Chapter 1
Relating material and space-time metrics within relativistic elasticity: a dynamical example

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Abstract Given a space-time and a continuous medium with elastic properties described by a 3-dimensional material space, one can ask whether they are compatible in the context of relativistic elasticity. Here a non-static, spherically symmetric spacetime metric is considered and we investigate the conditions for that metric to correspond to different 3-dimensional material metrics.

1.1 General results

Let (M,g) be a spacetime. The *material space* X is a 3-dimensional manifold endowed with a Riemannian metric γ, the *material metric*; points in X can then be thought of as the particles of which the material is made of. Coordinates in M will be denoted as x^a for $a = 0,1,2,3$, and coordinates in X as y^A, $A = 1,2,3$. The material metric γ is not a dynamical quantity of the theory and it roughly describes distances between neighboring particles in the relaxed state of the material.

The spacetime configuration of the material is said to be completely specified whenever a submersion $\psi : M \rightarrow X$ is given; if one chooses coordinate charts in M and X as above, then $y^A = y^A(x^b)$ and the physical laws describing the mechanical properties of the material can then be expressed in terms of a hyperbolic second order system of PDE. The differential map $\psi_* : T_pM \rightarrow T_{\psi(p)}X$ is then represented in

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the above charts by the rank 3 matrix $\left(y^A_b\right)_p, \quad y^A_b = \partial y^A/\partial x^b$ which is sometimes called *relativistic deformation gradient*. The kernel of ψ_* is spanned by a single timelike vector which we take as $\mathbf{u} = u^a\partial_a$, satisfying $y^A_b u^b = 0,\ u^a u_a = -1,\ u^0 > 0$. $\mathbf{u}$ is called the *velocity field of the matter*, and in the above picture in which the points in X are material points, the spacetime manifold M is then made up by the worldlines of the material particles.

The material space is said to be in a *locally relaxed state* at an event $p \in M$ if, at p, it holds $k_{ab} \equiv (\psi^*\gamma)_{ab} = h_{ab}$ where $h_{ab} = g_{ab} + u_a u_b$. Otherwise, it is said to be *strained*, and a measurement of the difference between k_{ab} and h_{ab} is the *strain*, whose definition varies in the literature; thus, while it can be defined simply as $S_{ab} = -\frac{1}{2}(k_{ab} - h_{ab}) = -\frac{1}{2}(k_{ab} - u_a u_b - g_{ab})$, we shall follow instead the convention in [1] and use $K_{ab} \equiv k_{ab} - u_a u_b$. The strain tensor determines the elastic energy stored in an infinitesimal volume element of the material space (or energy per particle), hence that energy will be a scalar function of K_{ab}. This function is called *constitutive equation* of the material, and its specification amounts to the specification of the material. We shall represent it as $v = v(I_1, I_2, I_3)$, where I_1, I_2, I_3 are any suitably chosen set of scalar invariants[1] associated with and characterizing K_{ab} completely. Following [1] we shall choose

$$I_1 = \frac{1}{2}(\mathrm{Tr}K - 4) \qquad I_2 = \frac{1}{4}\left[\mathrm{Tr}K^2 - (\mathrm{Tr}K)^2\right] + 3 \qquad I_3 = \frac{1}{2}(\det K - 1)\,. \quad (1.1)$$

Notice that for $K_{ab} = g_{ab}$ (equivalently $k_{ab} = h_{ab}$) the strain tensor S_{ab} is zero, in which case one has $I_1 = I_2 = I_3 = 0$.

The energy density ρ will then be the particle number density ε times the constitutive equation, that is $\rho = \varepsilon v(I_1, I_2, I_3) = \varepsilon_0 \sqrt{\det K}\, v(I_1, I_2, I_3)$ where ε_0 is the particle number density as measured in the material space, or rather, with respect to the volume form associated with $k_{ab} = (\psi^*\gamma)_{ab}$, and ε is that with respect to h_{ab}.

In the case of elastic matter, it can be seen using the standard variational principle for the Lagrangian density $\Lambda = \sqrt{-g}\rho$ (see for instance [2] or [3]) that decomposing the energy-momentum with respect to $\mathbf{u}$ (the velocity of the matter) yields $T_{ab} = \rho u_a u_b + p h_{ab} + P_{ab}$, where $h_{ab} = g_{ab} + u_a u_b$, $P_{ab} = h^m_a h^n_b (T_{mn} - 3p h_{mn})$, $\rho = T_{ab} u^a u^b$, $p = \frac{1}{3} h^{ab} T_{ab}$ which satisfy $h_{ab} u^b = 0$, $P_{ab} u^b = g^{ab} P_{ab} = 0$. The above energy-momentum tensor is of the diagonal Segre type $\{1,111\}$ or any of its degeneracies so that an orthonormal tetrad exists $\{u_a, x_a, y_a, z_a\}$ (with $u_a u^a = -1$, $x^a x_a = y^a y_a = z^a z_a = +1$ and the mixed products zero) such that:

$$T_{ab} = \rho u_a u_b + p_1 x_a x_b + p_2 y_a y_b + p_3 z_a z_b, \quad p = \frac{1}{3}(p_1 + p_2 + p_3),$$
$$h_{ab} = x_a x_b + y_a y_b + z_a z_b, \quad \text{etc.} \qquad (1.2)$$

One can show that the Dominant Energy Condition (DEC) is fulfilled if and only if $\rho \geq 0, \quad |p_A| \leq \rho, \quad A = 1,2,3$.

[1] Recall that one of the eigenvalues is 1, therefore, there exist three other scalars (in particular they could be chosen as the remaining eigenvalues) characterizing K^a_b completely along with its eigenvectors.

1.2 Spherical symmetry and material metrics

For a spherically symmetric spacetime, coordinates $x^a = t, r, \theta, \phi$ exist (and are non-unique) such that the line element can be written as

$$ds^2 = -a(r,t)dt^2 + b(r,t)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{1.3}$$

with a and b positive. This metric possesses three Killing vectors, namely $\xi_{\mathbf{1}} = -\cos\phi\ \partial_\theta + \cot\theta \sin\phi\ \partial_\phi$, $\xi_{\mathbf{2}} = \partial_\phi$ and $\xi_{\mathbf{3}} = -\sin\phi\ \partial_\theta - \cot\theta\cos\phi\ \partial_\phi$ which generate the 3-dimensional Lie algebra $so(3)$.

The existence of symmetries has some important consequences on physics (see [4]). For example, matter 4-velocity, pressure, density, anisotropic tensor all stay invariant along the above KVs, together with the projection tensor $h_{ab} = g_{ab} + u_a u_b$. One can also show that any timelike vector field $\mathbf{v}$ that remains invariant along the three Killing vectors is necessarily of the form $\mathbf{v} = v^t(t,r)\ \partial_t + v^r(t,r)\ \partial_r$.

Let us now consider in more detail the problem of elasticity in a spherically symmetric spacetime $(M, \bar{g})$ with associated material space $(X, \bar{\gamma})$. We shall demand that the submersion $\psi : M \longrightarrow X$ mentioned in section 1 preserves the KVs, that is $\psi_*(\xi_A) = \eta_A$ are also KVs on X. This implies that the metric $\bar{\gamma}$ is also spherically symmetric and therefore coordinates $y^A = (y, \tilde{\theta}, \tilde{\phi})$ exist with $y = y(t,r)$, $\tilde{\theta} = \theta$ and $\tilde{\phi} = \phi$, and are such that $\eta_A = \xi_A$ are KVs of the metric $\bar{\gamma}$. Thus, the line element $d\bar{s}^2$ of $\bar{g}$ is obtained from (1.3), with a and b substituted by $\bar{a}$ and $\bar{b}$, respectively. The line element of $\bar{\gamma}$ may be written as:

$$d\bar{\Sigma}^2 = f^2(y)(dy^2 + y^2 d\theta^2 + y^2 sin^2\theta d\phi^2), \tag{1.4}$$

This last expression is completely general, as any 3-dimensional spherically symmetric metric is necessarily conformally flat, as it is immediate to show. Notice also that the relation between $\bar{\gamma}$ and the flat material metric γ used in [1], is simply $\bar{\gamma}_{AB} = f^2(y)\gamma_{AB}$. Writing $\bar{k} = \psi^*(\bar{\gamma})$, one has:

$$\bar{k}^a_b = \begin{pmatrix} -f^2(y)(\dot{y}^2/\bar{a}) & -f^2(y)(\dot{y}y'/\bar{a}) & 0 & 0 \\ f^2(y)(\dot{y}y'/\bar{b}) & f^2(y)(y'^2/\bar{b}) & 0 & 0 \\ 0 & 0 & f^2(y)y^2/r^2 & 0 \\ 0 & 0 & 0 & f^2(y)y^2/r^2 \end{pmatrix}, \tag{1.5}$$

where a dot indicates a derivative with respect to t and a prime a derivative with respect to r. The velocity field of the matter, defined by the conditions $\bar{u}^a y^A_a = 0$, $\bar{g}_{ab}\bar{u}^a\bar{u}^b = -1$ and $\bar{u}^0 > 0$, can be expressed as $\bar{u}^a = \bar{\Gamma}\bar{a}^{-1/2}(1, -\dot{y}/y', 0, 0)$, with $\bar{\Gamma} \equiv [1 - (\bar{b}/\bar{a})(\dot{y}/y')^2]^{-\frac{1}{2}}$.

The projection tensor $\bar{h}^a_b = \delta^a_b + \bar{u}^a\bar{u}_b$ follows now easily from these expressions.

We will use the an orthonormal tetrad $\{\bar{u}, \bar{x}, \bar{y}, \bar{z}\}$, with $\bar{u}$ given above and such that the remaining vectors are eigenvectors of the pulled back material metric $\bar{k}^a_b$: $\bar{x}^a = \left(-(\bar{a}\bar{b}^{1/2})(\dot{y}/y')\bar{\Gamma}, \bar{\Gamma}/\sqrt{\bar{b}}, 0, 0\right)$, $\bar{y}^a = (0, 0, 1/r, 0)$, $\bar{z}^a = (0, 0, 0, 1/(r\sin\theta))$, so that $\bar{g}_{ab} = -\bar{u}_a\bar{u}_b + \bar{x}_a\bar{x}_b + \bar{y}_a\bar{y}_b + \bar{z}_a\bar{z}_b$. It is now immediate to see that the pressure tensor has the same eigenvectors as $\bar{k}_{ab}$ and can be written as $\bar{p}_{ab} = \bar{p}_1\bar{x}_a\bar{x}_b + \bar{p}_2(\bar{y}_a\bar{y}_b + \bar{z}_a\bar{z}_b)$. Therefore, (1.2) yields $\bar{T}_{ab} = \bar{\rho}\bar{u}_a\bar{u}_b + \bar{p}_1\bar{x}_a\bar{x}_b + \bar{p}_2(\bar{y}_a\bar{y}_b + \bar{z}_a\bar{z}_b)$, where $\bar{\rho}$ is the energy density, $\bar{p}_1$, the radial pressure and $\bar{p}_2$, the tangential pressure. These and other related issues are studies in depth in [4].

In order to know whether the spacetime metric $\bar{g}$ can be associated with different conformally related material metrics, it will be assumed that $g_{ab} = \bar{g}_{ab}$, with g and $\bar{g}$ associated, respectively, with a flat (γ)and a non flat ($\bar{\gamma}$) material metric, related by $\bar{\gamma} = f^2\gamma$.

Therefore the expressions relating the eigenvalues of $\bar{k}$ and k are: $\bar{s} = f^2y^2/r^2 = f^2 s, \quad \bar{\eta} = f^2 y'^2/(\Gamma^2 b) = f^2\eta$. These expressions are used to relate the invariants in (1.1), namely $\bar{I}_1,\ \bar{I}_2,\ \bar{I}_3$, with the corresponding ones $I_1,\ I_2,\ I_3$ through the conformal factor f, as follows:

$$\begin{aligned} &\bar{I}_1 = f^2\,(I_1 + 3/2) - 3/2, \quad \bar{I}_2 = f^4\,(I_1 + I_2 - 3/2) - f^2\,(I_1 + 3/2) + 3, \\ &\bar{I}_3 = f^6\,(I_3 + 1/2) - 1/2. \end{aligned} \tag{1.6}$$

The above expressions for $\bar{s}$ and $\bar{\eta}$ lead to the following relations

$$\bar{\rho} = f^3\frac{\bar{v}}{v}\rho \qquad \bar{\varepsilon} = \rho_0\bar{s}\sqrt{\bar{\eta}} = f^3\varepsilon. \tag{1.7}$$

Taking the above expressions for the invariants together with (1.7) one obtains

$$\frac{\partial\bar{\rho}}{\partial\bar{I}_1} = \frac{1}{f^2}\frac{\partial\rho}{\partial I_1} - \frac{\partial\rho}{\partial I_2}\left(\frac{1}{f^2} - \frac{1}{f^4}\right), \qquad \frac{\partial\bar{\rho}}{\partial\bar{I}_2} = \frac{1}{f^4}\frac{\partial\rho}{\partial I_2}, \qquad \frac{\partial\bar{\rho}}{\partial\bar{I}_3} = \frac{1}{f^6}\frac{\partial\rho}{\partial I_3}. \tag{1.8}$$

These expressions lead to the following relationship for the energy-momentum tensors:

$$\bar{T}^a_b = f^3\frac{\bar{v}}{v}T^a_b. \tag{1.9}$$

However the assumption on equal metric tensors leads to equal energy-momentum tensors, so that the following relation for constitutive equations must hold:

$$\bar{v} = \frac{1}{f^3}v. \tag{1.10}$$

It is now straightforward to conclude that $\bar{\rho} = \rho$.

References

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