RESEARCH ARTICLE

A penalty method and a regularization strategy to solve MPCC

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The goal of this paper is to solve Mathematical Program with Complementarity Constraints (MPCC) using nonlinear programming (NLP) techniques. This work presents two algorithms based on several nonlinear techniques such as Sequential Quadratic Programming (SQP), penalty techniques and regularization schemes. A set of AMPL problems were tested and the computational experience shows that both algorithms are effective.

1.   Introduction

Mathematical Program with Complementarity Constraints is an exciting new application of nonlinear programming techniques. The complementarity concept is related to the equilibrium notion and there exist lots of real situations that can be modeled as a MPCC in Engineering, Economics and Ecology. The complexity of the MPCC is caused by the disjunctive constraints which lead to some challenging issues that typically are the main concern in the design of efficient solution algorithms. The researchers have been spent lots of efforts studying the MPCC theory and proposing different algorithms to solve MPCC efficiently ([1], [4], [6], [8], [9] and [10]). One attractive way of solving MPCC is to consider its equivalent nonlinear programming formulation. However this formulation has no feasible point that satisfies the inequalities strictly, so the Mangasarian-Fromovitz constraint qualification is violated at any feasible point. Fletcher *et al.* [4] complements these numerical observation giving a theoretical explanation for the good performance of the SQP method - they show that SQP is guaranted to converge quadratically near a stationary point under relatively mild conditions. Ralph and Wright [8] described some properties of penalized and regularized nonlinear programming formulations of MPCC. Based on these promising results we propose two algorithms combining the SQP and the penalty and regularization strategies to solve MPCC.

This paper is organized as follows. The next section defines the MPCC problem. Section 3 introduces the penalty technique, the regularization scheme and the corresponding implemented algorithms in MATLAB environment. In Section 4 some numerical experiments to test the algorithms, the performance profiles and some final conclusions are carried out.

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2.   Problem definition

$$\begin{aligned}
min &\quad f(x) \\
s.t. &\quad c_i(x) = 0, \ i \in E, \\
&\quad c_i(x) \geq 0, \ i \in I, \\
&\quad 0 \leq x_1 \perp x_2 \geq 0,
\end{aligned} \qquad \text{(MPCC)}$$

where f and c are the nonlinear objective function and the constraint functions, respectively, assumed to be twice continuously differentiable. E and I are two disjoined finite index sets with cardinality p and m, respectively. $x = (x_0, x_1, x_2)$ is a decomposition of the variables into $x_0 \in \mathbb{R}^n$ (control variables) and $(x_1, x_2) \in \mathbb{R}^{2q}$ (state variables). $0 \leq x_1 \perp x_2 \geq 0 : \mathbb{R}^{2q} \to \mathbb{R}^q$ are the q complementarity constraints. The notation $x_1 \perp x_2$ means that $x_{1j}x_{2j} = 0$, for $j = 1, \ldots, q$, *ie*, the complementarity condition owns the disjunctive nature - $x_{1j} = 0$ or $x_{2j} = 0$, for $j = 1, \ldots, q$.

3.   Penalty technique and Regularization scheme

A way to deal with the complementarity constraints is to apply a penalty technique. Using the well known sequential penalty technique [2], the original problem (MPCC) is replaced by a nonlinear constrained optimization problem:

$$\begin{aligned}
Pen(r): \quad min &\quad P(x, r) \\
s.t. &\quad c_i(x) = 0, i \in E, \\
&\quad c_i(x) \geq 0, i \in I, \\
&\quad x_1 \geq 0, \quad x_2 \geq 0,
\end{aligned} \qquad \text{(Pen)}$$

where the penalty function is $P(x, r) = f(x) + r \sum_{j=1}^{q} (x_{1j} \, x_{2j})^2$ and r is the penalty parameter with $r > 0, r \to \infty$. In this approach only complementarity terms are penalized and a sequence of minimization problems is solved as far as r is incremented.

Ralph and Wright [8] present several regularization, also called relaxation, schemes and the corresponding properties in order to solve (MPCC). The same authors study a regularization scheme that is analyzed by Scholtes [10] where (MPCC) is approximated by the following NLP problem with a nonnegative scalar parameter t decreasing to zero:

$$\begin{aligned}
Reg(t): \quad min &\quad f(x) \\
s.t. &\quad c_i(x) = 0, i \in E, \\
&\quad c_i(x) \geq 0, i \in I, \\
&\quad x_1 \geq 0, \ x_2 \geq 0, \\
&\quad x_{1j}x_{2j} \leq t, \ \ j = 1, ..., q.
\end{aligned} \qquad \text{(Reg)}$$

The regularization scheme can be used by applying a NLP algorithm to $Reg(t)$ for a sequence of problems where t is positive and tends to zero. In this sequence, the result of each minimization, is the initial approximation of the next minimization.

Two algorithms each one with two iterative procedures were implemented in MATLAB - Quadratic Penalty Algorithm and Regularization Scheme Algorithm

denoted by A1 and A2, respectively. The inner iterative procedure is performed by the `fmincon` routine from MATLAB Optimization toolbox [5] which implements the SQP method. The external iterative procedure is based on the sequential penalty technique (A1) and on the regularization scheme suggested by Ralph [8] (A2).

Algorithm A1: Quadratic Penalty Algorithm

Initialization: r_0; Tolerances definition: r_{max}, k_{max}, ϵ_1, ϵ_2; Inner and external iterations counters: $it\_int, k = 0$;

Problem information (amplfunc): x_0, *lb*, *ub*, *cl*, *cu*, *cv*; Problem dimension: n, m, p, q;

REPEAT

    Built penalty function $P(x_k, r_k)$ and constraints;

    Run the MATLAB function: $[x, f, LAMBDA, output]=$`fmincon('function',....,'constraint')`;

    Lagrange multipliers estimation;

    Update x, r, k and it_int: $x_{k+1} \leftarrow x$; $r_{k+1} \leftarrow r_k \times \rho_1\,(\rho_1 > 1)$; $k \leftarrow k + 1$; $it\_int \leftarrow it\_int + output.iterations$;

UNTIL Stop criterium $(r_{max}, k_{max}, \epsilon_1, \epsilon_2)$

Algorithm A2: Regularization Scheme Algorithm

Initialization: t_0; Tolerances definition: t_{min}, k_{max}, ϵ_1, ϵ_2; Inner and external iterations counters: $it\_int, k = 0$;

Problem information (amplfunc): x_0, *lb*, *ub*, *cl*, *cu*, *cv*; Problem dimension: n, m, p, q;

REPEAT

    Built the constraints;

    Run the MATLAB function:$[x, f, LAMBDA, output]=$`fmincon('function',....'constraint')`;

    Update it_int, x: $it\_int \leftarrow it\_int + output.iterations$; $x_{k+1} \leftarrow x$;

    Lagrange multipliers update; Update t, k: $t_{k+1} \leftarrow t_k \times \rho_2\,(0 < \rho_2 < 1)$; $k \leftarrow k + 1$;

UNTIL Stop criterium $(t_{min}, k_{max}, \epsilon_1, \epsilon_2)$

In algorithm A1, the formula $(\lambda_\perp)_j = 2r\, x_{1j}\, x_{2j}$, $j = 1, \ldots, q$ is used to estimate the complementarity constraints Lagrange multipliers. The Algorithm A2 doesn't require the Lagrange multipliers estimation - they are updated using the output values of the `fmincon` routine ((MPCC) and (Reg) have the same number of constraints).

4. Numerical experiments

This section summarizes the numerical experiences using 85 AMPL test problems from MacMPEC [7] to test the Algorithms A1 and A2. The Algorithms A1 and A2 use $r = 10 \times r$ and $t = 0.05 \times t$ to update the penalty parameter and the regularization parameter, respectively. The performance profiles of Dolan and Moré [3] are used to analyse the relative performance of the two algorithms. The performance metrics are the number of internal and external iterations, respectively. The graphics of performance profiles are in Figures 1 and 2 using *log* scale.

The Algorithms A1 and A2 have two and one fails, respectively, *ie*, both algorithms present very good robustness. From the Figure 1 it is clear that the Algorithm A2 has the highest probability of being the optimal solver for more than 65% of the problems with respect to the internal iterations. Concerning to the external iterations, Figure 2, both codes present similar performance. In order to evaluate the algorithms efficiency the same problems set was solved using the `fmincon` routine directly. The original problem is converted to an NLP formulation where $x_1 \perp x_2$ is replaced by $x_{1j}x_{2j} = 0$, $j = 1, \ldots, q$. In this experience, denoted by A3 in Figure 3, twenty problems failed, compromising the robustness. A performance profile, comparing the internal iterations, for the three experiences is in the Figure 3 - it is clear that Algorithms A1 and A2 introduce significant improvements with respect to internal iterations.
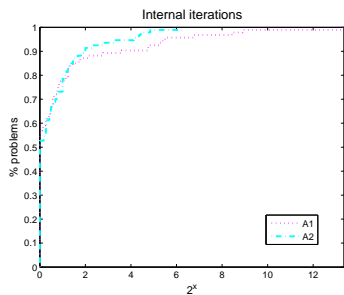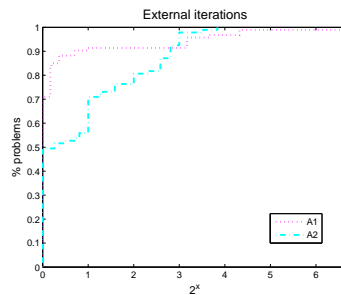
Figure 1. Internal iterations performance profile
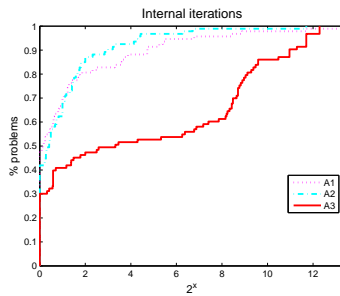

Figure 2. External iterations performance profile

Figure 3. A1 *vs* A2 *vs* A3

Two distinct algorithms were implemented in MATLAB, combining the SQP philosophy with the penalty technique and a regularization scheme, respectively. These algorithms aim at computing a local optimal solution. The numerical results are very promising - these nonlinear techniques combinations work very well, are very easy to implement and both algorithms present robustness and efficiency. The computational experience shows that penalty technique and regularization scheme are effective for solving MPCC.

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