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On Generalized Hypercomplex Laguerre-type Exponentials and Applications

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Abstract

In hypercomplex context, we have recently constructed Appell sequences with respect to a generalized Laguerre derivative operator. This construction is based on the use of a basic set of monogenic polynomials which is particularly easy to handle and can play an important role in applications. Here we consider Laguerre-type exponentials of order m and introduce Laguerre-type circular and hyperbolic functions.

1 Introduction

Hypercomplex function theory, renamed Clifford Analysis ([3]) in the 1980s, when it grew into an autonomous discipline, studies functions with values in a non-commutative Clifford Algebra. It has its roots in quaternionic analysis, developed mainly in the third decade of the 19th century ([12, 13]) as another generalization of the classical theory of functions of one complex variable compared with the theory of functions of several complex variables.

Curiously, but until the end of the 1990s the dominant opinion was that in Clifford Analysis only the generalization of Riemann's approach to holomorphic functions as solutions of the Cauchy-Riemann differential equations allows to define a class of generalized holomorphic functions of more than two real variables, suitable for applications in harmonic analysis, boundary value problems of partial differential equations and all the other classical fields where the theory of functions of one complex variable plays a prominent role. Logical, the methods employed during this period relied essentially on integral representations of those generalized holomorphic functions as consequence of the hypercomplex form of Stokes' integral formula, including series representations obtained by the development of the hypercomplex harmonic Cauchy kernel in series of Gegenbauer polynomials, for instance ([3]).

Only after clarifying the possibility of an adequate and equivalent concept of hypercomplex differentiability (thereby generalizing Cauchy's approach via complex differentiability to holomorphic functions) the systematical treatment of generalized holomorphic functions (also called *monogenic functions*) relying on the hypercomplex derivative allowed a more diversified and direct study of series representations by several hypercomplex variables and its applications, for example, for approximations of quasiconformal mappings ([14, 17, 19]).

But Clifford Analysis suffers from the drawback that the (pointwise) multiplication of monogenic functions as well as their composition are not algebraically closed in this class of functions. This causes serious problems for the use of corresponding formal power series, for the development of a suitable generating function approach to special monogenic polynomials, or for establishing relations to corresponding hypergeometric functions etc. It is also the reason why in the polynomial approximation in the context of Clifford Analysis almost every problem needs the development of different adapted polynomial bases (e.g. [1, 3, 4, 5, 11, 18]).

However, the analysis of all those possible different representations led in the past to a deeper understanding and the construction of a monogenic hypercomplex exponential function which plays the same central role in applications as the ordinary exponential function of a real or complex variable. Previous constructions of hypercomplex exponential functions and other special functions like, for instance, a monogenic Gaussian distribution function, are mainly relying on the Cauchy-Kovalevskaya extension principle or - with some restriction on the space dimension - the so-called Fueter-Sce mapping (see [2, 15, 22]). The latter connects holomorphic functions with solutions of bi-harmonic or higher order equations. The former is based on the analytic continuation of complex or, in general, Clifford Algebra valued functions of purely imaginary, respectively purely vectorial, arguments and therefore lacks the direct compatibility with the real or complex case.

The crucial idea for constructing a hypercomplex exponential function as shown in [11] (which stresses at the same time the central role of the hypercomplex derivative) was the construction of a monogenic hypercomplex exponential function as solution of an ordinary hypercomplex differential equation. The adequate multiple power series representations in connection with the concept of Appell sequences (c.f. [1, 4, 5, 11, 14, 17]) allowed, for instance, to develop new hypercomplex analytic tools for linking Clifford Analysis with operational approaches to special classes of monogenic hypercomplex polynomials or even to combinatorics.

After introducing in Section 2 the necessary notations from Clifford Analysis and the basic set of polynomials, Section 3 describes the operational approach to generalized Laguerre polynomials as well as to monogenic Laguerre-type exponentials, adapting the operational approach, developed by Dattoli, Ricci et al. ([6, 7, 8, 9]).

The study of corresponding Laguerre-circular and Laguerre-hyperbolic functions in the following Section 4 includes examples of their visualization and also their relationship with different types of Special Functions according to the dimension of the considered Euclidean space (see Table 5). The fact that the underlying Clifford Algebra is more general than the complex one allows a new approach to higher dimensions and reveals in our opinion new insides in the meaningful combination of different classes of Special Functions. Final remarks in Section 5 are illustrating our approach to the monogenic hypercomplex Gaussian distribution in comparison with another recently published and based on the Cauchy-Kovalevskaya extension.

2 Preliminary Results

2.1 Basic Notation

Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ with a non-commutative product according to the multiplication rules

$$e_k e_l + e_l e_k = -2\delta_{kl},\ k, l = 1, \cdots, n,$$

where δ_{kl} is the Kronecker symbol. The set $\{e_A : A \subseteq \{1, \cdots, n\}\}$ with

$$e_A = e_{h_1} e_{h_2} \cdots e_{h_r},\ 1 \leq h_1 < \cdots < h_r \leq n,\ e_\emptyset = e_0 = 1,$$

forms a basis of the 2^n-dimensional Clifford Algebra $\mathcal{C}\ell_{0,n}$ over $\mathbb{R}$. Let $\mathbb{R}^{n+1}$ be embedded in $\mathcal{C}\ell_{0,n}$ by identifying $(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}$ with the algebra's element $x = x_0 + \underline{x} \in \mathcal{A}_n := \mathrm{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset \mathcal{C}\ell_{0,n}$. Here $x_0 = \mathrm{Sc}(x)$ and $\underline{x} = \mathrm{Vec}(x) = e_1 x_1 + \cdots + e_n x_n$ are the so-called scalar resp. vector part of the paravector $x \in \mathcal{A}_n$. The conjugate of x is given by $\bar{x} = x_0 - \underline{x}$ and the norm $|x|$ of x is defined by

$|x|^2 = x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + \cdots + x_n^2$. Denoting by $\omega(x) = \frac{\underline{x}}{|\underline{x}|} \in S^n$, where S^n is the unit sphere in $\mathbb{R}^n$, each paravector x can be written as $x = x_0 + \omega(x)|\underline{x}|$.

We consider functions of the form $f(z) = \sum_A f_A(z)e_A$, where $f_A(z)$ are real valued, i.e. $\mathcal{C}\ell_{0,n}$-valued functions defined in some open subset $\Omega \subset \mathbb{R}^{n+1}$. The generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$, $n \geq 1$, is defined by

$$\overline{\partial} := \partial_0 + \partial_{\underline{x}}, \quad \partial_0 := \frac{\partial}{\partial x_0}, \quad \partial_{\underline{x}} := e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}.$$

C^1-functions f satisfying the equation $\overline{\partial} f = 0$ (resp. $f\overline{\partial} = 0$) are called *left monogenic* (resp. *right monogenic*). We suppose that f is hypercomplex differentiable in Ω in the sense of [14, 17], i.e. has a uniquely defined areolar derivative f' in each point of Ω (see also [19]). Then f is real differentiable and f' can be expressed by the real partial derivatives as $f' = \frac{1}{2}\partial$, where $\partial := \partial_0 - \partial_{\underline{x}}$ is the conjugate Cauchy-Riemann operator. Since a hypercomplex differentiable function belongs to the kernel of $\overline{\partial}$, it follows that in fact $f' = \partial_0 f$ like in the complex case.

2.2 A Basic Set of Polynomials

Following [4] we introduce now the main results concerning Appell sequences. Let U_1 and U_2 be (right) modules over $\mathcal{C}\ell_{0,n}$ and let $\hat{T} : U_1 \longrightarrow U_2$ be a hypercomplex (right) linear operator.

Definition 1. *A sequence of monogenic polynomials $(\mathcal{F}_k)_{k\geq 0}$ is called a $\hat{T}$-Appell sequence if $\hat{T}$ is a lowering operator with respect to the sequence, i.e., if*

$$\hat{T}\mathcal{F}_k = k\,\mathcal{F}_{k-1}, \ k = 1, 2, \ldots,$$

and $\hat{T}(1) = 0$.

Since the operator $\frac{1}{2}\partial$ defines the hypercomplex derivative of monogenic functions, the sequence of monogenic polynomials that is $\frac{1}{2}\partial$-Appell is the hypercomplex counterpart of the classical Appell sequence and it is simply called Appell sequence or Appell set.

Theorem 1 ([4], Theorem 1)**.** *A monogenic polynomial sequence $(\mathcal{F}_k)_{k\geq 0}$ is an Appell set if and only if it satisfies the binomial-type identity*

$$\mathcal{F}_k(x) = \mathcal{F}_k(x_0 + \underline{x}) = \sum_{s=0}^{k} \binom{k}{s} \mathcal{F}_{k-s}(\underline{x}) x_0^s, \ x \in \mathcal{A}_n. \tag{1}$$

In recent years, special hypercomplex Appell polynomials have been used by several authors and their main properties have been studied by different methods and with different objectives ([1, 5, 16]).

In this section, we consider a basic set of polynomials first introduced in [10] for $\mathcal{A}_2$-valued polynomials defined in 3-dimensional domains and later on generalized to higher dimensions in [11, 20]. The polynomials under consideration are functions of the form

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} T_s^k(n)\, x^{k-s}\, \bar{x}^s, \ n \geq 1 \tag{2}$$

where

$$T_s^k(n) = \frac{k!}{n_{(k)}} \frac{(\frac{n+1}{2})_{(k-s)} (\frac{n-1}{2})_{(s)}}{(k-s)!s!}, \tag{3}$$

and $a_{(r)}$ denotes the Pochhammer symbol, i.e. $a_{(r)} = \frac{\Gamma(a+r)}{\Gamma(a)}$, for any integer $r > 1$, and $a_{(0)} = 1$.

The case of the real variable $x = x_0$ (i.e. $\underline{x} = 0$) is formally included in the above definitions as the case $n = 0$ with

$$T_0^k(0) = 1 \quad \text{and} \quad T_s^k(0) = 0, \quad \text{for } 0 < s \leq k, \tag{4}$$

so that the polynomials (2) are defined in $\mathbb{R}^{n+1}$, for all $n \geq 0$.

The set $\{T_s^k(n),\ s = 0, \ldots, k\}$, resembles in a lot of aspects a set of non-symmetric generalized binomial coefficients. In fact, we can represent the first values of this set in the form of a triangular "matrix" (see Table 1).

Several intrinsic properties of this set can be obtained. For our purpose here, we highlight the following essential properties (see [10, 11, 20] and the references therein for details):

Table 1: The values of $T_s^k(n)$, for $k = 0, \dots, 3,\ s = 0, \dots, k$ and $n \geq 1$.

1			
$\frac{n+1}{2n}$	$\frac{n-1}{2n}$		
$\frac{n+3}{4n}$	$\frac{n-1}{2n}$	$\frac{n-1}{4n}$	
$\frac{(n+3)(n+5)}{8n(n+2)}$	$\frac{3(n-1)(n+3)}{8n(n+2)}$	$\frac{3(n-1)(n+1)}{8n(n+2)}$	$\frac{(n-1)(n+3)}{8n(n+2)}$

Property 1.

1. $(k-s)T_s^k(n) + (s+1)T_{s+1}^k(n) = kT_s^{k-1}(n)$, *for* $k \geq 1,\ s < k$.
2. $\sum_{s=0}^{k} T_s^k(n) = 1$.
3. *Denoting by* $c_k(n)$ *the alternating sum* $\sum_{s=0}^{k} (-1)^s T_s^k(n)$, *then for* $n \geq 1$ *and* $k = 1, 2, \dots,$

$$c_k(n) = \begin{cases} \frac{k!!(n-2)!!}{(n+k-1)!!}, & \text{if } k \text{ is odd} \\ c_{k-1}(n), & \text{if } k \text{ is even} \end{cases} \tag{5}$$

and $c_0(n) = 1$, *for* $n \geq 0$. *As usual, we define* $(-1)!! = 0!! = 1$.

It is clear, from the second property, that the polynomials $\mathcal{P}_k^n$ satisfy the normalization condition $\mathcal{P}_k^n(1) = 1$, for $k = 0, 1, \cdots$ and $n \geq 0$.

We can prove now directly the following fundamental result.

Theorem 2. *The sequence of polynomials* $\mathcal{P} := (\mathcal{P}_k^n)_{k \geq 0}$ *is an Appell sequence.*

Proof. If $n = 0$, from (2) and (4) we obtain $\mathcal{P}_k^0(x) = \mathcal{P}_k^0(x_0) = x_0^k$, which means that, in this case, $\mathcal{P}$ is the trivial classical Appell sequence. To obtain the hypercomplex counterpart we need to prove that

$$\tfrac{1}{2}\partial \mathcal{P}_k^n = k\mathcal{P}_{k-1}^n,\ k = 1, 2, \dots,\ n \geq 1. \tag{6}$$

Since $\mathcal{P}_k^n$ are monogenic polynomials, $\frac{1}{2}\partial \mathcal{P}_k^n = \partial_0 \mathcal{P}_k^n$. Furthermore

$$\begin{aligned} \partial_0 \mathcal{P}_k^n(x_0 + \underline{x}) &= \partial_0 \sum_{s=0}^{k} T_s^k(n)(x_0 + \underline{x})^{k-s}(x_0 - \underline{x})^s \\ &= \sum_{s=0}^{k-1} T_s^k(n)(k-s)(x_0 + \underline{x})^{k-s-1}(x_0 - \underline{x})^s \\ &\qquad + \sum_{s=1}^{k} T_s^k(n)(x_0 + \underline{x})^{k-s} s (x_0 - \underline{x})^{s-1} \\ &= \sum_{s=0}^{k-1} \left((k-s)T_s^k(n) + (s+1)T_{s+1}^k(n)\right)(x_0 + \underline{x})^{k-s-1}(x_0 - \underline{x})^s. \end{aligned}$$

Result (6) follows now at once from Property 1. $\square$

Table 2: The first polynomials of the Appell sequence $\mathcal{P}$

$$\mathcal{P}_0^n(x_0 + |\underline{x}|\boldsymbol{\omega}) = 1$$

$$\mathcal{P}_1^n(x_0 + |\underline{x}|\boldsymbol{\omega}) = x_0 + \tfrac{1}{n}|\underline{x}|\boldsymbol{\omega}$$

$$\mathcal{P}_2^n(x_0 + |\underline{x}|\boldsymbol{\omega}) = x_0^2 - \tfrac{1}{n}|\underline{x}|^2 + \tfrac{2}{n}x_0|\underline{x}|\boldsymbol{\omega}$$

$$\mathcal{P}_3^n(x_0 + |\underline{x}|\boldsymbol{\omega}) = x_0^3 - \tfrac{3}{n}x_0|\underline{x}|^2 + \tfrac{3}{n}\left(\tfrac{-1}{n+2}|\underline{x}|^3 + x_0^2|\underline{x}|\right)\boldsymbol{\omega}$$

Theorems 1 and 2 lead to the binomial-type formula

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k}\binom{k}{s} x_0^{k-s}\mathcal{P}_s^n(\underline{x}) = \sum_{s=0}^{k}\binom{k}{s} c_s(n)\, x_0^{k-s}\, \underline{x}^s, \tag{7}$$

which, in turn, can be used to derive immediately the following results:

Property 2.

1. $\mathcal{P}_k^n(x_0) = x_0^k$, *for all* $x_0 \in \mathbb{R}$.

2. $\mathcal{P}_k^n(\underline{x}) = c_k(n)\underline{x}^k$.

3. $\mathcal{P}_k^n(x) = \mathcal{P}_k^n(x_0 + \omega(x)|\underline{x}|) = u(x_0, |\underline{x}|) + \omega(x)v(x_0, |\underline{x}|)$, *where* u *and* v *are the real valued functions*

$$u(x_0, |\underline{x}|) = \sum_{s=0}^{[\frac{k}{2}]}\binom{k}{2s}(-1)^s c_{2s}(n)\, x_0^{k-2s}\, |\underline{x}|^{2s}$$

and

$$v(x_0, |\underline{x}|) = \sum_{s=0}^{[\frac{k-1}{2}]}\binom{k}{2s+1}(-1)^s c_{2s+1}(n)\, x_0^{k-2s-1}\, |\underline{x}|^{2s+1}.$$

The second result of Property 2 can be seen as the essential property which characterizes the difference to the complex case. Nevertheless, the polynomials $\mathcal{P}_k^1$ coincide, as expected, with the usual powers z^k, since we get from (5), $c_k(1) = 1$, for all k. Furthermore, observing that $\omega^2(x) = -1$, we can consider that $\boldsymbol{\omega} := \omega(x)$ behaves like the imaginary unit, which means that the last property gives a representation of $\mathcal{P}_k^n$ in terms of a scalar part and an "imaginary" part.

3 The Hypercomplex Laguerre Derivative Operator

3.1 Definition and Properties

The operational approach to the classical Laguerre polynomials as considered by Dattoli, Ricci and colaborators in a series of papers ([7, 8]) is based on the introduction of the so-called Laguerre derivative operator in the form $\frac{\partial}{\partial x}x\frac{\partial}{\partial x}$ acting on the set of $(\frac{x^k}{k!})_{k\geq 1}$. These authors have shown the role of the Laguerre derivative in the framework of the so-called *monomiality principle* and, in particular, its applications to Laguerre polynomials, Laguerre type exponentials, circular functions, Bessel functions, etc (see [6, 9, 21]).

In the recent paper [4], adapting the aforementioned operational approach, we have introduced the definition of hypercomplex Laguerre derivative operator

$${}_LD := \tfrac{1}{2}\partial\mathbb{E}, \tag{8}$$

based on a natural combination of the hypercomplex derivation operator $\frac{1}{2}\partial$ and the Euler operator

$$\mathbb{E} := \sum_{k=0}^{n} x_k \frac{\partial}{\partial x_k}.$$

This hypercomplex Laguerre derivative can be generalized by considering the operator

$$_{mL}D := \tfrac{1}{2}\partial\mathbb{E}^m, \;\; m \in \mathbb{N}. \tag{9}$$

It is possible to construct a $_{mL}D$-Appell sequence, based on the Appell property of the polynomials $\mathcal{P}_k^n$ introduced in (2) as next result confirms.

Theorem 3. *Let* $_mQ_k^n, \;\; m = 0, 1, \dots$ *denote the monogenic polynomials*

$$_mQ_k^n(x) := \frac{\mathcal{P}_k^n(x)}{(k!)^m}, \;\; k = 0, 1, \dots \tag{10}$$

For each $m \in \mathbb{N}$*, the sequence* $_m\mathcal{Q} := ({}_mQ_k^n)_{k\geq 0}$ *is a* $_{mL}D$*-Appell sequence.*

Proof. It is sufficient to prove the theorem for $m > 0$, since the $m = 0$ case reduces to the result (6) of Theorem 2. Using the fact that $(\mathcal{P}_k^n)_k$ is a sequence of homogeneous monogenic polynomials, we conclude from Euler's formula for homogeneous polynomials that

$$\mathbb{E}\left(\mathcal{P}_k^n\right) = k\mathcal{P}_k^n.$$

Thus $\mathbb{E}^m(\mathcal{P}_k^n) = k^m\mathcal{P}_k^n$ and

$$\mathbb{E}^m\left({}_mQ_k^n\right) = \mathbb{E}^m\left(\frac{\mathcal{P}_k^n}{(k!)^m}\right) = \left(\frac{k}{k!}\right)^m \mathcal{P}_k^n = \frac{\mathcal{P}_k^n}{((k-1)!)^m}.$$

Applying now the operator $\frac{1}{2}\partial$ and using the fact that $\mathcal{P}_k^n$ are $\frac{1}{2}\partial$-Appell, we obtain

$$\tfrac{1}{2}\partial\mathbb{E}^m({}_mQ_k^n) = \tfrac{1}{2}\partial\left(\frac{\mathcal{P}_k^n}{((k-1)!)^m}\right) = k\frac{\mathcal{P}_{k-1}^n}{((k-1)!)^m} = k\;{}_mQ_{k-1}^n.$$

□

3.2 Monogenic Laguerre-type Exponentials

We recall that we can define formally a monogenic exponential function associated with a $\hat{T}$-Appell sequence. In fact, if $\mathcal{F} := (\mathcal{F}_k)_{k\geq 0}$ is a sequence of homogeneous monogenic polynomials, the corresponding exponential function can be defined by

$$_{\hat{T}}\mathrm{Exp}_{\mathcal{F}}(x) := \sum_{k=0}^{\infty} \frac{\mathcal{F}_k(x)}{k!}. \tag{11}$$

The function $_{\hat{T}}\mathrm{Exp}_{\mathcal{F}}$ is an eigenfunction of the hypercomplex operator $\hat{T}$, i.e., it holds

$$\hat{T}({}_{\hat{T}}\mathrm{Exp}_{\mathcal{F}}(\lambda x)) = \lambda\;{}_{\hat{T}}\mathrm{Exp}_{\mathcal{F}}(\lambda x), \;\; \lambda \in \mathbb{R}.$$

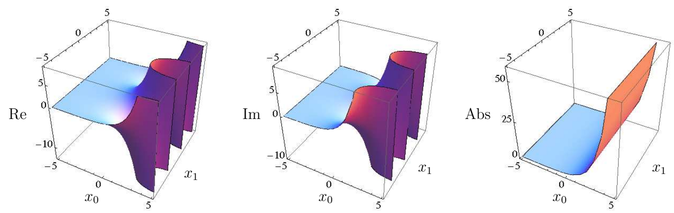
In other words, (11) constitutes a generalization of the classical exponential function. Considering, for example, the $_LD$-Appell sequence $_1\mathcal{Q}$ of monogenic polynomials $_1Q_k^n = \frac{\mathcal{P}_k^n}{k!}$, the corresponding exponential function is

$$_{LD}\mathrm{Exp}_{_1\mathcal{Q}}(x) := \sum_{k=0}^{\infty} \frac{{}_1Q_k^n(x)}{k!} = \sum_{k=0}^{\infty} \frac{\mathcal{P}_k^n(x)}{(k!)^2}. \tag{12}$$

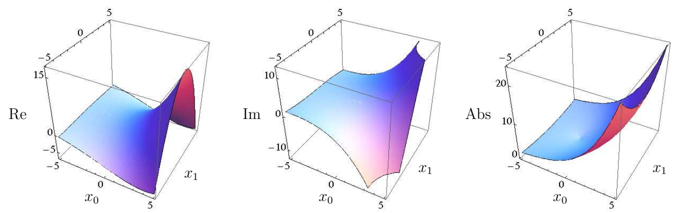
This Laguerre-type exponential (or L-exponential) can be generalized to an arbitrary m-th Laguerre-type exponential (or mL-exponential) as

$$_{mL}\mathrm{Exp}_n(x) :={}_{mLD}\mathrm{Exp}_{_m\mathcal{Q}}(x) = \sum_{k=0}^{\infty} \frac{\mathcal{P}_k^n(x)}{(k!)^{m+1}}, \tag{13}$$

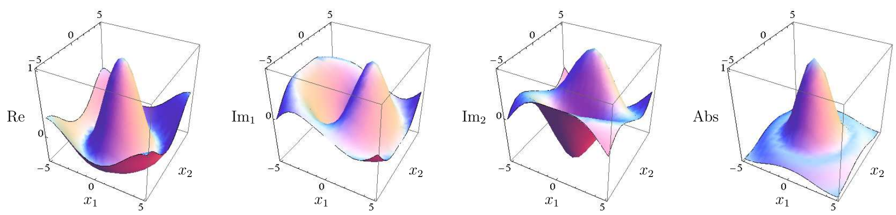
Table 3: Laguerre-type exponentials - examples



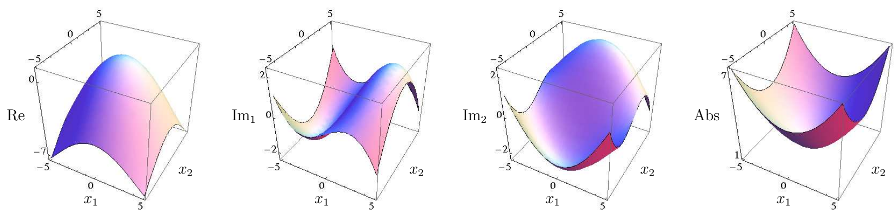
$_{0L}\mathrm{Exp}_1(x_0 + x_1 e_1)$



$_{1L}\mathrm{Exp}_1(x_0 + x_1 e_1)$



$_{0L}\mathrm{Exp}_2(x_1 e_1 + x_2 e_2)$



$_{1L}\mathrm{Exp}_2(x_1 e_1 + x_2 e_2)$

which is an eigenfunction of the operator (9).

Table 3 illustrates the differences from the classical complex case (the first row), for several mL-exponential functions.

Remark 1. *The $0L$-exponential coincides with the exponential function defined in [10, 11, 20] while the function ${}_{mL}\mathrm{Exp}_0$, corresponding to the real case, gives the m-th Laguerre-type exponential presented by Dattoli and Ricci in [9].*

Remark 2. *In [4], we have defined monogenic Laguerre polynomials in $\mathbb{R}^{n+1}$, which were obtained by applying the exponential operator $\mathrm{Exp}_n(-{}_LD)$ to the sequence $({}_1Q_k^n(-x))_{k\geq 0}$.*

4 L-circular and L-hyperbolic Functions

4.1 Definition

Recalling the definition of the ${}_{mL}\mathrm{Exp}_n$-function (13) and following [9] we can go further and define, in a natural way, the corresponding m-th Laguerre circular (or mL-circular) and m-th Laguerre hyperbolic (or mL-hyperbolic) functions. Hence

$$ {}_{mL}\mathrm{Cos}_n(x) := \sum_{k=0}^{\infty} \frac{(-1)^k \mathcal{P}_{2k}^n(x)}{((2k)!)^{m+1}} \quad \text{and} \quad {}_{mL}\mathrm{Sin}_n(x) := \sum_{k=0}^{\infty} \frac{(-1)^k \mathcal{P}_{2k+1}^n(x)}{((2k+1)!)^{m+1}}, \tag{14} $$

$$ {}_{mL}\mathrm{Cosh}_n(x) := \sum_{k=0}^{\infty} \frac{\mathcal{P}_{2k}^n(x)}{((2k)!)^{m+1}} \quad \text{and} \quad {}_{mL}\mathrm{Sinh}_n(x) := \sum_{k=0}^{\infty} \frac{\mathcal{P}_{2k+1}^n(x)}{((2k+1)!)^{m+1}}. \tag{15} $$

Examples of the above circular functions are given in Table 4.

4.2 Properties

We recall from last section that

$$ {}_{mL}D(\mathcal{P}_k^n) = k^{m+1}\mathcal{P}_{k-1}^n $$

and therefore

$$ {}_{mL}D^2(\mathcal{P}_k^n) = k^{m+1}(k-1)^{m+1}\mathcal{P}_{k-2}^n. $$

These two results together with (14) and (15) can be used to derive the following properties.

Property 3.

1. *The m-th circular functions (14) are solutions of the differential equation*

$$ {}_{mL}D^2 v + v = 0 $$

and satisfy the conditions

$$ {}_{mL}\mathrm{Cos}_n(0) = 1 \quad \textit{and} \quad {}_{mL}\mathrm{Sin}_n(0) = 0. $$

2. ${}_{mL}D\, {}_{mL}\mathrm{Cos}_n(x) = -{}_{mL}\mathrm{Sin}_n(x)$ *and* ${}_{mL}D\, {}_{mL}\mathrm{Sin}_n(x) = {}_{mL}\mathrm{Cos}_n(x)$.

Property 4.

1. *The m-th hyperbolic functions (15) are solutions of the differential equation*

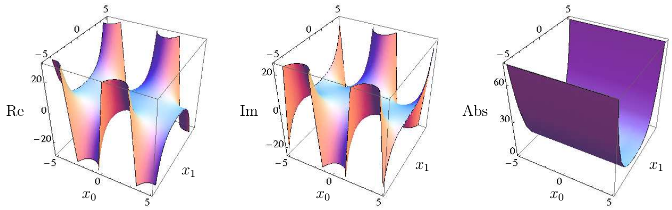
$$ {}_{mL}D^2 v - v = 0 $$

and satisfy the conditions

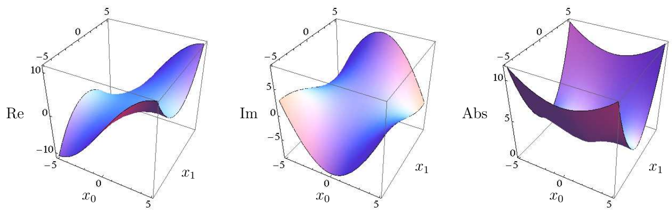
$$ {}_{mL}\mathrm{Cosh}_n(0) = 1 \quad \textit{and} \quad {}_{mL}\mathrm{Sinh}_n(0) = 0. $$

2. ${}_{mL}D\, {}_{mL}\mathrm{Cosh}_n(x) = {}_{mL}\mathrm{Sinh}_n(x)$ *and* ${}_{mL}D\, {}_{mL}\mathrm{Sinh}_n(x) = {}_{mL}\mathrm{Cosh}_n(x)$.

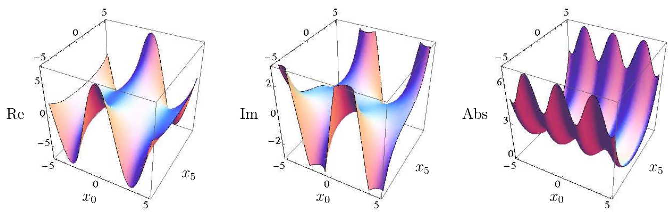
Table 4: L-circular functions - examples



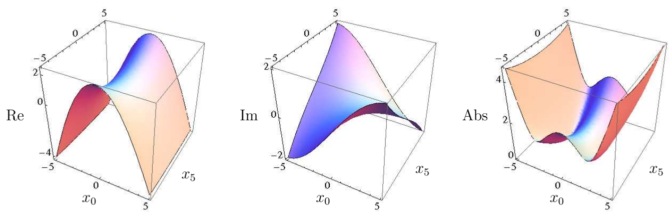
${}_{0L}\mathrm{Sin}_1(x_0 + x_1 e_1)$



${}_{1L}\mathrm{Sin}_1(x_0 + x_1 e_1)$



${}_{0L}\mathrm{Cos}_5(x_0 + x_5 e_5)$



${}_{1L}\mathrm{Cos}_5(x_0 + x_5 e_5)$

Table 5: Some examples of L-exponential, L-circular and L-hyperbolic functions

		$m=0$	$m=1$
${}_{mL}\mathrm{Exp}_n(\underline{x})$	$n=1$	$\cos(\lvert\underline{x}\rvert)+\boldsymbol{\omega}\sin(\lvert\underline{x}\rvert)$	$\mathrm{Ber}_0(2\sqrt{\lvert x_1\rvert})+\boldsymbol{\omega}\,\mathrm{Bei}_0(2\sqrt{\lvert x_1\rvert})$
	$n=2$	$J_0(\lvert\underline{x}\rvert)+\boldsymbol{\omega}J_1(\lvert\underline{x}\rvert)$	$I_0(\sqrt{2\lvert\underline{x}\rvert})J_0(\sqrt{2\lvert\underline{x}\rvert})+\boldsymbol{\omega}I_1(\sqrt{2\lvert\underline{x}\rvert})J_1(\sqrt{2\lvert\underline{x}\rvert})$
	$n>2$	$(\frac{2}{\lvert\underline{x}\rvert})^{\frac{1-n}{2}-1}\Gamma(\frac{n}{2})\left(J_{\frac{n}{2}-1}(\lvert\underline{x}\rvert)+\boldsymbol{\omega}J_{\frac{n}{2}}(\lvert\underline{x}\rvert)\right)$	${}_0\mathrm{F}_3\left(\ ;\frac{1}{2},1,\frac{n}{2};-\frac{\lvert\underline{x}\rvert^2}{16}\right)+\boldsymbol{\omega}\frac{\lvert\underline{x}\rvert}{n}\,{}_0\mathrm{F}_3\left(\ ;1,\frac{3}{2},\frac{n}{2}+1;-\frac{\lvert\underline{x}\rvert^2}{16}\right)$
${}_{mL}\mathrm{Cos}_n(\underline{x})$	$n=1$	$\cosh(\lvert\underline{x}\rvert)$	$\frac{1}{2}\left(J_0(\sqrt{2\lvert\underline{x}\rvert})+I_0(\sqrt{2\lvert\underline{x}\rvert})\right)$
	$n=2$	$I_0(\lvert\underline{x}\rvert)$	$\frac{1}{2\sqrt{\lvert\underline{x}\rvert}}\left(J_1(\sqrt{2\lvert\underline{x}\rvert})+I_1(\sqrt{2\lvert\underline{x}\rvert})\right)$
	$n>2$	$(\frac{2}{\lvert\underline{x}\rvert})^{\frac{n}{2}-1}\Gamma(\frac{n}{2})I_{\frac{n-2}{2}}(\lvert\underline{x}\rvert)$	$\frac{1}{2}\lvert\underline{x}\rvert^{\frac{1-n}{2}}\Gamma(n)\left(J_{n-1}(2\sqrt{\lvert\underline{x}\rvert})+I_{n-1}(2\sqrt{\lvert\underline{x}\rvert})\right)$
${}_{mL}\mathrm{Sin}_n(\underline{x})$	$n=1$	$\boldsymbol{\omega}\sinh(\lvert\underline{x}\rvert)$	$\boldsymbol{\omega}\frac{1}{2}\left(I_0(\sqrt{2\lvert\underline{x}\rvert})-J_0(\sqrt{2\lvert\underline{x}\rvert})\right)$
	$n=2$	$\boldsymbol{\omega}I_1(\lvert\underline{x}\rvert)$	$\boldsymbol{\omega}\left(\mathrm{Ber}_1(2\lvert\underline{x}\rvert)+\mathrm{Bei}_1(2\lvert\underline{x}\rvert)\right)$
	$n>2$	$\boldsymbol{\omega}(\frac{2}{\lvert\underline{x}\rvert})^{\frac{n}{2}-1}\Gamma(\frac{n}{2})I_{\frac{n}{2}}(\lvert\underline{x}\rvert)$	$\boldsymbol{\omega}\frac{\lvert\underline{x}\rvert}{n}{}_0\mathrm{F}_3\left(\ ;1,\frac{3}{2},\frac{n}{2}+1;-\frac{\lvert\underline{x}\rvert^2}{16}\right)$
${}_{mL}\mathrm{Cosh}_n(\underline{x})$	$n=1$	$\cos(\lvert\underline{x}\rvert)$	$\mathrm{Ber}_0(2\sqrt{\lvert x_1\rvert})$
	$n=2$	$J_0(\lvert\underline{x}\rvert)$	$I_0(\sqrt{2\lvert\underline{x}\rvert})J_0(\sqrt{2\lvert\underline{x}\rvert})$
	$n>2$	$(\frac{2}{\lvert\underline{x}\rvert})^{\frac{1-n}{2}-1}\Gamma(\frac{n}{2})J_{\frac{n}{2}-1}(\lvert\underline{x}\rvert)$	${}_0\mathrm{F}_3\left(\ ;\frac{1}{2},1,\frac{n}{2};-\frac{\lvert\underline{x}\rvert^2}{16}\right)$
${}_{mL}\mathrm{Sinh}_n(\underline{x})$	$n=1$	$\boldsymbol{\omega}\sin(\lvert\underline{x}\rvert)$	$\boldsymbol{\omega}\,\mathrm{Bei}_0(2\sqrt{\lvert x_1\rvert})$
	$n=2$	$\boldsymbol{\omega}J_1(\lvert\underline{x}\rvert)$	$\boldsymbol{\omega}I_1(\sqrt{2\lvert\underline{x}\rvert})J_1(\sqrt{2\lvert\underline{x}\rvert})$
	$n>2$	$\boldsymbol{\omega}(\frac{2}{\lvert\underline{x}\rvert})^{\frac{1-n}{2}-1}\Gamma(\frac{n}{2})J_{\frac{n}{2}}(\lvert\underline{x}\rvert)$	$\boldsymbol{\omega}\frac{\lvert\underline{x}\rvert}{n}\,{}_0\mathrm{F}_3\left(\ ;1,\frac{3}{2},\frac{n}{2}+1;-\frac{\lvert\underline{x}\rvert^2}{16}\right)$

When $x = \underline{x}$, i.e. $x_0 = 0$, it is possible to express Laguerre-type exponentials and related circular and hyperbolic functions, in terms of special functions, for particular values of m and n. Table 5 illustrates this situation.

Clearly,

$$_{mL}\mathrm{Exp}_n(\underline{x}) = {}_{mL}\mathrm{Cosh}_n(\underline{x}) + {}_{mL}\mathrm{Sinh}_n(\underline{x})$$

and

$$_{mL}\mathrm{Exp}_n(-\underline{x}) = {}_{mL}\mathrm{Cosh}_n(\underline{x}) - {}_{mL}\mathrm{Sinh}_n(\underline{x}).$$

Therefore we obtain the Euler-type formulas

$$_{mL}\mathrm{Cosh}_n(\underline{x}) = \frac{_{mL}\mathrm{Exp}_n(\underline{x}) + {}_{mL}\mathrm{Exp}_n(-\underline{x})}{2}$$

and

$$_{mL}\mathrm{Sinh}_n(\underline{x}) = \frac{_{mL}\mathrm{Exp}_n(\underline{x}) - {}_{mL}\mathrm{Exp}_n(-\underline{x})}{2}.$$

On the other hand, things are quite different with respect to the mL-circular functions (14). In fact, due to Property 2.2, we can state

$$_{mL}\mathrm{Exp}_n(\underline{x}) = {}_{mL}\mathrm{Cos}_n(\underline{x}) + \boldsymbol{\omega}\, {}_{mL}\mathrm{Sin}_n(\underline{x}),$$

just for the obvious case of $n = 1$.

5 Final Remarks

In Clifford Analysis, several different methods have been developed for constructing monogenic functions as series with respect to properly chosen homogeneous monogenic polynomials (see [1, 3, 4, 5, 11, 14, 17, 18]).

Behind the use of Appell polynomials in [5, 11, 20] is also the idea that, through the construction of a set of special polynomials with differential properties like x^n, also an easy to handle series representation should be obtained.

The same technique can be applied to obtain other generalized monogenic functions in higher dimensions. For example, mL-Gaussian functions in $\mathbb{R}^{n+1}$ can be obtained through the expansion

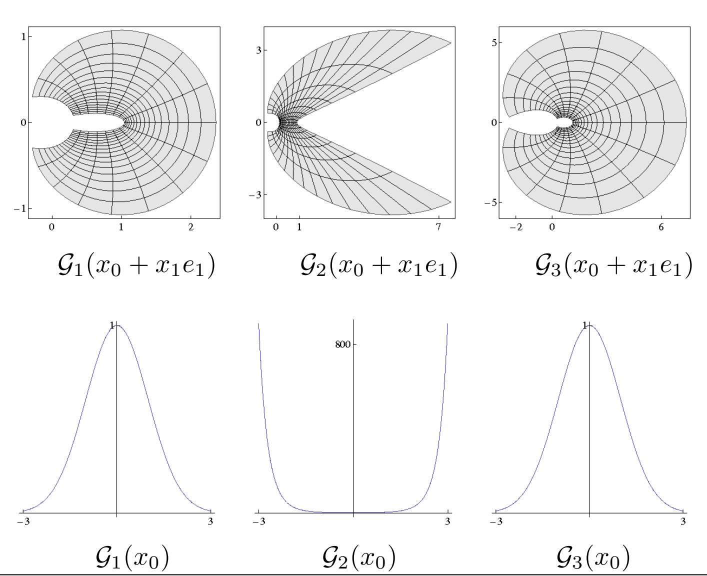
$$_{mL}G_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \mathcal{P}_{2k}^n(x)}{(k!)^{m+1}};\ m = 0, 1, \ldots. \tag{16}$$

Recently, a monogenic Gaussian distribution in closed form has been presented in [22] by using two well known techniques to generate monogenic functions: (i) the Cauchy-Kovalevskaya extension (see e.g. [3]) which consists of constructing a monogenic function in $\mathbb{R}^{n+1}$, starting from an analytic function in $\mathbb{R}^n$; (ii) Fueter's theorem (see [12, 13]) which can be used to generate a monogenic function in $\mathbb{R}^{n+1}$ by means of a holomorphic complex function. For the particular case $n = 3$, the authors obtained the following monogenic Gaussian distribution

$$G(x) = \begin{cases} \exp\left(\frac{x_0^2 - |\underline{x}|^2}{2}\right)\left(\cos(x_0|\underline{x}|) + \frac{x_0}{|\underline{x}|}\sin(x_0|\underline{x}|) + \right. \\ \qquad \left. \boldsymbol{\omega}\left(\sin(x_0|\underline{x}|) + \frac{\sin(x_0|\underline{x}|)}{|\underline{x}|^2} - \frac{x_0}{|\underline{x}|}\cos(x_0|\underline{x}|)\right)\right), & \text{for } \underline{x} \neq 0, \\ \\ \exp\left(\frac{x_0^2}{2}\right)(1 + x_0^2), & \text{for } \underline{x} = 0. \end{cases} \tag{17}$$

In Table 6, we first compare the classical complex Gaussian function ($\mathcal{G}_1$), the restriction $\mathcal{G}_2$ of the 4D-Gaussian function (17) and an approximation $\mathcal{G}_3$ to $_{0L}G_3$, obtained from (16), by computing the images of the rectangle $[-1.5, 1.5] \times [0.5, 2]$ in the complex plane, under the aforementioned Gaussian functions. Next, we include plots for the correspondent real Gaussian functions. This example reveals, once more, that a direct compatibility with the real and complex case can be obtained through the use of series expansions with respect to the polynomials $\mathcal{P}_k^n$. On the other hand, for (17) this compatibility is not visible.

Table 6: Gaussian functions



5.0.1 Acknowledgments

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