16 Reinforcement design using linear analysis

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Design of reinforced concrete structures can be described by the following consecutive steps:

1. Select the initial dimensions of all the structural elements using simple rules of thumb or experience. These dimensions should be able to satisfy the serviceability and ultimate limit states, and should fulfill the requirements for adequate site execution and any other requirement applicable (e.g. acoustic isolation, fire protection, etc.);
2. Perform a global structural analysis to calculate the internal forces (or stresses) due to the combination of loads defined in the codes. The method almost used exclusively today is the finite element method and the behavior of the structure is assumed to be linear elastic at this stage;
3. Verify concrete initial dimensions and calculate the reinforcement capable of resisting the calculated internal forces. At this stage, the ultimate capacity of the individual cross sections is considered, which is typically associated with non-linear constitutive laws.

The main advantage of the above process is that linear elastic finite element analysis is well established and is straightforward to apply. In addition, multiple load cases can be easily incorporated and reinforcement is placed in the locations where tensile stresses appear. These regions correspond to the initial crack locations, helping to control crack propagation.

Of course there are also some disadvantages in the process described, as stress redistribution can be difficult to incorporate, providing more expensive reinforcement arrangement, no real information is obtained about the collapse load of the structure, even if a lower bound estimate is obtained when ductility is enforced, and no real information is provided on inelastic phenomena as crack width, crack spacing or maximum deflection, even if they can be estimated for beam-type structural elements. A consequence of the process is that detailing guidelines need to be used to ensure ductility and serviceability demands.

Only in very few selected cases of structures with unusual size, shape, or complexity, a full nonlinear analysis of the previously designed structure would be made for assessment, tracing out the entire behavior through the uncracked, cracked, and ultimate stages. Such an analysis generally requires significant time for pre-processing, computation and post-processing, which is not compatible with cost and time demands. Also, as nonlinear analysis requires the definition of geometry and reinforcement, it should not be regarded as a design tool but, mostly, as an assessment tool.

The simulation of concrete walls, slabs, assemblages of walls and slabs and shells using finite element analysis is becoming a standard in structural analysis tools for building design, meaning that adequate methodologies for the design of these elements are necessary.

In this chapter, we subsequently discuss design methods for membrane states (walls), bending states (slabs) and combinations as may occur in spatial assemblages of plates and in shells. Hereafter they are shortly addressed as shells. Design of reinforced concrete elements subjected to membrane states has been developed since 1960s by authors like Baumann […], Braestrup and Nielsen […], only to name of few. This process resulted in formulas for reinforcement design and check of concrete strength in the CEB-FIB Model Code 1990 for Concrete Structures.

Reinforcement design for slabs and shells has also received attention in the Model Code 1990. For that purpose a three layer sandwich model was introduced. Pioneers of this approach are Gupta and Marti. The most preliminary version of Eurocode 2 suggested a different method on basis of the normal yield criterion. That result is alternatively referred to as Wood-Armer equations. It applies to slabs only and not for shells. The later version EN 1992-1-1:2004 removed this method. The version EN 1992-2:2005 of Eurocode 2 again included a solution in its appendices, returning to the three layer sandwich model. This solution has received a place in Part 2 of Eurocode 2 on Bridges. FIB has published in 2008 a *Practitioners' guide to finite element modeling of reinforced concrete structures*. This document also presents the three layer sandwich model. Readers interested in a more complete review of the historical development of the different methods are referred to […].

Here we will refer to the three layer sandwich model of Eurocode 2 and ***fib*** practitioners' guide as *basic model*. We denote it *basic* because the concept is very useful, but the working-out still permits improvement, because internal lever arms are only approximated. After the presentation of the basic model, we introduce an *advanced method* which meets in a consistent way all equilibrium conditions.

Both the basic and advanced model will first be discussed for cases of moderate transverse shear forces which can be carried by the concrete. After that we make the extension to slabs with larger transverse shear forces, which require transverse shear reinforcement, an extension we owe to Marti […].

16.1 Design of membrane states

Consider a membrane element with a thickness h, subjected to applied in-plane forces n_{xx}, n_{yy} and n_{xy}. The reinforcement consists of two orthogonal sets of rebars parallel to the x, y-axes. A_{sx} and A_{sy} are the needed reinforcement areas per unit length in this co-ordinate system. They are calculated from forces n_{sx} and n_{sy} respectively. The purpose of this Section is to find formulas for n_{sx} and n_{sy}

The applied forces will be resisted by the reinforcement and concrete contributions. It is assumed that the concrete is subjected to uni-axial compression n_c parallel to the cracking orientation, at an angle θ with the y-axis. The two rebar sets in Fig. 16.1b and the concrete struts of Fig. 16.1c together must carry the applied loads of Fig. 16.1a. For the sign convention of the applied n_{xx}, n_{yy} and n_{xy} loads we refer to Chapter 1. The forces n_{sx} and n_{sy} are always positive or zero, and the membrane force n_c in the concrete is negative or zero.

In the chosen x,y-coordinate system, the shear resistance of the reinforcement is zero and the state of stress of the concrete is uni-axial. The first principal membrane force is zero and the compressive force n_c occurs in the second principle direction. The stress state in Fig. 16.1a is equivalent with the combination of the states in Figs. 16.1b and 16.1c when the following equilibrium conditions are satisfied

$$\begin{aligned} n_{xx} &= n_{sx} + n_c \sin^2\theta \\ n_{yy} &= n_{sy} + n_c \cos^2\theta \\ n_{xy} &= -n_c \sin\theta\cos\theta \end{aligned} \qquad (16.1)$$

The condition holds that the second principal stress σ_c is smaller than the compressive strength f_c of concrete

$$n_c \geq -hf_c \qquad (16.2)$$

The applied forces are in the left member of Equ. (16.1) to (16.3), and the internal forces are in the right member. It should be reminded that n_c is negative, so n_{sx} is larger than n_{xx} and n_{sy} larger than n_{yy}. The cases of $\theta = 0$ and $\theta = \pi/2$ are trivial, meaning that only one set of reinforcement is needed, aligned with the axis y or x, respectively. Assuming that $\theta \neq 0$ and $\theta \neq \pi/2$, Equ. (16.1) to (16.3) can be recast such that the steel and concrete forces are in the left member and the applied forces in the right

$$\begin{aligned} n_{sx} &= n_{xx} + n_{xy} \tan\theta \\ n_{sy} &= n_{yy} + n_{xy} \cot\theta \\ n_c &= -\frac{n_{xy}}{\sin\theta\cos\theta} \end{aligned} \qquad (16.3)$$

Equ. (16.7) indicates that n_{xy} and θ must have the same sign, so that n_c is negative, or in compression. The total amount of reinforcement can be obtained from Equ. (16.5) and (16.6), and equals,

$$n_{sx} + n_{sy} = n_{xx} + n_{yy} + n_{xy}(\tan\theta + \cot\theta) \tag{16.4}$$

Note that the last term in this equation is always positive, as n_{xy} and θ have the same sign. Thus, the minimum amount of reinforcement corresponds to $\theta = \pm\pi/4$. For these values of θ, noting that the reinforcement must be always subjected to tension, i.e. $n_{sx} \geq 0$ and $n_{sy} \geq 0$, Equ. (16.5) and (16.6) give $n_{xx} \geq -|n_{xy}|$ and $n_{yy} \geq -|n_{xy}|$ respectively. Or else the θ value must be changed. Therefore four different cases of reinforcement have to be considered.

Case 1 Reinforcement in *x*- and *y*-direction needed

For this case it holds

$$\begin{aligned} &n_{xx} \geq -|n_{xy}|, && n_{yy} \geq -|n_{xy}|; \\ &n_{sx} = n_{xx} + |n_{xy}|, && n_{sy} = n_{yy} + |n_{xy}| \\ &\theta = \pm\frac{\pi}{4}, && n_c = -2|n_{xy}| \end{aligned} \tag{16.5}$$

Case 2 Only reinforcement in *y*-direction needed

For this case the following equations hold

$$\begin{aligned} &n_{xx} < -|n_{xy}| \;\rightarrow\; n_{sx} = 0 \\ &\tan\theta = -\frac{n_{xx}}{n_{xy}}; \\ &n_{sy} = n_{yy} - \frac{n_{xy}^2}{n_{xx}}, \quad n_{sy} \geq 0 \;\rightarrow\; n_{yy} \geq \frac{n_{xy}^2}{n_{xx}} \\ &n_c = n_{xx} + \frac{n_{xy}^2}{n_{xx}} \end{aligned} \tag{16.6}$$

Case 3 Only reinforcement in *x*-direction needed

For this case the following equations hold

$$
\begin{aligned}
& n_{yy} < -|\,n_{xy}\,| \;\; \rightarrow n_{sy} = 0 \\
& \tan\theta = -\frac{n_{xy}}{n_{yy}} \\
& n_{sx} = n_{xx} - \frac{n_{xy}^2}{n_{yy}}, \quad n_{sx} \geq 0 \quad \rightarrow n_{xx} \geq \frac{n_{xy}^2}{n_{yy}} \\
& n_c = n_{yy} + \frac{n_{xy}^2}{n_{yy}}
\end{aligned}
\tag{16.7}
$$

Case 4 No reinforcement needed in any direction

No cracking occurs and the stress state is biaxial compression. In the concrete two principle membrane forces n_{c1} and n_{c2} are present.

$$
\begin{aligned}
& \begin{cases} n_{xx} < -|\,n_{xy}\,| \\ n_{yy} < \dfrac{n_{xy}^2}{n_{xx}} \end{cases} \quad or \quad \begin{cases} n_{yy} < -|\,n_{xy}\,| \\ n_{xx} < \dfrac{n_{xy}^2}{n_{yy}} \end{cases} \\
& n_{sx} = 0, \quad n_{sy} = 0 \\
& n_{c1,c2} = \frac{n_{xx} + n_{yy}}{2} \pm \sqrt{\left(\frac{n_{xx} - n_{yy}}{2}\right)^2 + n_{xy}^2}
\end{aligned}
\tag{16.8}
$$

Rebar design and check on concrete stress

The four cases are summarized in Fig. 16.2. The formulas correspond to the optimum direction of concrete compression, i.e., the θ value leading to the minimum amount of reinforcement. The reinforcement design then obtained from

$$
a_{sx} = \frac{n_{sx}}{f_{syd}}, \qquad a_{sy} = \frac{n_{sy}}{f_{syd}} \tag{16.9}
$$

where a_{sx} and a_{sy} are steel areas per unit length and f_{syd} is the design yield strength of the reinforcement. The concrete stress is given by

$$
f_c = -n_c / h \tag{16.10}
$$

which must be checked against the design compressive strength f_{cd}. For this strength we can apply the Model Code 1990 or the Practitioners guide of ***fib***.

Model Code 1990

The Model Code 1990 recommends

$$\begin{array}{ll} \text{Case 1 to 3} & f_c \leq \ \ f_{cd2} \\ \text{Case 4} & f_c \leq K\, f_{cd1} \end{array} \tag{16.11}$$

where

$$\begin{aligned} f_{cd1} &= 0.85\left[1-\frac{f_{ck}}{250}\right] f_{cd} \\ f_{cd2} &= 0.60\left[1-\frac{f_{ck}}{250}\right] f_{cd} \\ K &= \frac{1+3.65\alpha}{\left(1+\alpha\right)^2}, \qquad \alpha = \frac{\sigma_2}{\sigma_1} \end{aligned} \tag{16.12}$$

Here f_{cd} is the design strength of the concrete, f_{ck} is the characteristic strength of the concrete, and σ_1 and σ_2 are the two principal compressive stresses. These formulas are based on experimental studies on biaxial concrete behaviour of Kupfer.

fib practitioners'guide

The practitioners'guide of ***fib*** recommends following planned changes to the ACI code. The proposed formula for the concrete strength is

$$f_{cd} = 0.85\beta f_{ck} / \gamma_c \tag{16.13}$$

where the factor 0.85 accounts for the variation between the in-situ and cylindrical strengths, β accounts for influence of transverse tensile strain, f_{ck} is the characteristic compression strength, and γ_c is the partial safety factor. The formula for β is

$$\beta = \frac{1}{0.8+170\varepsilon_1} \tag{16.14}$$

Herein ε_1 is the major principle strain normal to the direction of the concrete struts. For this strain the yield strain of the steel reinforcement might be chosen, so $\varepsilon_1 = f_{syd} / E$, where E is the Young's modulus of steel.

Remark

At this point, it should be pointed out that the non-continuous variation of concrete compressive strength between Cases 3 and 4, or between Cases 2 and 4 does not seem acceptable. This gains special relevance as Equ.(16.10) corresponds practically to an absolute minimum of cracked reinforced concrete. However, this seems to be the price to pay for a simplified design approach.

16.2 Design of slabs. Normal moment yield criterion

Similarly to the case of an element in membrane state, dimensioning of slabs and shells from internal forces obtained in a finite element analysis is based on an equilibrium model at ultimate state. While careful consideration of the limited ductility of concrete is important in the dimensioning of membrane elements, such a concern is lower for slabs because such structures are typically under-reinforced. Failure is usually governed by yielding of reinforcement, with the exception of point loads, which may result in brittle punching failures in slabs and in shells without transverse reinforcement.

The stress resultants acting in a slab are the bending moments m_{xx} and m_{yy} and twisting moments m_{xy}. For the derivation of the design equations a set of orthogonal axes is chosen in directions x and y, giving moments per unit length m_{xx}, m_{yy} and m_{xy}, such that m_{yy}> m_{xx}. Basis of the derivation is the normal moment yield criterion. It states that
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………………………………………………………………………… …………
and the manifestation of the equations reminds of the formulas for membrane states. The resulting formulas for the design moments are presented in Fig. 16.4. Reinforcement is provided in the x- and y-directions to resist design ultimate moments m_{xb}, m_{xt}, m_{yb} and m_{yt}. The subscripts b and t indicate bending moments giving tension in the slab bottom and slab top, respectively. The bottom is at the positive z-side of the slab middle plane, and the top at the negative side. The shown equations are used in many software packages for slab reinforcement design. Often even only the top-left corner of the figure is used.

Evaluation

The use of the equations of Fig. 16.4 is *discouraged* for a number of reasons. The equations are not able to take into consideration transverse shear forces, do not check for concrete crushing, and do not fulfill equilibrium. It is strongly recommended to use the three layer sandwich model, which applies for slabs and shells. This is the subject of Section 16.3 and Section 16.4.

16.3 Slab and shell elements. Basic model

The problem to be discussed in the present Section is the design of a shell element of thickness h, subjected to combined membrane forces and bending moments and where the directions of the principal flexural and membrane forces do not, in general, coincide. A slab element is a special case of the stated problem. Fig. 16.5 shows the applied forces and moments. These forces and moments have to be in equilibrium with the tensile forces in the reinforcement and the compressive forces in the concrete. We choose a set of x,y,z-axes as we did in the Chapters 3 and 4, where z is pointing downward, see Fig. 16.5. The reinforcement consists again of a mesh of orthogonal rebars parallel to the x,y-axes, now placed in a upper and lower layer. We refer to the upper or top layer by the subscript t (negative z-side) and to the lower or bottom layer by b (positive z-side).

The formulation of this problem is identical to the one in Section 16.1 for membrane states. Again, the total resistance of the element is obtained by adding the concrete and reinforcement contributions. We model the shell element as a three layer sandwich, shown in Fig. 16.6. The outer layers are covers of the sandwich and the inner layer is the core. The cover layers provide resistance to the in-plane effects of flexure and membrane loading, while the core provides a shear transfer between the covers. The thickness of the covers is a and the distance between the middle planes of the covers is d_v.

Dependent on its size the transverse shear force has impact on the amount of the reinforcement in the covers. For small values they do not, otherwise they do. To decide whether the shear forces are small we must consider the maximum shear force v_o as specified in Eq.(4.24).

$$v_o = \sqrt{v_x^2 + v_y^2} \tag{16.15}$$

which acts in the direction of an angle β_o with the x-axis, defined by Equ.(4.23).

$$\tan \beta_o = \frac{v_y}{v_x} \tag{16.16}$$

This shear force is small if it is below the shear cracking resistance $d_v \tau_{c,red}$, where $\tau_{c,red}$ is the nominal strength of the slab without transverse reinforcement. Then the core will remain uncracked. For the value $\tau_{c,red}$ we may apply ENV 1992-1-1, which provides

$$\tau_{c,red} = 0.25 f_{ctd} \left(1.6 - d_v\right)\left(1.2 + 40\rho_l\right) + 0.15\sigma_{cp} \tag{16.17}$$

Here, f_{ctd} is the design tensile strength, d_v is the internal lever arm in meters, ρ_l is the percentage of longitudinal reinforcement and σ_{cp} is the in-plane normal compressive stress. If significant tensile membrane forces are applied to the element, $\tau_{c,red}$ should be taken equal zero. Provided that no significant tensile membrane forces exist, the expression in Equ.(16.15) can be simplified to a lower bound, neglecting the positive effect of the longitudinal reinforcement.

$$\tau_{c,red} = 0.30 f_{ctd} \left(1.6 - d_v\right) \tag{16.18}$$

16.3.1 Basic model. No cracking due to transverse shear.

We start with the case of small shear forces. Then the core layer is supposed not to crack and is able to carry transverse shear forces. Fig. 16.6b depicts the sandwich model for this case. The need for reinforcement needs only be investigated for the combination of membrane forces and bending and twisting moments. It is an important decision which thickness is assigned to the top and bottom layer. In the basic model these thicknesses are equal to each other. Further it is assumed that all reinforcement layers are positioned in the middle plane of the outer sandwich layers. Therefore one lever arm d_v applies for both directions x and y. The membrane forces in the external layers are given by

$$\begin{aligned} n_{xxt} &= \frac{m_{xx}}{d_v} + \frac{n_{xx}}{2}, \quad & n_{xxb} &= -\frac{m_{xx}}{d_v} + \frac{n_{xx}}{2} \\ n_{yyt} &= \frac{m_{yy}}{d_v} + \frac{n_{yy}}{2}, \quad & n_{yyb} &= -\frac{m_{yy}}{d_v} + \frac{n_{yy}}{2} \\ n_{xyt} &= \frac{m_{xy}}{d_v} + \frac{n_{xy}}{2}, \quad & n_{xyb} &= -\frac{m_{xy}}{d_v} + \frac{n_{xy}}{2} \end{aligned} \tag{16.19}$$

Using the expressions provided above for the cover membrane elements, we can obtain the final expressions for the forces per unit width for the reinforcement design in shells

$$\begin{aligned}
n_{sxt} &= \frac{m_{xx}}{d_v}+\frac{n_{xx}}{2}+\left|\frac{m_{xy}}{d_v}+\frac{n_{xy}}{2}\right| \\
n_{syt} &= \frac{m_{yy}}{d_v}+\frac{n_{yy}}{2}+\left|\frac{m_{xy}}{d_v}+\frac{n_{xy}}{2}\right| \\
n_{sxt} &= -\frac{m_{xx}}{d_v}+\frac{n_{xx}}{2}+\left|-\frac{m_{xy}}{d_v}+\frac{n_{xy}}{2}\right| \\
n_{syt} &= -\frac{m_{yy}}{d_v}+\frac{n_{yy}}{2}+\left|-\frac{m_{xy}}{d_v}+\frac{n_{xy}}{2}\right|
\end{aligned} \tag{16.20}$$

From these forces reinforcement percentages can be derived

$$\rho_{xt}=\frac{n_{sxt}}{h\,f_{syd}},\qquad \rho_{yt}=\frac{n_{syt}}{h\,f_{syd}},\qquad \rho_{xb}=\frac{n_{sxb}}{h\,f_{syd}},\qquad \rho_{yb}=\frac{n_{syb}}{h\,f_{syd}} \tag{16.21}$$

where f_{syd} is the design yield stress of steel.

16.3.2 Basic model. Cracking due to transverse shear

If the transverse shear forces are high enough to produce cracking of the sandwich core, additional reinforcement is required. Here we follow the approach as proposed by Marti […]. An alternative proposal can be found in EN 19922-1-1:2004. The core is treated like the web of a girder of flanged cross-section running in the β_o-direction of the maximal shear force. Fig. 16.6c depicts that concrete struts in the core come into being under an angle θ with the middle plane. To ensure equilibrium additional membrane forces must occur in the upper and lower cover. Choosing $\theta = 45^o$ leads to additional membrane forces in both covers of size v_o in the direction of the maximal shear force. The choice of 45^o for the crack angle in the core is conforming to the traditional Morsch truss for reinforced concrete beams. Decomposing the additional membrane force in the covers to membrane forces in *x*- and *y*-direction leads to the following expressions

$$\begin{aligned}
n_{xxt} &= \frac{m_{xx}}{d_v}+\frac{n_{xx}}{2}+\frac{v_x^2}{2v_0}, \qquad & n_{xxb} &= -\frac{m_{xx}}{d_v}+\frac{n_{xx}}{2}+\frac{v_x^2}{2v_0} \\
n_{yyt} &= \frac{m_{yy}}{d_v}+\frac{n_{yy}}{2}+\frac{v_y^2}{2v_0}, \qquad & n_{yyb} &= -\frac{m_{yy}}{d_v}+\frac{n_{yy}}{2}+\frac{v_y^2}{2v_0} \\
n_{xyt} &= \frac{m_{xy}}{d_v}+\frac{n_{xy}}{2}+\frac{v_x v_y}{2v_0}, \qquad & n_{xyb} &= -\frac{m_{xy}}{d_v}+\frac{n_{xy}}{2}+\frac{v_x v_y}{2v_0}
\end{aligned} \tag{16.22}$$

Using the expressions provided above for membrane elements the final expressions for the design of reinforcement obtained for shells are

$$
\begin{aligned}
n_{sxt} &= \frac{m_{xx}}{d_v}+\frac{n_{xx}}{2}+\frac{v_x^2}{2v_0}+\left|\frac{m_{xy}}{d_v}+\frac{n_{xy}}{2}+\frac{v_x v_y}{2v_0}\right| \\
n_{syt} &= \frac{m_{yy}}{d_v}+\frac{n_{yy}}{2}+\frac{v_y^2}{2v_0}+\left|\frac{m_{xy}}{d_v}+\frac{n_{xy}}{2}+\frac{v_x v_y}{2v_0}\right| \\
n_{sxt} &= -\frac{m_{xx}}{d_v}+\frac{n_{xx}}{2}+\frac{v_x^2}{2v_0}+\left|-\frac{m_{xy}}{d_v}+\frac{n_{xy}}{2}+\frac{v_x v_y}{2v_0}\right| \\
n_{syt} &= -\frac{m_{yy}}{d_v}+\frac{n_{yy}}{2}+\frac{v_y^2}{2v_0}+\left|-\frac{m_{xy}}{d_v}+\frac{n_{xy}}{2}+\frac{v_x v_y}{2v_0}\right|
\end{aligned}
\qquad (16.23)
$$

From these forces reinforcement percentages are derived by the formulas in Equ.(16.17-1). Transverse reinforcement is needed with a percentage ρ_z given by

$$
\rho_z = \frac{v_0}{d_v\, f_{syd}} \qquad (16.18)
$$

In practical problems, it is recommended to increase the slab or shell thickness so that transverse reinforcement is avoided.

16.3.3 Evaluation

The basic sandwich model is simple to apply, but definitely is an approximation of reality. We mention that:

- It is assumed that the core does not contribute in transferring membrane forces, which for reasons of compatibility cannot be correct.
- The basic model works with equal thickness for the two outer layers of the sandwich. In general this need not be the case.
- It is assumed that both reinforcement layers in an outer layer are positioned in the middle plane of the cover, which is physically impossible.
- The angle θ of the membrane crack direction in both outer layers has been tacitly assumed to be ±45°. The same applies to the angle β_o for the cracks in the core due to the transverse shear force.

It is a consequence of some suppositions that equilibrium only is satisfied in an approximate manner, the deviance being of an increasing degree for higher reinforcement percentages. Particularly in case of large twisting moments the method is unsafe.

An improvement is made if the angles θ and β_o are not fixed to ±45°. In next Section 16.4 we present an advanced sandwich model. It also starts from the supposition of a three layer sandwich and division of force transfer, such that the covers carry the membrane forces, bending and twisting moments and the core carries the shear forces. For the rest the shortcomings of the basic model are fully repaired. We assign its own plane to each reinforcement layer, permit the thickness of covers to differ, have crack angles freely adapt, and rigorously satisfy equilibrium conditions. So a consistent set of suppositions lays the foundation of the advanced model.

16.4 Formulation of the advanced three-layer model

In the consistent model, the internal lever d_v is not assumed *a priori* and it is not equal in all directions, being calculated using an iterative process. Four different cases must be analyzed and treated separately: (a) reinforcement needed in both outer layers; (b) reinforcement needed only in the bottom layer; (c) reinforcement needed only in the top layer; (d) no need for reinforcement. The complete formulation of the problem, the software code and validation can be found in [….]. The described phenomena are simple but the resulting equations are reasonably complex, leading to an indeterminate system of nonlinear equations.

The geometry of the advanced model is shown in Fig. 16…. We introduce different distances h_{xt}, h_{yt}, h_{xb} and h_{yb} for the four reinforcement layers to the middle plane of the slab. The thickness of the outer layers is a_t and a_b, respectively. The core between these layers has thickness h_c. As done for the basic model, we define resisting reinforcement forces n_{sxt}, n_{syt}, n_{sxb} and n_{syb}. The two forces for the x-direction are summed to n_{sx} and for the y-direction to n_{sy}. Correspondingly resisting reinforcement moments m_{sx} and m_{sy} are defined. For the concrete top and bottom layer we introduce resisting forces n_{ct} and n_{cb} , respectively, and resisting concrete moments m_{ct} and m_{cb}. Here, subscripts s and c indicate steel and concrete, respectively, and subscripts t and b indicate again top and bottom external layer, respectively.

Case1. Reinforcement in both outer layers.

In case reinforcement is needed in the outer layers, the resisting forces and moments for the reinforcement in the x- and y- directions are given by

$$\begin{aligned} n_{sx} &= n_{sxt} + n_{sxb} \\ n_{sy} &= n_{syt} + n_{syb} \end{aligned} \tag{16.25}$$

$$
\begin{aligned}
m_{sx} &= -n_{sxt} h_{xt} + n_{sxt} h_{xb} \\
m_{sy} &= -n_{syt} h_{yt} + n_{syt} h_{yb}
\end{aligned}
\tag{16.26}
$$

and for the concrete by

$$
\begin{aligned}
n_{ct} &= -a_t f_c \\
n_{cb} &= -a_b f_c
\end{aligned}
\tag{16.27}
$$

$$
\begin{aligned}
m_{ct} &= -\frac{1}{2}(h - a_t) n_{ct} \\
m_{cb} &= \frac{1}{2}(h - a_b) n_{cb}
\end{aligned}
\tag{16.28}
$$

Equ. (16.23)-(16.26) provide the internal forces and moments. Equilibrium with the applied set of forces and moments leads to

$$
\begin{aligned}
n_{xx} &= n_{sx} + n_{ct} \sin^2 \theta_t + n_{cb} \sin^2 \theta_b \\
n_{yy} &= n_{sy} + n_{ct} \cos^2 \theta_t + n_{cb} \cos^2 \theta_b \\
n_{xy} &= -n_{ct} \sin \theta_t \cos \theta_t - n_{cb} \sin \theta_b \cos \theta_b
\end{aligned}
\tag{16.29}
$$

$$
\begin{aligned}
m_{xx} &= m_{sx} + m_{ct} \sin^2 \theta_t + m_{cb} \sin^2 \theta_b \\
m_{yy} &= m_{sy} + m_{ct} \cos^2 \theta_t + m_{cb} \cos^2 \theta_b \\
m_{xy} &= -m_{ct} \sin \theta_t \cos \theta_t - m_{cb} \sin \theta_b \cos \theta_b
\end{aligned}
\tag{16.30}
$$

Equations (16.27) to (16.29) correspond to the membrane forces, while equations (16.30) to (16.32) correspond to bending equations. If $\theta_t \neq 0, \pi/2$ and $\theta_b \neq 0, \pi/2$, (16.24), (16.26), (16.29) and (16.32) give

$$
\begin{aligned}
-n_{ct} &= \frac{(h - a_t) n_{xy} - 2m_{xy}}{h_c \sin 2\theta_t} \\
-n_{cb} &= \frac{(h - a_b) n_{xy} + 2m_{xy}}{h_c \sin 2\theta_b}
\end{aligned}
\tag{16.31}
$$

Reinforcement will be given upon solving (16.27) to (16.32). The objective is to calculate the forces in the reinforcement n_{sxt}, n_{syt}, n_{sxb} and n_{syb}. The other unknowns are a_t, a_b, θ_t and θ_b. Therefore the system of six equations contains eight unknowns. This means that the values of θ_t and θ_b should be chosen so that the total amount of reinforcement is minimized. The values of $\theta_t = \theta_b = \pi/4$ and $a_t = a_b = 0.2h$ can be assumed as an initial guess. Setting the values of θ to $\pi/4$ is obvious, as this value minimizes the total reinforcement in membrane elements. Setting $a=0.2h$ is a usual value for beam sections. The values are then adjusted by an iterative procedure until equilibrium is fulfilled. The reader is referred to […] for a full description of the iterative method.

Compressive crushing is checked by enforcing that $a_t + a_b \leq h$ and tensile reinforcement is calculated by Equ.(16.17-1), assuming yielding of reinforcement.

Case 2. Reinforcement in bottom layer only.

In case of biaxial compression in the top layer, reinforcement in the top layer is not needed. We indicate the concrete top layer membrane forces by n_{cxxt}, n_{cyyt} and n_{cxyt}. The forces and moments that the reinforcement resists in the x,y-directions are given by

$$\begin{aligned} n_{sx} &= n_{sxb} \\ n_{sy} &= n_{syb} \\ m_{sx} &= n_{sxb} h_{xb} \\ m_{sy} &= n_{syb} h_{yb} \end{aligned} \tag{16.32}$$

and by the concrete bottom layer are

$$\begin{aligned} n_{cb} &= -a_b f_c \\ m_{cb} &= \frac{1}{2}(h - a_b) n_{cb} \end{aligned} \tag{16.33}$$

Equilibrium with the applied set of forces and moments yields

$$\begin{aligned} n_{xx} &= n_{sx} + n_{cxt} + n_{cb} \sin^2 \theta_b \\ n_{yy} &= n_{sy} + n_{cyt} + n_{cb} \cos^2 \theta_b \\ n_{xy} &= n_{cxyt} - n_{cb} \sin \theta_b \cos \theta_b \end{aligned} \tag{16.34}$$

$$\begin{aligned} m_{xx} &= m_{sx} + m_{cxt} + m_{cb} \sin^2 \theta_b \\ m_{yy} &= m_{sy} + m_{cyt} + m_{cb} \cos^2 \theta_b \\ m_{xy} &= m_{cxyt} - m_{cb} \sin \theta_b \cos \theta_b \end{aligned} \tag{16.35}$$

with

$$m_{cxt} = -\frac{1}{2}(h - a_t)\, n_{cxt}$$
$$m_{cyt} = -\frac{1}{2}(h - a_t)\, n_{cyt} \tag{16.36}$$
$$m_{cxyt} = -\frac{1}{2}(h - a_t)\, n_{cxyt}$$

In the current case there are still eight unknowns. However, one extra equation must be added to the six equations of equilibrium, representing the biaxial state of stress in the concrete top layer

$$n_{ct} = -a_t\, f_c = \frac{n_{cxt} + n_{cyt}}{2} - \sqrt{\left(\frac{n_{cxt} + n_{cyt}}{2}\right)^2 + n_{cxyt}^2} \tag{16.37}$$

Here, f_c has a higher value than the uni-axial compressive strength of cylinders due to biaxial confinement. Nevertheless there are eight unknowns and seven equations, meaning that θ_b should be chosen so that the total amount of reinforcement is minimized.

Case 3. Reinforcement in top layer only

The case of biaxial compression in the bottom layer is identical to the case of biaxial compression in the top layer, with a rotation of indices. Therefore establishing the equilibrium equations requires no additional explanation.

Case 4. No reinforcement at all

Finally, in the case of biaxial compression in top and bottom layers, there is no need of reinforcement and the solution is unique. Assuming that the concrete top layer membrane forces are n_{cxt}, n_{cyt} and n_{cxyt} respectively in the x,y-direction and as shear force, and the concrete bottom layer membrane forces are n_{cxb}, n_{cyb} and n_{cxyb} with a similar meaning, the equilibrium equations might be written as

$$n_{xx} = n_{cxt} + n_{cxb}$$
$$n_{yy} = n_{cyt} + n_{cyb} \tag{16.38}$$
$$n_{xy} = n_{cxyt} + n_{cxyb}$$

$$\begin{aligned} m_{xx} &= m_{cxt} + m_{cxb} \\ m_{yy} &= m_{cyt} + m_{cyb} \\ m_{xy} &= m_{cxyt} + m_{cxyb} \end{aligned} \tag{16.39}$$

with

$$\begin{aligned} m_{cxt} &= -\frac{1}{2}\left(h - a_t\right) n_{cxt}, \quad & m_{cxb} &= \frac{1}{2}\left(h - a_b\right) n_{cxb} \\ m_{cyt} &= -\frac{1}{2}\left(h - a_t\right) n_{cyt}, \quad & m_{cyb} &= \frac{1}{2}\left(h - a_b\right) n_{cyb} \\ m_{cxyt} &= -\frac{1}{2}\left(h - a_t\right) n_{cxyt}, \quad & m_{cxyt} &= \frac{1}{2}\left(h - a_b\right) n_{cxyt} \end{aligned} \tag{16.40}$$

The principle concrete compression forces in each layer may be calculated according to

$$\begin{aligned} n_{c,t} &= \frac{n_{cxt} + n_{cyt}}{2} \pm \sqrt{\left(\frac{n_{cxt} - n_{cyt}}{2}\right)^2 + n_{cxyt}^2} \\ n_{c,b} &= \frac{n_{cxb} + n_{cyb}}{2} \pm \sqrt{\left(\frac{n_{cxb} - n_{cyb}}{2}\right)^2 + n_{cxyb}^2} \end{aligned} \tag{16.41}$$

and the layer thickness may be calculated according to the MC90 as

$$a_t = -\frac{n_{ct,\max}}{K f_{cd1}}, \qquad a_b = -\frac{n_{cb,\max}}{K f_{cd1}} \tag{16.42}$$

As shown above there are eight unknowns and eight equations (the six equilibrium equations and two equations to check the maximum compressive stress in the layers), meaning that the problem is determined.

16.5 Applications on element level

In this Section we illustrate the use of the basic and advanced model for two elements. The first one is subjected to a combination of a membrane force and bending moment. The second is a slab element subjected to a twisting moment.

16.5.1 Element with membrane force and bending moment

An element is subjected to an applied set of a bending moment and membrane shear force given by m_{xx} = 235 kNm/m and n_{xy} = 1806 kN/m. The material properties of concrete and steel are f_c = 41.8 MPa and f_{sy} = 492 MPa. The location of the reinforcement is given by $h_{xt} = h_{xb}$ = 0.122 m and $h_{yt} = h_{yb}$ = 0.100 m. This element is chosen because an experimental result of Kirsher and Collins is available.

The top row in Table 16.1 shows that in total 111.4 cm^2/m reinforcement is applied in the element in the test. Not all the reinforcement yielded at failure. The second row in the table is the prediction of the needed reinforcement basis of a non-linear analysis by an iterative computer program with optimization. This provided a minimum amount of reinforcement equal to 68.6 cm^2/m [...]. The third row presents the results of the basic sandwich model and the fourth row of the advanced model.

Table 16.1 – Reinforcement for single shell element

Method	Reinforcement areas (cm^2/m)				
	x – top	*y* – top	*x* – bottom	*y* – bottom	total
Experiment	41.8	13.9	41.8	13.9	111.4
Nonlinear	0.0	14.1	37.6	16.9	68.6
Basic mod.	0.0	15.7	39.9	18.4	74.0
Advanced	0.0	16.6	36.8	17.9	71.3

For the basic model an average distance of layer centres to the middle plane of the element of 0.111 m is chosen. Therefore d_v = 0.222 m. On basis of Equ. (16.12) to (16.14) we find the following

$$
\begin{aligned}
n_{xxb} &= \ \ 235/0.222 = \ \ 1059 \text{ kN/m} \\
n_{xxt} &= -235/0.222 = -1059 \text{ kN/m}
\end{aligned}
$$

$$
\begin{aligned}
n_{yyb} &= 0 \\
n_{yyt} &= 0
\end{aligned}
$$

$$
\begin{aligned}
n_{xyb} &= 1806/2 = 903 \text{ kN/m} \\
n_{xyt} &= 1806/2 = 903 \text{ kN/m}
\end{aligned}
$$

Using the expressions for membrane elements it is possible to obtain, for the top layer (x reinforcement not needed)

$$n_{sxt} = 0$$
$$n_{syt} = n_{yyt} - \frac{n_{xyt}^2}{n_{xxt}} = 0 - \frac{903^2}{-1059} = 770 \text{ kN/m}$$

$$n_{ct} = -1059 + \frac{903^2}{-1059} = -1829 \text{ kN/m}$$

$$a_{sxt} = 0$$
$$a_{syt} = \frac{n_{syt}}{f_{sy}} = \frac{770}{492 \times 10^3} \times 10^4 = 15.7 \text{ cm}^2\text{/m}$$

$$a_t = -\frac{n_{ct}}{f_c^{eff}} - \frac{-1829}{0.6 \times 41.8 \times 10^3} = 0.073 \text{ m}$$

Note that the value of the effective compressive strength was here assumed as $0.6f_c$. Similarly, for the bottom layer

$$n_{sxb} = n_{xxb} + \left|n_{xyb}\right| = 1059 + 903 = 1962 \text{ kN/m}$$
$$n_{syb} = n_{xyb} + \left|n_{xyb}\right| = 0 + 903 = 903 \text{ kN/m}$$
$$n_{cb} = -2\left|n_{xyb}\right| = -2 \times 903 = 1806 \text{ kN/m}$$

$$a_{sxb} = \frac{n_{sxb}}{f_{sy}} = \frac{1962}{492 \times 10^3} \times 10^4 = 39.9 \text{ cm}^2\text{/m}$$
$$a_{sxt} = \frac{n_{sxt}}{f_{sy}} = \frac{903}{492 \times 10^3} \times 10^4 = 18.4 \text{ cm}^2\text{/m}$$
$$a_b = -\frac{-1806}{0.6 \times 41.8 \times 10^3} = 0.072 \text{ m}$$

The advanced sandwich model requires five iterations, provides the thickness of the layers equal to 0.072m and 0.075m for the top and bottom respectively, and the reinforcement results as given in the third row of Table 16.1. It can be seen that the results are almost the same as the basic sandwich model. If the results of the nonlinear analysis are assumed as reference values, the basic sandwich model provides +8% and the advanced sandwich model provides +4% of the total reinforcement.

16.5.2 *Slab element with twisting moment*

A slab element is subjected to pure torsion by an applied twisting moment. The value of the twisting moment is one time chosen m_{xy} = 42.5 kNm/m and one time m_{xy} = 101.5 kNm/m. These values are chosen because results of a test are available. Marti […] obtained them for a lightly reinforced (0.25%) and a severely reinforced (1.0%) element, respectively. The material properties are f_c = 44.4 MPa, and f_{sy} = 479 MPa for the light reinforcement and f_{sy} = 412 MPa for the severe reinforcement. The location of the reinforcement is given by $h_{xt} = h_{xb}$ = 0.073 m and $h_{yt} = h_{yb}$ = 0.084 m, for the light one, and $h_{xt} = h_{xb}$ = 0.066 m and $h_{yt} = h_{yb}$ = 0.082 m, for the severe one. Column 'Experiment' in Table 16.2 provides the reinforcement existing in the element, which is the same in *x*- and *y*-direction, and in the top and bottom layer (5 cm^2/m for light and 20 cm^2/m for severe reinforcement).

The slab method on basis of the normal moment yield criterion violates equilibrium as different reinforcements are calculated for each direction. This is in agreement with the formulation, as different lever arms are found for each reinforcement direction, but equilibrium requires the forces in all reinforcements to be the same. The basic and advanced sandwich models fulfill equilibrium correctly.

The normal moment method provides a reasonable (conservative) value of reinforcement for the small twisting moment but an unacceptable, unsafe value of reinforcement for the large twisting moment. The reinforcement found is in *x*-direction about 27% less than the required value and 19% less in *y*-direction. The basic model is very safe for the small moment, but equally unsafe for the large moment. The prediction by the advanced sandwich model is exact for the small moment and only 3% too low for the large one, which is very satisfactory. The reason for the bad predictions is that the location of the resultant for the forces in the concrete are incorrectly calculated and the interaction between the different forces in reinforcement and concrete are neglected. Therefore, the equations on basis of the normal moment yield criterion and the basic sandwich model should be used with much precaution, or not used at all. Obtained results from these models must be distrusted if high reinforcement ratio's are obtained.

Table 16.2 – Reinforcement for slab elements due to pure torsion.

Test	Reinforcement areas (cm^2/m)

	Experi-ment	Normal mo-ment yield crite-rion	Basic sand-wich model	Advanced sandwich model
Small twisting mo-ment	5.0	A_{sx} = 5.0 A_{sy} = 5.3	5.6	5.0
Large twisting mo-ment	20.0	A_{sx} = 14.6 A_{sy} = 16.3	16.2	19.4

16.6 Applications on structural level

16.6.1 Deep beam

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…………………………………………………………………………………………
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16.6.2 Slab

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16.7 Message of the Chapter

- Reinforcement in a membrane state (wall) can be designed including the effect of shear forces. Four different cases must be considered, ranging from the need to apply reinforcement in two directions to no reitforcement at all.

- Design of reinforcement in a slab on basis of the normal moment yield criterion leads to simple, easy to apply formulas. However, no check on concrete crushing is at disposal and equilibrium is not satisfied.

- The three layer sandwich model for slabs and shell elements leads both to design of reinforcement, a check on concrete crushing, and includes a reinforcement design method for transverse shear forces.

- Two variants of the sandwich model are known, basic and advanced.. The basic model is easy to apply. However, it is an unsafe approximation when large twisting moments occur. The advanced model consistently accounts for the real geometry of the element and yields safe results.

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