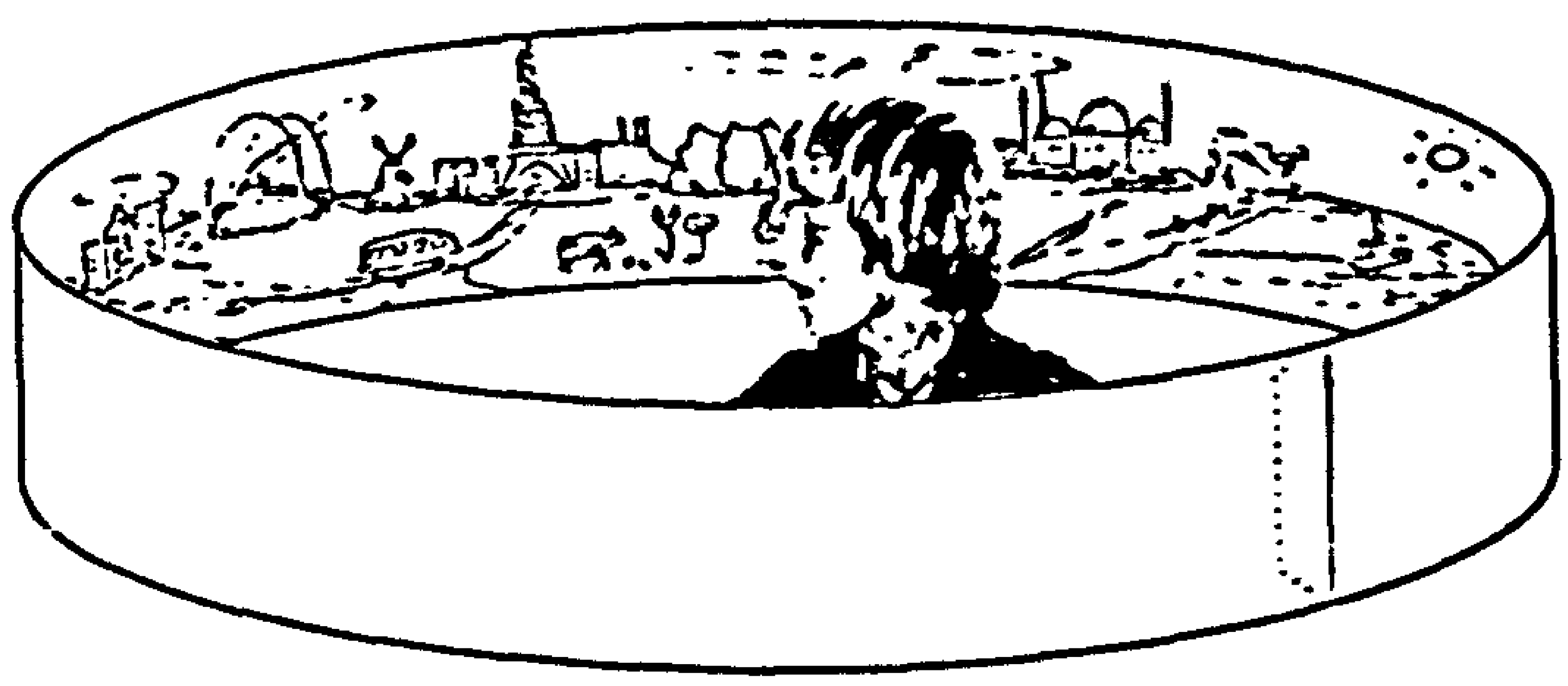


figure 1

The screen is cylindrical and the observer stands at the centre (figure 1). A straight line on a plane, running towards a plane horizon, behaves as shown in figure 2.

The view of the picture in figure 1 and the view of figure 2 apparently do not differ much, although the horizon in figure 1 is supposed to be a horizon on earth (which is a sphere) and the horizon in figure 2 is the horizon of a large plane.

There are developments in secondary school education which predict a more dominant role for visual geometry in the geometry curriculum. The book "Achtergronden van het Nieuwe Leerplan Wiskunde *12 - 16* " (Backgrounds for the new curriculum of Mathematics 12 - 16) has appeared, edited by the Freudenthal Instituut RU Utrecht/SLO Enschede, aiming to renew secondary school mathematics in The Netherlands for 12 - 16 year olds (author: Abels & others). The new curriculum must be implemented at all secondary schools in The Netherlands. In the new curriculum the traditional presentation of a cube is condemned (Abels, 1992, page 58) for being drawn strangely confusing in that some angles are drawn to conform to

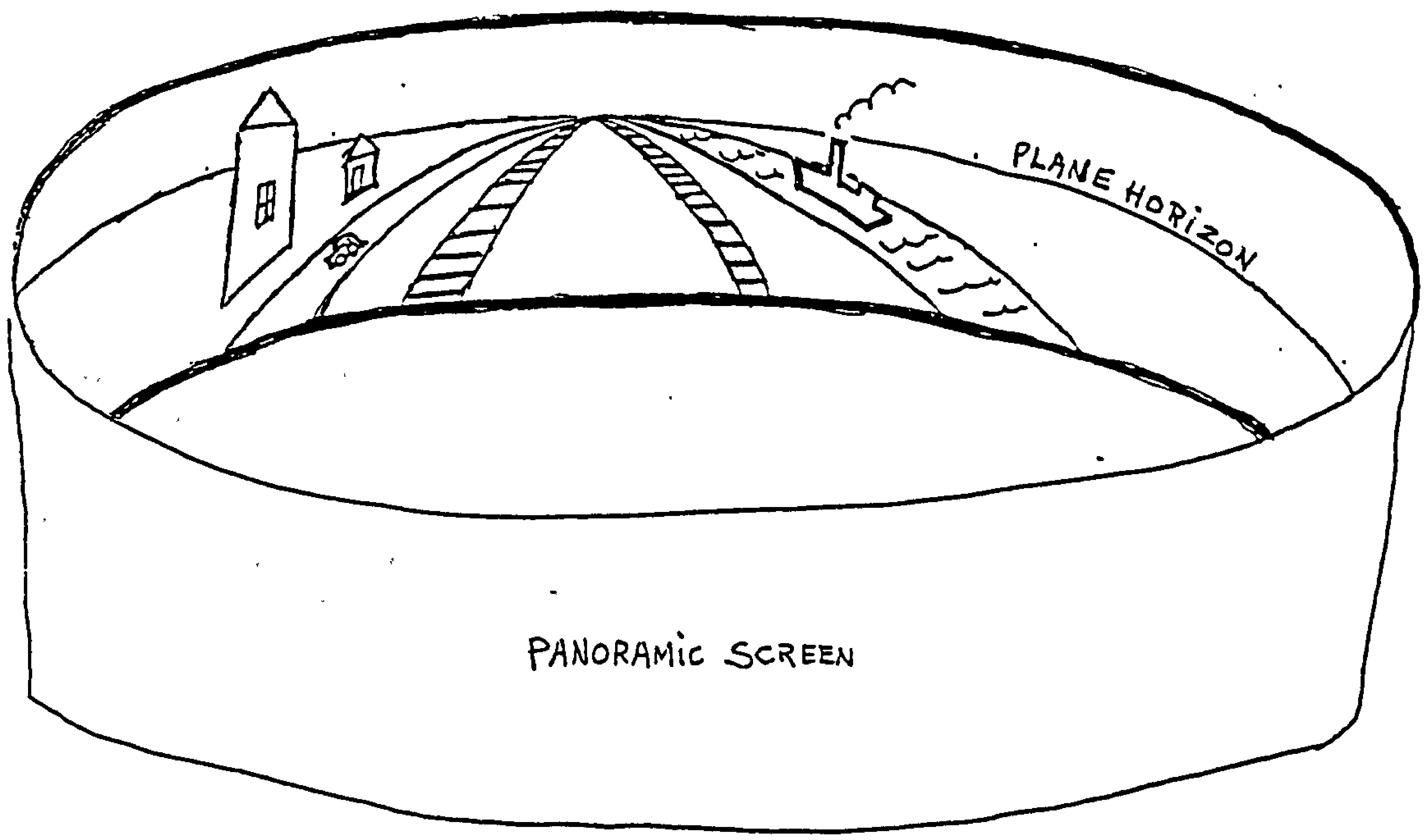
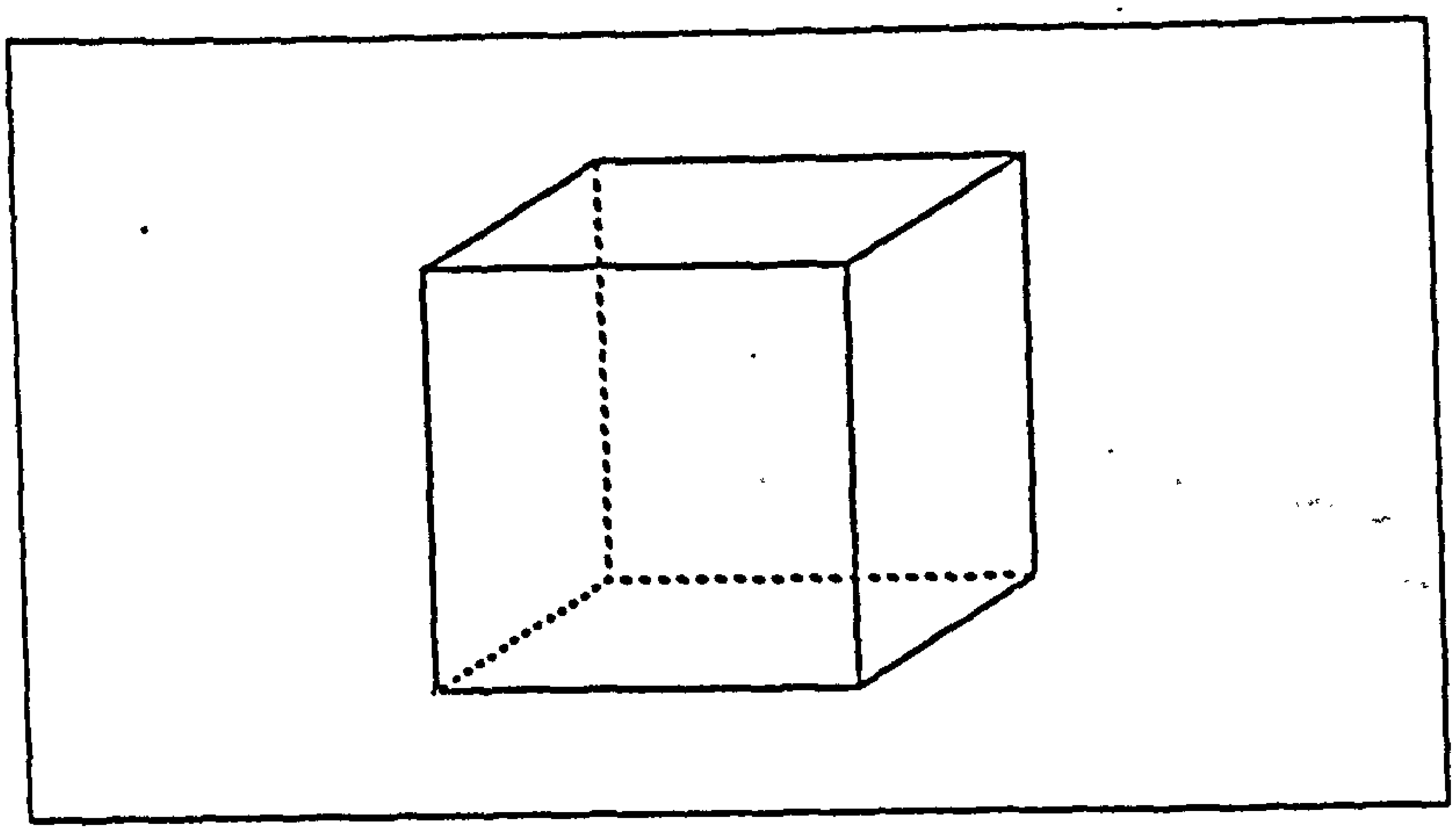


figure 2

reality but others are not (see figure 3, which is a photocopy from Abels, 1992, page 58). There is no reference to the visual beauty of the cube in figure 3; nothing about Magic Realism.

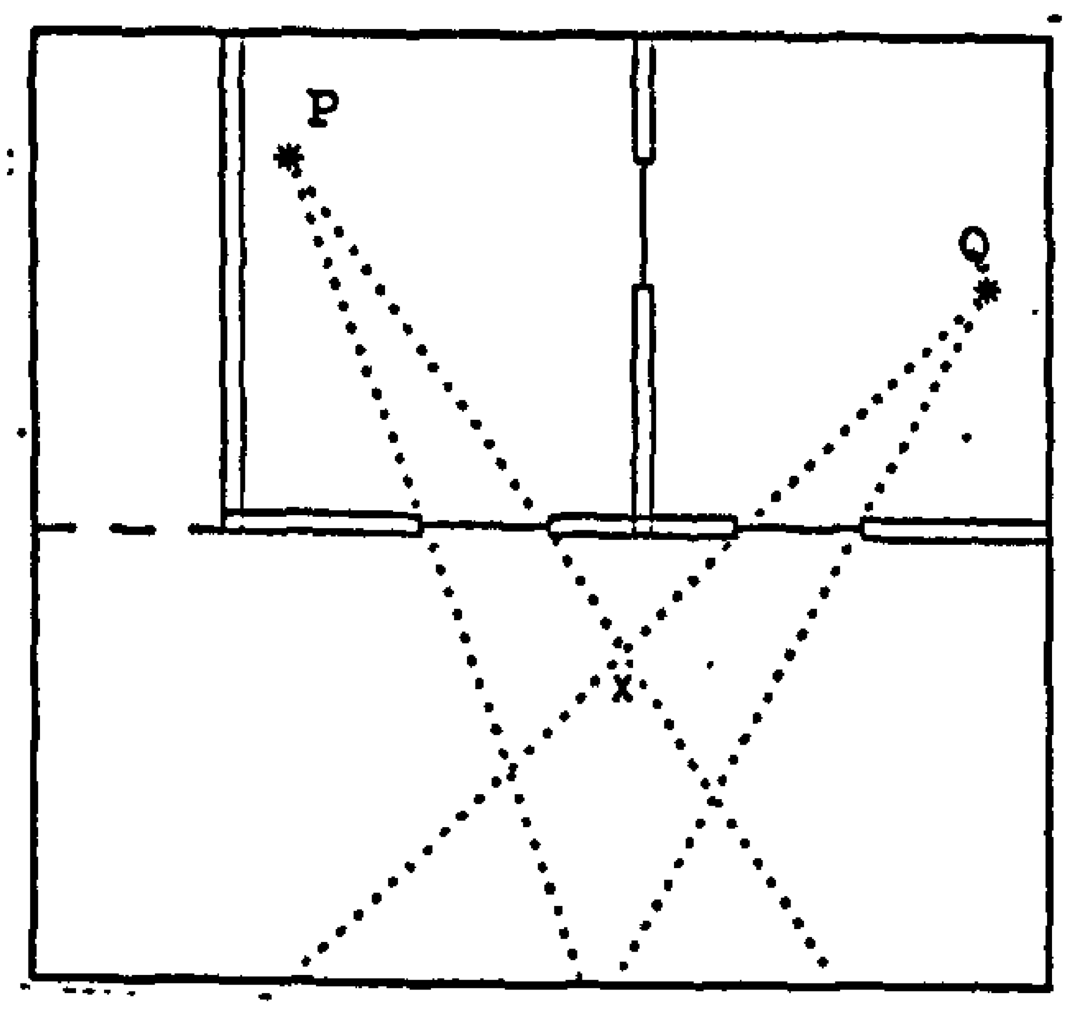
figure 3



I would suggest that we should not abolish the old-fashioned presentation of figure 3 completely, because it sticks to the mind as a result of its beauty. Why should this beauty be wasted by ignorance or thoughtless attempts to change everything?

Further, there are the so-called 'lines of sight' which denote the straight lines seemingly emanating from the eye and hitting the objects. In figure 4 we have two persons P and Q looking out of different windows but observing part of the same area. The letters P and Q are added by me to the original picture, which is a photocopy, taken from *Abels*, 1992,page 28.

figure 4



From the figures 5 & 6 (photocopies, taken from *Abels*, pages 32 & 33) it becomes quite clear what 'lines of sight' are. For some children it is a real discovery when they are watching from such a point of view and observe the objects coinciding.

figure 5

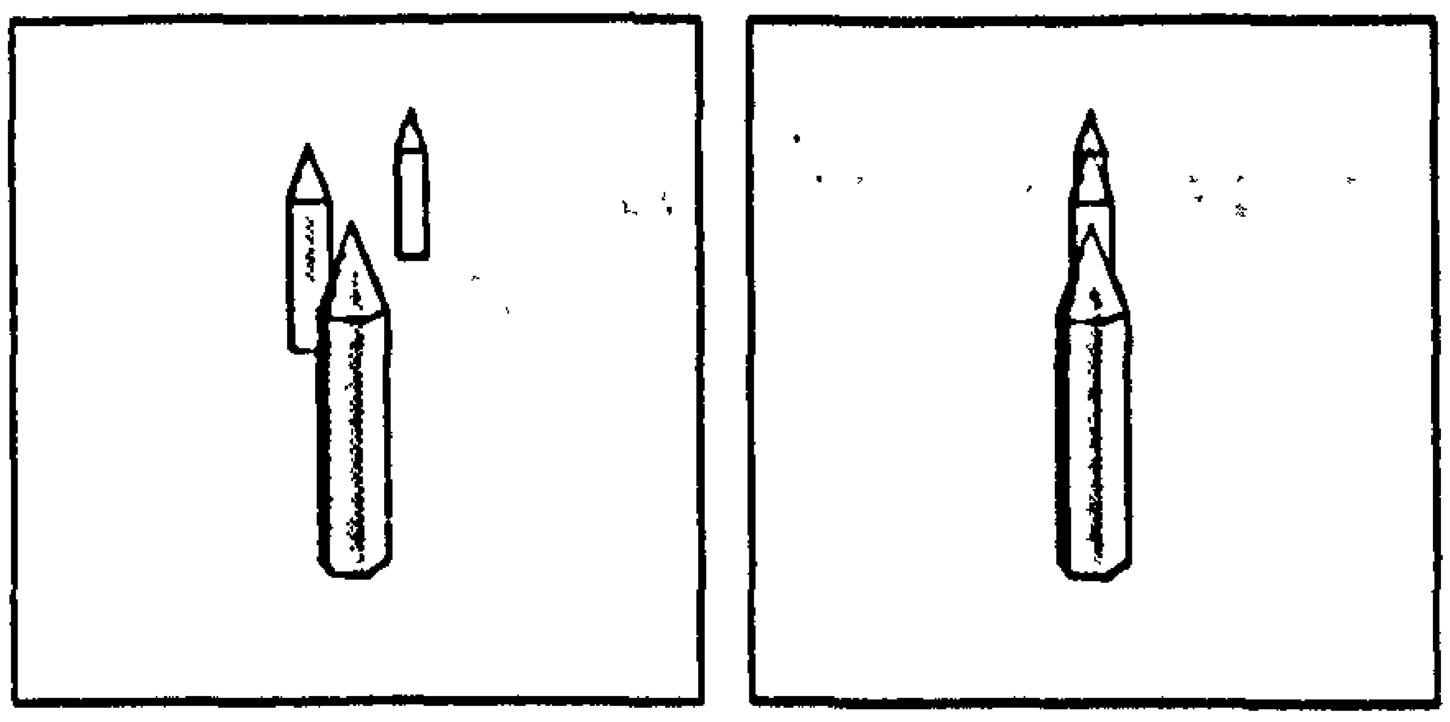
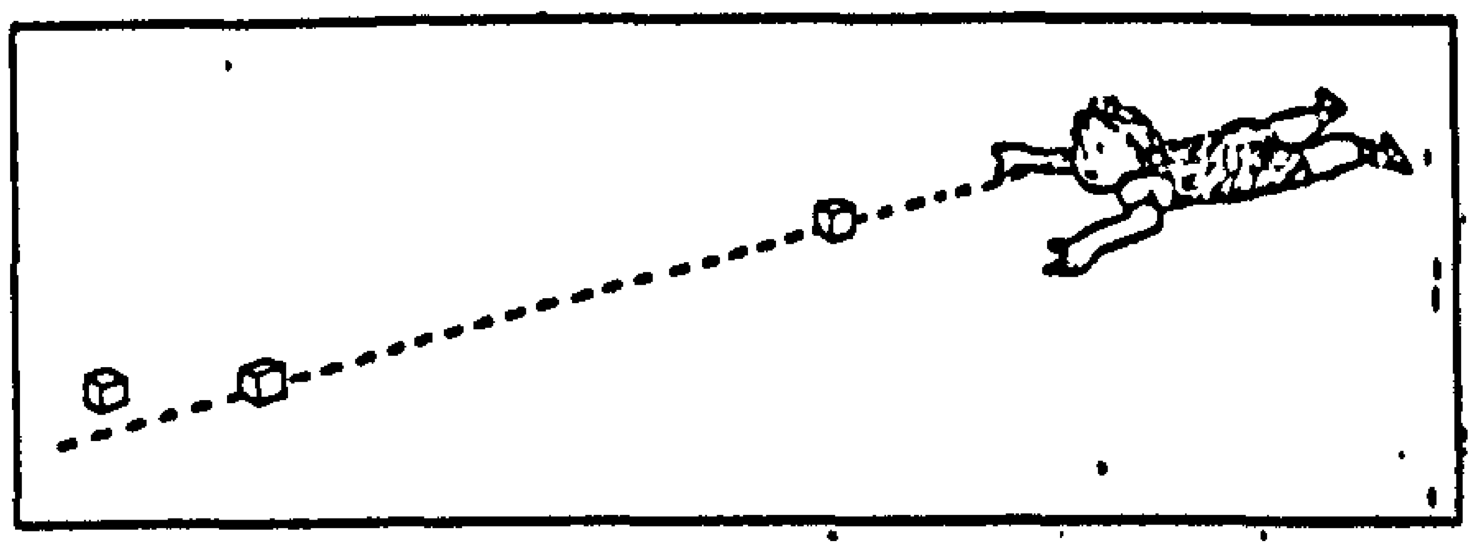


figure 6

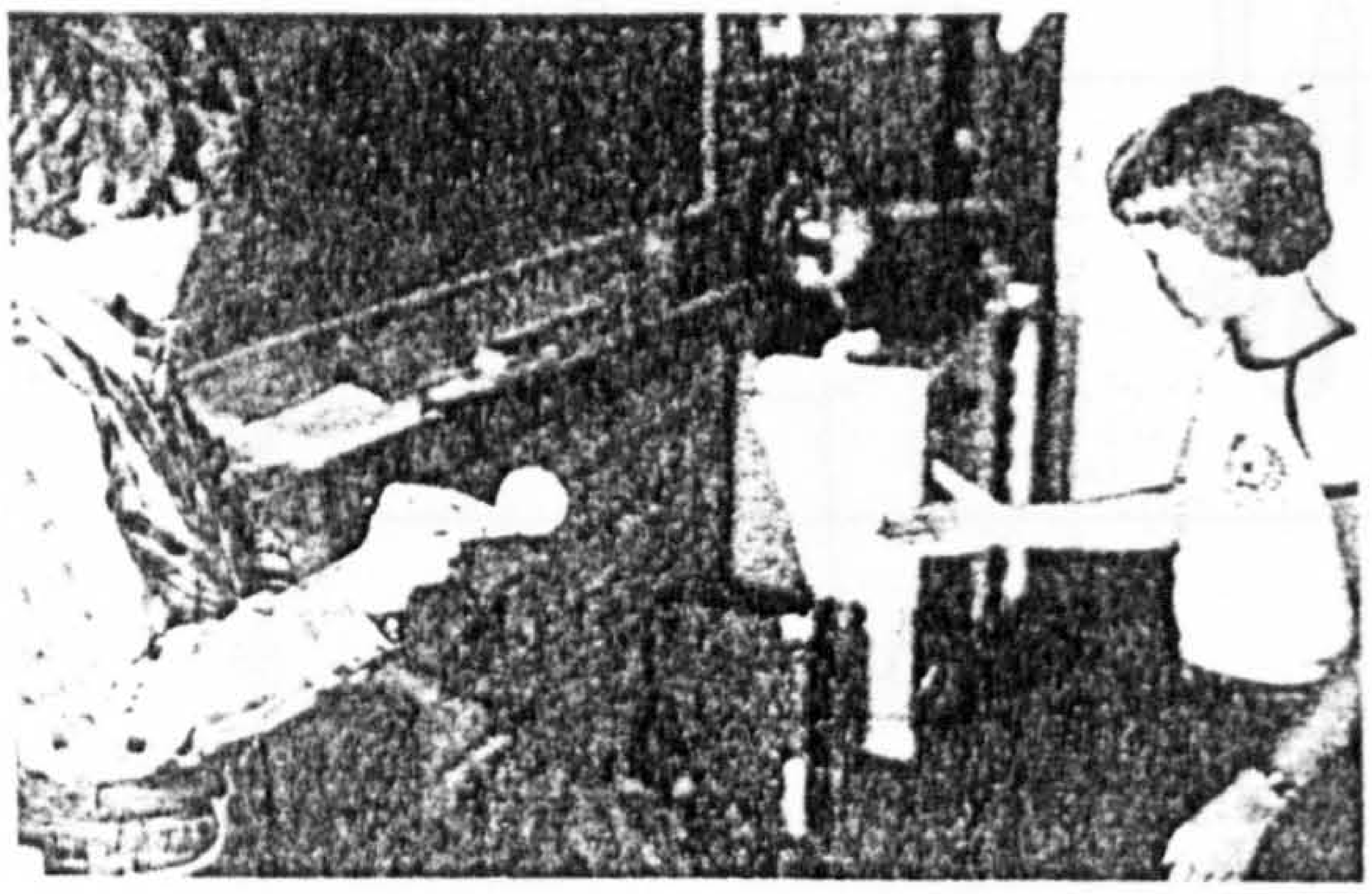


In figure 7 there is an exercise: although you are not standing in the meadow, you can still check whether the eye can see the cock on top of the tower. The decision is made by drawing a straight line between eye and cock.

figure 7

The drawing of figure 7 is a photocopy, taken from Abels, 1992, page 33. Somewhere, at some moment, the pupil should have to state explicitly that it is necessary to draw a straight line from the eye towards the treetop or to the cock on top of the tower. Then the next question could be: from where is it just possible to see the cock over the treetop? We can also have light beams emanate from the cock and ask where the shadow of the tree will be. In this case there might be pupils who do not draw straight lines but they think that the lamp (or the sun) drops something dark behind the tree. So it is useful to demonstrate a shadow-line (or light-line) in space as is done in figure 8.

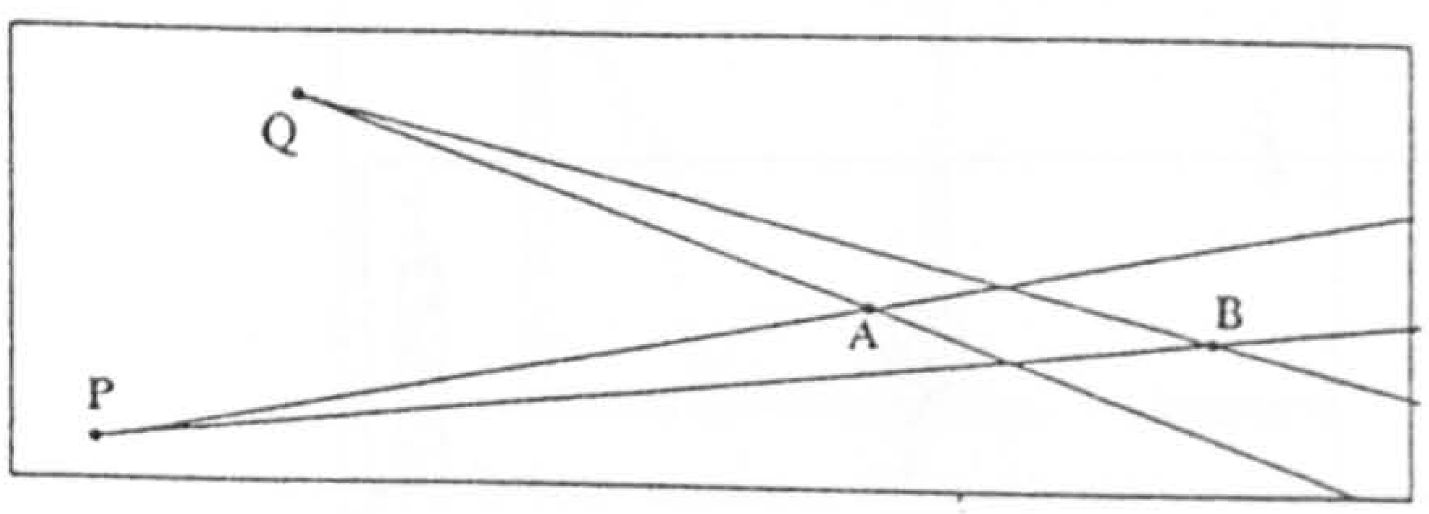
figure 8



Somebody places a small disk in the sunlight on a sunny day. Another pupil catches the shadow and puts it on the ground. The hand follows a straight line. Figure 8 is a photocopy, taken from *Abels*, 1992, page 34 and my text, concerning figure 8, is a extract from the text in *'Achtergronden'*.

The spot where the observer stands is, of course, very important for the view which the observer will have. For the person P in figure 9 the point A is at the left side of B. For Q, however, A is at the right side of B. This explains that when you look at a tree in the landscape, while you are sitting in a moving train, the landscape seems to rotate about the tree.

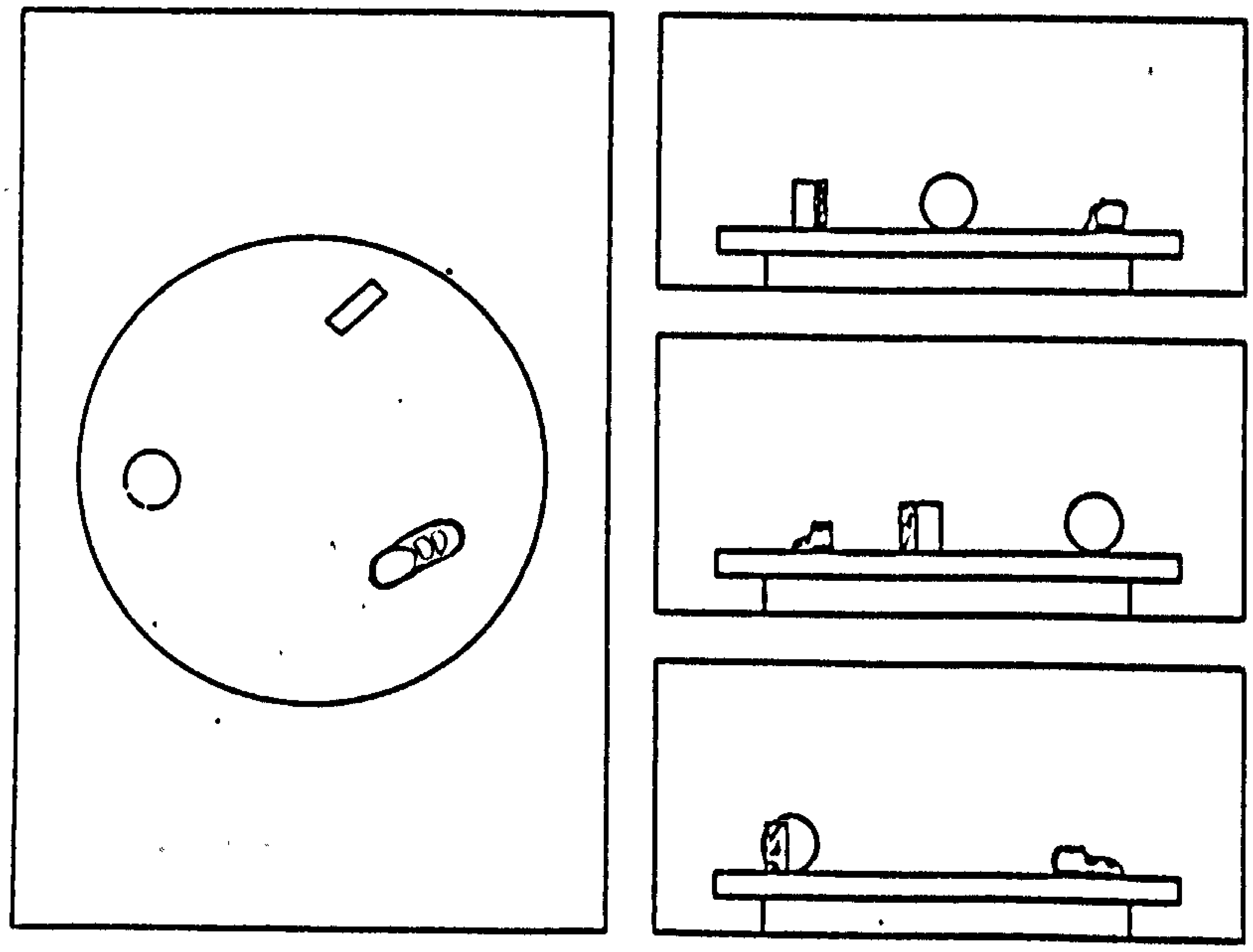
figure 9



In figure 10 the pupil has to indicate with arrows from where the pictures at the right hand side could have been produced. It is an elementary exercise in spatial imagination.

Figures 9 - 12 are photocopies, taken from Abels 1992, pages 35 - 40, and the associated text is again an extract from the text, concerning these figures, in that book (Achtergronden).

figure 10



These exercises, taken from Abels, 1992, are very adequate means to sustain and enhance the spatial imagination of the pupils. No computations have been involved so far.

In "Achtergronden" it is also explained that an object becomes visually smaller when it moves away from the eye (figure 11); the 'angle of sight' becomes smaller.

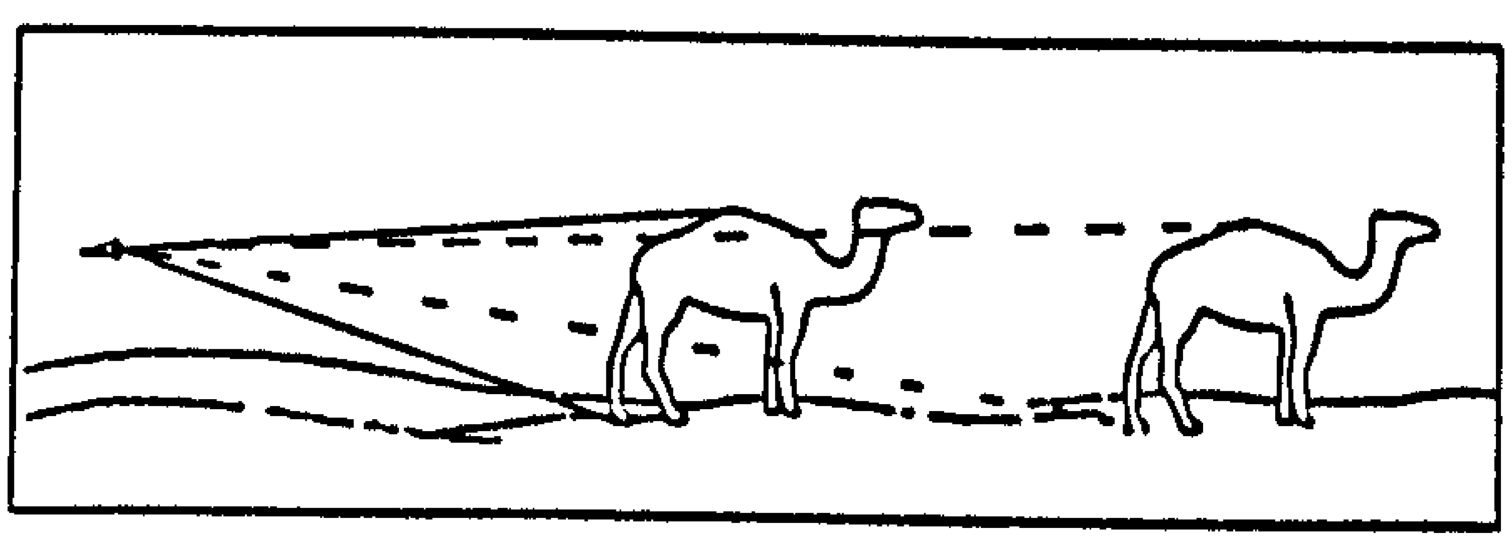
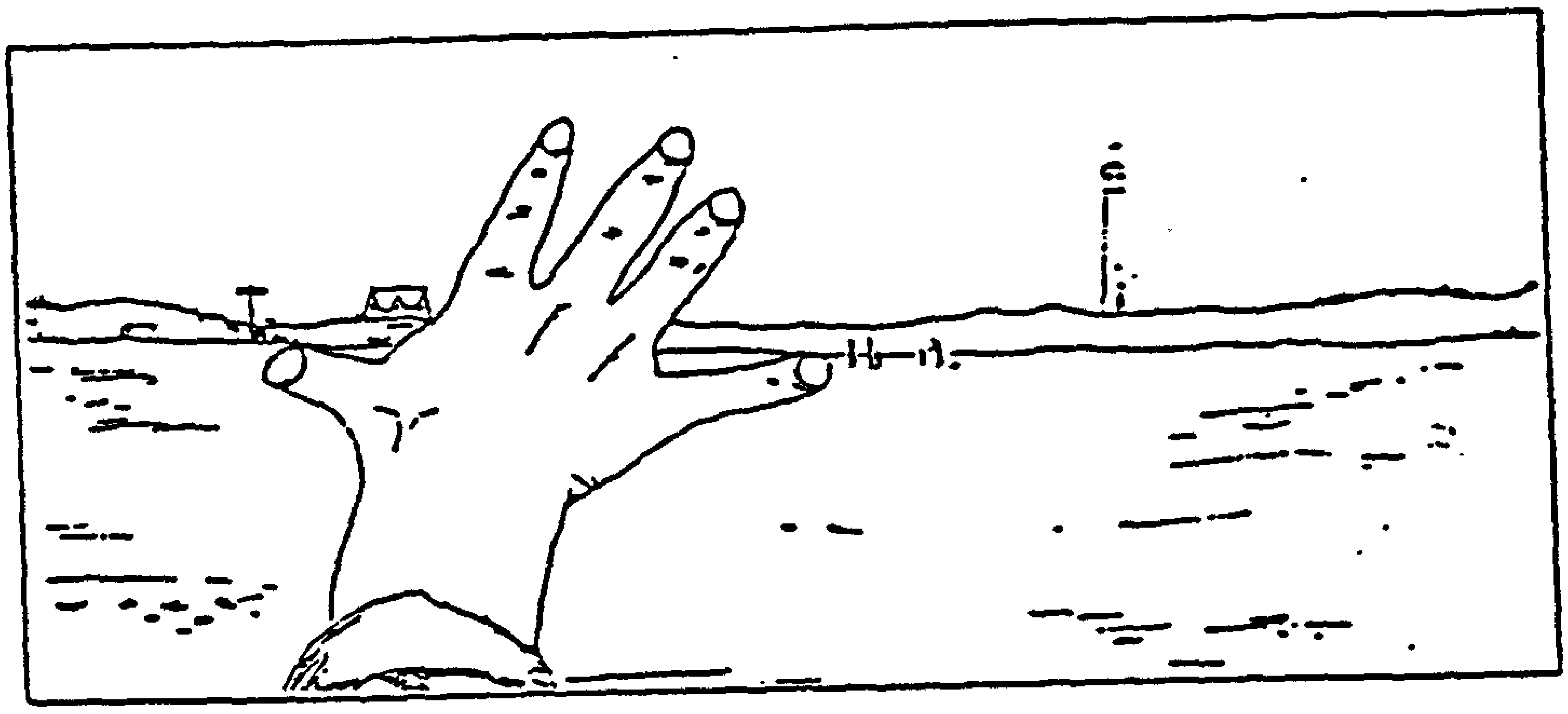


figure 11

figure 12



Our hand can be used to measure an angle of about 20 degrees as demonstrated in figure 12; the arm should be stretched out.

In "Achtergronden" a construction of Brunelleschi (1377 - 1446 AD) is demonstrated and it is the projection of what the eye observes (figure 13) on a transparent screen (figure 14). The object is an Olympic rostrum.

figure 13

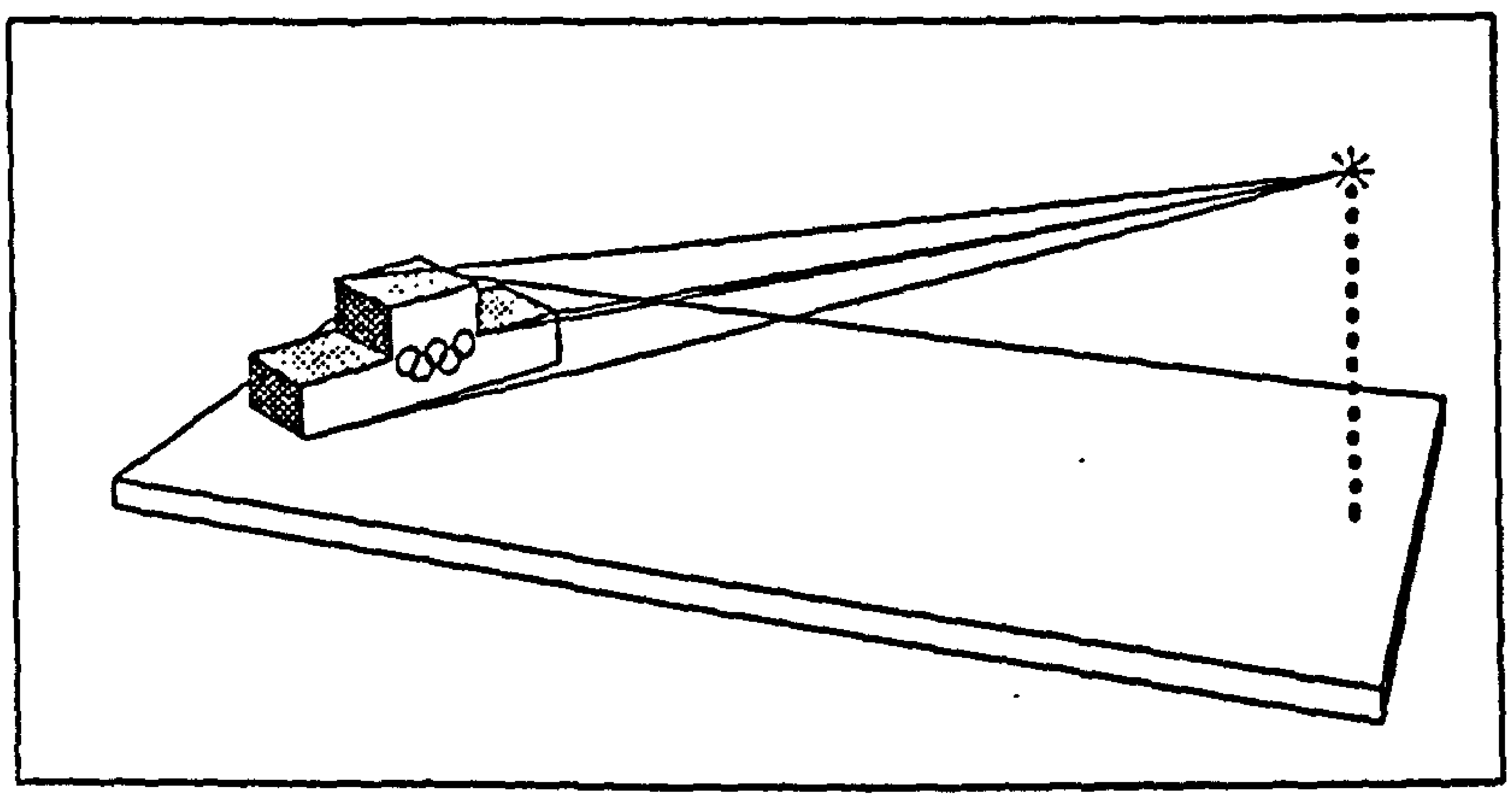
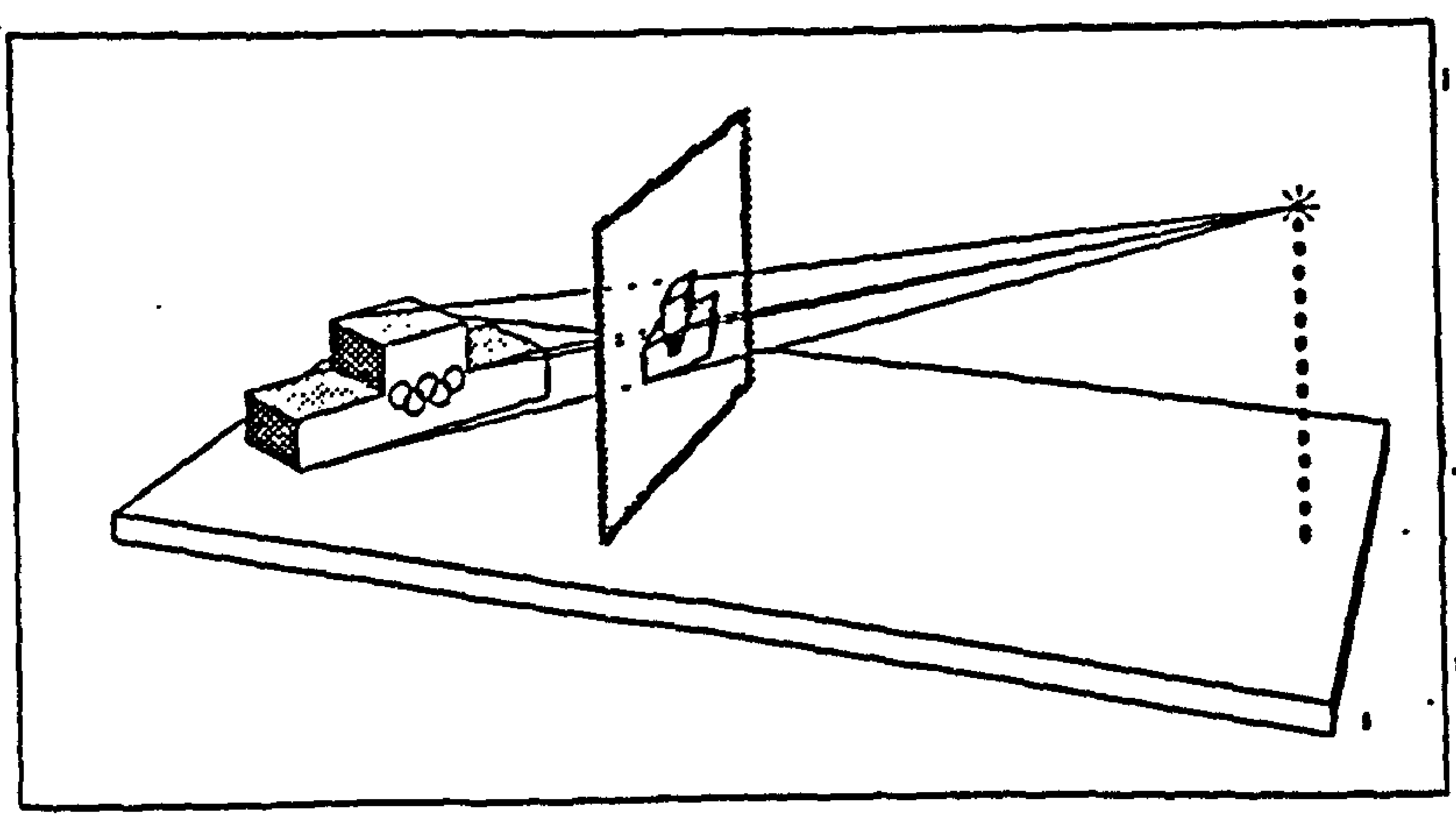
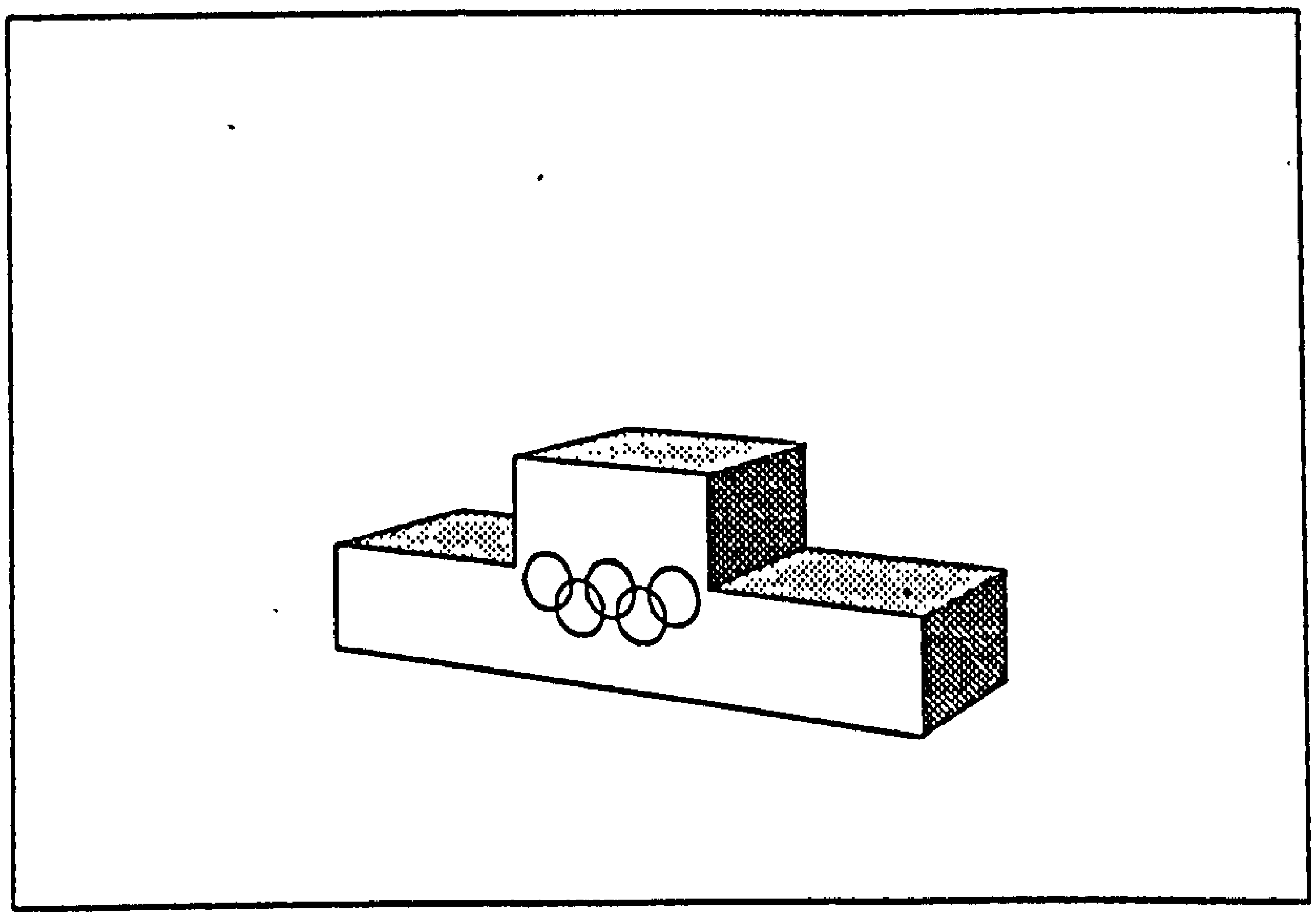


figure 14



The picture of the projection on the screen is shown in figure 15; it provides an accurate image of what the eye observes.

figure 15



The way this projection is implemented is demonstrated in figure 16. Figures 13 - 16 are all photocopies, taken from Abels, 1992, pages 48 - 51.

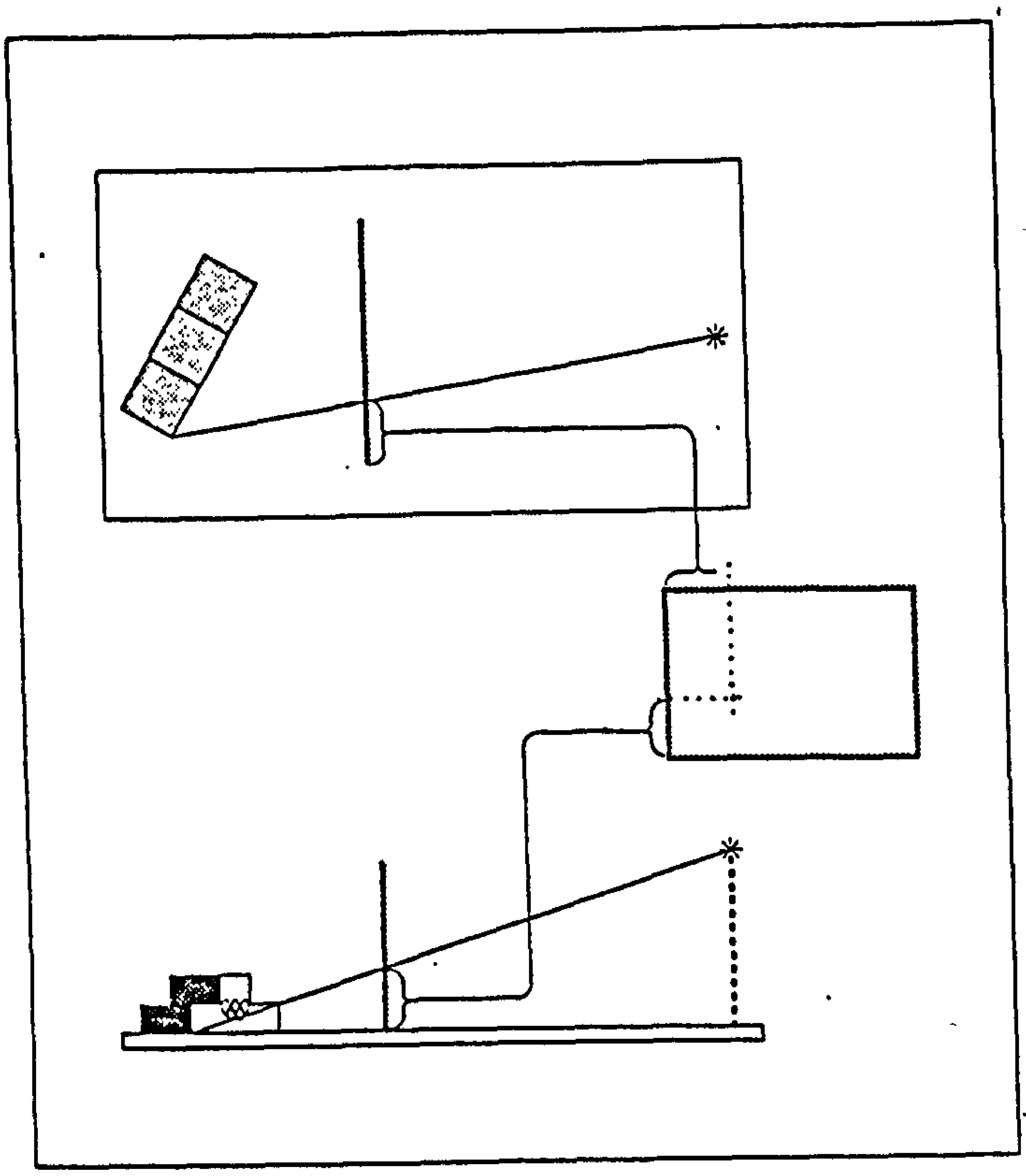


figure 16

In figure 17 the shadow of the piles should be drawn in the colour blue; the tops of the shadows will be equidistant if the work is carried out accurately.

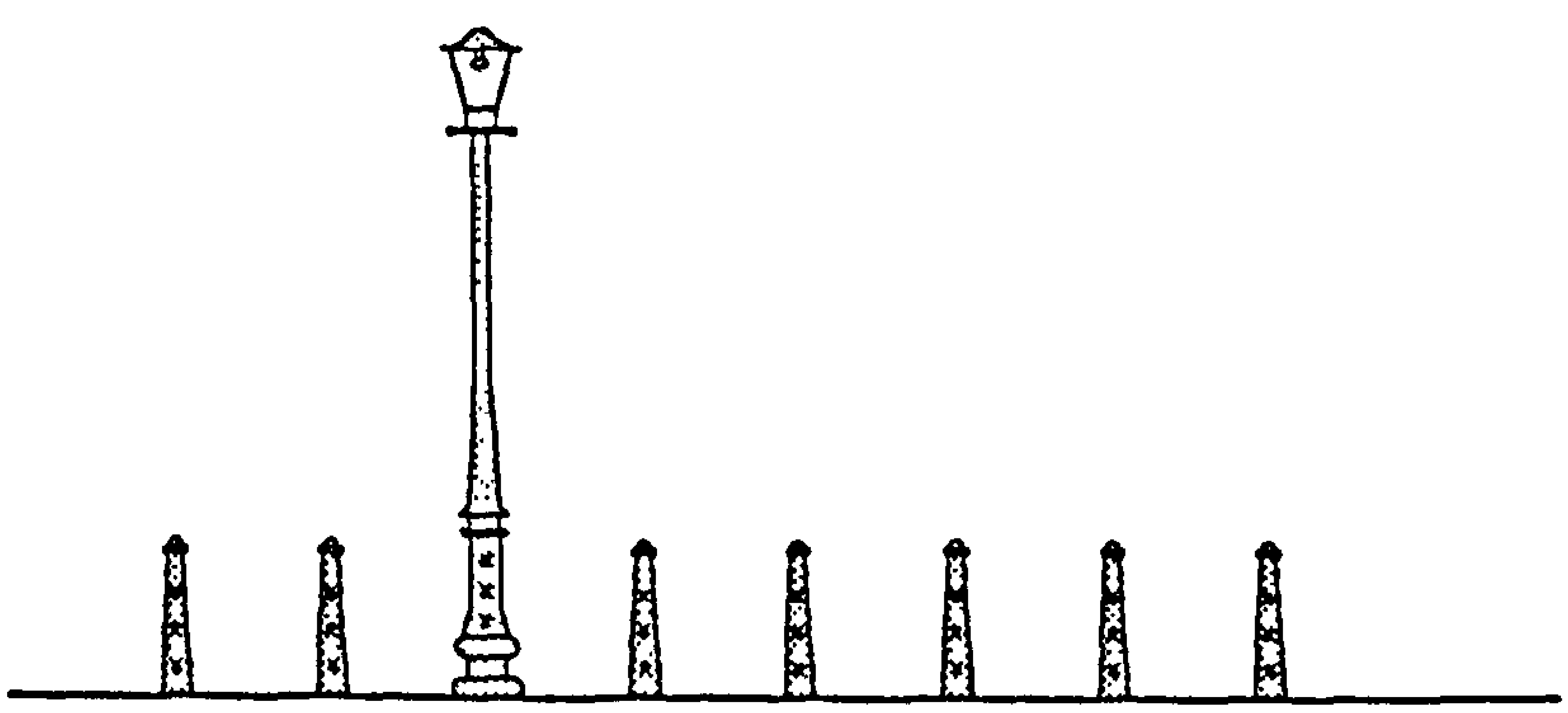


figure 17

Now for the sake of completeness six different forms of projection are displayed in figure 18. In these images different practical situations may be depicted.

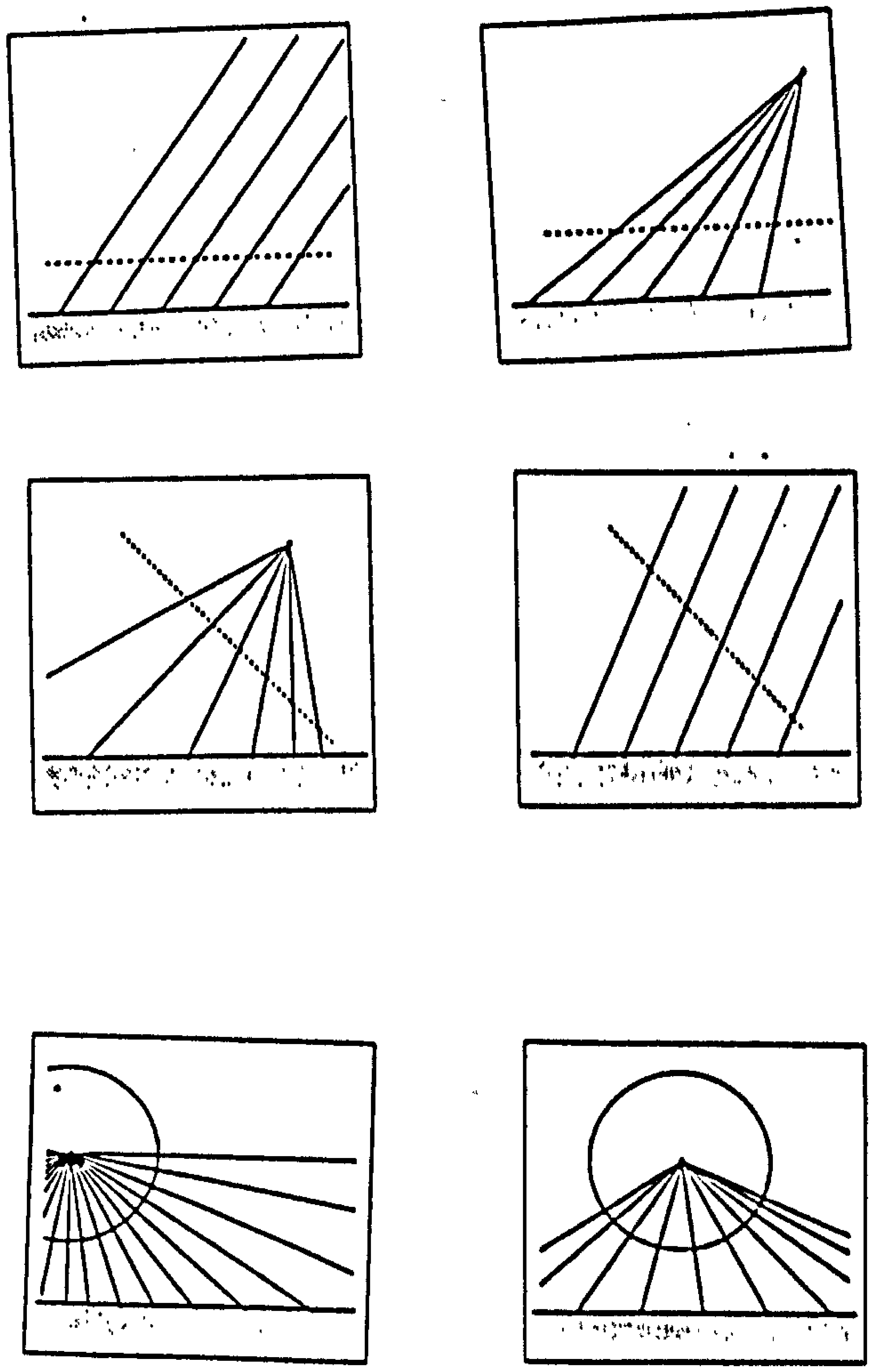


figure 18

Again figures 17 & 18 are photocopies, taken from *Abels, 1992, page 52.*

To summarise, we can conclude that the geometry curriculum of "Achtergronden" provides much material which is a valuable background for 'Educational Geometry', because it encourages the pupil to realise that the perception of objects and drawings is an important and indispensable part of the science of geometry.

In the next Chapter an alternative approach for the geometry education of 12 - 16 year olds will be discussed. It was designed by Dr P.M. Van Hiele, a Dutch teacher of mathematics, and his wife Dr D. Van Hiele Geldof, who also was a teacher of mathematics. The curriculum is called: 'The Van Hiele Model'. It has proved to be valid in the classroom.

CHAPTER XIII

The Van Hiele Model

In the summer of 1988 I visited the United States and in New York I met a group of people who were working on an edition of Monograph Number 3 of the Journal for Research in Mathematical Education. The title of the book was to be: The Van Hiele Model of Thinking in Geometry among Adolescents. Four people were working on the Project: David Fuys, Dorothy Geddes, C. James Lovett and Rosamund Tischler. The same year, 1988, the book was published.
In the summer the group was still working on the book and I was invited to have dinner at Rosamund Tischler's home and make the acquaintance of the members of the group. The Project was carried out at the City University of New York.
I happened to meet Dr Pierre Van Hiele in person several times: he lived near my hometown, The Hague in The Netherlands. In the nineteen-sixties I was a teacher of Mathematics; and textbooks written by Van Hiele were on the curriculum in The Netherlands. He came from a Montessori background and much of that approach could be recognised in his books. In those years he was working on a special approach to Geometry Education, which would later be called The Van Hiele Model. From the Monograph 3, I have taken a survey of the Van Hiele levels.

Reading from the Monograph I quote: *"According to the Van Hieles, the learner, assisted by appropriate instructional experience, passes through the following five levels, where the learner can not achieve one level of thinking without having passed through the previous levels"*. (Monograph 3, 1988, page 5).

Now follows a summing up of the levels and a description of the contents. The summing up is again taken from Monograph 3, quoted above.

"LEVEL 0. The student identifies, names, compares, and operates on geometric figures (e.g. triangles, intersecting or parallel lines) according to their appearance.

LEVEL 1. The student analyses figures in terms of their components and relationships among components and discovers properties/rules of a class of shapes empirically (e.g. by folding, measuring, using a grid or diagram).

LEVEL 2. The student logically interrelates previously discovered properties/rules by giving or following informal arguments.

LEVEL 3. The student proves theorems deductively and establishes interrelationships among networks of theorems.

LEVEL 4. The student establishes theorems in postulational systems and analyses/compares these systems'
(Monograph 3, 1988, page 5).

On page 7 of the Monograph Pierre Van Hiele is quoted. There he states that failure to progress in geometry might be the result of a language-barrier - the teacher using the language of a higher level than the pupil possesses.

Also on page 7 we find the phases, which will allow the student to progress from one level to the next. The examples, given in the following quotation, refer to the transition from level 0 to level 1.

"INFORMATION. The student gets acquainted with the working domain (e.g., examines examples and non - examples).

GUIDED ORIENTATION. The student does tasks involving different relations of the network that is to be formed (e.g. folding, measuring, looking for symmetry).

EXPLICITATION (means: making explicit). *The student becomes conscious of the relations, tries to express them in words, and learns technical language which accompanies the subject matter (e.g., expresses ideas about properties of figures).*

FREE ORIENTATION. The student learns, by doing more complex tasks, to find his/her own way in the network of relation(e.g. knowing the properties of one kind of shape, investigates these properties for a new shape, such as kites).

INTEGRATION. The student summarises all that he/she has learned about the subject, then reflects on his/her actions and obtains an overview of the newly formed network of relations now available (e.g. properties of a figure are summarised)."
(Monograph, 1988, page 7).

For a description of the Van Hiele levels 3 and 4, I quote again and the following quotation is a photocopy from Monograph 3:

Level 3: Student establishes, within a postulational system, theorems and interrelationships between networks of theorems.

Level 3 Descriptors	Level 3: Sample Student Responses
The student	Note: This study was not designed to include an in-depth investigation of students using level 3 type of thinking. However, listed below are some proposed student responses which in the Project's view would be indicative of level 3 thinking.
1. recognizes the need for undefined terms, definitions, and basic assumptions (e.g., postulates).	1. Student gives examples of axioms, postulates, and theorems in Euclidean plane geometry and describes how they are related.
2. recognizes characteristics of a formal definition (e.g., necessary and sufficient conditions) and equivalence of definitions.	2. Student identifies sufficient properties for defining a shape (e.g., parallelogram) and derives other properties from the sufficient ones. Student proves that two sets of properties are equivalent for defining a shape (e.g., parallelogram).
3. proves in an axiomatic setting relationships that were explained informally on level 2.	3. Student proves the sum of the angles of a triangle equals 180° in rigorous way (e.g., using the parallel postulate, saws and ladders, and theorems about angle addition).
4. proves relationships between a theorem and related statements (e.g., converse, inverse, contrapositive).	4. Student proves that if a triangle is isosceles, then its base angles are congruent, and conversely. Using proof by contrapositive, student proves that medians of a triangle do not bisect each other.
5. establishes interrelationships among networks of theorems.	5. Student recognizes the role of saws and ladders in various theorems involving properties of quadrilaterals and area rules.
6. compares and contrasts different proofs of theorems.	6. Student gives proofs via Euclidean geometry and via coordinate geometry (or vector geometry) that the diagonals of a parallelogram bisect each other and compares the two methods of proof. Student compares alternate proofs of the Pythagorean Theorem.
7. examines effects of changing an initial definition or postulate in a logical sequence.	7. Starting with "Two lines perpendicular to the same line are parallel," the student investigates how to prove other parallel line theorems.
8. establishes a general principle that unifies several different theorems.	8. Student proves the following relationship for the area of figures whose vertices lie on two parallel lines: area = midline x height.
9. creates proofs from simple sets of axioms frequently using a model to support arguments.	9. Student gives proofs of theorems in a finite geometry.
10. gives formal deductive arguments but does not investigate the axiomatics themselves or compare axiomatic systems.	10. Student does not examine independence, consistency or completeness of a set of axioms.

Level 4: *Student rigorously establishes theorems in different postulational systems and analyzes/compares these systems.*

Level 4 Descriptors	
The student	
1. rigorously establishes theorems in different axiomatic systems (e.g., Hilbert's approach to foundations of geometry).	
2. compares axiomatic systems (e.g., Euclidean and non-Euclidean geometries); spontaneously explores how changes in axioms affect the resulting geometry.	
3. establishes consistency of a set of axioms, independence of an axiom, and equivalency of different sets of axioms; creates an axiomatic system for a geometry.	
4. invents generalized methods for solving classes of problems.	
5. searches for the broadest context in which a mathematical theorem/principle will apply.	
6. does in-depth study of the subject logic to develop new insights and approaches to logical inference.	

(Monograph 3, 1988, pages 70 & 72).

The Project of the Group of the City University of New York has produced three Modules for children of the age 12 - 15. In the following quotation from these modules, geometric topics are treated:

"MAJOR CHARACTERISTICS OF MODULES

MODULE 1. Basic geometric concepts (parallelism, angle, congruence, properties of quadrilaterals, etc.)

MODULE 2. Angle measurement; angle sums for triangles, quadrilaterals, pentagons; angle relationships in triangles and parallelograms (i.e. exterior angle, opposite angles).

MODULE 3. Area measurement; area of rectangles, triangles, parallelograms, trapezoids and figures whose vertices lie on two parallel lines."
(Monograph 3, 1988, page 11)

Further, I have looked for subjects common to the Van Hiele Model and Educational Geometry. In Activity 2 of Module 2, I found the production of tiling and grids. This is a subject I have also used in my sections on group theory (Chapter IX).

Back in 1988 I had several talks with Dr Van Hiele. He told me that he was not interested in the education in geometry beyond the age of 16. This means that the Van Hiele Model and Educational Geometry really have very little in common.

From contemporary geometry curricula , we will take a journey back in time and we will go back to antiquity where the concepts of Euclid and Plato have a lot to tell us about the essential characteristics of optics, cosmology and geometry. The next 3 chapters will be dedicated to ancient knowledge.

CHAPTER XIV

Euclid's Optics

The famous ancient Greek geometer, Euclid, provides in his 'Optics' a surprising view of the visual geometry of his time. Therefore I think it is useful and fruitful to consider Euclid's ideas about the subject of optics.

It might be interesting to compare Euclid's 'Optics' with the introduction of a horizon in paintings of the fifteenth century. In my opinion the use of such a horizon is not allowable in the context of Euclid's designs. So the up-to-then customary and valid principles of Euclid were challenged in the fifteenth century by the acceptance of a new issue such as the horizon.

It has even been tried to suggest that Euclid had discussed the issue of the visual appearance of a very large plane in his Optics. On page 9 of Optics (Euclid, 1959, page 9) it is stated that a large plane seems visually concave. This statement apparently has been inserted in medieval times, according to a footnote at the bottom of the paragraph.

Although this statement was not written by Euclid, it shows that the issue was considered important enough in medieval times to warrant using Euclid as a witness. This discussion about the visual shape of a very large plane foreshadows the emergence of a horizon on the canvas of the painters. It concerns the shape of a very large plane which I have called 'saucer'.

Despite the emergence of new ideas, Euclid has survived as a geometer and up to recent times his principles were taught to pupils in secondary schools as the exclusive truth about geometry. Only during the last decades has it been attempted to replace Euclid's books and principles by something new.

According to Euclid's concepts, 'rays of vision' emanated from the eye, hitting objects and thus perceiving them. The 'rays of vision' formed a cone with the vertex in the eye and the base of the cone resting on the surface of the perceived objects.

In Proposition III (Euclid, 1959, page 3) Euclid states that if an object is too far away from the eye, it will no longer be perceived. And in Proposition IX (Euclid, 1959, page 8) he asserts that a rectangle, observed from a distance, will be perceived as deformed so that we see the image of figure 2 while looking at figure 1 from a distance.

figure 1

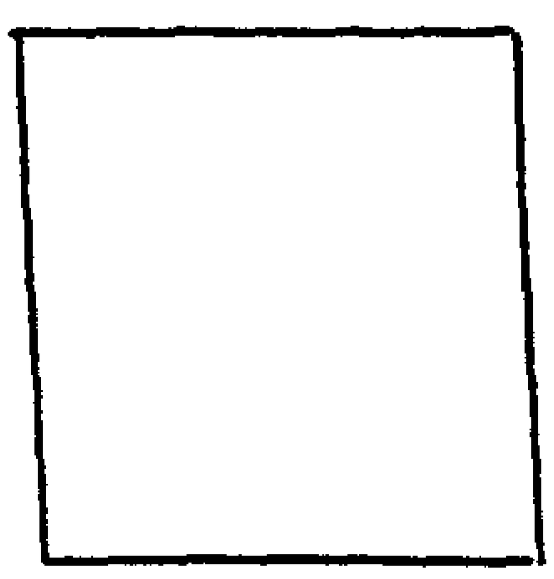
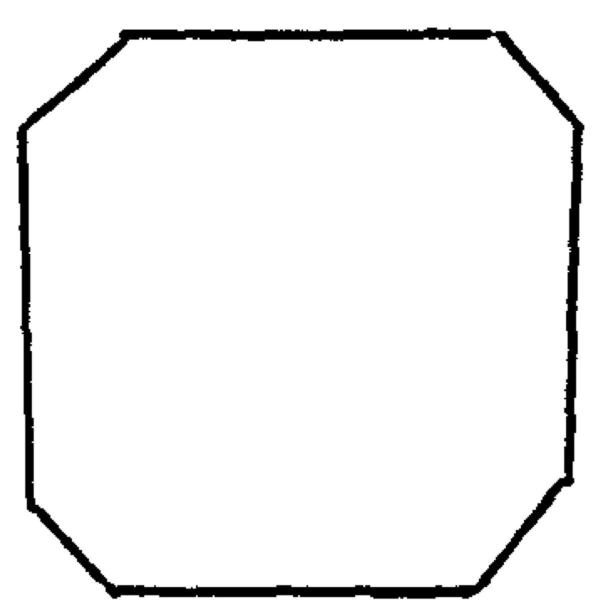


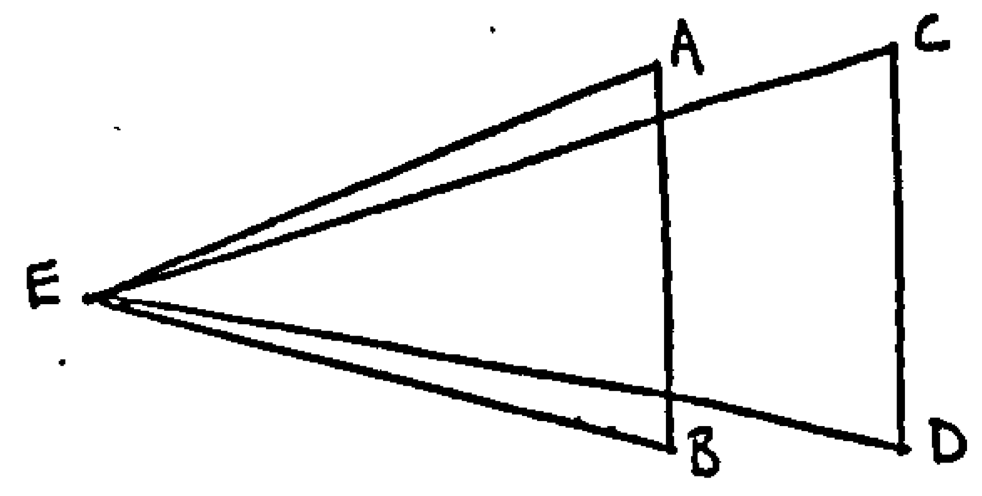
figure 2



Apparently the horizon was a hazy matter to Euclid so that its use might not be acceptable in his geometry. Euclid drew his pictures by visualising the 'rays of vision'. To give an example, in Proposition

II (Euclid, 1959, page 2) he says that two parallel line segments at different distances from the eye (E) are perceived as having unequal size. So, AB seems larger than CD, seen from E (figure 3). EA, EB, EC and ED are the 'rays of vision'.

figure 3



Referring to the picture of figure 3, we might depict Euclid's concept of 'rays of vision' in figure 4 where the rays are emanating from the eye E (figure 4). So the triangle ABC in figure 5 is 'assessed' by the 'rays of vision' emanating from the eye E.

figure 4

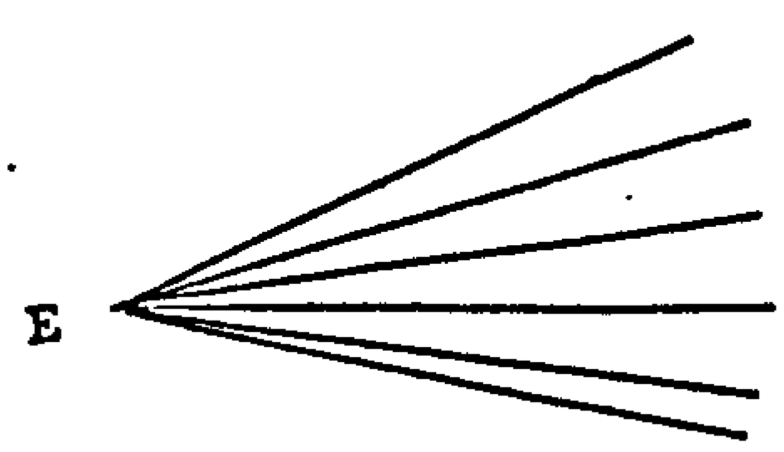
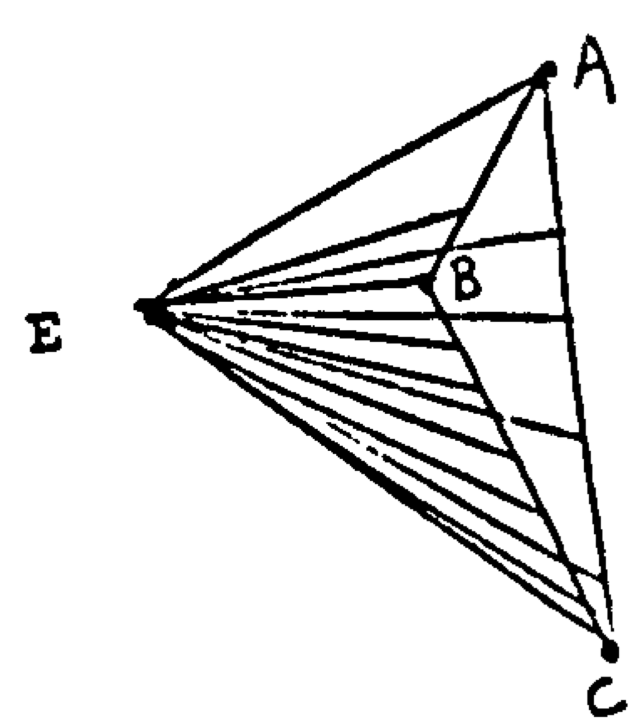
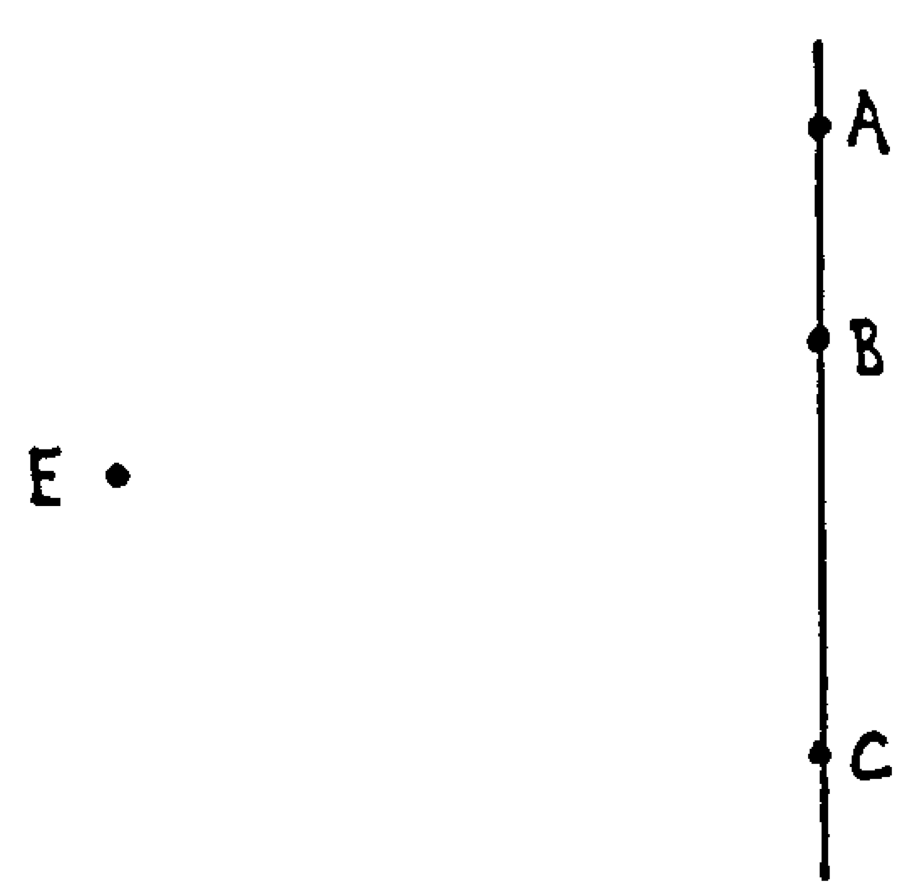


figure 5



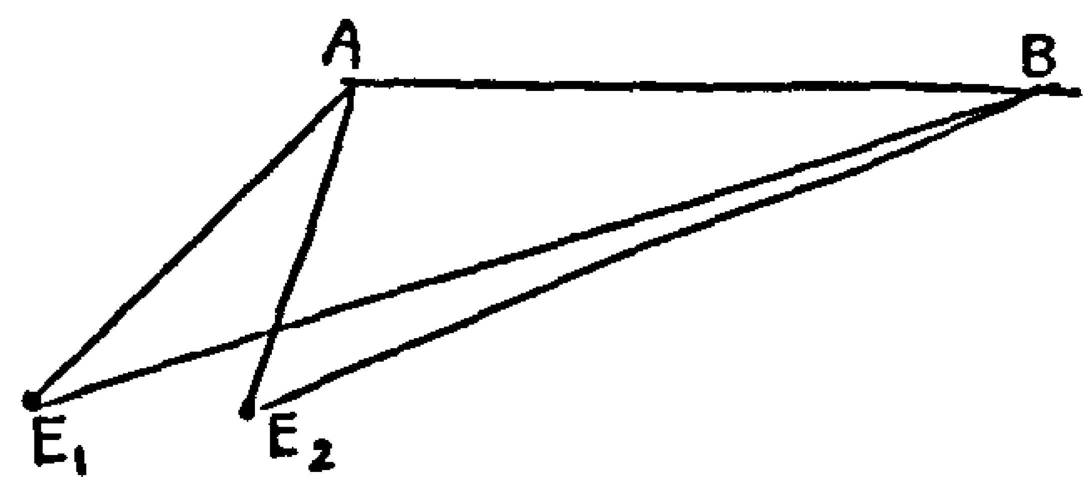
What does the eye perceive in figure 5? A 2-dimensional world is displayed in figure 5 with the eye E only looking along the surface of a plane. Figure 5 shows a flat world. The eye might perceive the following in figure 5 (see figure 6):

figure 6



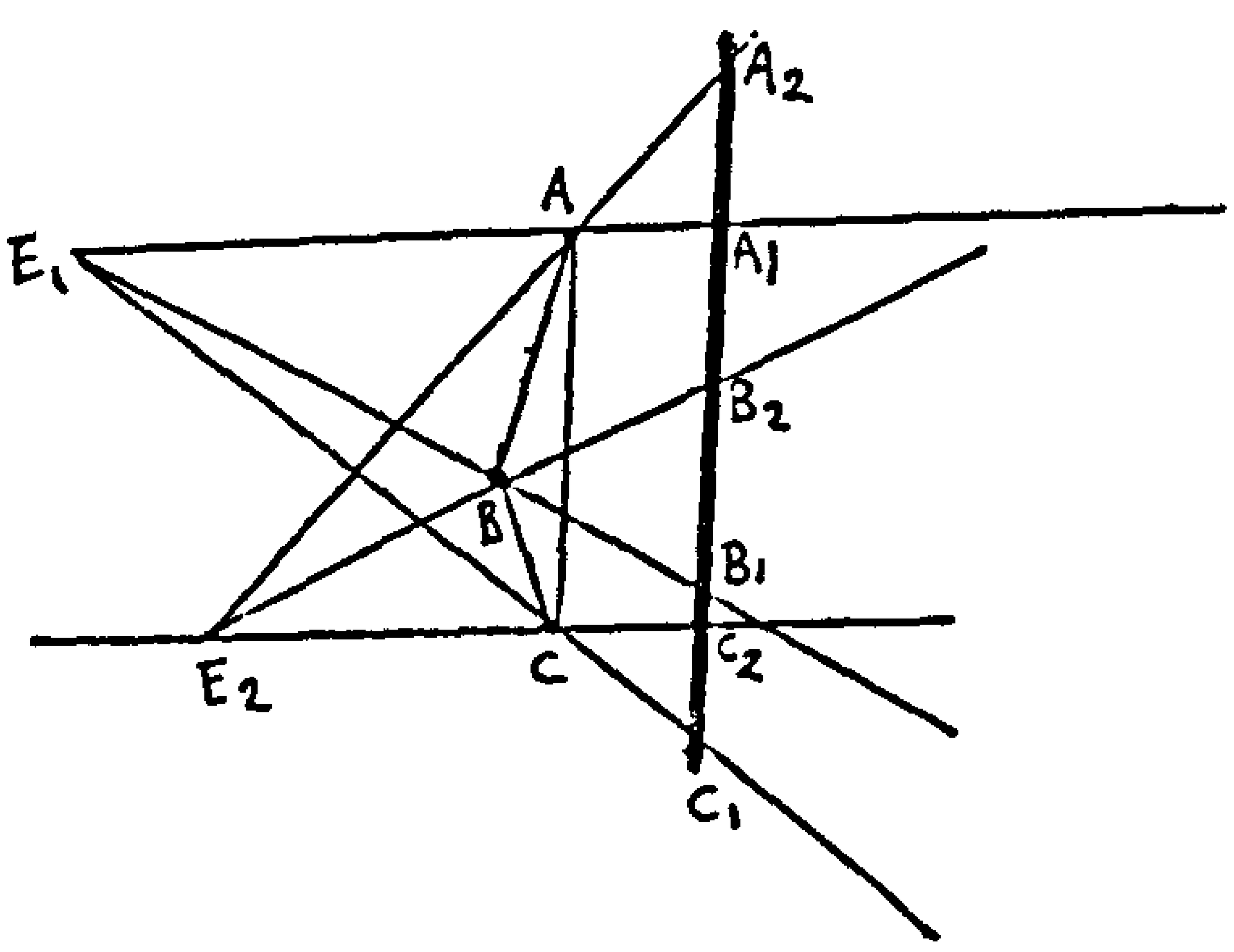
In figure 5 the eye does not watch in depth; one might assume that it does not distinguish distances. Watching with two eyes however may provide a sense of distance. This subject is not mentioned by Euclid in his Optics. In his Proposition IV (Euclid, 1959, page 3) Euclid states that if equal line segments lying on the same line are shown, the more remote segments would look smaller. That would for instance mean that the line segment AB in figure 7 seems smaller seen by E_1 than by E_2. In that case a person P with two eyes E_1 and E_2 would see incongruent images and thus see double.

figure 7



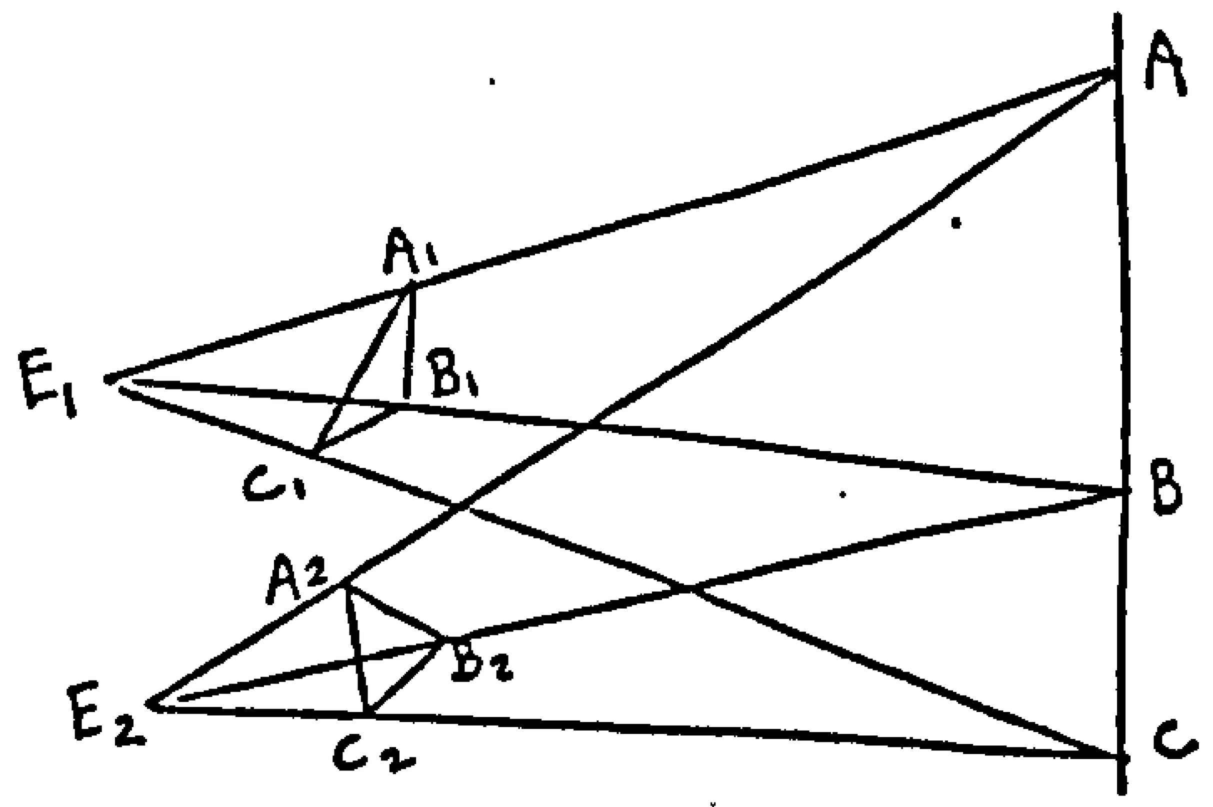
And if, in figure 8, the person P observes a triangle ABC with two eyes E_1 and E_2, P would look at the following: (figure 8)

figure 8



In order to see one image, the triangle $A_1B_1C_1$ has to be doubled so that P will see only one image, ABC (figure 9):

figure 9



It seems that we are running into trouble; but actually we are near to Desargues' theorem on perspective triangles (figure 10):

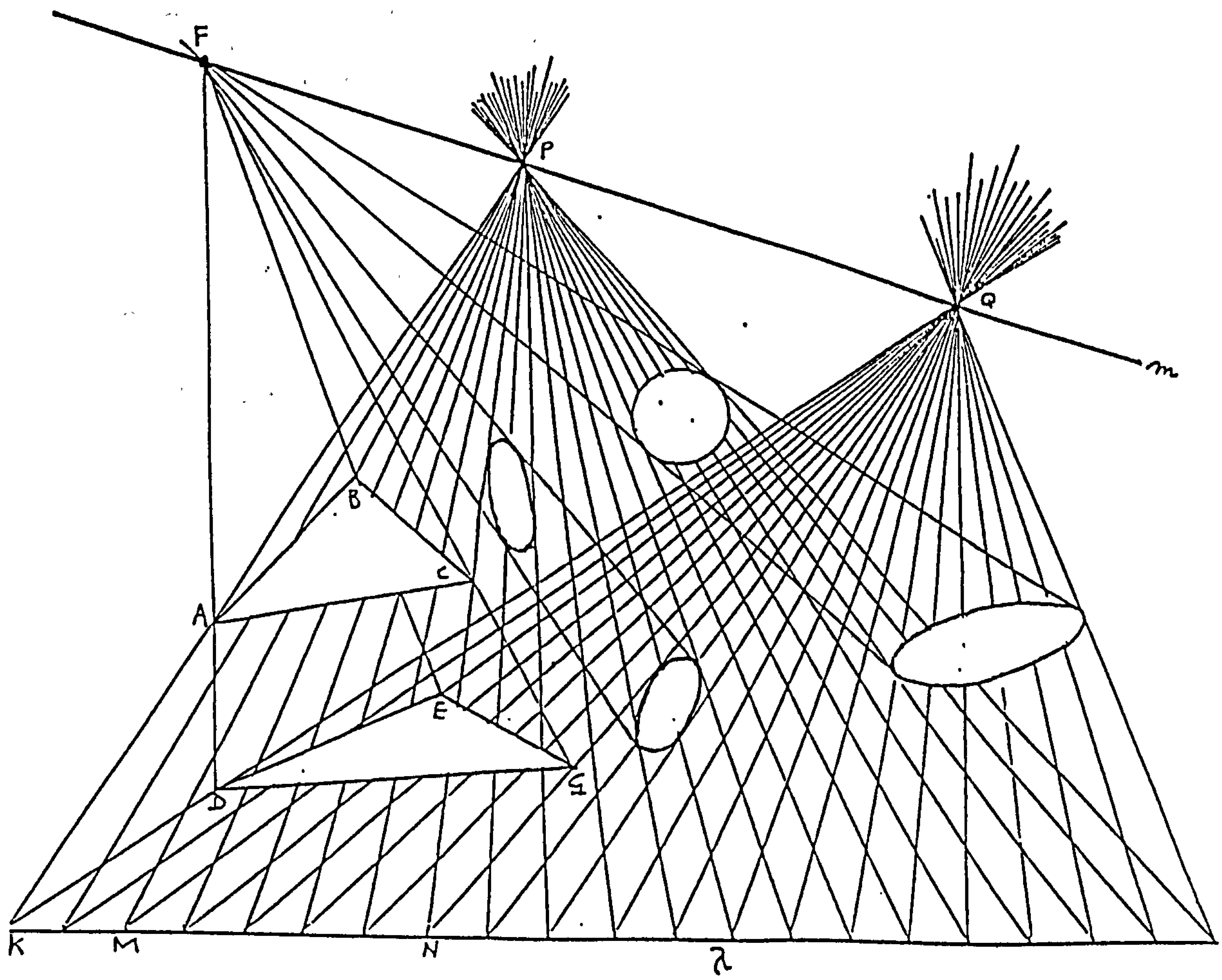


figure 10

In the picture of figure 10 we can apply Desargues' theorem. We notice that the edges AB and DE of the two triangles intersect on the line λ and so do the edgesDG, AC, and BC, EG. Consequently the straight lines AD, BE and CG will meet at one point F.

How does this relate to Euclid's views? The points P and Q (figure 10) are considered as the two eyes of one person. From the eyes, 'rays of vision' are emanating. From the eye P the triangle ABC is perceived and it is projected on the line λ as KMN. The same procedure is taken for the eye Q, from where the triangle DEG is projected on λ and it also becomes KMN.

The person, looking through the eyes P and Q , perceives KMN as a figure common to the perception of the two eyes and consequently it is seen in depth and the distances from K, M and N to the person are determined by the use of two different eyes. This means that, in a 2-dimensional world, it is possible to distinguish depth. The 2-dimensional world consists of two coinciding planes (P,λ) and (Q, λ) .

This, of course, seems rather far from Euclid's world. The above interpretation of what 2-dimensional observation with two eyes means, is no more than a product of my imagination. I have tried to find some roots for my 'Projections' in the antique world of ancient Greece. Euclid's 'rays of vision' seem somewhere related to the 'rays', emanating from P and Q in figure 10, the picture of which was taken from my 'Projections'.

There is an additional application. We observe in figure 10 that from point F even one 3-dimensional picture of the combined triangles ABC and DEG can be obtained. So, by using two 2-dimensional universes, (P,λ) and (Q,λ) which are glued together to a double plane, it becomes possible to see the two triangles as only one triangle in a 3-dimensional universe.

This is surprising. By doubling R^2 we are transgressing the 2-dimensional universe and arrive at 3-dimensional perception (from F). In figure 10 some concepts of Euclid are applied but it is also shown in figure 10 how perception with two eyes might work.

It is another remarkable fact that Euclid states that no geometrical object can wholly be observed from one position. When we watch such objects, there are always parts of it we do not see (Euclid, 1959, page 2, Proposition I).

This statement can be used in the case of the plane horizon which we observe in figure 11. The straight line FEDS (figure 11) may be seen as a visual straight line; we computed that it must be seen as curved but nobody seems to notice. Why? Possibly because the eye works as a search light; only a small part of the object is noticed and it is too small to notice the curvature of the prolonged line. This could be an explanation of the fact that one normally does not perceive that a so-called straight line in most cases is visually curved; and the explanation has its roots in Euclid's proposition I.

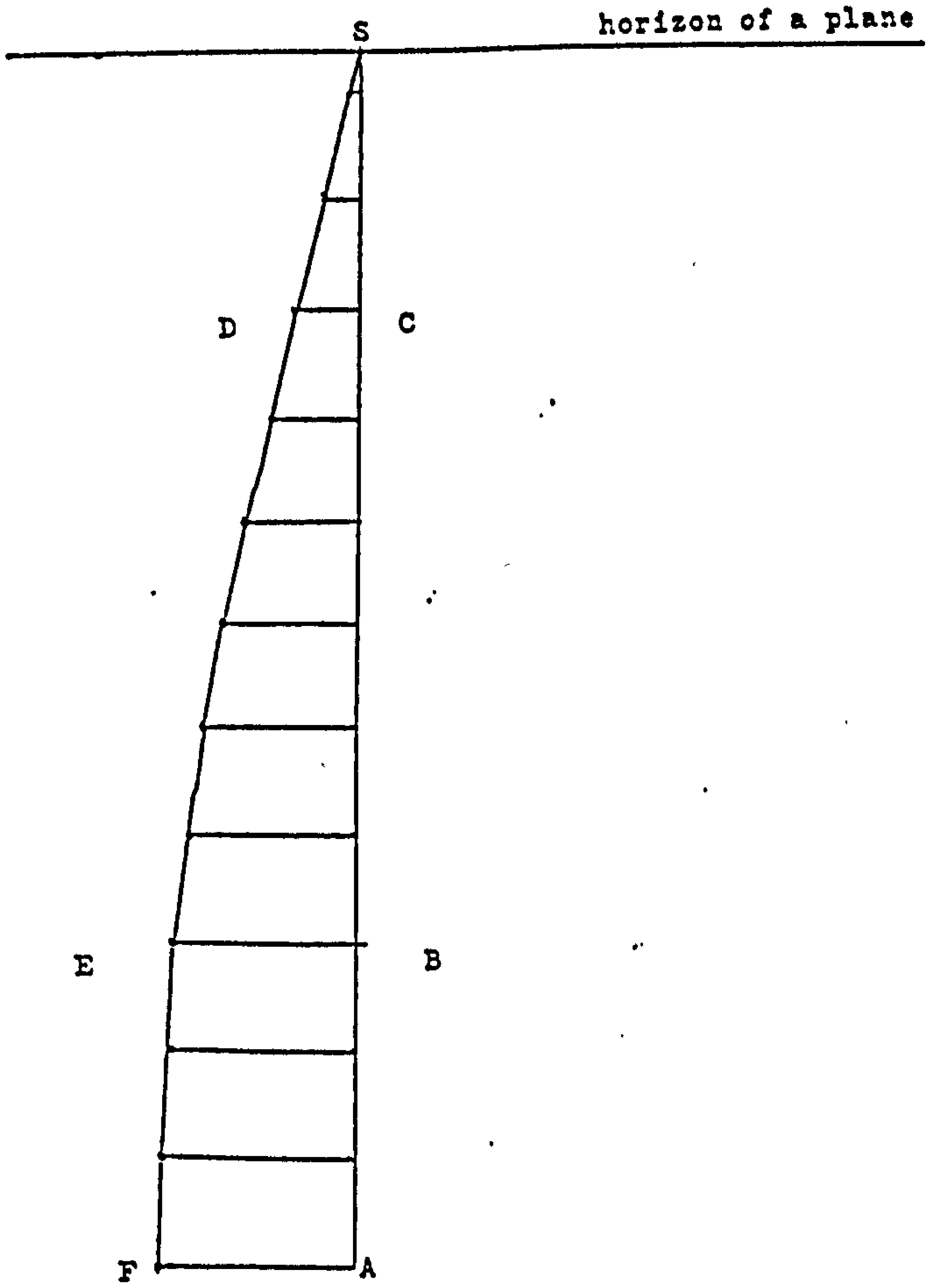


figure 11

I stated that the use of a horizon seems impossible within the framework of Euclid's world. Let us have a closer look.

In figure 12 a person is watching from the eyes P and Q along the 'rays of vision' emanating from the two eyes.

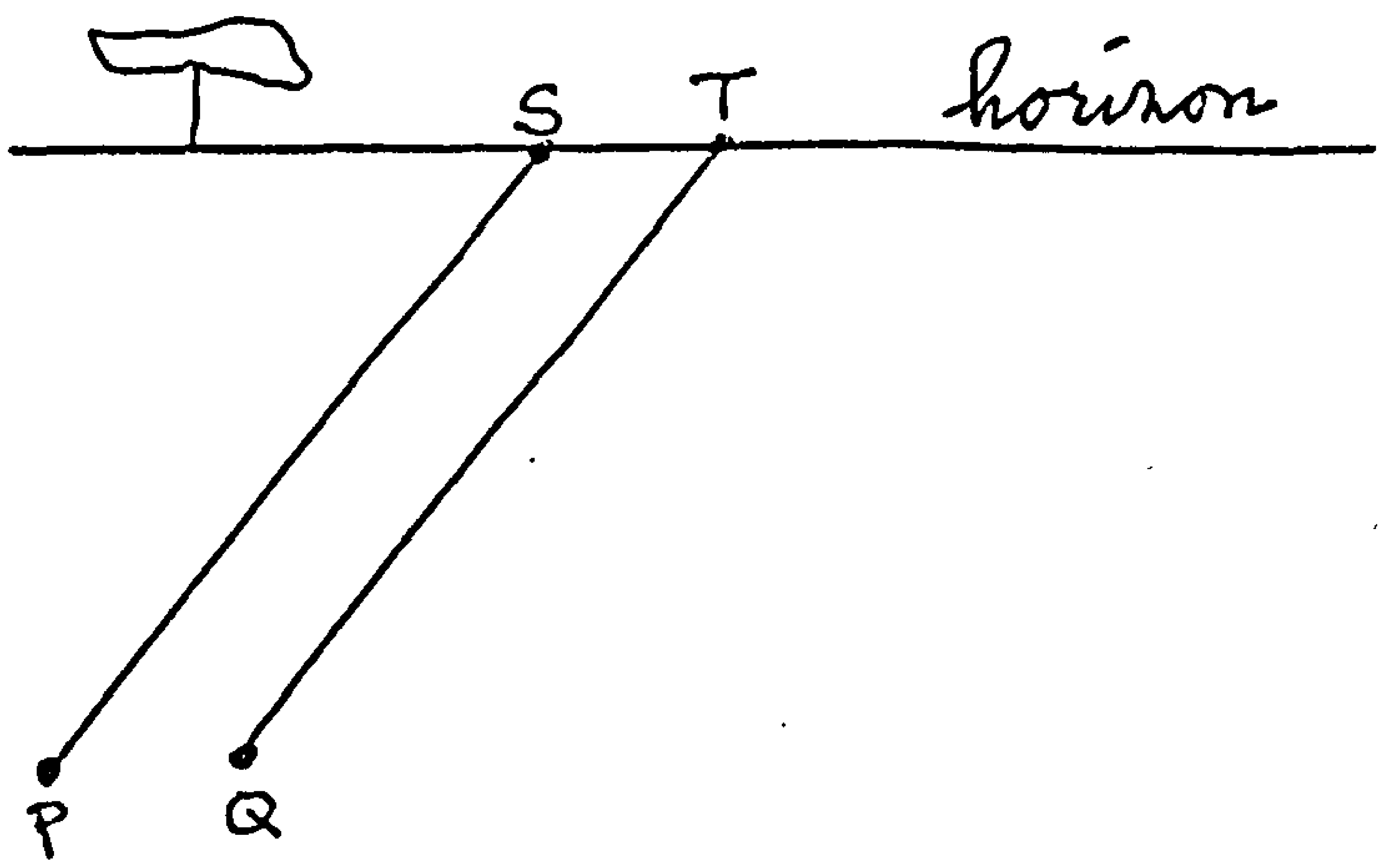


figure 12

The 2 rays, PS and QT, are not parallel because they are directed at different points of the horizon. However, the lines PS and QT are drawn parallel, so that they will not meet. This yields a contradiction, as we saw before in Chapter VI, section 6.2. This problem can be solved by Euclid's assertion that in the neighbourhood of the horizon things become hazy and thus there is no contradiction, simply because perception is fading when we try to observe objects which are far away.

One could conjecture that Euclid's 'rays of vision' can be sometimes represented by what I have called: 'visual straight lines'. Is that allowable ?

There is the famous fifth postulate of Euclid which concerns two parallel straight lines and implies that such straight lines will never meet, however far produced. This means of course that Euclid was thinking about what infinity means in terms of distance.

I will now try to find some concept of infinity, avoiding the 'hazy' horizon, so that this new concept of infinity might relate more closely to Euclid's 'Optics'. I also want to avoid the situation of figure 12. Now look at figure 13:

figure 13

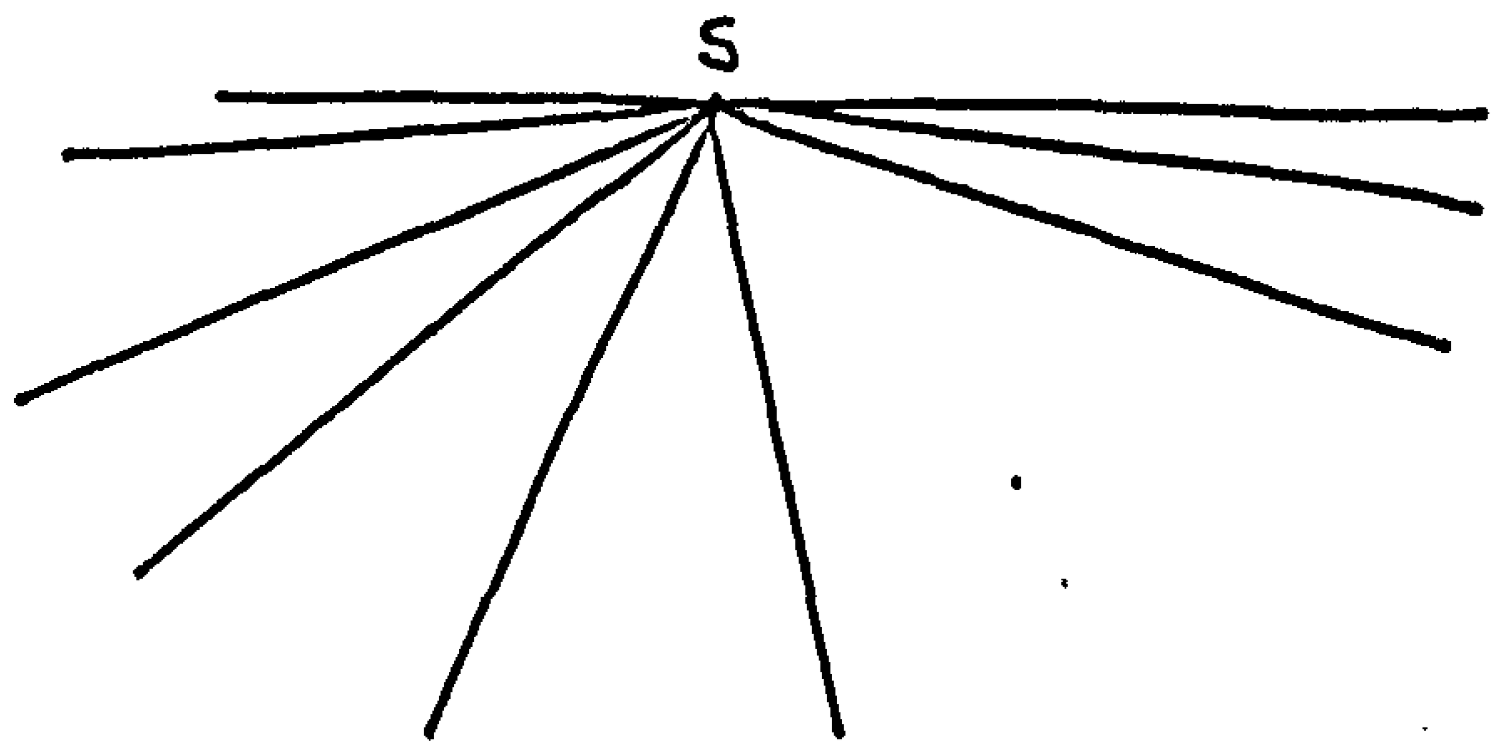
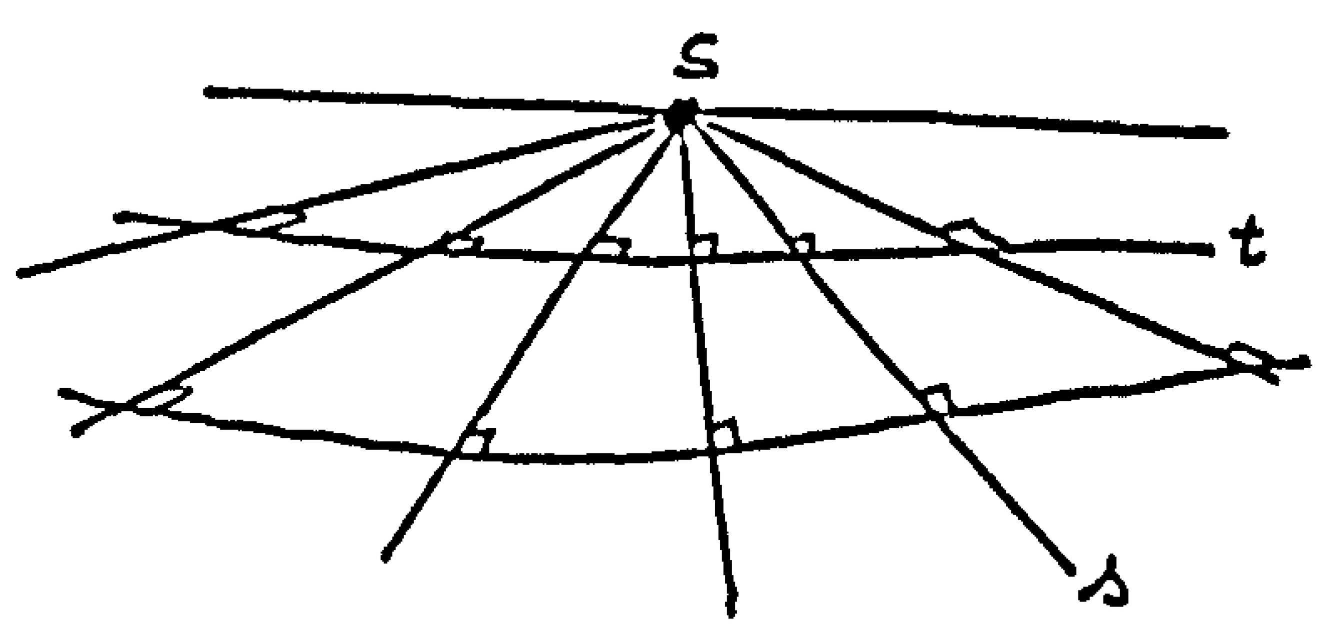


Figure 13 shows parallel lines, heading for their infinite point S. And these parallel lines cover the plane. One may not speak about a horizon; the only geometrical objects in figure 13 are a set of parallel lines.

Now think of straight lines ⊥ to the lines of figure 13. Such lines will not reach point S and even in the old situation (with a horizon) such lines orthogonal to the lines of figure 13 will not reach any visual point of the 'horizon' (figure 14).

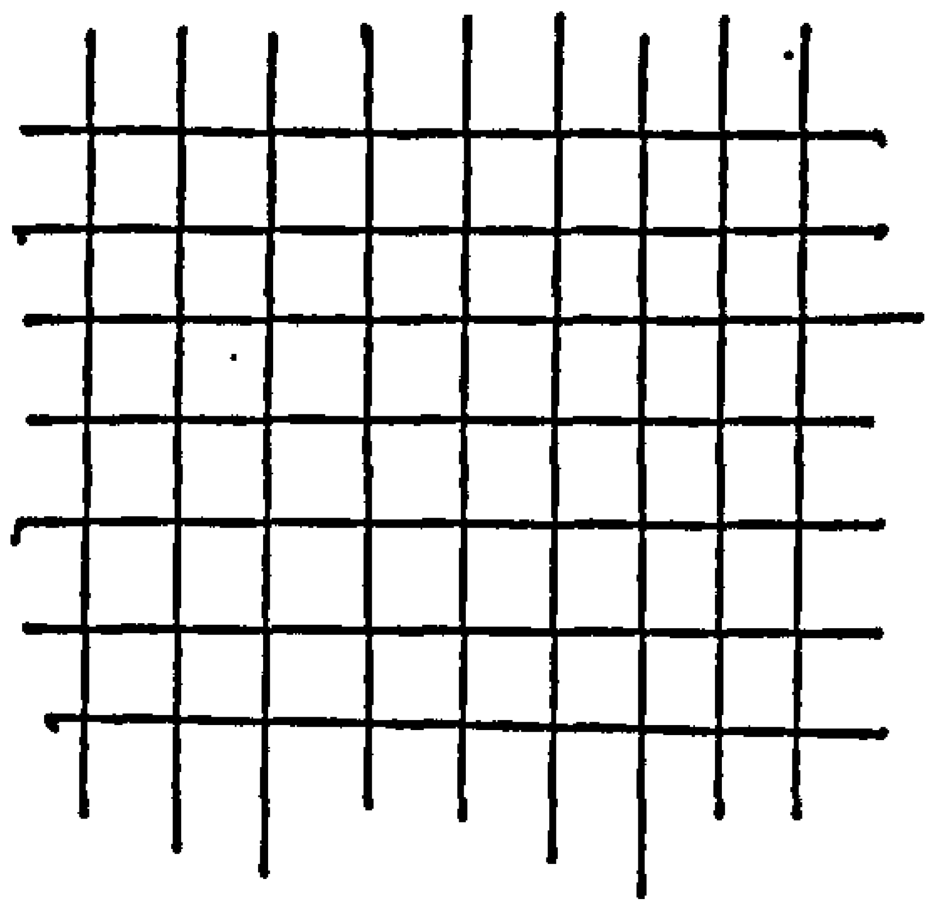
In figure 14 we observe two sets of mutually perpendicular lines: set S and set T. The plane of figure 14 is covered with two sets of straight lines and the plane is not supposed to contain other lines.

figure 14



So the orthogonal axial system of Descartes (figure 15) can be derived from the rejection of a horizon and designs without a horizon (figures 13 and 14).

figure 15



The introduction of a point S in figures 13 & 14 is a step towards the picture of a global straight line, but this, again, is far from what can be found in Euclid's Optics.

One might ask whether it is allowable to relate the thoughts of an ancient Greek to the results of Desargues' theorem as I did in figure 10 where two 2-dimensional universes provided a 3-dimensional view (from F). But is this not what we are performing daily? Concepts and notions of the past are applied in a modern world. And that is what I have tried to do with Euclid's ideas, bringing them into contact with my ideas in 'Projections'.

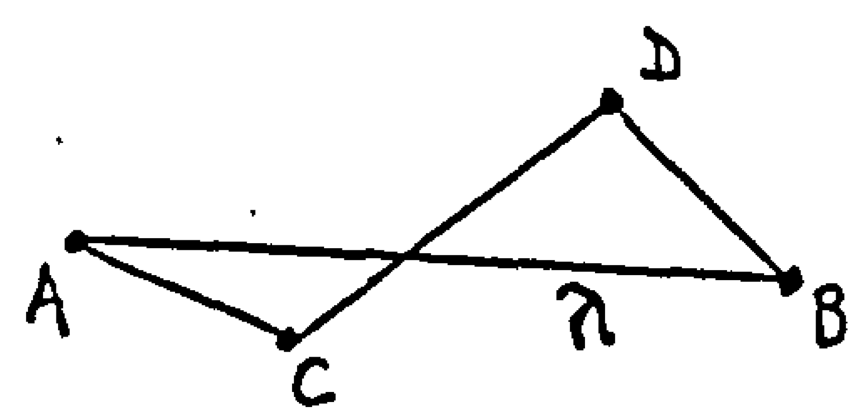
Above, I mentioned 'common sense'. This 'common sense' seems to provide a solution to the question of what precisely a straight line is. It is, according to many opinions, 'the shortest connection between two points'. This definition is widely accepted. However, such a definition might be seen as educationally invalid because it bars the way to a further consideration of the subject.

The belief that the statement 'a straight line is the shortest connection between two points' holds good, prevents the student from rethinking the issue. Now look at figure 17:

figure 16

figure 17

figure 18



There is a line AB (λ) drawn along a ruler between the points A and B. According to common sense, λ is a straight line. We have then to prove that there is no shorter connection than λ between A and B. Only then we can agree that λ is straight. This means that we can not define a straight line between A and B until we have proved that there is no shorter connection than along λ (figure 17). How can we prove that? We draw another connection ACDB between A and B (figure 18).

It has to be proved that ACDB is longer between A and B than λ. Let us assume that the length of ACDB is AC + CD + DB where AC, CD and DB are straight line segments. To start with AC, one might ask: what the length of the straight line segment AC is. The answer has to be that it is the shortest connection between A and C. How do we prove that a line, drawn along a ruler from A to C, is the shortest connection? Then we have to draw a line AEFC between A and C and we must demonstrate that it is longer than the ruled line AC (m). See figure 19:

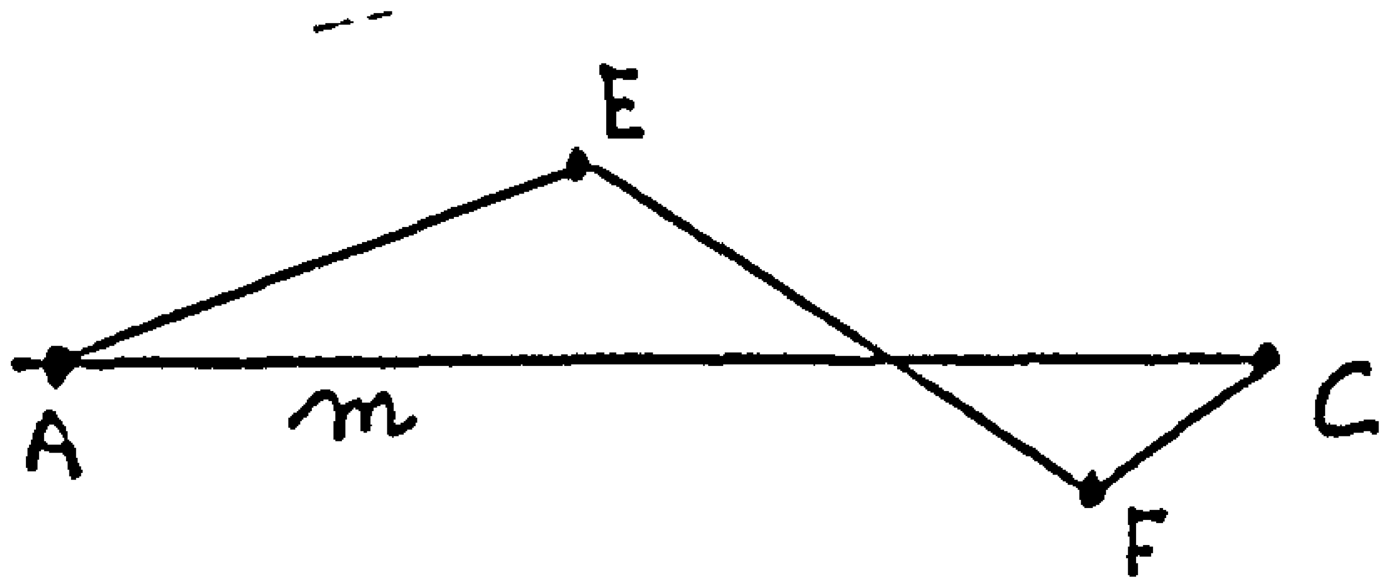


figure 19

It will be evident that continuing in this way will not yield success. Despite this, the 'common sense' definition of a straight line being the 'shortest connection between two points' is unbeatable. Moreover, that definition holds good for practical work; and as a practical instruction it is very successful. The approach of the 'shortest connection' between two points is practically very useful but theoretically it is problematic.

The result could be that many people are inclined to prefer practical solutions and want to keep far away from theoretical issues because such issues are felt to be an annoyance which undermines a sense of certainty. The following story confirms this.

"It happened that I was teaching geometry to a class of adults and I stated that it was not at all sure that line pieces of a certain length existed. My audience was indignant and one of the students asked whether she might explain the matter. I agreed. She then started to convey to the other students already long-known ideas such as: 'a straight line is the shortest connection'. Her words were very much approved by the other students. Later these students reported to the principal that I had not prepared my lessons well and that a man who pretended to be a teacher of mathematics should be aware, to say the least, that the shortest distance between two points was infallibly a straight line. They apparently considered such statements as basic knowledge in mathematics: a teacher who cannot subscribe to such statements should be sent back to college to learn the truth.

My conclusion was that my students only wanted to hear things they had already known for a long time and that new and challenging issues were not welcome. I said this to the principal when he showed me the negative reports of my students. The principal answered that I should have been aware of the fact that these students would behave badly if confronted with issues such as I had offered and that I should have presented other, less controversial items. After many years I am inclined to admit that the principal was indeed right."

Now we return to Euclid. During the nineteen-seventees Euclid's impact on geometry and geometry education was challenged.

The idea was that Euclid had dominated the territory of geometry long enough and the time had come to welcome new and refreshing approaches. However, this new approach comprised another system of axioms, written by Hilbert. But if you replace one system of axioms by another, an outsider may not be convinced that any renewal is achieved. Most people will not be aware that there is any difference between such systems of axioms.

I do not think that a departure from systems of the past should be too abrupt. There are many good and useful things in Euclid; and that should not be denied. This is what I have tried to show when I made a connection between Euclid's Optics and my 'Projections'. It was not my aim to demonstrate that Euclid would have been in favour of my artistic constructions. Neither am I trying to present Euclid as a predecessor of my ideas. Nor do I intend to behave like the medieval author who smuggled his own concepts into the edition of Euclid's "Optics".

About Euclid it is said that all geometers are standing on the shoulders of a giant. That giant is Euclid and in this position the head of a dwarf may still be higher than Euclid's head.

It is conceivable that much geometry is involved in a topic like 'optics'. So optics could be seen as a subject related to geometry and the geometer can borrow optical concepts for use in geometry. But that is the same as I have tried to do with art. Visual art can be presented in such a way that it becomes a source of information for the education of geometry. Therefore optics and art can both serve as a meansto support the teaching of geometry. For that reason I have sought a connection between optics and visual art and I think I found such a connection in the configuration of figure 10, where the eyesight of two eyes is situated in an artistic configuration, using Desargues' theorem. In figure 10 it has all come together.

Thus the configuration of figure 10 can be used as a discussion of binocular sight, displaying the purely geometrical theorem of Desargues; and as an example of visual art.

Here Euclid's idea of 'rays of vision' is very useful, just to focus attention on the subject of eyesight; and it is very similar to the modern concept of 'lines of sight'. It is also of historical interest, revealing the valuable fact that people many centuries ago were already contemplating questions similar to those we ask ourselves today.

There is also the matter of the horizon, absent from ancient art. However, when we realise that the visual imagination was much discussed in ancient Greece (see Chapter XV), it becomes conceivable that the possibility of drawing horizons may have been considered and rejected in ancient times. According to Euclid's "Optics", the perception of regions far away became a difficult matter because objects became less clear if observed from afar. One can easily understand that the horizon might have been considered too problematic to deal with.

The possibility of the discussion of the horizon in ancient times is purely imaginative but it has led to the construction of figures 14 and 15 where no horizon is used and the horizon is replaced by a system of mutually parallel lines, in figure 14 visually heading for one point S. Here I have tried to find a connection between Euclid's Optics and the modern concept of 'ideal' point, without suggesting in any way that Euclid was involved in the construction of'ideal' points.

These constructions of figures 10, 14 and 15 are products of my own imagination but I have tried to show that from Euclid's concepts elements can be taken that fit comfortably into the construction of 'Educational Geometry'.

CHAPTER XV

Optical Practice

In everyday life one sometimes faces problems which are related to optics. In ancient times, one major practical problem seems to have been that a statue had to be designed so that the viewer did not get a distorted impression. Look at figure 1:

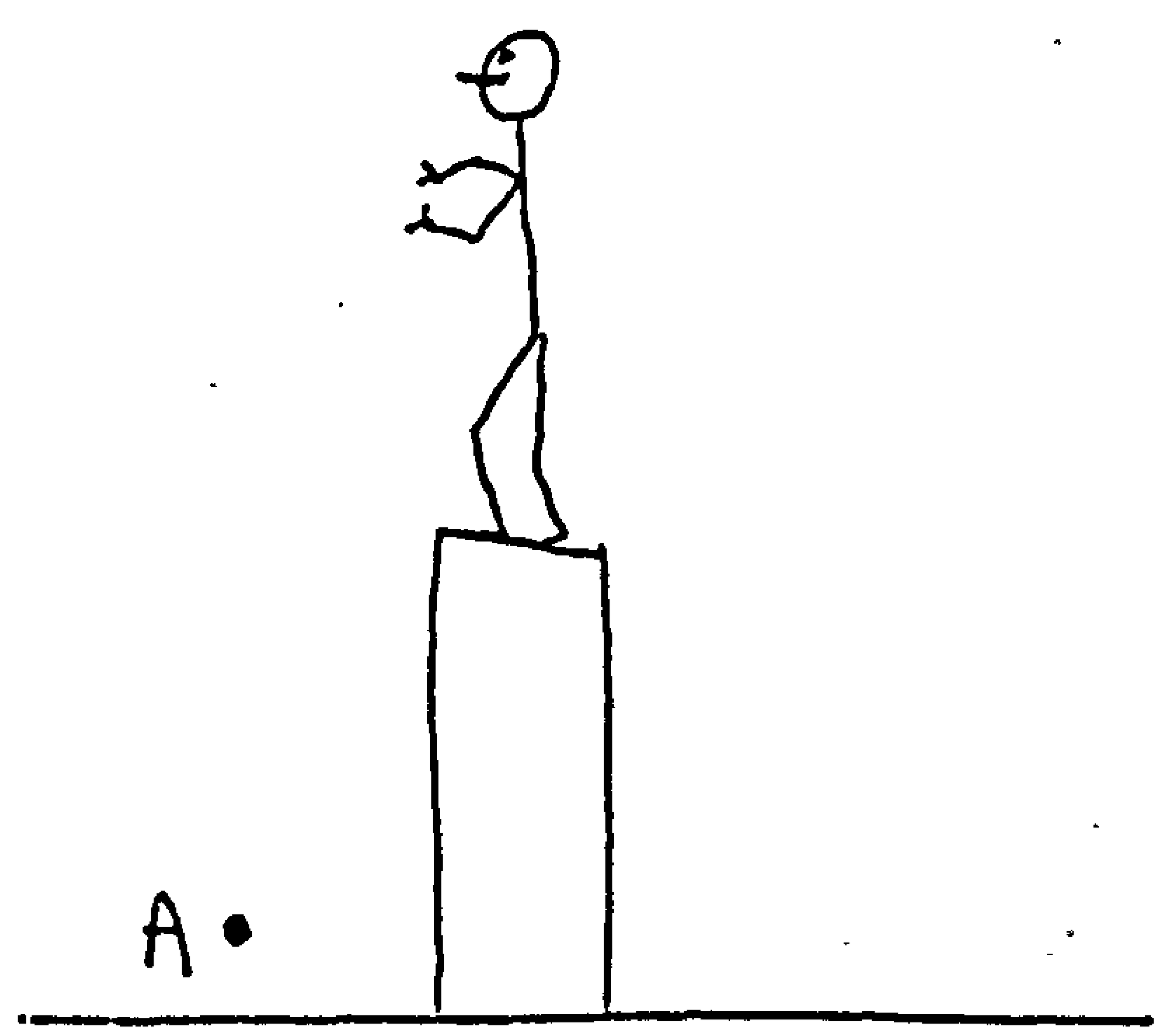
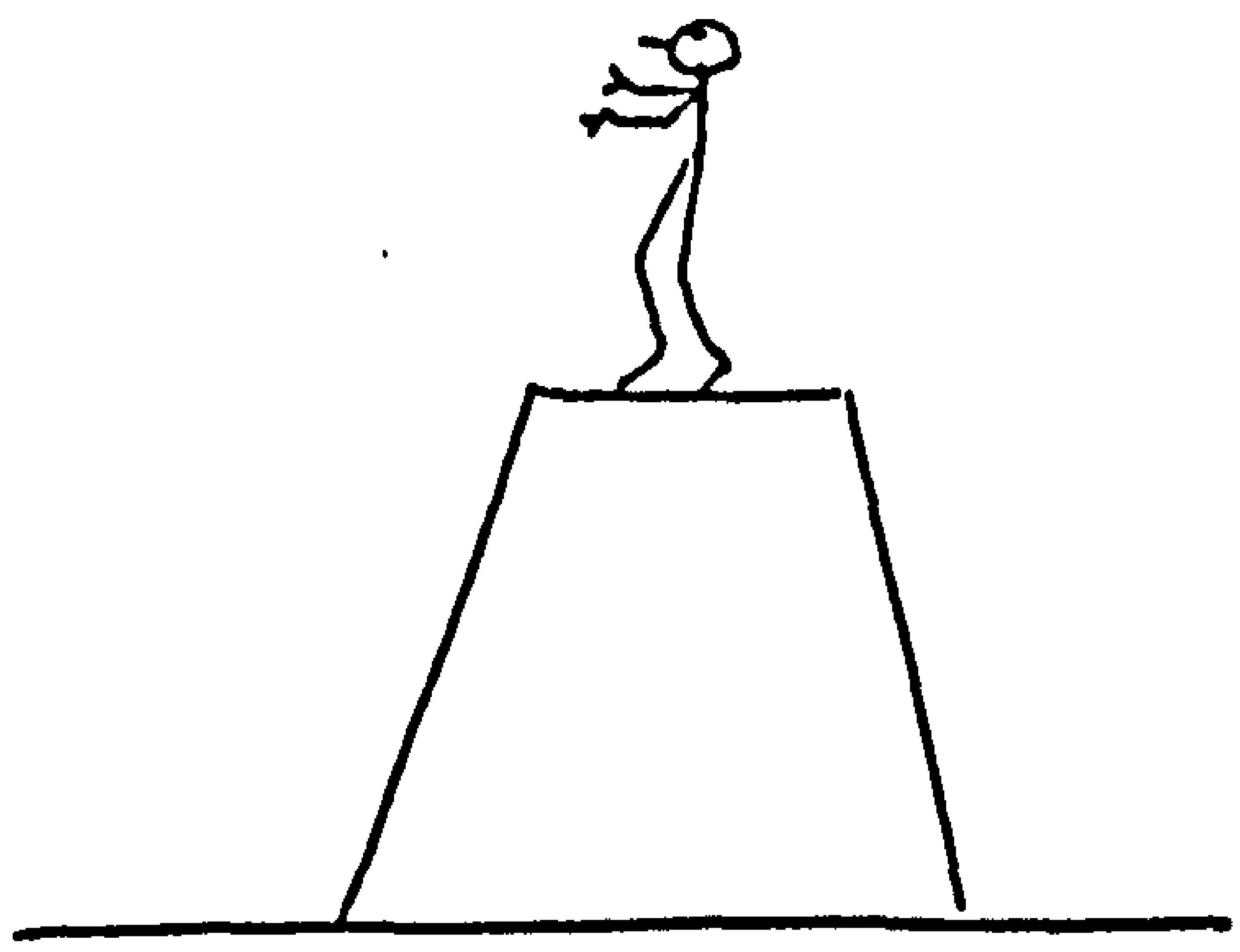


figure 1

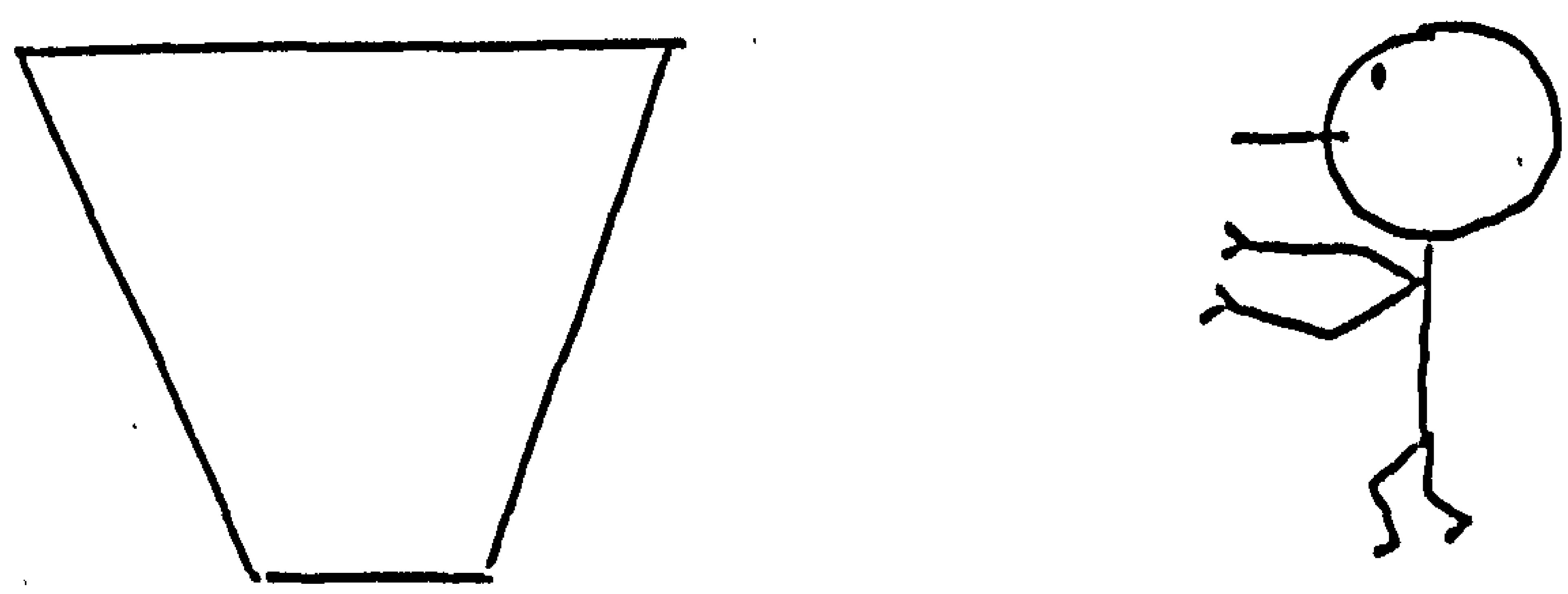
Viewer A looks at the statue, but the image which he gets from the statue appears to be distorted: figure 2 shows what he really perceives:

figure 2



To eliminate the distortion, the statue has to be moulded in a way which, when studied in close up, seems to be distorted (figure 3). However, now viewer A will see the image as shown in figure 1.

figure 3



Already in ancient times there was a wide knowledge about visual distortions. One could say that visual geometry was much discussed then. The following text is taken from Kim Veltman: Studies on Leonardo da Vinci, Munchen 1986, Deutscher Kunstverlag.

" *3. Ancient Methods.*

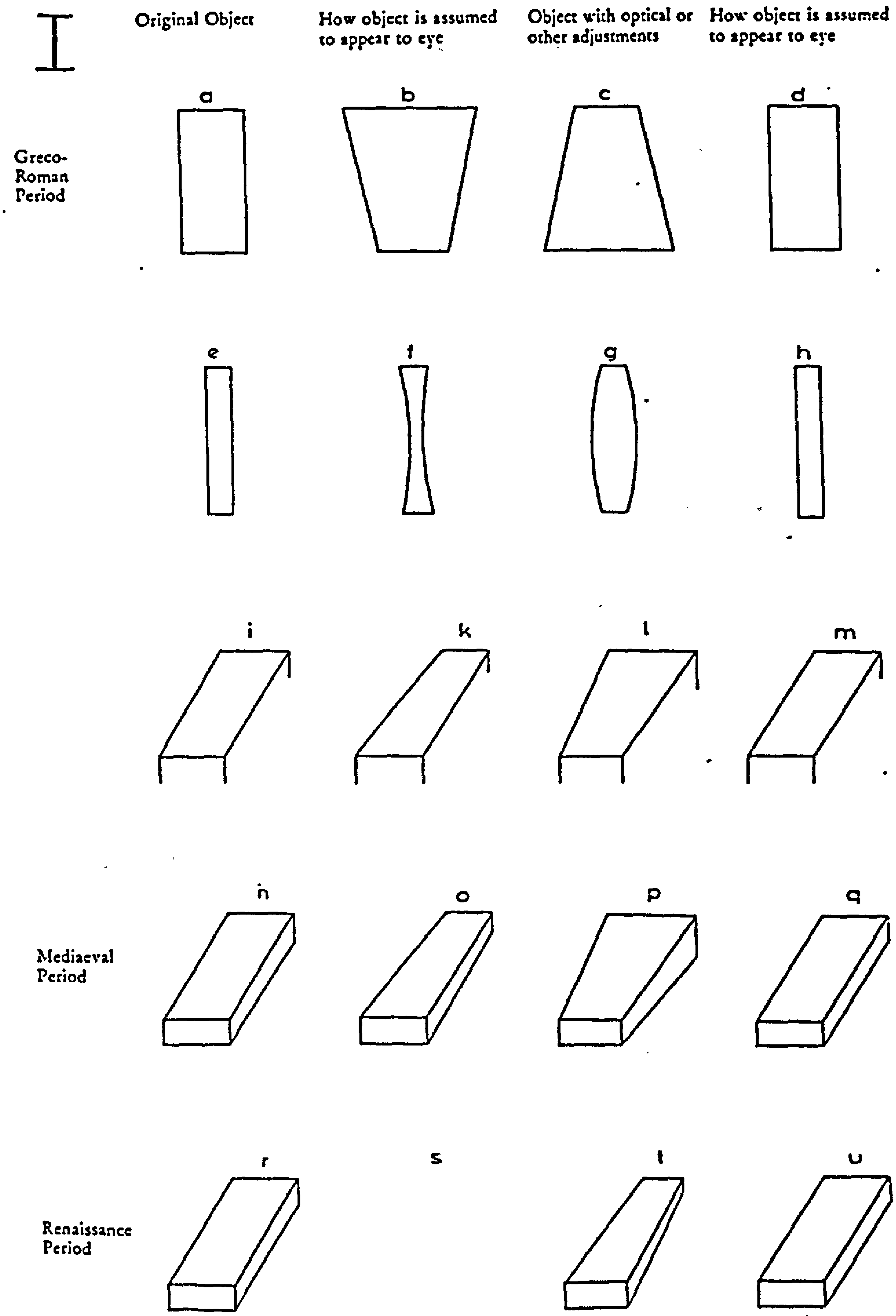
Various records of this ideal of optical adjustments among the Greeks and Romans have come down to us. Among them is Tsetses' story[7] of Phidias and Alcamenes, who both competed in making a statue of Minerva that was to stand on a high pillar. Alcamenes made an entirely realistic likeness. Phideas made a likeness distorted by optical adjustments that found great criticism from the crowd until the statue was set in its place. Phideas then won the competition.

No less instructive are examples from architecture provided by the Roman polymath Vitruvius, for they reveal how architects interpolated subjectively estimated optical adjustments into their objective designs. Vitruvius points out that if a door[8] were constructed without adjustments (Chart I a), the viewer would assume that an adjustment had already been made and therefore interpret it as spreading outwards from the top (I b). Compensation was therefore required to account not only for effects of distance but also for assumed earlier adjustments introduced to compensate for these effects of distance. As a result, it was required to construct a door which converged towards the top (I c) in order that it be seen as a normal door[9] (I d).

A similar logic governed the famous principle of 'entasis' in columns. If the columns were constructed without adjustments (I e) they would, according to Vitruvius, appear bi-concave in shape (I f). Hence it was required that one construct a column bi-convex[11] in shape (I g) in order that it appear normal to the eye.

These examples reveal how the Ancients compensated and even over-compensated for the effects of distance."

(Veltman, 1986, page 34).

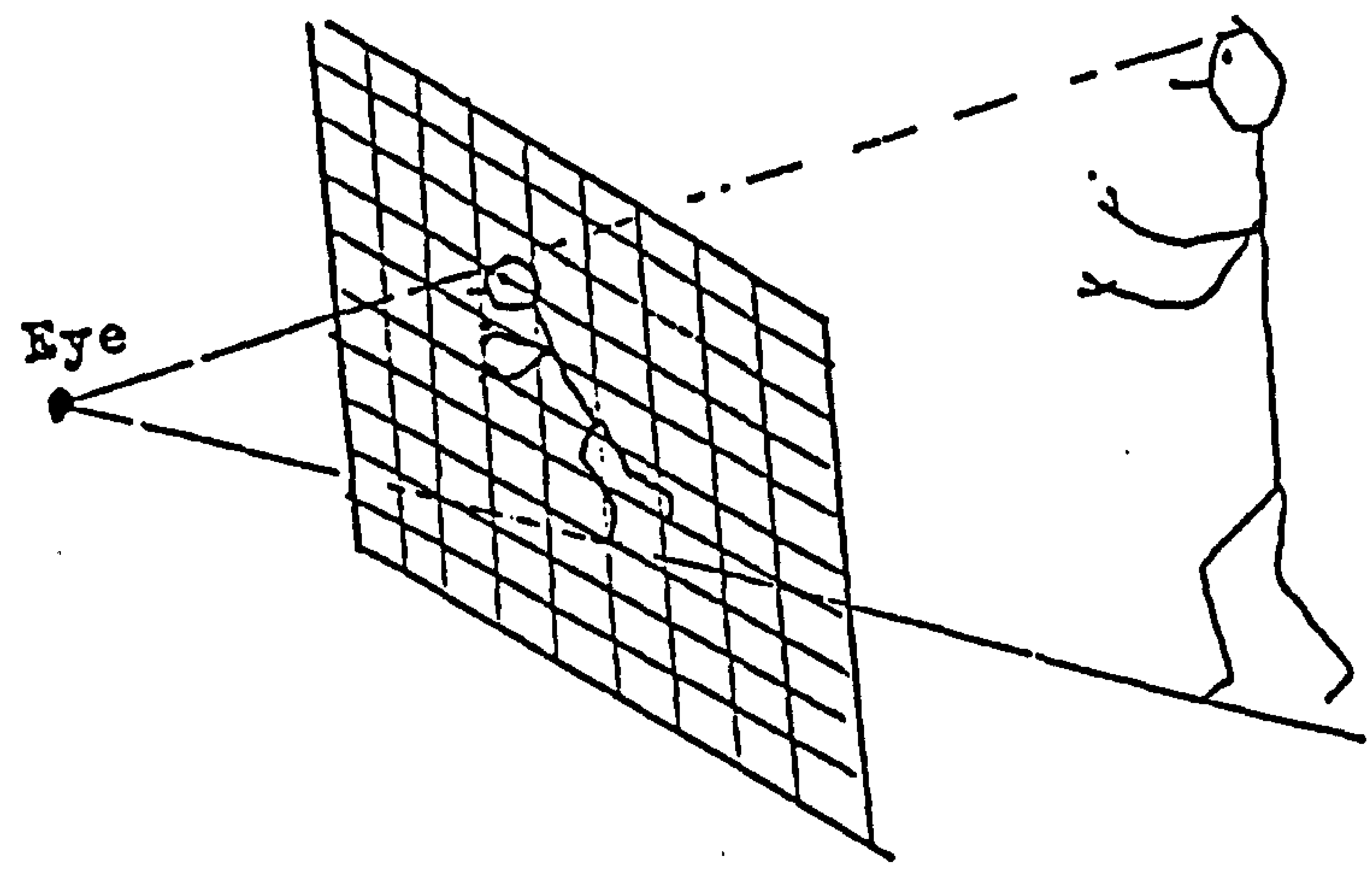


The text and images of this page are photocopied from Veltman, 1986, page 35.

More than thousand years later Albrecht Dürer and Leonardo da Vinci also had to cope with visual distortions.

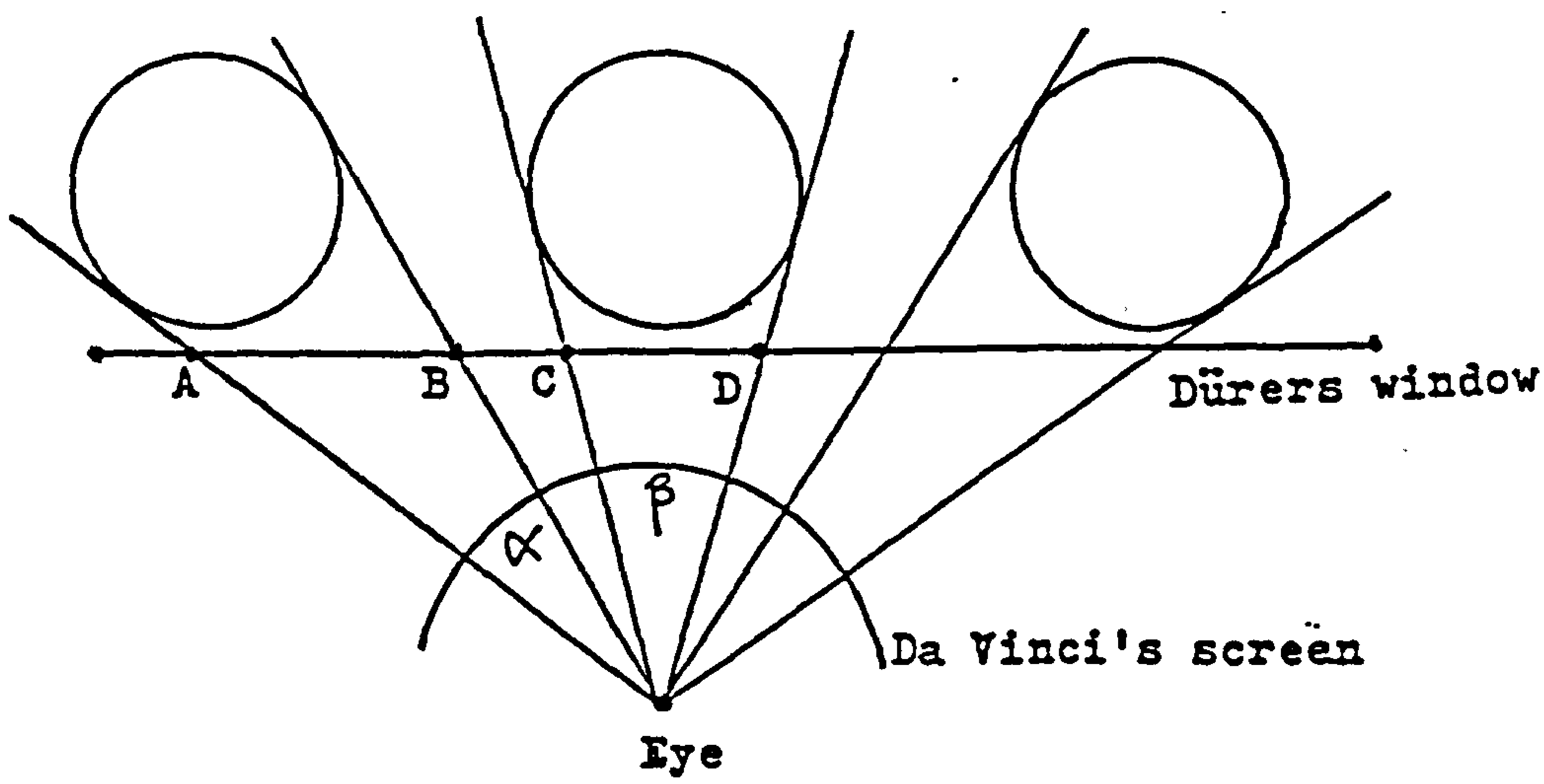
Albrecht Dürer (1471 - 1528) took a window and made a drawing on the glass of what he observed through it (figure 4):

figure 4



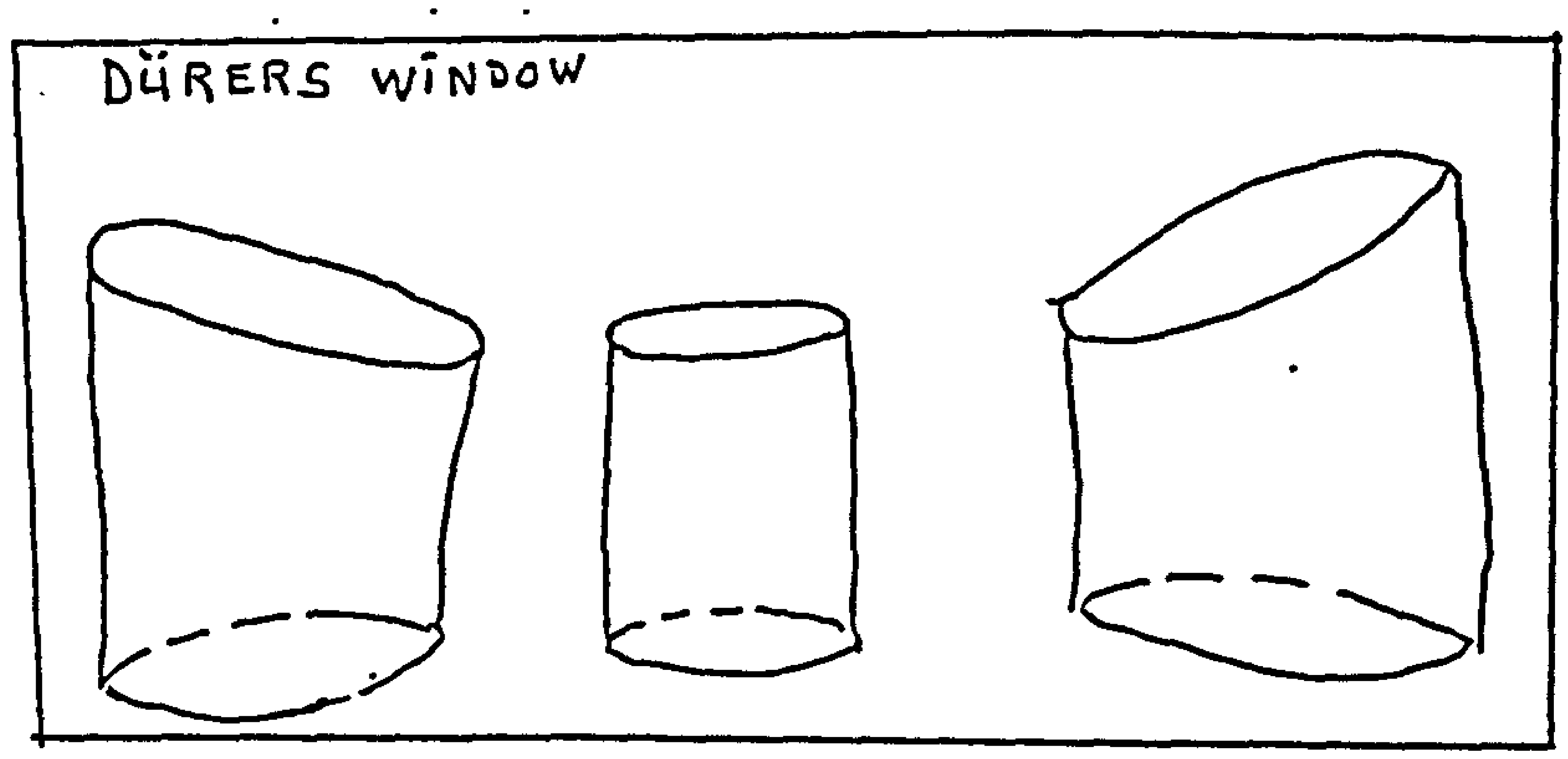
In this way perspective was introced into painting and the horizon was depicted.
An interesting problem emerges: the objects, drawn near the edge of Dürer's window, are deformed and larger than their equals depicted in the centre. The situation is sketched in figure 5 (the three circles are congruent).
Because of that distortion Leonardo da Vinci (1452 - 1519) chose a bowed window (figure 5). The reader is invited to check with a ruler that AB is larger than CD on Dürer's window. On Da Vinci's screen the angle α is smaller than β.

figure 5



The three cylinders of figure 6 might be congruent, although depicted as distorted on Dürer's window.

figure 6



Moreover, the part of the cylinder close to the edge of the window will be depicted larger than the part close to the centre of the window.

One might ask what it means when an image is undistorted.

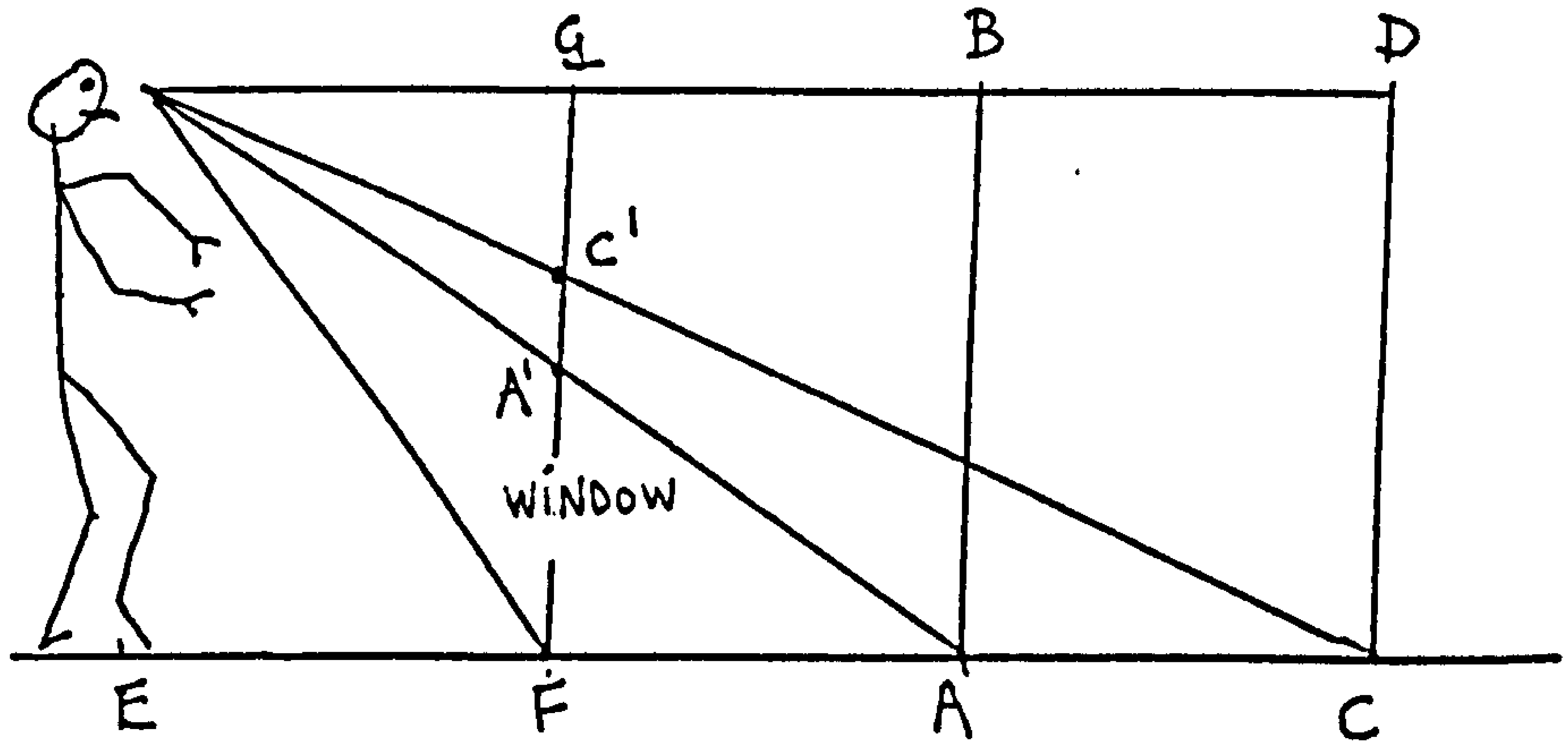
In ancient Greece the answer was not simple as we have seen. We could simply state that an image is undistorted when you do not notice a distortion. However, what is seen as undistorted by one person may look distorted if observed by somebody else. Apparently it is a matter of taste what the correct visual image of an object is.

Inverse Distance/Size Ratio

In perspective there is the so-called inverse distance/size ratio. Look at figure 7: an observer watches an object FG. Now the object travels away from the observer towards the position AB. On the spot of FG a window is erected. Now the projection of AB on the window FG is A'G. If we assume that EA = 2 EF then the image A'G = ½ AB (size). Had we taken the distance EC = 3 EF then the image C'G= 1/3 CD.

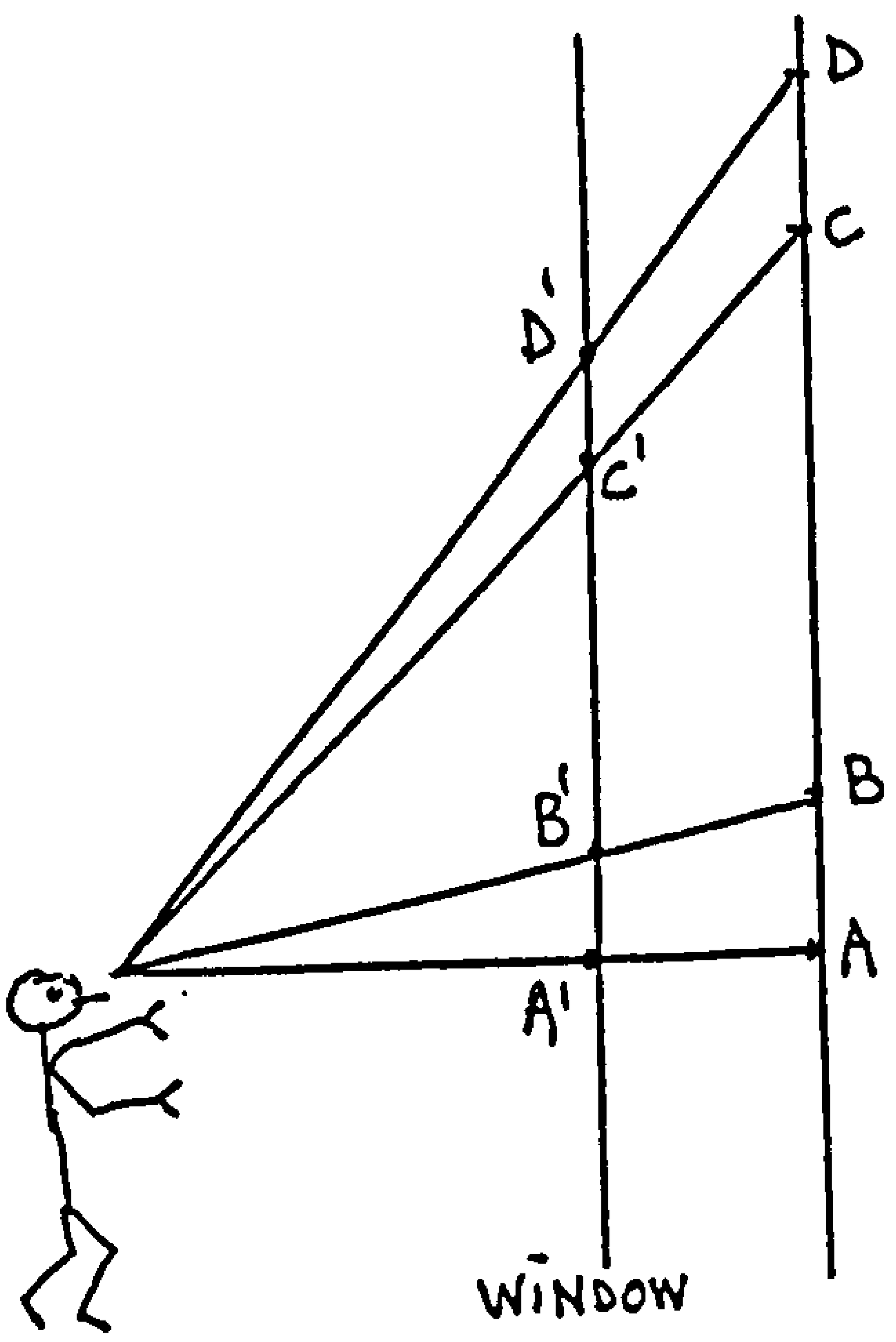
So, when the distance of the object becomes N times the distance EF, then the image on the window becomes 1/N times the size of FG.

figure 7



The inverse distance/size ratio is only valid when the object travels perpendicularly to the window surface. If the object travels parallel to the window surface (see figure 8) then the size of the image on the window does not change. So, if AB = CD, then A'B' = C'D'.

figure 8



Educational Validity In Antiquity

The story of Phidias and Alcamenes, narrated by Tzetzes (Veltman, 1986, page 34) demonstrates two things. First that the matter of visual distortions drew wide attention in ancient Greece. However, on the

copied representations I (Veltman) we see that also in Roman times visual distortions were widely discussed and that the ancients even over-compensated for the consequences of such distortions.
Looking at image I, I note that such images as shown there are difficult to create in drawings in which a horizon is applied. When a horizon is involved, the shape of the depicted objects is almost predetermined by the demands of parallel lines, meeting at a point on such a horizon.
However, in ancient Greece, by compensating for the effects of distance, the impact of a horizon is practically abolished and there is no room for such a phenomenon as a horizon. It seemed the aim of the ancients to present the drawing of natural objects in such a way that the distance of the observer from the object played no role. Every object had a fixed image which should be presented from far or from close and the distortions had to be compensated for.
So the implementation of a horizon in the late Middle Ages meant dealing quite differently with distance from the way the ancients had dealt with it.
The second issue demonstrated by the story of Tzetzes is that the crowd, criticising Phidias' work, demonstrated much educational invalidity, which was a result of the crowd's 'common sense'.
It might be said that the way in which the ancients handled visual distortions was a kind of visual geometry in its own right. One is inclined to say that these are not scientific problems. I conjectured before that the way in which an image can be seen as a distortion is largely dependent on taste and we are not used to calling that 'science'.
However, in Part I of the thesis we saw that it is not wrong to consider visually curved lines as straight. Take Escher's Pond as an example; and in the light of modern non- Euclidean geometries (like S^2 and Escher's Pond) it might well be scientific to accept that a visually straight line, drawn along a ruler, may be seen as curved.

Did the ancient Greeks have any approach to space, that we might recognise as 'scientific'? The ancient Greek philosopher Plato displayed an early cosmology in his book *Timaeus*. This will be the topic of the next chapter.
I will quote some parts of Timaeus in an English translation, comprising the famous 'gold analogy', and the discussion of the contents will be based on a doctoral thesis by Keimpe Algra, Utrecht, 1995. The doctoral thesis is named: "Concepts of Space in Greek Thought".
The explanation of Plato's cosmology has been much contested throughout history so I think it is necessary to seek an expert opinion like Algra's on the subject in order to lead the discussion. That does not mean that one has to accept Algra's opinion but I have used his statements to sketch a stage play, in which a geometer M is lecturing about topology in antiquity, but is vehemently attacked by some members of his audience. The material for this discussion, which is in the next Chapter (XVI) is largely borrowed from Algra's doctoral thesis.

CHAPTER XVI

Plato's Cosmology

One day a professional geometer, M, was phoned by a friend, N, who was no mathematician, but yet much interested in mathematical topics. N told M that he had regular meetings with a group of interested people who invited guest speakers to lecture on an issue. Such a guest speaker had of course to be an expert on his subject. N said that the group had talked about topology and that there had been lively discussions. One of the members of the group had wondered whether there were roots of topology to be found in ancient knowledge. Because nobody had been able to answer the question, N had put forward the idea of inviting the geometer M to lecture about topology in antiquity. And the chairman of the group had eagerly agreed.
M was very honoured by the invitation and without hesitation he accepted. His friend N was very happy and he sent a note to the members of the group saying that the lecture by M was going to take place and he fixed a date and an hour. The geometer M, however, became apprehensive. He was, after all, not so sure

that he would be able to find interesting material or even if he was going to find any material at all. But he could not go to the meeting just to admit that he had not been able to find anything noteworthy on the subject.
For the rest of the week M spent his days in the University Library researching; but sadly his efforts appeared to be wasted because he found nothing. Desperately he consulted a book about Euler's Characteristic and looked through the pages concerning solids. Then he was struck by the properties of Platonic bodies. Once at home he phoned his colleague P, who was an amateur philosopher, and he asked him whether Plato had said anything about topology. "I am not quite sure," P replied, "but I know that Plato wrote an early cosmology which could be helpful for your lecture." M thanked P sincerely. Now he could make a start. First of all he sent every member of the discussion group a copy of the text of a lecture on Euclid's *Optics* which he had given not long before. It was largely the same as we have read in Chapter XIV of this thesis.
Then came the lecture. M made an introduction by speaking as follows:

"In his book, the ancient philosopher Plato gives a remarkable account of the way the universe is organised. In those days people assumed that there were four elements: earth, water, air, and fire. By lowering the density of earth, it transformed into water. Proceeding by lowering the density further, one got air and finally fire. Nowadays we have other views of this process, mainly as a result of the development of chemistry. So it is useful to ask whether Plato's cosmology has any actual importance other than for the historians.
It is often argued that mathematics does not change too much as the time goes by. Mathematicians have no other choice than to change their subject if they want something new and satisfy their sense of fashion. From the point of view of 'visual geometry' it turns out that the ancient Greeks already considered differences between what you look at and abstract reflections of it. As far as Plato is concerned, his philosophy created a clear dichotomy between eternal, invariant ideas and the floating, unstable realm of the sensible world.
In this philosophy, the universe is split into two. On the one hand, there is the abstract part where the ideas are which cannot be apprehended by our senses. These eternal beings can not be seen, heard, touched, smelt or tasted. They belong to the realm of God and the only thing we can do is try to understand a small part of it. With our intelligence we are able to gain something of this knowledge. On the other hand, there is the everyday reality of the ever changing phenomena of this world, where nothing is stable and everything changes or fades away. Appearances come and go rapidly.
These two worlds, the abstract one of the ideas and the floating sensible world, are not only separated but seem to contrast each other."

After this brief introduction, there was a pause. M had made copies of certain parts of Plato's book 'Timaeus'and he handed a copy to each member of the audience. They were asked to read through the text so that the rest of the lecture would be understandable. M had checked beforehand that this text could be easily understood without the knowledge of other works of Plato or other ancient authors.

Now follows the text of the copies M had handed to his audience. The copies are taken from the Desmond Lee translation of Plato's 'Timaeus', published by Penguin Classics in 1965.

16. The receptacle of becoming.

"We must start our new description of the universe by making a fuller subdivision than we did before; we then distinguished two forms of reality - we must now add a third. Two were enough at an earlier stage, when we postulated on the one hand an intelligible and unchanging model and on the other a visible and changing copy of it. We did not distinguish a third form, considering two would be enough; but now the argument compels us to try to describe in words a form that is difficult and obscure. What must we suppose its power and nature to be? In general terms, it is the receptacle and, as it were, the nurse of all becoming and change. But true as this is, it needs a great deal of further clarification, and that is difficult, among other reasons, because it requires a preliminary discussion about fire and other elements."

18. The receptacle compared to a mass of plastic material upon which differing impressions are stamped. As such it has no definite character of its own.

"Let me try to explain the point again more clearly. Suppose a man modelling geometrical shapes of every kind in gold, and constantly remoulding each shape into another. If anyone were to point to one of them and ask what it was it would be much the safest, if we wanted to tell the truth, to say that it was gold and not to speak of the triangles and other figures as being real things, because they would be changing as we spoke; we should be content if they even admit of a qualitative description with any certainty. The same argument applies to the natural receptacle of all bodies. It can always be called the same because it never alters its characteristics. For it continues to receive all things, and never itself takes a permanent impress
from any of the things that enter it; it is a kind of neutral plastic material on which changing impressions are stamped by the things which enter it, making it appear different at different times. And the things which pass in and out of it are copies of the eternal realities, whose form they take in a wonderful way that is hard to describe - we will follow this up some other time. For the moment we must make a threefold distinction and think of that which becomes, that in which it becomes, and the model which it resembles. We may indeed use the metaphor of birth and compare the receptacle to the mother, the model to the father, and what they produce between them to their offspring; and we may notice that, if an imprint is to present a very complex appearance, the material on which it is to be stamped will not have been properly prepared unless it is devoid of all the characters which it is to receive. For if it were like any of the things that enter it, it would badly distort any impression of a contrary or entirely different nature when it received it, as its own features would shine through. So anything that is to receive in itself every kind of character must be devoid of all character. Manufacturers of scent contrive the same initial conditions when they make liquids which are to receive the scent as odourless as possible; and those who set about making impressions in some soft substance, make its surface as smooth as possible and allow no impression at all to remain visible in it. In the same way that which is going to receive properly and uniformly all the likeliness of the intelligible and eternal things must itself be devoid of all character. "
(Plato, 1965, pages 66,68 and 69)

It took the members of the group some time to digest all this information. For some of them it was the first time they had read anything written by Plato; but others had already studied Plato's works thoroughly so that they finished reading quickly. This latter group had already started a conversation about the contents of the text. It appeared that an understanding of what precisely the receptacle could be created a division amongst them. One party saw the receptacle more as an object through which things were created but the receptacle was not seen as a space. The other party believed that the receptacle served to produce a modelling of the phenomena and that its character was space-like. However, before the discussion could become vehement, M continued his lecture as follows.

"For the further interpretation of Plato's text I will no longer rely on my personal views. Interpretations of Plato's cosmology vary widely and there are many arguments about the correct explanation of Plato's 'Timaeus.' So I took Keimpe Algra's doctoral thesis: "Concepts of Space in Greek Thought", E.J. Brill, 1995 and I used it for the preparation of this. Algra is a historian specialising in antiquity and he is an associate professor at the University of Utrecht in The Netherlands.
Algra states that Plato's descriptions of the receptacle are not entirely coherent. One might suppose that Plato had a spatial receptacle in mind, a 'something-in-which' (Algra, 1995, page 77). However, this issue has divided scholars ever since the days of Aristotle. There is a party of scholars who assume that the receptacle is 'something-out-of -which', indicating that the receptacle is a kind of matter.
So the controversy is focused on whether the receptacle may be considered as space or as matter. The question 'matter or space' becomes even more urgent when it is assumed that the labels 'space' and 'matter' are incompatible. The 'matter' party underlines that from the 'gold analogy' it must be clear that the receptacle has a function rather than being a space. So it is matter and it has a 'thing-like' character. The 'matter' party further argues that the terminology 'in-which' only indicates that the receptacle receives all things and thus has no spatial meaning.
The 'space' party of course is opposed to this and uses the following arguments:
(1) matter is corporal
(2) space is absolute (Algra, 1995, page 84)
From these two points the 'space' party infers that it is incompatible for the receptacle to be space and matter at the same time. The 'gold analogy', where gold is seen as a 'moulding stuff', is no more than a metaphor. The 'space' party further states that Plato uses the label 'in-which' many times in Timaeus, but the label 'out-of-which' is only used in the context of the 'gold analogy'. So, according to the view of the 'space' party, Plato refers more often to 'space' than to 'matter'.

Algra quotes Guthrie who states that at some places in Timaeus the labels 'in-which' and 'out-of-which' are used synonymously. In English the term 'in-which' has an ambiguous meaning, and the same is true for the Greek term. An example is provided: A clay bust is 'in-space' but also modelled 'in-clay'. Both English and Greek allow one to speak of the object being made 'in' that material and also to talk about the manufacturing of an object 'out-of' pre-existing material.
Algra makes the question quite clear in the following sentence: "the idea that it is the 'form' of the final product which is worked out 'in' clay whereas that which is said to be made 'out of clay' is not the form but the final product." (Algra, 1995, page 90).
Consecutively Algra says that, talking about the 'gold analogy', the gold is both the 'in-which' and the 'out-of-which' but it does not have these qualifications 'with respect to the same things'. (Algra, 1995, page 91).
The foregoing interpretation of Plato's receptacle, asserting that the labels 'in-which' and 'out-of-which' need not exclude each other, is Algra's view on Plato's third factor."

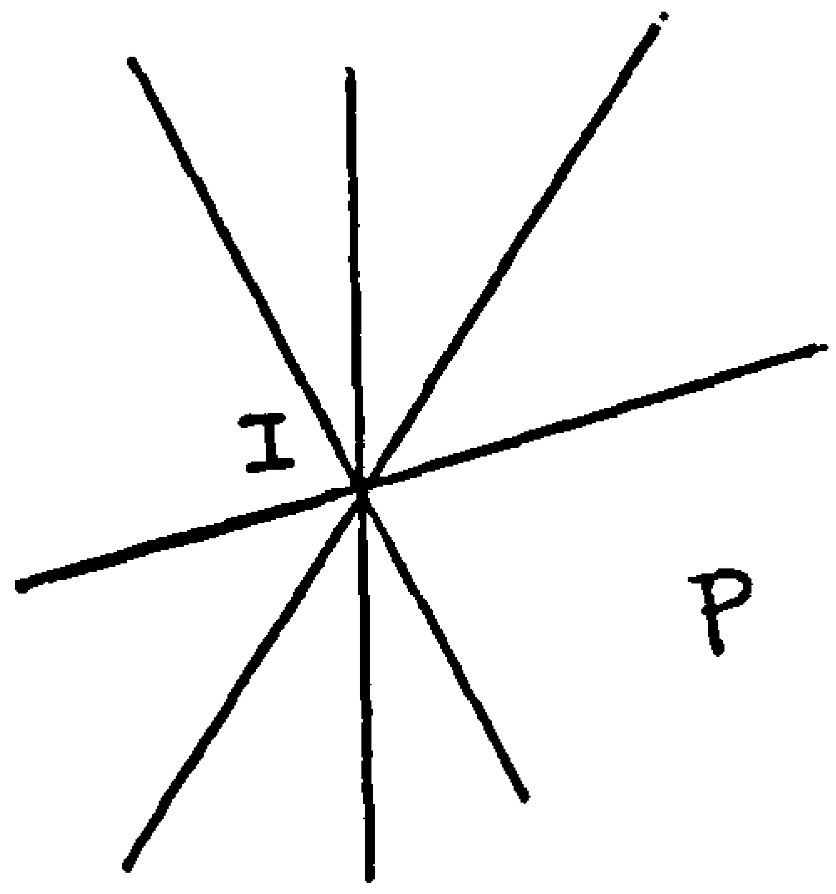
There was a short break before M continued:

"At this point I will not proceed to follow the discussion in Algra's doctoral thesis on the working and status of the third factor because I think we have now collected sufficient material to be able to connect the above discussion between the 'matter' and 'space' parties with our subject which is 'topology in antiquity'.
What I have been trying to do is to find roots of topology in antiquity. One may talk about different topological spaces. I hope to make clear that the above dual interpretation of Plato's receptacle as an 'in-which' or an 'out-of-which' can be illustrated by means of the dual concept of a straight line as 'local' or global'.
The global straight line will be seen as an 'in-which' and the local straight line as an 'out-of-which'.
First let us focus on the local straight line; it is what we call a 'visual straight line', (see Definition (1)).
Such a line piece could be seen as build up from 'out-of-points'. The points are the material from which the 'visual straight line' is made. So we are supposed to have a set of points and out of these points the visual straight line is manufactured.
Secondly let us look at the global straight line. We saw that such a line can be represented by just two single points. It would not be correct to say that these two points provide the material to build a global line. Many more points are required. Nevertheless the global straight line is fully determined just by the choice of two points. So it would be correct to say that the global straight line is modelled 'in two points', referring to the gold analogy of Plato where the geometrical figures were modelled 'in gold'.
The global straight line is build up 'in two points' and not 'out-of-two points'. It refers to the modelling, to the form of the straight line, not to the material 'out-of-which' the global straight line is manufactured.
Two components are essential for a straight line: the material and the form. We may say that a straight line consists of points. Then however nothing refers to the form. We can not hand somebody a set of points saying, "This is a straight line." So the modelling of the straight line is a matter of basic importance. That is reflected in the presentation of a global straight line of which it is implicitly assumed that it is modelled; but the visual presentation of such a modelling consists of just two points 'in-which' the line is built up.
This example: the straight line 'out-of-points' or build up 'in-points' is an illustration which I have produced myself to show you the difference between the two competing opinions concerning Plato's receptacle. It may be noticed that Algra's assertion that the terms 'in -which' and 'out-of-which' are not referring to the same things, has fully come true. In one case we are talking about just two points (the 'in-which' of the global straight line) and in the other case about an infinite number of points (the 'out-of-which' of the local straight line).

You may now remark that we still have not discussed how the above differences between 'local' and 'global' geometry are connected to the question of 'topology in antiquity'; and after all this was the topic we were supposed to discuss now. I ask you to be patient. It will not be long before we arrive at the topological issue.

A second illustration of the differences between the 'matter' party and the 'space' party of Plato's receptacle will be given with the help of a flat plane which can be denoted by R^2. We know it very well from our secondary school education. We remember from Euclid's *Optics* that a plane could be formed which consisted of mutually parallel straight lines. At a later stage the parallel lines were replaced by lines going through one point and for that point an 'ideal' point of the set of parallel lines could be chosen (figure 1).

figure 1

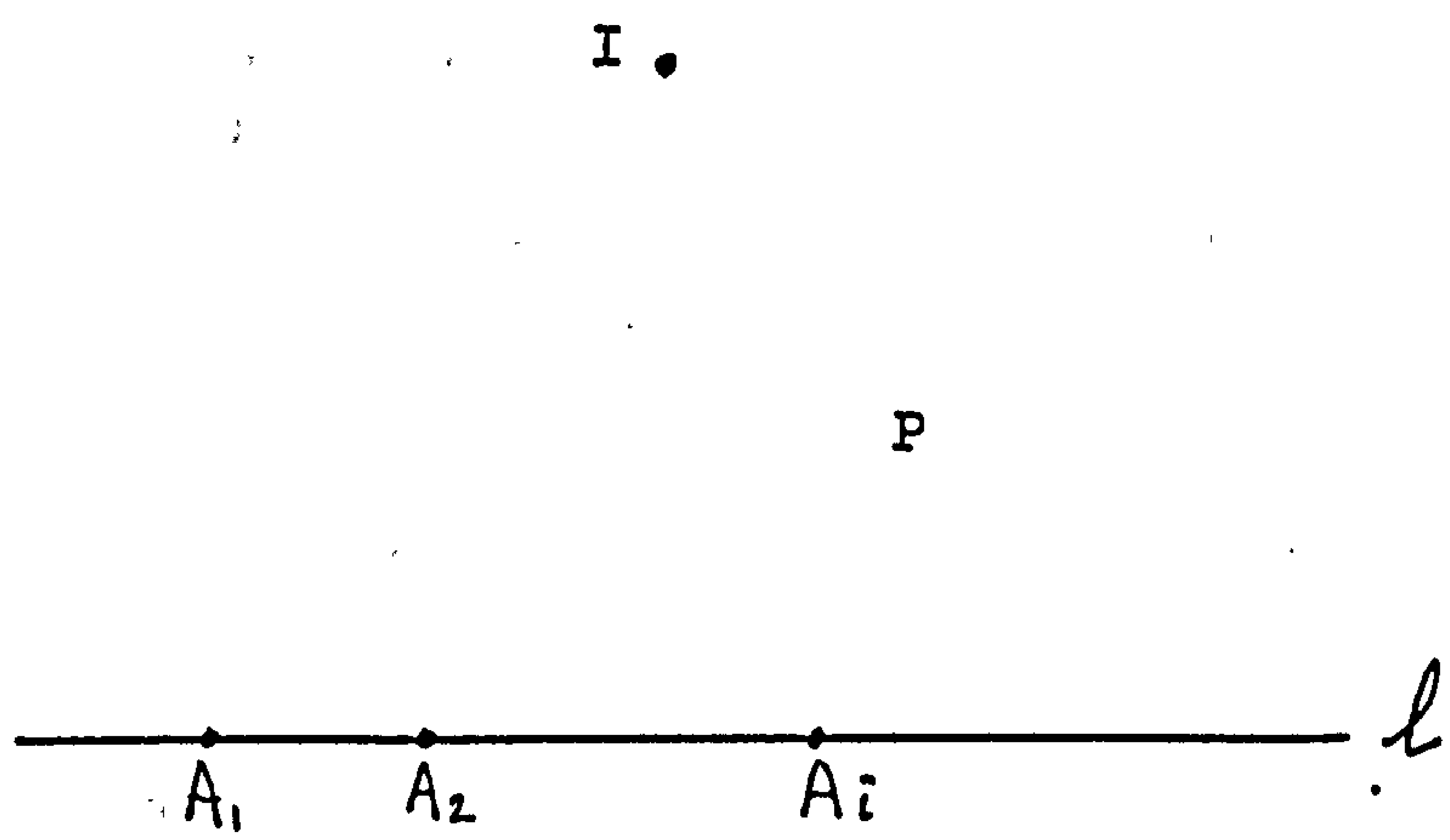


I = ideal point

P = plane

The (straight) lines in figure 1 are supposed to be parallel because they meet at one point which is the ideal point of all the mutually parallel lines. We may state that all the straight lines of figure 1 lie in one plane P. This plane P has emerged as a modelling of those parallel lines and therefore we say that plane P is built up 'in parallel lines'. So this is the 'in-which' approach to a plane P.
It is, however, also possible to draw straight lines in the plane P which are not going through the 'ideal' point I. Now P can be seen as manufactured 'out-of' a set of straight lines which need not be parallel at all. Here we have the 'out-of' approach to the plane P.
There are some objections to be made. In figure 1 there are not only lines; there is also a point I. However that point I is 'only' an 'ideal' point which cannot be pinpointed anywhere and is rather a help for the modelling of the plane than that it can be considered as a point that can be pinpointed specifically and actually.
In figure 2 a presentation is displayed of the plane P in the two situations 'in-which' and 'out-of-which'. One more objection can be made. In figure 1 the plane P is a geometrical object but the material 'out-of-which' the plane is made (the parallel lines) is also geometric. This is a serious objection. In the 'gold analogy' the geometrical figures and the material 'gold' are not of the same kind and do not belong to the same 'stuff'; but in our examples of 'local' and 'global' straight lines and the plane P of figure 1, the modelled 'object' and the material are both geometric objects. Therefore I do not think that Plato would have approved of my example but, on the contrary, he might vehemently have tried to bring it down. I should apologise for my shortcomings but I have done my best to provide you with an example of what could be seen as the first topological issue in antiquity - the receptacle of Plato and its controversial explanations.
In figure 2 a visual presentation of the plane P 'out-of-lines' and 'built-up-in-lines' is provided."

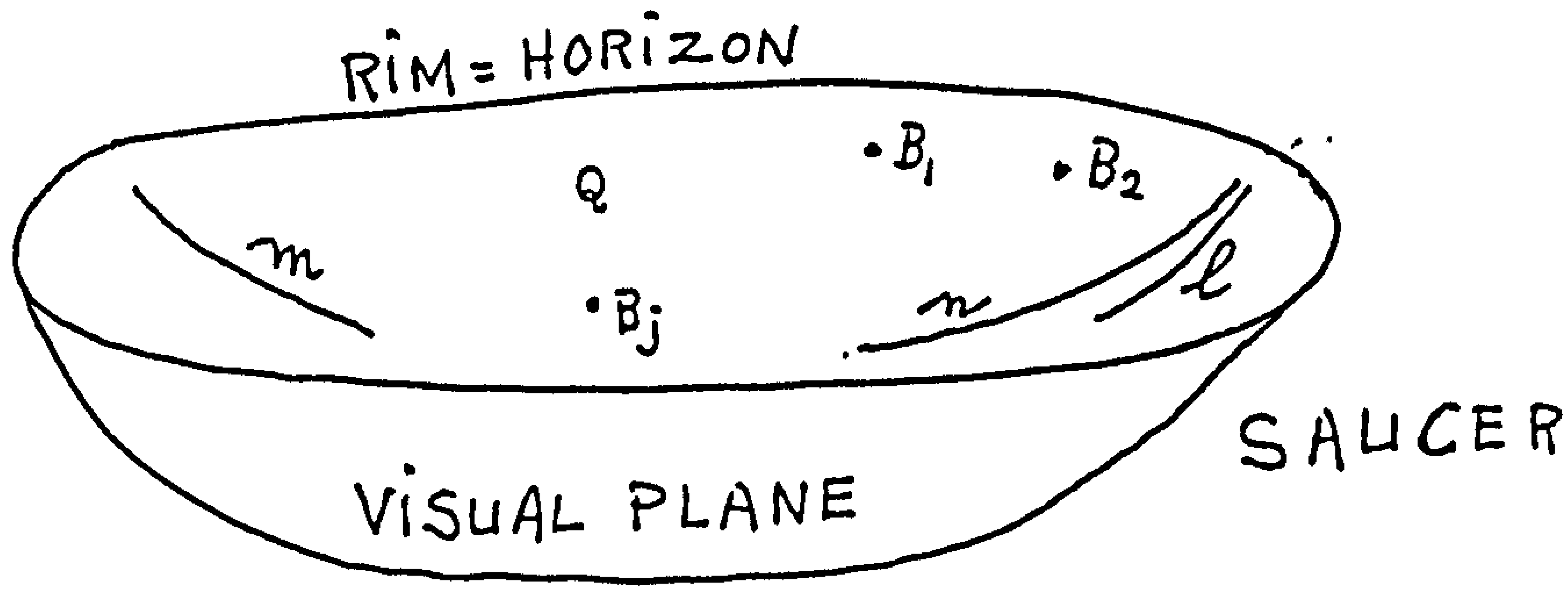
<u>MODELLED</u>: PLANE P



ℓ = LINE MADE OUT OF POINTS A_i
P = PLANE BUILT UP IN GLOBAL LINES IA_i
I = IDEAL POINT

FIGURE 2

<u>FINAL</u> <u>PRODUCT</u>: PLANE Q



Q = PLANE MADE OUT OF LOCAL LINES ℓ, m, n
OR POINTS B_j

At the end of his lecture, M unfolded figure 2 and he pointed out that plane P was from a different topological structure than the plane Q.
P could be considered to be 'built-up' of parallel lines so that in plane P there was no line except for the already existing mutually parallel set. Plane Q was less structured; lines of all direction were scattered over the plane. M now declared that he had found roots of topology in antiquity and that these roots could be found in Plato's cosmology in Timaeus by virtue of the contesting explanations of it.

M thanked his audience and they all had a drink. There was a pause and after that the audience had a chance to ask questions.

The chairman of the group asked M how he could possibly be so mad as to think that Plato had foreseen the controversy over the explanation of his concept of the receptacle and that it was unfair to Plato to maintain on these grounds that there were roots of topology to be found in his Timaeus.

M answered that he had chosen the side of the 'space' party so that it was quite natural to connect Plato's receptacle to considerations about space and that then of course topological issues would inevitably emerge, for instance when the question was put concerning how such a space could be modelled. Moreover, already during Plato's life the concept of the 'receptacle' was fiercely discussed and contested. The 'Space' party and the 'Matter' party must have emerged then so that the topological aspect was already there.

The chairman answered that M unfortunately had superimposed his own insane concepts on the innocent philosopher Plato. The chairman stated that he had chosen the side of the 'matter' party and that he could not conceive that somebody would dare to abuse Plato in the way M had done.

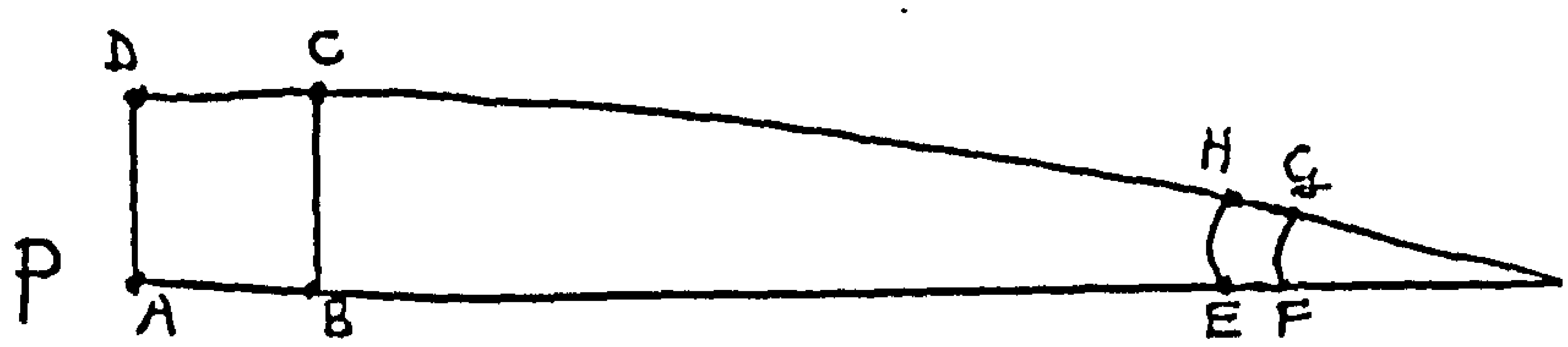
Now M's friend N, who had chosen the 'space' party, shouted to the chairman that he only fostered a snobbish and pathetic interest in the subject of 'topology' because he had never been able to collect sufficient marks in geometry when he was a schoolboy.
Therefore he played down every geometric issue and all the geometers whenever he got the opportunity. He had only agreed to invite M to lecture on such a difficult subject to be able to bring him down.

The chairman then challenged N to discuss the accurate meaning of the Greek words of Plato's Timaeus. The chairman knew very well that N had never studied Greek or Latin.

Finally the discussions ended and M was allowed to react. He said the following: 'I have been accused of abusing Plato and his Timaeus by irrelevantly extracting so-called topological issues from his texts. Even if you are right, then will you please not forget that you are doing such things yourself almost every day?'

After these words he drew the following image on the blackboard (figure 3).

figure 3



He continued: 'You see square ABCD in front of person P. Let us assume that it is one square metre. Now the square travels along the straight line AB towards the position EFGH.

I have drawn it to demonstrate how the square EFGH is perceived by P and I can prove that it is true.

Nevertheless you dare call that visually distorted square EFGH still a square metre?
You must be out of your mind. What you are doing is abusing visual geometry.' After this scientific argument we will consider some issues concerning the composition of a thesis and we will look at an overview of the sources consulted (Chapter XVII)

CHAPTER XVII *Considerations*

17.1. Considerations about the composition of a thesis.

Once one has decided to take up scientific research, one's attitude towards existing literature changes; suddenly one has to be careful. The sources available are split into two groups. The first group consists of books which are judged inadequate since their scientific level is considered too poor. One should painstakingly avoid quoting from such books or mentioning the names of their authors.
The second group consists of the approved and welcome sources which contain celebrated results from the past or refer to a future in scientifically acceptable environments at a high intellectual level. These approved sources form a general background literature.

When I started my investigations, the relationship between the researcher and the existing literature was a matter of deep concern for me. The standard procedure would be to read extensively and intensively at the beginning of the research period to gather as much information as possible. After having subsumed all this new material, one is expected to have arrived at such a high level that it is possible to focus on a specific subject. One has to be aware of the related topics and know how other researchers consider that subject. Only then may fruitful research be initiated.
In my case there was an extra dimension and that is the fact that I have already lectured in both mathematics and mathematics education for many years. So I knew many sources from the past by experience. Some of these sources are now offered to new generations and to me too as fresh and essential.
It is not always an advantage when you have to state that you have gained knowledge from practical experience. Soon you may be considered too old-fashioned to appreciate what is going on nowadays. Your history is turned against you. It sounds better when you can say that you are too young to remember.

So a problem I was facing at the start of my research was that I would have to consult sources which I knew too well and, moreover, that I would have to draw fresh conclusions from these memories. It is a fact that much that is considered by new generations as a fresh approach, sounds obsolete to one who has seen it before. But subsequently you arrive at the 'common sense' opinion that nobody is allowed to state that he has seen it before, that particular problems were noted long before, and that therefore it would be wise to learn from the past and avoid the errors which occur time and time again.

However, somebody who says that it has already been tried before unsuccessfully will inevitably be considered as unfit for his job!

These are some of the issues one has to cope with at the start of a research period. I had to focus on a subject that I could recognise for myself as really new and not something that would be seen as a renewal by everybody except by myself. Especially in the area of renewals, there are ossified topics, which are stubbornly presented as brand new. Moreover, the feared image of the old revolutionary may emerge, who is the only one who does not know that he is outdated.

Starting the research, could I expect to be able to extract something new from the sources available? A new danger was looming. If a reader works his way through much literature, by the end he has been thoroughly influenced by it, but the new knowledge may not yet serve as a background: it is still too fresh. Moreover, you might conjecture that something that is presented as new and challenging by the sources, could appear to be old-fashioned and obsolete when it is further investigated.
It is a well-known fact that when you investigate something, the experience of it will always be different from what you expected before you started the work. So you might walk into some kind of trap and it will be difficult to extricate yourself because too much work has already been done on the unsuitable subject. This is a trap indeed. How do you know beforehand that the results of your work will be acceptable or interesting? Will any result be achieved ? Starting research is a gamble the outcome of which is uncertain. Indeed it is sometimes heard that the investigator is disappointed with the results of the research. In the beginning one might think that a certain issue is worth investigating but then gradually one might lose interest as the work proceeds. This is not uncommon and there is a tragic aspect to it. Sometimes it may even be better not to reach your goal because the final successful achievement might yield little more than a feeling of dismay.
These are the hazards which threaten the investigator starting his research.

Thus, in my case I had the advantage that a large part of the literature to be read was already known by me. But what is that advantage? As we saw above, your memories may be used against you. I saw it as an advantage that the old literature was no longer new to me so that I would not be unduly influenced by reading it. Besides this, I have had so much experience in the field to be investigated, that a reasonable background was already there at the start of my research. Thus, it might enable me to choose a suitable subject,

There is still another issue. How does Scientific Geometry relate to the subject of Geometry Education? The matter seems to be simple, but it is not. Some teachers are too academic. Is it desirable for a teacher to have a level of knowledge far above the level of his pupils? Is that a stimulating situation for those pupils, facing a teacher who knows much more than they will probably ever be able to master? Will they be discouraged?

On the one hand, pupils may be eager to tell stories to their acquaintances and relatives about the seemingly inconceivably high scientific level of their teachers. On the other hand, they need a teacher who understands their difficulties and so his level should not be too far above the pupils'. I will not try to solve these problems or even think of a solution. But what was indeed important for my research was the question whether the educator is qualified and entitled to assess the scientific geometry he tries so desperately to convey to his audience.

It has become a central issue of my work. Should the educator only be seen as a soldier in the army of scientists, fighting on the outskirts of the territory with the only task to defend it and recruit new members for the military of the Science? Normally a soldier is not expected to think about the orders of his chiefs but he just has to obey and carry them out.

I conjecture that there are strong ties between the nature and character of the material (chosen from the vast area of scientific geometry) and the way it has to be conveyed to pupils and students. However, one should be careful not to try to reverse the positions and install the Geometry Educator as an expert and an inspector of the territory of Geometric Science.
Such a reversal of positions is not uncommon, however. It can be observed in secondary schools nowadays. In former decades the principal of the school used to be a learned teacher with high qualifications in his field of study. There was a real distance between him and the rest of the school population.

This has changed. In many cases a teacher has been deliberately appointed as a headmaster with lower qualifications to underline the unity of the school community. In my experience, this need not be a problem: a poorly qualified headmaster may still be an excellent principal.

Again, I do not think that this approach is to be recommended in the case of Geometers and Geometry Educators. However, the present situation is not entirely satisfactory either.

I now return now to the problem I sketched above concerning what kind of literature would be appropriate for me to start my period of research. The bulk of the existing sources seems only to support the existing practice in which the Geometry Educator is seen as a foot-soldier fighting at the borders of the territory of the Science of Mathematics.

Now in secondary schools it may be difficult to teach scientific geometry to the pupils: on the one hand, most of them are still too young; and on the other hand they may be less interested. However, after secondary school, there are adults with a lively interest in scientific geometry, but for those people, if they do not want to attend college or attend vocational lectures, there are few opportunities to enhance their knowledge of the subject.

This is a largely ignored area of Geometry Education: teaching people who have completed their secondary school education and wish to foster their interest in scientific geometry. My first efforts were to find some literature on that special subject but there was nothing useful for my purpose.

How should one teach scientific geometry to adults?

In such cases it is customary to take a well-known secondary school textbook and hand it to the mature student. Such a secondary school textbook may be rather embarrassing because it was written for young people and the student may feel offended by the childish way in which things are explained. The textbook may be written in such a way that the reader is not expected to understand the material without a thorough explanation from the teacher. Mature readers easily get annoyed by such an approach although it is quite normal for children.

I think one may assume, without any investigation, that there are adults who would like to take up some kind of study of scientific geometry, if they had the opportunity. But now an obstacle is presented by their lack of proficiency in the customary mathematical skills of carrying out proper computations and the application of formulae.Again it is customary in such cases to offer an extract from secondary school exercises to the mature student who, after some unsuccessful computations, will end his attempts for good.

This is a regrettable situation and one should try to improve it. It is a fact that, normally, efforts to convey some knowledge of geometry are restricted exclusively to primary or secondary school populations, or university students. There is almost nothing for adults; and there is little literature for the education of adults interested in geometry except for vocational purposes; but that kind of geometry is rather specialised and the subject is only taught as far as it is useful for a specific profession. It is also true that most vocational students are not primarily interested in scientific geometry but rather in study for their profession. Nevertheless it represents a valuable piece of geometry education, even if inaccessible or uninteresting to the general public.

Concerning the literature to be read by me as a starting research student: I found nothing. Gradually it became clear to me that I would be one of the few authors who has to try to produce his own adequate material.

Once I had identified my aim, things had to be arranged. As there was no literature on the subject of Geometry Education for interested adults, except for vocational purposes, I faced the fact that I had not only to produce a curriculum, but also the geometrical contents. This meant that a new type of scientific geometry had to be offered, not the secondary or primary school specimen, because that was too embarrassing for adults. Real geometrical issues of a demanding scientific level had to be offered so that the mature student would find them attractive to study.

The next problem was: if I was not able to produce such a new type of geometry, then it was also useless to work on a new curriculum; that would only mean a rearrangement of obsolete secondary school knowledge; obsolete to the mature students to whom the secondary school knowledge had already been offered at school.

So, contrary to expectations, my task was not to start reading and exploring sources, because there was no literature on my subject. Therefore I skipped the initial phase of consulting books and other literature and started looking for suitable subjects for my future students. Thus I worked on the display of an unprecedented Geometry Education. What I have developed is demonstrated in this thesis and I have called it: 'Educational Geometry'. Its nature is explained in Part I of the thesis.

However, the question of background reading had not disappeared and as I progressed with the issue of 'Educational Geometry', it became clear that some items of this new scientific geometry could fruitfully be discussed with the help of existing literature. Fischbein's book 'Intuition in Science and Mathematics' appeared to be particularly relevant. The question of 'intuition' in geometry turned out to be an important issue in my thesis and from the works of Spinoza and Descartes I extracted satisfactory philosophical support for my approach. Spinoza's definition of 'intuition' was a particular breakthrough in the deadlock I felt after reading works of E. Fischbein, C. Jung and others on the subject.

More background reading was provided by the Dutch edition of "Achtergronden", (Abels, 1992) (in English: "Backgrounds", 1992), intended to renew the secondary school Mathematics curriculum in The Netherlands. This book contains a rich variety of attractive and useful exercises for pupils of ages 12 - 16. I have given an extract from the section on geometry in "Achtergronden" in Chapter XII. Although the exercises in "Achtergronden" are generally not suitable to serve in 'Educational Geometry', the main thought of emphasising the visual approach is a major support for my work.

There is also a historical dimension. Already in antiquity people were discussing questions about the visual distortions of statues, buildings and so on. Architects, sculptors and other people studied the subject and found elaborate solutions. Kim Veltman's study of Leonardo da Vinci contained a lot of information about how to handle visual distortions. Leonardo da Vinci wrote on perspective and Dürer's window is a celebrated example of renewal in painting. The introduction of a horizon also stems from history and we can study the paintings of, for instance, Gerard Houckgeest (1651) who plays with artificial horizons.

We know that Euclid was the great geometer of antiquity. He wrote a book on Optics, which provides precious information about how the ancients looked at the world in which we live. Another celebrated author of antiquity is Plato, whose style is approachable and who managed to create a very special atmosphere in his books.
His 'Cosmology' is quoted in my thesis and it is described in his 'Timaeus'.
These authors from antiquity still have an impact on Science in our day. Euclid's Geometry has been on the secondary school curriculum since antiquity until quite recently, only some decades ago. And Plato even described examples of geometry education. He asserted that you could teach people only by questioning them. A slave is handed a geometrical problem and he finds the solution himself after questioning which pushes him in the right direction. However, no knowledge is conveyed: the slave only has to answer questions. It is not my aim to elaborate on Plato's approach here, but it shows that in antiquity the issues of geometry and geometry education were already important.

A Bibliography usually provides an indication of the background from which the author has written his work. In my case I emphasise that I have given many years of teaching service to Mathematics, including Geometry. I have attended many conferences on geometry education and I have also seen applications in practice - for instance, of the Van Hiele Model and Papy's approach and of course the Wiskobas arithmetic and geometry. I believe that we are standing on the threshold of a fascinating new area in Geometry, also due to the development of travelling in space. To cope with that challenge, I designed the 'Educational Geometry' which is based on the application of visual Art as an educational means and also on the differences between Local and Global Geometry. Local and Global Geometry are topics from Differential Geometry, in which I took a Master's Degree in 1986 at the University of Aberdeen, Scotland. I then detected that the differences between Local and Global Geometry are of great educational significance if implemented in the right way and consequently I developed the visual approach, presented in this thesis. On page 61 of my Master's thesis a spherical surface is demonstrated with a visually curved line λ, which on that surface is straight (Blok, 1986, page 61). See figure 1.

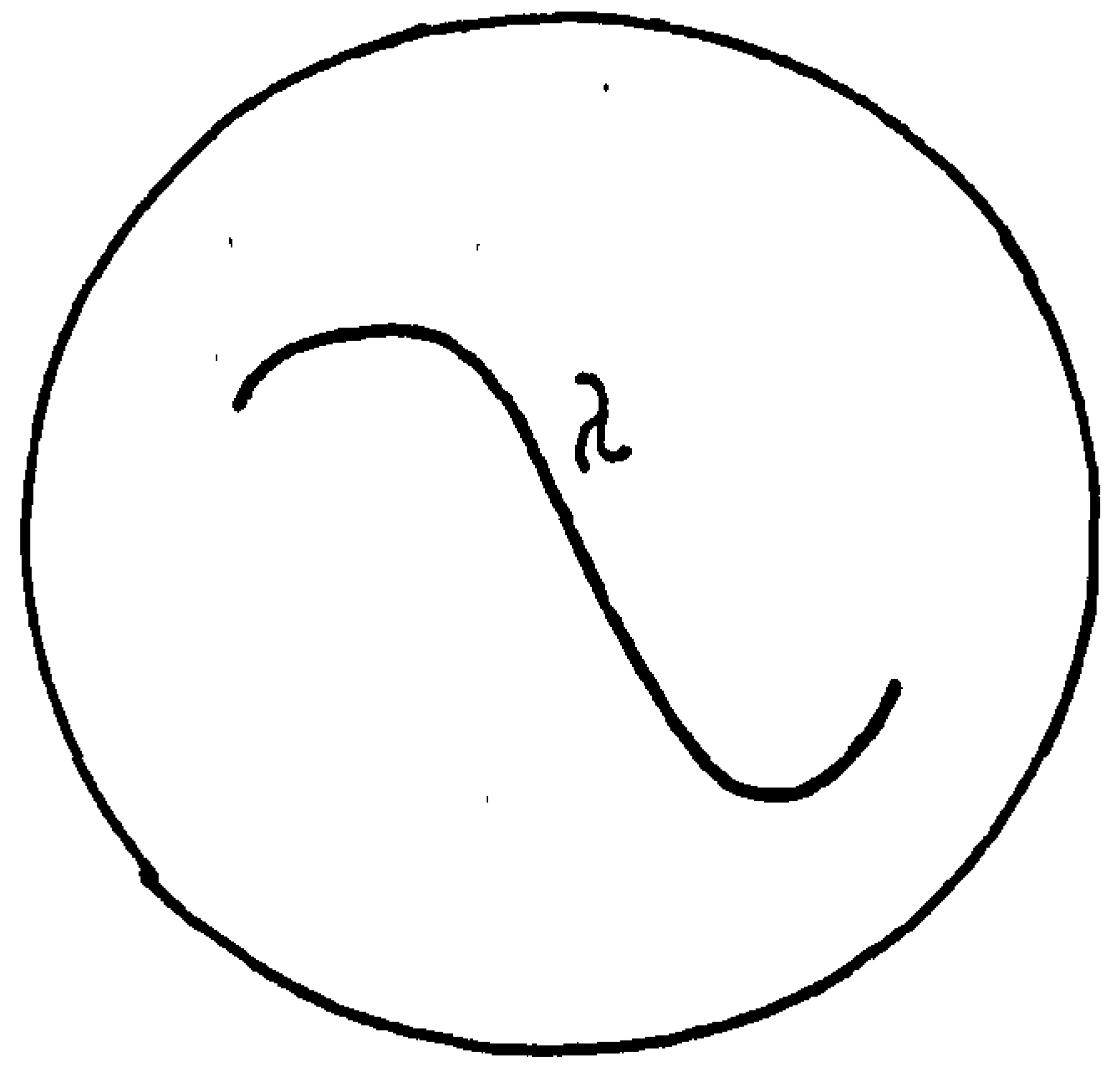


figure 1

This picture in figure 1 yields the notion that a straight line may be visually curved.

Differential Geometry is based on the application of Calculus and the use of a co-ordinate system. But these ingredients are largely unsuitable for elementary education so that I looked for another means. Reading *The Life and Worksof M.C. Escher* I found a picture called 'Cirkellimiet 'which seemed to me a suitable means of explaining things to the students (see Bool , 1993). I am very interested in Art, being chairman of an Art Foundation in the Netherlands.

So now we have it all together: local and global geometry, the visually curved straight line, and the display of visual art. As far as I know my approach is unprecedented. I have nowhere seen the development of visual art with the purpose of serving geometry education. There are painters and sculptors who produce geometrical art but I can not say that their approach is educational. The Italian painter Leon Battista Alberti stated in 1435 AD.: " *Mathematics measures the size and form of things in the mind alone, completely detached from the material. We, however, talking more about visual things, express ourselves in rougher terms.*" (Abels, 1992 page 94).

17.2. Overview of the sources consulted.

I will now indicate what I particularly learned or distilled from the books of the Bibliography.

To begin with, there are the books which are used as sources of information. They comprise mathematical or scientific texts, and quotations have been taken from these sources (for instance Spinoza's Ethics).These background sources are mentioned below. The numbers refer to the numbering of the Bibliography.

3. Algra, 5. Blok, 6. Boertien, 7. Bool, 8. Bor, 9. Descartes,10. Dijkhuis, 11. Emmer, 14. Goffree, 15. Goffree, 16. Heege, 17. Hicks, 19. James, 20. Jansen, 23. Kelfkens, 24. Kemp, 27. Krogt, 28 Lang, 29. Leen, 30.Liebeck, 31. Menton, 32. Molenbroek, 33. Monograph 3, 34. Mulder, 35. Nauta, 36. NCTM, 37. Noss, 38. Papy,. 39. Piaget, 40. Plato, 41. Spinoza, 42. Struys, 43. Treffers, 44. Turkstra, 48. Veen, 49. Veltman, 50. Verwey, 51 Vip, 52. Vries, 55. Westendorp, 56. Wijdenes, 57. Woestenenk, 58. Wolff.

In the books mentioned above a wide variety of subjects is treated, which are all in a way related to the subject of visual geometry. It should be seen as background material, which has been used as a source.

Now we arrive at the works that are more directly related to the design of my thesis and I will indicate in this way their relevance for my work.

1. Abels, 1992. In this book a new curriculum for secondary school geometry in the Netherlands is described and it has a large section on visual geometry. This is relevant, although the notion of a horizon is not emphasised. However, a fresh approach to visual geometry can be found in this book.

2. Alders, 1969. Secondary School textbooks, which I used in the sixties when I started my career as a teacher of mathematics. Visual geometry is considered as 'intuitive' in these kind of books and no special attention is paid to the subject.

4. Barrau, 1918. Barrau's remarks about the 'thorn hedge' of elementary education in geometry are important. This has been an incentive for me to leave out a treatment of the axioms of geometry.

12. Euclid, 1959. Euclid is so important in geometry that his views, displayed in his Optics, can simply not be ignored. These views clarify his treatment of geometry in some respects. I even tried to use some of my drawings as material for his concepts.

13. Fischbein, 1987. Essentially I can not agree with his views on intuition but the reading of his book has inspired me to think thoroughly about the subject.

22. Jung, 1973. He has written much about psychology and about the explanation of dreams. Although eventually I have other views than Jung's on that subject, his work is a rich source of material and his approach is fascinating. It is said that Jung is hard to understand for people who have not yet reached the age of 50. In my case that appeared to be true. At the age of 40 I still found the books of Jung essentially obscure and a riddle. But, arriving at the magic border of 50, suddenly the difficulties disappeared and I understood quite easily what he had to say. However, now that the age of 50 is already some way behind me, I begin to find difficulty again in understanding his ideas. Perhaps Jung was just an author writing for a certain age group, those of the age of 50!

23. Kelfkens. One of the authors of the textbook on mathematics I use at the secondary school where I teach. Regrettably, the size of the geometry curriculum will be diminished in the future.

25. Koenen, 1887. A display of visual geometry education in the nineteenth century. Suitable to compare with 'Educational Geometry'.

26. Kok,1992. Provides background information for the teacher who has to implement the 12-16 curriculum. The section on geometry provides a great improvement for visual geometry taught at secondary schools.

47. Van Hiele Model, 1988. The Van Hiele Model provides a complete curriculum for geometry to be taught to 12 - 16 year olds. His approach makes me think a bit of the Montessori School, where Dr Van Hiele was a teacher back in the nineteen-fifties.
Van Hiele's method certainly is fruitful in practice.

As I said before, many of these books had already been studied by me or used for my work as a teacher before I started the research for this thesis. However, since the emergence of 'Educational Geometry' resulting from my investigations, I have looked upon these texts in a different way. They have been a help to find new ways and detect new paths of learning; but the result of such an investigation also changes the way one thinks about what the texts are conveying. Although indispensable as background literature, the books consulted did not show a clear direction towards 'Educational Geometry'. The next Chapter will provide a final conclusion to the thesis and this will focus on a summary of what has been achieved.

CHAPTER XVIII

18.1. Final Conclusion

The visual geometry presented in this thesis is based on practical perceptions but the theoretical background of geometry as a science has not been ignored. Although observations are paramount in the development of visual geometry, no conclusions are drawn from the appearances which contradict the result of theoretical considerations. The means used have been carefully chosen with the purpose of conveying knowledge in a minimum of time, avoiding educational pitfalls. The territory of theoretical knowledge is protected against rash developments. The means, as for instance the application of visual art, only serve the goal of a better understanding of geometry.

The theoretical background is nowhere displayed explicitly because it could hinder the students' development and even stop the educational process when the difficulties of mastering certain computational or analytical skills become a bar to further study.

So I have tried to find alternatives to the customary presentation of geometry to pupils and students; and these alternatives can be found within the realm of visual art and the restricting of geometry to the study of visual aspects.

I expect that much of the material presented in this thesis may serve as educational material for students. A curriculum for 'Educational Geometry' is indicated in Chapters VIII, IX and X which deal with duality, group theory, and topology. But also the consideration of Part I of the thesis, about non-Euclidean geometries (for instance, Escher's Pond), and the differences between local and global geometry are very suitable to form part of the curriculum. Moreover, most of the issues of the above- mentioned Chapters in Part II have been introduced in Part I.

The choice of the curriculum taken from the Chapters VIII, IX and X indicates the level required of students studying 'Educational Geometry'. It means of course that this Educational Geometry is only accessible to a limited public, selected by their ability to understand the issues and study the topics adequately.

I think this is the best way to introduce something as unprecedented as Educational Geometry is: from the highest level, which is offered first to a few people, to a level appropriate to a more general public. The borders of this intended curriculum are best protected in this way so that the contents cannot be changed by development at lower levels. It is a hierarchic structure which in this case seems to me to be the best one.

Although it is surely possible to insert something of the curriculum of Educational Geometry into the customary ones, one has to be careful, I think. In my opinion art long ago found a place in visual geometry. The depiction of an ordinary circle, for instance, must initially have been a product of art but at a later stage it was no longer recognised as art because the use of it had become common practice.

It may be put forward as an objection against Educational Geometry that the realms of visual art and visual geometry are quite different and a merger of the two is not desirable at all. This is correct, I assume. But what I have in mind is products of art especially designed to serve the purposes of geometry. These drawings must be geometric in the first place and artistic in the second place.

Educational Geometry should be educationally valid so that it does not bar further and deeper study of the science of geometry. May the beauty of visual images be an incentive for many to take up the study of geometry.

Now I will describe how, according to my ideas, Educational Geometry might be taught at secondary schools and beyond in The Netherlands.

18.2. Educational Geometry in Secondary Schools and beyond (in The Netherlands).

A DUTCH SECONDARY SCHOOL CURRICULUM (14 - 18) OF EDUCATIONAL GEOMETRY, DESIGNED PROVISIONALLY; A TEXTBOOK ON HIGHER EDUCATIONAL GEOMETRY; AND THE PREPARATION OF TEACHERS OF EDUCATIONAL GEOMETRY IN THE NETHERLANDS.

THE CURRICULUM. (14 - 18)

It should be admitted that the work required to produce a design for a complete curriculum would take some years of intensive work. So the following contribution to a secondary school curriculum for Educational Geometry comprises no more than a list followed by some comments of the ideas which I have in mind. This list is not based on investigations but just provides some ideas, taken from the body of the thesis or from history.
For each item the pupils' age is mentioned at which I think it is appropriate to offer the mentioned topic.

1. Symmetry as an artistic concept. (16)
2. Architecture in cultural and in practical perspective. (15)
3. Drawings of M.C.Escher, in relation to non euclidean geometry. (17)
4. Problem of the four colours. (14)
5. Squaring the circle (16)
6. Duality (16 -18)
7. Triangulations (15 - 18))
8. R. Descartes (18)
9. The straightness of a straight line (16).
10. Projections (17 -18)

1. Symmetry as an artistic concept. (16)

On page 118 of the thesis I stated that symmetry might be considered as a purely artistic concept, following the consideration that the drawing of parallel lines is not unequivocally possible. So symmetry is not accepted as an issue in visual geometry. One might remark however, that, although symmetry is rejected as an issue in visual geometry, the notion of symmetry is of great importance. One only has to look at the symmetrical build up of the human body, or at the symmetry in 'right and left'.

At secondary school level it is quite acceptable to show the pupils for instance the symmetry in the ornaments of the Alhambra, in Sevilla in Spain. The visual appearance of these ornaments may also lead to the application of group theory on the possible movements of the ornament. It should be carried out free of computations.
This must be considered as an introduction to visual geometry, and to group theory. The symmetry will not be seen as an activity of visual geometry, but it foreshadows the rejection of symmetry at a later stage. One can not reject symmetry before it has been studied. Group theory will keep its validity; it will not be rejected later.

We are, so to say, assessing the borders of visual geometry by studying symmetry and at a higher level it can be explained, why symmetry is not accepted as a concept in visual geometry.

In my view, the products of visual art in the Alhambra are a rich source from which material can be extracted showing symmetry and group theory to the pupils without requiring computations. In this way, together with the Arab cultural background of the Alhambra a good subject of study may emerge.

2. Architecture in cultural and practical perspective. (15)

An interesting task can be found in the building of models to scale. The student will build a model to scale of a theatre in which, for instance, a piece of music will be performed. The theatre has to meet acoustic demands and the design of the interior should be such that the audience gets the best of the music. The students should try to become aware of the architectural practice in the building of theatres. Besides this the study of a cultural stream in which the building of a theatre can be embedded, may, together with practical work, lead to an attractive whole.

3. Drawings of M.C.Escher , in relation to non euclidean geometry. (17)

It is possible to talk about non euclidean geometry in secondary school education. In the beginning it might be explained that for instance on a sphere a geometry exists, which makes us think of the real situation in which we live on the surface of a sphere, called: planet earth. Further, the fact may be referred to, that on the surface of the moon a geometry will be found, which is different to the geometry on earth.

Next it should be explained precisely, what a euclidean geometry is. And after that it will be possible to study pictures and products of visual art of M.C.Escher, and look at different properties of Escher's geometry, which may vary from picture to picture. The drawing 'Cirkellimiet', which I have discussed in my thesis, might be handed to the pupils with the request to describe the behaviour of the fish. It should be stated that these fish live in a world of their own, similar to our world. The question may even arise what kind of image real fish perceive, when they look at human beings.

Escher's drawings has been the subject of a Master's thesis (Struys, 1986). In that thesis the group structure of some symmetrical figures in Escher's drawings has lead to an investigation by two students of the University of Utrecht. Examples of this are shown on page 145 (figure 7) and on page 148 (figure 13) of the thesis. It could lead to an interesting geometrical report of the visual representations in the works of M.C.Escher. Escher was not a mathematician, which means that his works have been created without knowledge of formal geometry. He shows that non mathematicians can produce artistic pictures, which are of a high geometrical level. So, a careful study of some of his works may be paramount for the development of visual geometry. It should also be mentioned that not all of Escher's drawings are valuable for geometry education so that one has to be careful in choosing examples from his work.

So, group theory and non euclidean geometry may be issues, related to Escher's art. Practical exercises can be extracted anyhow from the group structure of some of Escher's ornaments (Struys, 1986).

4. Problem of the four colours (14).

The pupils will receive a map of Europe which shows on white paper the borders of the European countries. Moreover, the pupils will have four different coloured pencils at their disposal and they are asked to colour the different countries such, that no two equal colours have the same border. After this exercise has been completed, the four colour problem will be mentioned, which states that any map, however complicated, can be coloured in this way with no more than 4 different colours. After that a map of Africa will be provided, with three coloured pencils this time. The pupils will be told that leaving the paper white is also a way of colouring it. After that a survey will be given of the history of the problem, the work of famous map-designers and the attempts to solve the four colour problem. The pupils will be told that the problem finally has been solved because people have detected that essentially there is a limited number of cases and all these cases have been solved by the computer. Examples can be provided that apparently different cases are essentially the same. Taking in consideration the young age of the pupils, the examples should be simple and convincing.

5. **Squaring the circle. (16)**

The first approach will be to hand the pupils two circles, of different diameter, drawn on a grid paper. Of both circles the area will be computed by counting the squares of the grid inside the circle. The formula: circle area = $\pi \cdot r^2$ will be provided. Contrary to the basic assumptions of visual geometry a formula is provided, but it is a formula, even in use in primary school curricula, so it is not secondary, but primary school material and thus there is no objection against the use of it. With help of this results the value of π can be approached. It is also possible to divide the length of the circumference of the circle by its diameter, in which case also the number π will appear.

After this the pupils will receive a lecture on the number π, and they are told that it has been proved that it is impossible to square any circle. Finally the pupils will be given an account of the various fruitless attempts to find the squaring of the circle and how it has become a symbol of carrying out useless work.

6. **Duality (16 -18)**

The topic of duality has been described in detail in Chapter IV, section 4.3. and in Chapter VIII, sections 8.1. to 8.4. of the thesis. With the help of that description a course can be designed, starting at the age of 16 and ending at 18. The first approach, say in Chapter IV, section 4.3. would be appropriate for 16 year olds and the sections 8.1. to 8.3. can be taught to 17 or 18 year olds. Section 8.4. of Chapter VIII could be beyond secondary school purposes.

7. **Triangulations (15 - 18)**

Like duality, the topic of triangulations and the Euler Characteristic has been described in detail in the thesis. Simple triangulations can already been carried out by 15 year olds and the pupils should know that triangulations are used to produce maps of unknown areas in the interest of the production of geographical maps. Practical work in triangulating can be carried out.
The Euler Characteristic can already be offered at an age of 15, I assume, but the topic of dissections should be restricted to pupils of 18 years old. The required minimal dissections are a bit beyond secondary school purposes, I think.

8. **R.Descartes (18)**

As we know, the Cartesian axial system is nowadays in use in many parts of mathematics and geometry.
It may be interesting to read Descartes' philosophical work and follow his thoughts where he is developing such an axial system. Also interesting are his observations which lead to the conclusion that people generally prefer obscure and difficult issues which are far away from the simple and beautiful things which surround them and he calls that a failure of mortals.

9. The straightness of a straight line. (16)

Already when the pupils are at the age of 16 it should be possible to show them that the drawing of straight lines brings a lot of problems. Photographic pictures, as I took, standing on the railway (page 22 of the thesis, figure 4) demonstrate that clearly. Also the travellers, lost in the desert (page 33 of the thesis) could be interesting material for these pupils. The advantage of offering the issue at a relatively young age of 16 is, that the pupils' concepts are not yet quite fixed so that is not that difficult for them to accept that straight lines sometimes must be pictured curved.

10. Projections (17 - 18).

The interviews on Projections (Chapter V and Chapter VIII, section 8.5. of the thesis) can be carried out by questioning secondary school pupils of the age of 17 or above 17 years old. It may worthwhile to note the pupils' reaction to the questioning, after the interviews.

So far the list of suggested items.

The above examples show that sufficient material will be available for the Educational Geometry curriculum (14 - 18). To produce a more comprehensive curriculum, more material must be collected. To that end a group of people must be formed, who are going to design such a curriculum. Besides the curriculum, a textbook for secondary school use is, of course, needed. A provisional textbook can be written by one or more authors, in co-operation with the above mentioned group.

This work (designing a curriculum and writing a provisional textbook for secondary school purposes) could take about three years, and after that period the actual lessons in the classroom may start. The group will also provide tests for the examining of the pupils. Preferably these tests will consist of practical work and an oral examination. The practical part may consist of work in a workshop, the oral examination has to make clear how the pupil looks at the topic.

The legal procedure for the introduction of a new topic in a secondary school curriculum also is a matter of importance, but I will not discuss that issue here.

Here ends the discussion of the curriculum of Educational Geometry in secondary schools in The Netherlands.

TEXTBOOK ON HIGHER EDUCATIONAL GEOMETRY

There still is the issue of a textbook on higher Educational Geometry, beyond secondary school level. It would comprise philosophical items about the straightness of a straight line, the minimal dissections of Chapter X of the thesis , the theorems of Desargues, Pascal, and more. Although I am convinced that such a textbook would be necessary in the future, I think that for the time being the thesis as such would be sufficient as a provisional textbook on higher Educational Geometry.

THE PREPARATION OF TEACHERS OF EDUCATIONAL GEOMETRY (HOLLAND)

Special certificates could be awarded to those teachers, who have proved to be familiar with the topic of Educational Geometry and who display creativity in the teaching of Educational Geometry in the classroom .

During the first period of introduction of the new curriculum of Educational Geometry every teacher who feels a certain sympathy with the topic, may be allowed to teach it during one year. If the school is not satisfied with his \ her performance, the experiment will not be prolonged.

The other teachers who have shown a certain skill in Educational Geometry, may then proceed for another two years of probation in which they can enhance their proficiency on the topic and try to find their own ways in producing items, in which philosophy and \ or visual art serves to display geometrical concepts.

The provisional curriculum, established by the above mentioned group, will be a guide to the teachers who are lecturing Educational Geometry and they will also use the textbook, concerning the teaching of Educational Geometry at secondary schools.

After this the teacher will not automatically be qualified to teach Educational Geometry. To become qualified, it will not be sufficient for instance that a candidate has a degree in mathematics and a degree in philosophy. The essential point is a creative linking of these two topics to the benefit of Educational Geometry.

The candidate will have to show the ability to use items, taken from visual art or from philosophy, in such a way that a concept of visual geometry is displayed. As an example one could take the picture 'Cirkellimiet' of M.C.Escher. Three years of teaching plus the production of a picture like 'Cirkellimiet' will convince the examiners that the candidate can be awarded a certificate of qualification to teach Educational Geometry.

The creative linking of philosophy and \ or visual art to geometry education will not consist of applied geometry. For instance the building of a perfect model to scale of a music theatre which displays all the necessary requirements of such a building, can not be accepted as an item to qualify a teacher because the building of a model to scale has nothing to do with the display of a concept of geometry. Only when an item from philosophy and \ or art can be considered as an obvious display of a geometrical concept, then it will be considered as a contribution to the understanding of geometry and make the examiners decide to confer a qualification upon the candidate.

The examination may be carried out by an independent body, empowered by the government, after which a diploma can be given to the candidate. Alternatively, the qualification may be awarded by a university as part of a degree course or after an examination for a university diploma. These three options should be possible.
So three years of teaching on probation combined with a successful examination at the end of the period will qualify the candidate to teach Educational Geometry. The requirements for the qualification as part of a degree course will be determined by the university.

18.3. Summary

Now we come to a final summary (S) of the conclusion to this thesis:

(S): Educational Geometry provides a valid path of geometric learning which leads to an acceptable scientific level. Essentially it is both non-verbal and educationally valid, without requiring arithmetical skills. The presentation is based on Art. Items from Duality, Group Theory and Topology have been chosen as the basis of a curriculum.

Below, this summary will be denoted by (S) and it will be scrutinised how the chapters of my thesis relate to (S).
In the Introduction a summing up was given of what the reader should know even before proceeding. The purpose of this is to prevent the reader from making quick and erroneous conclusions which at a later stage might be an obstacle. Drawing wrong conclusions would not be the fault of the reader: it would be understandable because the approach is quite unprecedented, so that nobody may be supposed to know what the thesis is about before having studied the body of the text.

In that Introduction, the absence of arithmetical and analytical requirements was already highlighted as in (S). In Chapter I a necessary definition of 'visual geometry' was given and this appeared to be central to the development of a curriculum of Educational Geometry. From that definition, steps had to be taken toward the discussion of the notion of 'educational validity'.
'Educational validity' and 'non verbal education' are central issues in the thesis and they are mentioned in (S). More definitions in Chapter II provide a background for thinking about the difference of 'local' and 'global' geometry . These considerations about 'local' and 'global' geometry give rise to fruitful discussions without the need of the introduction of formulae or the necessity of calculations to be carried out. In this way Chapters I and II play their role in preparing the above summary (S).
Visual art, as an alternative to the arithmetical approach, appears in Chapter III. There, Escher's Pond provides the opportunity to blend visual art and scientific geometry in a powerful picture, which educationally is a help to the understanding of non-Euclidean geometry.
In Chapter IV the proposed curriculum is preceded by the treatment of the subjects 'Duality' and 'Euler Characteristic'. In Chapter V, a piece of Projective Geometry, disguised as visual art, is displayed. Fifteen people are interviewed about this example of geometry and the answers of the interviewees provide an idea of the way in which Educational Geometry might be taught. The interviews are not seen as statistical data, but they represent an example of how geometry might be taught in the future. They also embody an acceptable scientific level, as mentioned in (S). Some background thinking is provided in Chapter VI. Basic notions, like 'straight lines', give rise to philosophical considerations about the concept of 'straightness'. Finally, with the help of Spinoza's book 'Ethics', an intuitive base for 'straightness' is found. At the end of Chapter VI, the impact of statistical methods on the outcome of intuitive statements is discussed. An example is provided, in which one intelligent interviewee might be right against 99 people of average intelligence, who might be wrong. Intuitive knowledge is seen by me as coming from a more or less mystical origin. Quantitative methods, like statistics and calculations, are of no relevance.
This also provides a link with the summary (S), where arithmetical skills are ignored. However, in the case of the Euler Characteristic, the student should be able to count vertices, faces and edges. In Chapter VIII, the designed curriculum is demonstrated by means of a lesson on 'Duality', some historical notes, and a renewed study of 'Projections', which was introduced in Chapter V. The curriculum is further demonstrated in the Chapters IX and X. This is summed up in (S).

Applications of Educational Geometry at secondary school level are discussed in Chapter XVIII, section 18.2. . It differs of course, in some respects, from existing current methods. These differences have been scrutinised in the Chapters XII and XIII of the thesis.

In Chapters XIV, XV and XVI issues from the Classical world are related to my research. Euclid's Propositions about 'optics' are related to the artistic drawings in 'Projections'. In Chapter XV the visual distortions, which arise from the Classical point of view, are scrutinised; they add to our view of visual geometry. In Chapter XVI Plato's cosmology is used as a model to demonstrate topological issues, mentioned in (S), as applications of developments in Chapters III and IV on the difference between 'local' and 'global' geometry.
The thesis presents a coherent whole, which is presented in the summary (S). I believe that this thesis contains an unprecedented and viable branch of geometry education.

Epilogue

At the end of my thesis I wish to thank Douglas French, my supervisor at the University of Hull, for his stimulating and understanding tuition. I have learned so much from him.

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