

EXPONENTIAL ENERGY DECAY FOR THE KADOMTSEV-PETVIASHVILI (KP-II) EQUATION

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ABSTRACT. In this paper we study the exponential decay of the energy of the externally damped Kadomtsev-Petviashvili (KP-II) equation. Our main tool is the classical dissipation-observability method. We use multiplier techniques to establish the main estimates, and obtain exponential decay result.

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1. INTRODUCTION

Consider the initial value problem (IVP) associated with the Kadomtsev-Petviashvili (KP) equation,

$$\begin{cases} (u_t + u_{xxx} + uu_x)_x = \alpha u_{yy}, & (x, y) \in (0, L) \times (0, L), t \in \mathbb{R} \\ u(x, y, 0) = u_0(x, y), \end{cases} \quad (1.1)$$

where $u = u(x, y, t)$ is a real valued function, $L > 0$ and $\alpha = \pm 1$. This model was derived by Kadomtsev and Petviashvili [8] to describe the propagation of weakly nonlinear long waves on the surface of a fluid, when the wave motion is essentially one-directional with weak transverse effects along y -axis. Equation (1.1) is known as KP-I or KP-II equation depending whether $\alpha = 1$ or $\alpha = -1$. In this paper we consider the KP-II equation, that is (1.1) with $\alpha = -1$.

The KP-II equation is a two dimensional generalization of the Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + uu_x = 0, \quad x, t \in \mathbb{R}, \quad (1.2)$$

which arises in modeling the evolution of one dimensional surface gravity waves with small amplitude in a shallow channel of water. The KdV model is a widely studied model which arises in various physical contexts and has a very rich mathematical structure.

It is customary to work with (1.1) for $(x, y) \in \mathbb{R}^2$, without any boundary conditions. In this case the L^2 norm, the energy,

$$E(u(t)) := \int_{\mathbb{R}^2} |u|^2 dx dy$$

is a conserved quantity. However, since $(x, y) \in (0, L) \times (0, L)$, if we impose suitable boundary conditions energy may be dissipated. These conditions will be discussed in Section 3. If in addition to these boundary conditions a weak damping is imposed we obtain an exponential decay of the energy. As a damping term, we take a non-negative function $a(x, y)$ and consider

the following damped KP-II equation:

$$\begin{cases} (u_t + u_{xxx} + uu_x + au)_x = \alpha u_{yy}, & (x, y) \in (0, L) \times (0, L), t \in \mathbb{R} \\ u(x, y, 0) = u_0(x, y). \end{cases} \quad (1.3)$$

If $a > 0$ everywhere, an easy computation shows exponential decay. If, however, we allow a to vanish the exponential decay requires additional work. In the case of the KdV equation this problem was studied in [13]. In this paper we extend these results to the KP-II equation by introducing suitable boundary conditions and using the unique continuation principle proved in [12]. We note that our techniques do not rely on the Holmgren's Uniqueness Theorem but in a unique continuation principle. Therefore it works both for linear and non-linear equations.

KP models are extensively investigated in the recent literature see for example [2], [4], [5], [6] [7], [9], [11], [12], [14], [15], [16] and references therein, for issues such as local and global well-posedness, gain of regularity and unique continuation principles. As we are interested in the case when the initial data has sufficiently high Sobolev regularity, the question of well-posedness follows, for instance, by the semi-group theory. So, we will omit the details of this aspect.

The plan of the paper is as follows: firstly, in Section 2 we give a compact presentation of the dissipation-observability method which is the basis for many decay results. Then in Section 3 we establish, using multiplier techniques, the main technical estimates which allow us to address the exponential dissipation of energy for the KP-II equation.

2. DISSIPATION - OBSERVABILITY METHOD

In this section we put forward a general method to prove energy decay that follows from the energy dissipation law and an observability inequality.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Let A be a linear operator, and B a non-linear operator with domain dense in $L^2(\Omega)$. Suppose that u is a solution to the evolution equation in $L^2(\Omega)$*

$$u_t = Au + B(u), \quad (2.1)$$

under suitable boundary conditions. Suppose that the evolution associated to (2.1) satisfies a semigroup property and that the energy $E(u(t)) := \int_{\Omega} u^2 dx$ is dissipated according to

$$\frac{d}{dt}E(u(t)) = -Q(u) \leq 0, \quad (2.2)$$

where $\int_{\Omega} u(Au + B(u)) = -Q(u)$. Assume that $\forall T > 0$ there exists $C > 0$ such that the following observability inequality

$$E(u(\cdot, 0)) \leq C \int_0^T Q(u(\cdot, s)) ds \quad (2.3)$$

holds. Then the energy E decays exponentially, i.e, there exists $\alpha > 0$ such that $\forall t \geq 0$,

$$E(u(t)) \leq CE(u(0))e^{-\alpha t}. \quad (2.4)$$

Proof. Integrating (2.2) in $(0, T)$ we get,

$$E(u(T)) = E(u(0)) - \int_0^T Q(u(s)) ds. \quad (2.5)$$

Now, multiplying (2.5) by C , adding to itself and using the observability inequality (2.3), yields

$$\begin{aligned} (1 + C)E(u(T)) &= E(u(0)) - C \int_0^T Q(u(s)) ds + CE(u(0)) - \int_0^T Q(u(s)) ds \\ &\leq CE(u(0)) - \int_0^T Q(u(s)) ds \\ &\leq CE(u(0)). \end{aligned} \quad (2.6)$$

From (2.6) we obtain

$$E(u(T)) \leq \frac{C}{1 + C}E(u(0)). \quad (2.7)$$

Therefore, for some $0 < \alpha < 1$ we have $E(u(T)) \leq \alpha E(u(0))$. Hence, the semigroup property implies the conclusion of the Theorem. \square

Lemma 2.2. *Let u be as in Theorem 2.1. Then the following estimate holds*

$$E(u_0) \leq \frac{1}{T} \int_0^T E(u(t)) + 2 \int_0^T Q(u). \quad (2.8)$$

Proof. Multiplying (2.1) by $(T - t)u$ and integrating we obtain

$$\int_0^T \int_{\Omega} (T - t)uu_t dxdt = \int_0^T \int_{\Omega} (T - t)u(Au + B(u)) dxdt. \quad (2.9)$$

After integrating by parts in the t variable, (2.9) yields

$$\frac{1}{2} \int_{\Omega} \left[\left(\int_0^T u^2 \right) - Tu_0^2 \right] = - \int_0^T (T - t)Q(u),$$

i.e.,

$$-\frac{T}{2}E(u_0) + \frac{1}{2} \int_0^T E(u)dt + \int_0^T (T - t)Q(u)dt = 0. \quad (2.10)$$

Now the desired estimate (2.8) follows from (2.10). \square

Corollary 2.3. *Suppose that*

$$\int_0^T E(u(t)) \leq C(T) \int_0^T Q(u). \quad (2.11)$$

Then the observability inequality (2.3) holds.

3. NONLINEAR KP-II - EXPONENTIAL ENERGY DISSIPATION

Let $a(x, y) \geq a_0 > 0$ almost everywhere in the complement of a compact non-empty proper subset Θ of $\Omega := (0, L) \times (0, L)$. We assume that for some $\delta > 0$, $\Theta \subset (\delta, L - \delta) \times (\delta, L - \delta)$, so that we can apply the UCP, using an extension technique.

Consider the damped KP-II model

$$\begin{cases} u_t + u_{xxx} + \partial_x^{-1}u_{yy} + uu_x + a(x, y)u = 0, & (x, y) \in (0, L) \times (0, L), t \in \mathbb{R}, \\ u(L, y, t) = 0 = u(0, y, t), & u(x, L, t) = 0 = u(x, 0, t), \\ u_x(L, y, t) = 0, \\ u(x, y, 0) = u_0(x, y), \end{cases} \quad (3.1)$$

where the operator ∂_x^{-1} is defined as $\partial_x^{-1}f(x, y, t) = g(x, y, t)$ with $g(L, y, t) = 0$ and $g_x(x, y, t) = f(x, y, t)$.

Define the energy as

$$E(u(t)) := \frac{1}{2} \int_0^L \int_0^L u^2(x, y, t) dx dy. \quad (3.2)$$

As we prove in the next proposition, the energy is a decreasing function of t . Our main objective is to show that the decay is exponential in time, by using the dissipation-observability method.

Proposition 3.1. *Suppose u solves (3.1), and let E be given by (3.2). Then*

$$\frac{d}{dt}E(u(t)) = -\frac{1}{2} \int_0^L [u_x^2(0, y, t) + (\partial_x^{-1}u_y)^2(0, y, t)] dy - \int_0^L \int_0^L a(x, y)u^2(x, y) dx dy \leq 0. \quad (3.3)$$

Proof. We have

$$\begin{aligned} \frac{d}{dt}E(u(t)) &= \frac{1}{2} \frac{d}{dt} \int_0^L \int_0^L u^2(x, y, t) dx dy \\ &= \int_0^L \int_0^L uu_t(x, y, t) dx dy \\ &= \int_0^L \int_0^L u(-u_{xxx} - \partial_x^{-1}u_{yy} - uu_x - a(x, y)u) dx dy. \end{aligned} \quad (3.4)$$

Now observe that

$$\begin{aligned} - \int_0^L \int_0^L uu_{xxx} dx dy &= \int_0^L \left[\int_0^L u_x u_{xx} dx - uu_{xx} \Big|_{x=0}^L \right] dy \\ &= \int_0^L \left[\frac{1}{2} \int_0^L (u_x)_x^2 dx \right] dy \\ &= \frac{1}{2} \int_0^L \left[u_x^2 \Big|_{x=0}^L \right] dy \\ &= -\frac{1}{2} \int_0^L u_x^2(0, y, t) dy. \end{aligned} \quad (3.5)$$

Define v by $u_y = v_x$, then

$$\begin{aligned}
-\int_0^L \int_0^L u \partial_x^{-1} u_{yy} dx dy &= -\int_0^L \int_0^L u v_y dy dx \\
&= \int_0^L \left[\int_0^L u_y v dy - uv \Big|_{x=0}^L \right] dx \\
&= \int_0^L \int_0^L v_x v dx dy \\
&= \frac{1}{2} \int_0^L \int_0^L (v^2)_x dx dy \\
&= \frac{1}{2} \int_0^L \left[v^2(L, y, t) - v^2(0, y, t) \right] dy \\
&= -\frac{1}{2} \int_0^L v^2(0, y, t) dy,
\end{aligned} \tag{3.6}$$

where we have used $v = 0$ at (L, y, t) .

Also, integrating by parts yields

$$-\int_0^L \int_0^L u^2 u_x dx dy = 0. \tag{3.7}$$

Now, using (3.5), (3.6) and (3.7) in (3.4) we obtain (3.3). \square

Now we state and prove the main result of this work that deals with the exponential decay of energy of the nonlinear KP-II equation.

Theorem 3.2. *Given $M > 0$, let u be a solution of (3.1) with data $u_0 \in H^s(\mathbb{R}^2)$, $s \geq 3$, satisfying $\|u_0\|_{L^2(\Omega)} \leq M$, and let $E(u(t))$ be the energy as defined in (3.2). Then the energy $E(u(t))$ decays exponentially.*

Proof. The proof of this Theorem follows from Theorem 2.1, using Corollary 2.3 with

$$Q(u) = \frac{1}{2} \int_0^L \left[u_x^2(0, y, t) + (\partial_x^{-1} u_y)^2(0, y, t) \right] dy + \int_0^L \int_0^L a(x, y) u^2(x, y) dx dy$$

and the estimate

$$\int_0^T \int_0^L \int_0^L |u|^2 \leq \tilde{C} \left\{ \int_0^T \int_0^L \left[|u_x(0, y, t)|^2 + |\partial_x^{-1} u_y(0, y, t)|^2 \right] dy dt + 2 \int_0^T \int_0^L \int_0^L a(x, y) |u|^2 dx dy dt \right\}, \tag{3.8}$$

which holds for some $\tilde{C} > 0$ independent of solution u to (3.1) with initial data u_0 satisfying $\|u_0\|_{L^2(\Omega)} \leq M$ for any given $M > 0$, as will be shown in Lemma 3.5 below. \square

Now we prove the following result which will be used in the proof of (3.8).

Lemma 3.3. *Let u be a solution of (3.1). Then the following estimate holds:*

$$\|u\|_{L^2(0,T;H^1(\Omega))}^2 \leq \frac{2L}{3} \|u_0\|_{L^2(\Omega)}^2 + \frac{4CT}{81} \|u_0\|_{L^2(\Omega)}^3. \quad (3.9)$$

Proof. Multiply equation (3.1) by xu and integrate on $(0, L) \times (0, L) \times (0, T)$. The resulting identity is composed of five terms, that we simplify next:

$$\begin{aligned} \int_0^T \int_0^L \int_0^L xuu_t dx dy dt &= \frac{1}{2} \int_0^T \int_0^L \int_0^L x \frac{d}{dt} (u^2) dt dx dy \\ &= \frac{1}{2} \int_0^L \int_0^L xu^2(x, y, T) dx dy - \frac{1}{2} \int_0^L \int_0^L xu_0^2(x, y) dx dy; \end{aligned} \quad (3.10)$$

the next term is

$$\begin{aligned} \int_0^T \int_0^L \int_0^L xuu_{xxx} &= \int_0^T \int_0^L \left[- \int_0^L (xu)_x u_{xx} dx + xuu_{xx} \Big|_0^L \right] dy dt \\ &= \int_0^T \int_0^L \left[- \int_0^L uu_{xx} dx - \int_0^L xu_x u_{xx} dx \right] dy dt \\ &= \int_0^T \int_0^L \left[\int_0^L u_x^2 dx - uu_x \Big|_0^L + \int_0^L (xu_x)_x u_x dx - xu_x^2 \Big|_0^L \right] dy dt \\ &= \int_0^T \int_0^L \left[\int_0^L u_x^2 dx + \int_0^L u_x^2 dx + \int_0^L xu_{xx} u_x dx \right] dy dt \\ &= \int_0^T \int_0^L \left[2 \int_0^L u_x^2 dx - \int_0^L (xu_{xx})_x u dx + xuu_{xx} \Big|_0^L \right] dy dt \\ &= \int_0^T \int_0^L \left[2 \int_0^L u_x^2 dx - \int_0^L u_{xx} u dx - \int_0^L xu_{xxx} u dx \right] dy dt \\ &= \int_0^T \int_0^L \left[2 \int_0^L u_x^2 dx + \int_0^L u_x^2 dx - u_x u \Big|_0^L - \int_0^L xu_{xxx} u dx \right] dy dt, \end{aligned} \quad (3.11)$$

which, from (3.11), yields

$$\int_0^T \int_0^L \int_0^L xuu_{xxx} dx dy dt = \frac{3}{2} \int_0^T \int_0^L \int_0^L u_x^2 dx dy dt; \quad (3.12)$$

for the next term, as earlier, we set $u_y = v_x$, and obtain

$$\int_0^T \int_0^L \int_0^L xu \partial_x^{-1} u_{yy} dx dy dt = \frac{1}{2} \int_0^T \int_0^L \int_0^L v^2 dx dy dt. \quad (3.13)$$

Also, integrating by parts we get,

$$\int_0^T \int_0^L \int_0^L xuuu_x dx dy dt = \frac{1}{3} \int_0^T \int_0^L \int_0^L x(u^3)_x dx dy dt = -\frac{1}{3} \int_0^T \int_0^L \int_0^L u^3 dx dy dt. \quad (3.14)$$

Finally, the last term is simply

$$\int_0^T \int_0^L \int_0^L xa(x, y)u^2 dx dy dt. \quad (3.15)$$

Now, adding (3.10), (3.12), (3.13), (3.14) and (3.15) we get

$$\begin{aligned} & \frac{1}{2} \int_0^L \int_0^L xu^2(x, y, T) dx dy - \frac{1}{2} \int_0^L \int_0^L xu_0^2(x, y) dx dy + \frac{3}{2} \int_0^T \int_0^L \int_0^L u_x^2 dx dy dt \\ & + \int_0^T \int_0^L \int_0^L xa(x, y)u^2 dx dy dt + \frac{1}{2} \int_0^T \int_0^L \int_0^L v^2 dx dy dt - \frac{1}{3} \int_0^T \int_0^L \int_0^L u^3 dx dy dt = 0. \end{aligned} \quad (3.16)$$

Or,

$$\begin{aligned} \frac{3}{2} \int_0^T \|u\|_{H^1(\Omega)}^2 dt &= \frac{1}{2} \int_0^L \int_0^L xu_0^2(x, y) dx dy - \frac{1}{2} \int_0^L \int_0^L xu^2(x, y, T) dx dy \\ &\quad - \int_0^T \int_0^L \int_0^L xa(x, y)u^2 dx dy dt - \frac{1}{2} \int_0^T \int_0^L \int_0^L v^2 dx dy dt \\ &\quad + \frac{1}{3} \int_0^T \int_0^L \int_0^L u^3 dx dy dt, \end{aligned} \quad (3.17)$$

which yields,

$$\begin{aligned} \int_0^T \|u\|_{H^1(\Omega)}^2 dt &= \frac{1}{3} \int_0^L \int_0^L xu_0^2(x, y) dx dy - \frac{1}{3} \int_0^L \int_0^L xu^2(x, y, T) dx dy \\ &\quad - \frac{2}{3} \int_0^T \int_0^L \int_0^L xa(x, y)u^2 dx dy dt - \frac{1}{3} \int_0^T \int_0^L \int_0^L v^2 dx dy dt \\ &\quad + \frac{2}{9} \int_0^T \int_0^L \int_0^L u^3 dx dy dt. \end{aligned} \quad (3.18)$$

Since $a(x, y) \geq 0$, we obtain from (3.18) that

$$\int_0^T \|u\|_{H^1(\Omega)}^2 dt \leq \frac{L}{3} \|u_0\|_{L^2(\Omega)}^2 + \frac{2}{9} \int_0^T \int_0^L \int_0^L u^3 dx dy dt. \quad (3.19)$$

We have the following Gagliardo-Nirenberg type inequality (see [3] or [10])

$$\|u\|_{L^3(\Omega)} \leq C \|u\|_{H^1(\Omega)}^{\frac{1}{3}} \|u\|_{L^2(\Omega)}^{\frac{2}{3}} \leq C \|u\|_{H^1(\Omega)}^{\frac{1}{3}} \|u\|_{L^2(\Omega)}^{\frac{2}{3}}. \quad (3.20)$$

Now using (3.20) and (3.3) the last term in (3.19) can be controlled by

$$\begin{aligned} \int_0^T \int_0^L \int_0^L u^3 dx dy dt &\leq \int_0^T \|u\|_{H^1(\Omega)} \|u\|_{L^2(\Omega)}^2 dt \\ &\leq C \|u_0\|_{L^2(\Omega)}^2 \int_0^T \|u\|_{H^1(\Omega)} dt \\ &\leq C \|u_0\|_{L^2(\Omega)}^2 \sqrt{T} \|u\|_{L^2(0,T;H^1(\Omega))}. \end{aligned} \quad (3.21)$$

Substituting (3.21) in (3.19) yields

$$\begin{aligned} \|u\|_{L^2(0,T;H^1(\Omega))}^2 &\leq \frac{L}{3} \|u_0\|_{L^2(\Omega)}^2 + C \frac{2}{9} \sqrt{T} \|u_0\|_{L^2(\Omega)}^2 \|u\|_{L^2(0,T;H^1(\Omega))} \\ &\leq \frac{L}{3} \|u_0\|_{L^2(\Omega)}^2 + \frac{\left(\frac{2}{9} C \sqrt{T} \|u_0\|_{L^2(\Omega)}^2\right)^2}{2} + \frac{\left(\|u\|_{L^2(0,T;H^1(\Omega))}\right)^2}{2} \\ &\leq \frac{L}{3} \|u_0\|_{L^2(\Omega)}^2 + \frac{2CT}{81} \|u_0\|_{L^2(\Omega)}^4 + \frac{1}{2} \|u\|_{L^2(0,T;H^1(\Omega))}^2. \end{aligned} \quad (3.22)$$

Therefore, from (3.22) we get

$$\|u\|_{L^2(0,T;H^1(\Omega))}^2 \leq \frac{2L}{3} \|u_0\|_{L^2(\Omega)}^2 + \frac{4CT}{81} \|u_0\|_{L^2(\Omega)}^4, \quad (3.23)$$

which yields the required result. \square

In addition to the Lemma 3.3, unique continuation principle (UCP) for the KP-II equation (1.1) will also be used in the proof the estimate (3.8). The UCP for the KP-II equation (1.1) was established in [12] whose precise statement is given in the following theorem.

Theorem 3.4 ([12]). *Let $u \in C(\mathbb{R}; H^s(\mathbb{R}^2))$ be a solution to the IVP associated with the KP-II equation with $s > 0$ large enough. If there exists a non trivial time interval $I = [-T, T]$ such that for some $\beta > 0$*

$$\text{supp } u(t) \subseteq [-\beta, \beta] \times [-\beta, \beta], \quad \forall t \in I,$$

then $u \equiv 0$.

In the following lemma we prove the main estimate (3.8) used in the proof of Theorem 3.2.

Lemma 3.5. *given $M > 0$, there exists a constant $\tilde{C} > 0$ such that the following estimate holds*

$$\int_0^T \int_0^L \int_0^L |u|^2 \leq \tilde{C} \left\{ \int_0^T \int_0^L [|u_x(0, y, t)|^2 + |\partial_x^{-1} u_y(0, y, t)|^2] dy dt + 2 \int_0^T \int_0^L \int_0^L a(x, y) |u|^2 dx dy dt \right\}, \quad (3.24)$$

for all solutions u of (3.1) with initial data u_0 satisfying $\|u_0\|_{L^2(\Omega)} \leq M$.

Proof. We prove it by contradiction with use of the Lemma 3.3 and the unique continuation principle stated in Theorem 3.4.

Suppose that (3.24) is false. Then there exists a sequence of solution u_n of (3.1) such that

$$\lim_{n \rightarrow \infty} \frac{\|u_n\|_{L^2(0, T; L^2(\Omega))}^2}{\int_0^T \int_0^L [|\partial_x u_n(0, y, t)|^2 + |\partial_x^{-1} \partial_y u_n(0, y, t)|^2] dy dt + 2 \int_0^T \int_0^L \int_0^L a(x, y) |u_n|^2 dx dy dt} = +\infty. \quad (3.25)$$

Let

$$\lambda_n = \|u_n\|_{L^2(0, T; L^2(\Omega))} \quad \text{and} \quad v_n(x, y, t) = \frac{1}{\lambda_n} u_n(x, y, t).$$

From Lemma 3.3, we have that λ_n is a bounded sequence for $\|u_n(0)\|_{L^2(\Omega)} \leq M$. Therefore, extracting a subsequence if necessary, we can assume that $\lambda_n \rightarrow \lambda \geq 0$.

We notice that v_n solves

$$\begin{cases} (v_n)_t + (v_n)_{xxx} + \partial_x^{-1} (v_n)_{yy} + \lambda_n (v_n) (v_n)_x + a(x, y) (v_n) = 0, & (x, y) \in \Omega, t \in \mathbb{R}, \\ (v_n)(L, y, t) = 0 = (v_n)(0, y, t), & (v_n)(x, L, t) = 0 = (v_n)(x, 0, t), \\ (v_n)_x(L, y, t) = 0, \end{cases} \quad (3.26)$$

with initial data $\frac{1}{\lambda_n} u_n(x, y, 0)$. Moreover,

$$\|v_n\|_{L^2(0, T; L^2(\Omega))} = 1 \quad (3.27)$$

and from (3.25)

$$\int_0^T \int_0^L [|\partial_x v_n(0, y, t)|^2 + |\partial_x^{-1} \partial_y u_n(0, y, t)|^2] dy dt + 2 \int_0^T \int_0^L \int_0^L a(x, y) |v_n|^2 dx dy dt \rightarrow 0, \quad (3.28)$$

as $n \rightarrow \infty$.

In view of Lemma 2.2, $v_n(x, y, 0)$ is bounded in $L^2(\Omega)$. Thus, combining with an analogue of (2.8), we have

$$\|v_n(\cdot, \cdot, t)\|_{L^2(\Omega)} \leq M, \quad \forall 0 \leq t \leq T. \quad (3.29)$$

Now, from Lemma 3.3,

$$\|v_n\|_{L^2(0, T; H^1(\Omega))}^2 \leq C, \quad (3.30)$$

for all $n \in \mathbb{N}$.

Estimates (3.27) and (3.30) yield

$$(v_n)_t = -(v_n)_{xxx} - \partial_x^{-1}(v_n)_{yy} - \lambda_n v_n (v_n)_x - a(x, y) v_n, \quad (3.31)$$

is bounded in $L^2(0, T; H^{-2}(\Omega))$. Note that the non-linear term, is bounded in $L^2(0, T; L^p(\Omega))$ for all $1 \leq p < 2$, which is bounded in $L^2(0, T; H^{-2}(\Omega))$.

Since the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, by Rellich's theorem, it follows that $\{v_n\}$ is relatively compact in $L^2(0, T; L^2(\Omega))$. By extracting a subsequence we may deduce that

$$v_{n_j} \rightharpoonup v, \quad \text{weakly in } L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-2}(\Omega)) \quad (3.32)$$

and

$$v_{n_j} \rightarrow v, \quad \text{strongly in } L^2(0, T; L^2(\Omega)). \quad (3.33)$$

Since $\|v_{n_j}\|_{L^2(0, T; L^2(\Omega))} = 1$, it follows that

$$\|v\|_{L^2(0, T; L^2(\Omega))} = 1. \quad (3.34)$$

By weak lower semicontinuity of convex functionals we have (see (3.28))

$$\begin{aligned} 0 &= \liminf_{j \rightarrow \infty} \left\{ \int_0^T \int_0^L [|\partial_x v_{n_j}(0, y, t)|^2 + |\partial_x^{-1} \partial_y v_{n_j}(0, y, t)|^2] dy dt + 2 \int_0^T \int_0^L \int_0^L a(x, y) |v_{n_j}|^2 dx dy dt \right\} \\ &\geq \int_0^T \int_0^L [|\partial_x v(0, y, t)|^2 + |\partial_x^{-1} \partial_y v(0, y, t)|^2] dy dt + 2 \int_0^T \int_0^L \int_0^L a(x, y) |v|^2 dx dy dt. \end{aligned} \quad (3.35)$$

From this we conclude that $a(x, y)v \equiv 0$ in $\Omega \times (0, T)$. Since $a(x, y) > 0$ in Θ^c , so in particular $v \equiv 0$ in $\Theta^c \times (0, T)$. We will show that $v \equiv 0$ in $\Omega \times (0, T)$.

Note that the limit v satisfies

$$v_t + v_{xxx} + \partial_x^{-1} v_{yy} + \lambda v v_x = 0, \quad (3.36)$$

where $\lambda \geq 0$ is the limit of λ_n as $n \rightarrow \infty$.

In any case, whether $\lambda = 0$ or $\lambda > 0$, we will use the UCP discussed earlier to conclude that $v \equiv 0$ in $\Omega \times (0, T)$. To be able to apply the UCP, we must show that v is sufficiently regular. Let $Z := (\delta, L - \delta) \times (\delta, L - \delta)$ and define a function,

$$w(x, y, t) = \begin{cases} v(x, y, t), & (x, y, t) \in Z \times (0, T), \\ 0, & (x, y, t) \in \{\mathbb{R}^2 - Z\} \times (0, T). \end{cases} \quad (3.37)$$

Because $\Theta \subset (\delta, L - \delta) \times (\delta, L - \delta)$, this extension is as smooth as v . Furthermore w satisfies

$$\begin{cases} w_t + w_{xxx} + \partial_x^{-1} w_{yy} + \lambda w w_x = 0, & (x, y, t) \in \mathbb{R}^2 \times (0, T) \\ w(x, y, 0) = \phi(x, y), \end{cases} \quad (3.38)$$

where

$$\phi(x, y) = \begin{cases} v(x, y, 0), & (x, y) \in Z, \\ 0, & (x, y) \in \mathbb{R}^2 - Z. \end{cases} \quad (3.39)$$

Note that, ϕ is a compactly supported function in $H^s(\mathbb{R}^2)$, $s \geq 3$. So, by the regularization property of the KP-II equation (see [9]), the IVP (3.38) has a smooth solution w . Therefore, by the unique continuation property (Theorem 3.4), we conclude that $w \equiv 0$ in $\Omega \times (0, T)$. Consequently we conclude that $v \equiv 0$ in $\Omega \times (0, T)$ which contradicts (3.34). Hence (3.24) must be true. \square

REFERENCES

- [1] J. Bourgain, *On the Compactness of the Support of Solutions of Dispersive Equations*, IMRN International Mathematics Research Notices 1997, No. 9 437–447.
- [2] J. Bourgain, *On the Cauchy problem for the Kadomtsev-Petviashvili equation*, GAFA, **3** (1993) 315–341.
- [3] A. Friedman, *Partial Differential Equations*, Holt, Rinehart and Winston, New York, 1976.
- [4] R. J. Íorio Jr., W. V. L. Nunes, *On equations of KP-type*, Proc. Roy. Soc., (1998) 725–743.
- [5] P. Isaza, J. Meija, *Global solution for the Kadomtsev-Petviashvili equation (KP-II) in anisotropic Sobolev spaces of negative indices*, Electronic Journal of Differential Equations, **2003** (2003) 1–12.
- [6] P. Isaza, J. Meija, *Local and Global Cauchy problem for the Kadomtsev-Petviashvili (KP-II) equation in Sobolev spaces of negative indices*, Comm. PDE, **26** (2001) 1027–1057.
- [7] P. Isaza, J. Meija, *The Cauchy problem for the Kadomtsev-Petviashvili equation (KP-II) in Sobolev spaces H^s , $s > 0$* , Differential and Integral Eqns., **14** (2001) 529–557.
- [8] B. B. Kadomtsev, V. I. Petviashvili, *On the stability of solitary waves in weakly dispersive media*, Sov. Phys. Dokl., **15** (1970) 539–549.
- [9] J. L. Levandosky, *Gain of regularity for the KP-II equation*, Indiana Univ. Math. J., **49** (1) (2000) 353–403.
- [10] F. Linares, G. Ponce, *Introduction to nonlinear dispersive equations*, Universitext, Springer, New York, 2009.
- [11] L. Molinet, J. C. Saut, N. Tzvetkov, *Global well-posedness for the KP-I equation*, Math. Ann. **324** (2002) 255–275.
- [12] M. Panthee, *Unique continuation property for the Kadomtsev-Petviashvili (KP-II) equation*, Electronic Journal of Differential Equations, Vol 2005(2005), No. 59, 1–12.
- [13] G. Perla Menzala, C. F. Vasconcellos, E. Zuazua *Stabilization of the Korteweg-de Vries equation with localized damping*, Quart. Appl. Math. 60 (2002), no. 1, 111–129.
- [14] H. Takaoka, *Global well-posedness for the Kadomtsev-Petviashvili II equation*, Discrete Contin. Dynam. Systems **6** (2000) 483–449.
- [15] H. Takaoka, N. Tzvetkov, *On the local regularity of Kadomtsev-Petviashvili-II equation*, IMRN, International Mathematics Research Notices **2** (2001) 77–114.

- [16] N. Tzvetkov, *Global low regularity solutions for Kadomtsev-Petviashvili equation*, Differential and Integral Equations **13** (2000) 1289–1320.

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