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Laguerre derivative and monogenic Laguerre polynomials: an operational approach

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Abstract

Hypercomplex function theory generalizes the theory of holomorphic functions of one complex variable by using Clifford Algebras and provides the fundamentals of Clifford Analysis as a refinement of Harmonic Analysis in higher dimensions. We define the Laguerre derivative operator in hypercomplex context and by using operational techniques we construct generalized hypercomplex monogenic Laguerre polynomials. Moreover, Laguerre-type exponentials of order m are defined.

1 Introduction

In the theoretical background of any differential, integral or functional equation Special Functions are omnipresent. The use of symbolic algebraic calculation simplifies the detection of interesting unknown properties and implies that the amount of papers written about Special Functions and related subjects is enormously increasing. From this general point of view it seems also natural to ask for the role and the application of Special Functions in Clifford Analysis. Since the theory of *monogenic* or *Clifford holomorphic* functions (see [1, 2]) has its origin in complex function theory one can expect that almost everything from Special Function theory should have its counterpart in Clifford Analysis and that therefore a deeper study would not be interesting. A first superficial look may confirm those positions, but two arguments exist that are in our opinion essential for a different judgement about the usefulness of studies on Special Functions in Clifford Analysis. First, it seems that the possible contribution to a different dealing with functions in several real variables could enrich the multidimensional theory of Special Functions. Second, the use of the non-commutative Clifford Algebra promises results which cannot be obtained in the usual multidimensional commutative setting.

After introducing very briefly the necessary notations in Section 2 we prepare from Section 3 to Section 6 the operational approach to generalized Laguerre polynomials as well as to Laguerre-type exponentials based on our results from [3] and [4] about a special Appell sequence of polynomials and its connection with Bessel functions.

The special sequence of monogenic Appell type polynomials which we use in this framework stresses the central role of the hypercomplex derivative in questions of this type. This fact has been confirmed by several authors, who meanwhile started to work with Appell sets in the context of Clifford Analysis also in connection with other problems ([5, 6, 7]).

Our interest in Appell sequences in the Clifford setting has been motivated by the search for an exponential function due to the connection with a corresponding generating exponential function. But the idea to ask for the existence of a generalized holomorphic exponential function was from the beginning on in Clifford Analysis a principal question and reflects the main differences with the complex case ([2, 8, 9, 10]).

The definition of the hypercomplex Laguerre derivative in Section 6 and its application for obtaining monogenic Laguerre polynomials in Section 7 stresses once more the usefulness of the concept of a hypercomplex derivative in connection with the adaptation of the operational approach, developed by Dattoli, Ricci et al. ([11, 12, 13, 14]). Consequently, it leads us in Section 8 directly to the definition of a monogenic Laguerre-type exponential, including its relationship with different types of Special Functions according to the dimension of the considered Euclidean space (Theorem 7).

2 Basic notation

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ with a non-commutative product according to the multiplication rules

$$e_k e_l + e_l e_k = -2\delta_{kl}, \ k, l = 1, \ldots, n,$$

where δ_{kl} is the Kronecker symbol. The set $\{e_A : A \subseteq \{1, \ldots, n\}\}$ with

$$e_A = e_{h_1} e_{h_2} \ldots e_{h_r}, \ 1 \leq h_1 < \cdots < h_r \leq n, \ e_\emptyset = e_0 = 1,$$

forms a basis of the 2^n-dimensional Clifford Algebra $\mathcal{C}\ell_{0,n}$ over $\mathbb{R}$. Let $\mathbb{R}^{n+1}$ be embedded in $\mathcal{C}\ell_{0,n}$ by identifying $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ with $x = x_0 + \underline{x} \in \mathcal{A} := \text{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset \mathcal{C}\ell_{0,n}$. Here $x_0 = \text{Sc}(x)$ and $\underline{x} = \text{Vec}(x) = e_1 x_1 + \cdots + e_n x_n$ are the so-called scalar resp. vector parts of the paravector $x \in \mathcal{A}$. The conjugate of x is given by $\bar{x} = x_0 - \underline{x}$ and the norm $|x|$ of x is defined by $|x|^2 = x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + \cdots + x_n^2$.

We consider functions of the form $f(z) = \sum_A f_A(z) e_A$, where $f_A(z)$ are real valued, i.e. $\mathcal{C}\ell_{0,n}$-valued functions defined in some open subset $\Omega \subset \mathbb{R}^{n+1}$. The generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$, $n \geq 1$, is defined by

$$\overline{\partial} := \partial_0 + \partial_{\underline{x}}, \quad \partial_0 := \frac{\partial}{\partial x_0}, \quad \partial_{\underline{x}} := e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}.$$

C^1-functions f satisfying the equation $\overline{\partial} f = 0$ (resp. $f \overline{\partial} = 0$) are called *left monogenic* (resp.*right monogenic*). We suppose that f is hypercomplex differentiable in Ω in the sense of [15, 16], i.e. has a uniquely defined areolar derivative f' in each point of Ω (see also [17]). Then f is real differentiable and f' can be expressed by the real partial derivatives as $f' = \frac{1}{2}\partial$, where $\partial := \partial_0 - \partial_{\underline{x}}$ is the conjugate Cauchy-Riemann operator. Since a hypercomplex differentiable function belongs to the kernel of $\overline{\partial}$, it follows that in fact $f' = \partial_0 f$, like in the complex case.

3 Monogenic Appell sequences with respect to hypercomplex linear operators

Let U_1 and U_2 be (right) modules over $\mathcal{C}\ell_{0,n}$ and let $\hat{T} : U_1 \longrightarrow U_2$ be a hypercomplex (right) linear operator.

Definition 1. *A sequence of monogenic polynomials $(\mathcal{F}_k)_{k\geq 0}$ is called a $\hat{T}$-Appell sequence if $\hat{T}$ is a lowering operator with respect to the sequence, i.e., if*

$$\hat{T}\mathcal{F}_k = k\,\mathcal{F}_{k-1}, \ k = 1, 2, \ldots,$$

and $\hat{T}(1) = 0$.

Taking into account that the operator $\frac{1}{2}\partial$ defines the hypercomplex derivative of monogenic functions, the sequence of monogenic polynomials that is $\frac{1}{2}\partial$-Appell is the hypercomplex counterpart of the classical Appell sequence and it is simply called Appell sequence or Appell set.

Theorem 1. *A monogenic polynomial sequence $(\mathcal{F}_k)_{k\geq 0}$ is an Appell set if and only if it satisfies the binomial-type identity*

$$\mathcal{F}_k(x) = \mathcal{F}_k(x_0 + \underline{x}) = \sum_{s=0}^{k} \binom{k}{s} \mathcal{F}_{k-s}(\underline{x}) x_0^s. \tag{1}$$

Proof. Let $(\mathcal{F}_k)_{k\geq 0}$ be an Appell sequence of monogenic polynomials. Then, for any $x \in \mathcal{A}$, we have

$$\left(\frac{1}{2}\partial\right)^s \mathcal{F}_k(x) = \frac{k!}{(k-s)!}\mathcal{F}_{k-s}(x), \; s \leq k. \tag{2}$$

In the paravector variable $x = x_0 + \underline{x}$, let $\underline{x}$ be fixed and consider the real part x_0 varying in $\mathbb{R}$ such that $G_k(x_0) := \mathcal{F}_k(x_0 + \underline{x})$ is a polynomial of degree k in x_0. From the Taylor expansion of G, we obtain

$$G_k(x_0) = \sum_{s=0}^{k} \frac{G_k^{(s)}(0)}{s!} x_0^s. \tag{3}$$

Due to (2) and the monogeneity of the polynomials $\mathcal{F}_k$ $(k = 0, 1, 2, \dots)$, the s-th derivative of G_k, for $s \leq k$, can be written as

$$G_k^{(s)}(x_0) = \partial_0^s \mathcal{F}_k(x_0 + \underline{x}) = (\frac{1}{2}\partial)^s \mathcal{F}_k(x_0 + \underline{x}) = \frac{k!}{(k-s)!}\mathcal{F}_{k-s}(x_0 + \underline{x}). \tag{4}$$

The final result (1) follows now at once, by considering $x_0 = 0$ in (4) and using (3). Conversely, the hypercomplex derivative of (1) provides

$$\begin{aligned}\frac{1}{2}\partial \mathcal{F}_k(x) &= \partial_0 \mathcal{F}_k(x_0 + \underline{x}) = \sum_{s=1}^{k} \binom{k}{s} s \mathcal{F}_{k-s}(\underline{x}) x_0^{s-1} \\ &= k \sum_{s=0}^{k-1} \binom{k-1}{s} \mathcal{F}_{k-1-s}(\underline{x}) x_0^s = k\mathcal{F}_{k-1}(x_0 + \underline{x}) = k\mathcal{F}_{k-1}(x).\end{aligned}$$

□

Remark 1. *The Cauchy-Kovalevskaya extension (CK) (see e.g. [1]) is an important technique to construct a monogenic function F in $\mathbb{R}^{n+1}$, starting from an analytic function f in $\mathbb{R}^n$. This extension is given by*

$$F(x) := \mathit{CK}(f)(\underline{x}) = \sum_{j=0}^{\infty} (-1)^j \frac{x_0^j}{j!} \left(\partial_{\underline{x}}\right)^j f(\underline{x}).$$

For monogenic Appell sets, the Cauchy-Kovalevskaya extension coincides with the binomial-type identity. In fact, if the sequence $(\mathcal{F}_k)_{k\geq 0}$ is monogenic and Appell, we have for each $k \geq 1$, $-\partial_{\underline{x}}\mathcal{F}_k(x) = \partial_0 \mathcal{F}_k(x) = k\mathcal{F}_{k-1}(x)$, which implies that

$$(-1)^j \partial_{\underline{x}}^j \mathcal{F}_k(\underline{x}) = \frac{k!}{(k-j)!}\mathcal{F}_{k-j}(\underline{x}), \; j \leq k$$

and we get

$$\mathit{CK}(\mathcal{F}_k)(\underline{x}) = \sum_{j=0}^{\infty} (-1)^j \frac{x_0^j}{j!} \left(\partial_{\underline{x}}\right)^j \mathcal{F}_k(\underline{x}) = \sum_{j=0}^{k} \binom{k}{j} x_0^j \mathcal{F}_{k-j}(\underline{x}).$$

4 Appell sequence in terms of a paravector variable and its conjugate

In this section, we consider a special set of monogenic basis functions defined and studied to some extend with respect to its algebraic properties in [3, 4, 18], namely functions of the form[1]

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} T_s^k(n)\, x^{k-s}\, \bar{x}^s, \tag{5}$$

where

$$T_s^k(n) = \frac{k!}{(n)_k} \frac{(\frac{n+1}{2})_{(k-s)}(\frac{n-1}{2})_{(s)}}{(k-s)!s!},\ n \geq 1 \tag{6}$$

and $a_{(r)}$ denotes the Pochhammer symbol, i.e. $a_{(r)} = \frac{\Gamma(a+r)}{\Gamma(a)}$, for any integer $r > 1$, and $a_{(0)} := 1$. The case of a real variable will formally be included in the above definitions as the case $n = 0$ with $T_0^0(0) = 1$ and $T_s^k(0) = 0$, for $0 < s \leq k$.

It can be proved, under the additional (but natural) condition $\mathcal{P}_k^n(1) = 1$ that the sequence $\mathcal{P} := (\mathcal{P}_k^n)_{k\in\mathbb{N}}$ is Appell, i.e.

$$\frac{1}{2}\partial\mathcal{P}_k^n = k\mathcal{P}_{k-1}^n. \tag{7}$$

Other important properties of such a sequence can also be obtained. For our purpose here, we highlight the following essential results:

1. Denoting by $c_k(n),\ n \geq 1$ the alternating sum $\sum_0^k (-1)^s T_s^k(n)$, then for $n \geq 1$, $k = 1, 2, \ldots$

$$c_k(n) := \sum_0^k (-1)^s T_s^k(n) = \begin{cases} \dfrac{k!!(n-2)!!}{(n+k-1)!!}, & \text{if } k \text{ is odd} \\[2ex] c_{k-1}(n), & \text{if } k \text{ is even} \end{cases} \tag{8}$$

and $c_0(n) = 1$, $n \geq 0$. As usual, we define $(-1)!! = 0!! = 1$.

2. The binomial-type formula (1) can be written as

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s} \mathcal{P}_s^n(\underline{x}) = \sum_{s=0}^{k} \binom{k}{s} c_s(n)\, x_0^{k-s}\, \underline{x}^s. \tag{9}$$

If x is real, i.e. $x = x_0$, (9) leads to $\mathcal{P}_k^0(x_0) = x_0^k$ and if $x_0 = 0$, i.e. $x = \underline{x}$, we obtain the essential property which characterizes the difference to the complex case

$$\mathcal{P}_k^n(\underline{x}) = c_k(n)\underline{x}^k.$$

Using these properties, we can easily compute the first polynomials, $n \geq 1$:

$$\mathcal{P}_0^n(x) = 1, \qquad \mathcal{P}_1^n(x) = x_0 + \frac{1}{n}\underline{x},$$

$$\mathcal{P}_2^n(x) = x_0^2 + \frac{2}{n}x_0\underline{x} + \frac{1}{n}\underline{x}^2, \qquad \mathcal{P}_3^n(x) = x_0^3 + \frac{3}{n}x_0^2\underline{x} + \frac{3}{n}x_0\underline{x}^2 + \frac{3}{n(n+2)}\underline{x}^3.$$

[1] Special monogenic polynomials in terms of x and $\bar{x}$ similar to (5) have been considered before in [19] and some subsequent papers of the same authors. But the authors have been concerned with the extension of the theory of basic sets of polynomials in one complex variable, as introduced by J. M. Whittaker and B. Cannon, to the setting of Clifford analysis. At the time of publication of [19] the concept of hypercomplex differentiability ([16]) or the corresponding use of the hypercomplex derivative, first published in [15], have not been available for the investigation of Appell sequences of monogenic polynomials. In the quaternionic framework ($n = 2$) and using a completely different approach, these polynomials were also obtained in [20] as part of complete polynomial bases systems.

We observe that in the complex case ($n = 1$), the polynomials $\mathcal{P}_k^1$ coincide, as expected, with the usual powers z^k. In fact, from (8), we get $c_k(1) = 1$, for all k. Then, the binomial-type formula (9) permits to state that

$$\mathcal{P}_k^1(x) = \sum_{s=0}^{k} \binom{k}{s} x_0^{k-s} \underline{x}^s = (x_0 + e_1 x_1)^k \simeq z^k.$$

5 Monogenic exponential function

As mentioned in the introduction the existence of a generalized holomorphic exponential function was a principal question from the beginning of Clifford Analysis. At that time only the Riemann approach to generalized holomorphic (monogenic) functions as solutions of $\overline{\partial} f = 0$ or $f\overline{\partial} = 0$ was at hand and was even considered as the only one possible approach (see [1, 2]). But distinguishing a generalized holomorphic exponential function $f = f(x),\ x \in \mathbb{R}^{n+1}$, among all functions in the kernel of $\overline{\partial}$ is not possible, if one relies only on this fact or has in mind to conserve only some formal algebraic structure. In this context it seems remarkable, that one cannot find any remark on a regular quaternionic exponential function in the work of Fueter, one of the founders of hypercomplex function theory (see [21]).

Nevertheless, if the characterization should naturally be based on some analogy with the ordinary complex exponential function $f(z) = e^z,\ z \in \mathbb{C}$, then several different approaches are at our disposal which could lead to a generalized exponential function. First, one could demand a formal series representation similar to $f(z) := \sum_{k=0}^{\infty} \frac{z^n}{k!}$. But the fact that only for the complex case $n = 1$, an integer power of x belongs to the set of monogenic functions, causes problems by taking directly this usual series expansion as a defining relation.

Secondly, one could look for a monogenic function which fulfills the functional equation $f(z + w) = f(z)f(w)$. But to conserve the functional equation in the higher dimensional case in this form is illusorily, because the set of monogenic functions is not closed with respect to multiplication.

A third possibility would be through an analytic continuation approach (Cauchy-Kowalevskaya extension) starting from the exponential function with real argument and asking for a monogenic function with restriction to the real axis equal to the exponential function with a real argument. Indeed, the first attempts towards a meaningful definition of an exponential function in the context of Clifford Analysis have been [8, 9] and both papers rely on the Cauchy-Kowalevskaya extension approach (see also [1]).

But with the hypercomplex derivative at our disposal it was natural to define a generalized holomorphic exponential function f as a solution of the simple first order differential equation $f' = f$, with $f(0) = 1$. Here, of course, the derivative means the hypercomplex derivative $f' = \frac{1}{2}\partial f$.

This approach is also well motivated by the fact that hypercomplex differentiability is granted for our type of solutions of generalized Cauchy-Riemann systems. In [3, 4, 5] we have chosen exactly this approach, but still combined with the idea that through the construction of a set of special polynomials with differential properties like x^n also an easy to handle series representation should be obtained. Therefore we combined the differential equation approach with the construction of Appell sequences of holomorphic polynomials as will be described in the following.

Giving a $\hat{T}$-Appell sequence $\mathcal{F} := (\mathcal{F}_k)_{k\geq 0}$ of homogeneous monogenic polynomials, we can define formally a corresponding monogenic exponential function as

$${}_{\hat{T}}\mathrm{Exp}_{\mathcal{F}}(x) := \sum_{k=0}^{\infty} \frac{\mathcal{F}_k(x)}{k!}. \tag{10}$$

The function ${}_{\hat{T}}\mathrm{Exp}_{\mathcal{F}}$ is an eigenfunction of the hypercomplex operator $\hat{T}$, i.e., it holds

$$\hat{T}({}_{\hat{T}}\mathrm{Exp}_{\mathcal{F}}(\lambda x)) = \lambda\ {}_{\hat{T}}\mathrm{Exp}_{\mathcal{F}}(\lambda x),\ \lambda \in \mathbb{R}.$$

In other words, (10) constitutes a generalization of the classical exponential function. For example, considering the Appell set $\mathcal{P} := (\mathcal{P}_k^n)_{k\in\mathbb{N}_0}$ defined by (5)-(6), the corresponding exponential function is

$${}_{\frac{1}{2}\partial}\mathrm{Exp}_{\mathcal{P}}(x) = \sum_{k=0}^{\infty} \frac{\mathcal{P}_k^n(x)}{k!}. \tag{11}$$

Following [3], we denote the exponential function (11) in $\mathbb{R}^{n+1}$, simply by Exp_n. It follows immediately from the above definition that Exp_n is an eigenfunction of the operator $\frac{1}{2}\partial$ and, in addition, $\mathrm{Exp}_n(xt)$ is an *exponential generating function* of the sequence $\mathcal{P}$.

With $\omega(x) := \dfrac{\underline{x}}{|\underline{x}|}$ and $\omega^2 = -1$ as the equivalent for the imaginary unit i we can prove the following result (c.f.[3]):

Theorem 2. *The Exp_n-function can be written in terms of Bessel functions of the first kind, $J_a(x)$, for orders $a = \frac{n}{2} - 1, \frac{n}{2}$ as*

$$\mathrm{Exp}_n(x_0 + \underline{x}) = e^{x_0}\Gamma\left(\frac{n}{2}\right)\left(\frac{2}{|\underline{x}|}\right)^{\frac{n}{2}-1}\left(J_{\frac{n}{2}-1}(|\underline{x}|) + \omega(x)J_{\frac{n}{2}}(|\underline{x}|)\right).$$

The exponential function in $\mathbb{R}^{n+1}$ defined above, permits to consider the exponential operator

$$\mathrm{Exp}_n(\lambda\hat{T}) = \sum_{k=0}^{\infty}\frac{\mathcal{P}_k^n(\hat{T})}{k!}\lambda^k, \;\; \lambda \in \mathbb{R} \tag{12}$$

as a multidimensional counterpart in hypercomplex function theory of the usual exponential operator $e^{\lambda Q} = \sum_{k=0}^{\infty}\frac{Q^k}{k!}\lambda^k$.

6 Hypercomplex Laguerre derivative operator

The operational approach to the classical Laguerre polynomials as considered in [13, 14] is based on the introduction of the so-called Laguerre derivative operator in the form $\frac{\partial}{\partial x}x\frac{\partial}{\partial x}$ acting on the set of $(\frac{x^k}{k!})_{k\geq 1}$. Its generalization to the case of paravector valued functions of a paravector variable can be realized by a natural combination of the hypercomplex derivation operator $\frac{1}{2}\partial$ and the Euler operator

$$\mathbb{E} := \sum_{k=0}^{n} x_k\frac{\partial}{\partial x_k}.$$

Therefore we use the paravector representation of $x = (x_0, x_1, \ldots, x_n)$ and $y = (y_0, y_1, \ldots, y_n)$ of $\mathbb{R}^{n+1}$,

$$x = x_0 + x_1e_1 + \cdots + x_ne_n, \quad y = y_0 + y_1e_1 + \cdots + y_ne_n \in \mathcal{A}.$$

The usual Euclidean inner product in $\mathbb{R}^{n+1}$ can be expressed by the scalar part of the product of x and $\bar{y}$, i.e.

$$\langle x, y\rangle = \mathrm{Sc}(x\,\bar{y}).$$

Substituting now y by the operator $\bar{\partial}$, we obtain the following form of the Euler operator

$$\langle x, \bar{\partial}\rangle = \mathrm{Sc}(x\,\partial) = \mathbb{E}.$$

Using the fact that $\mathcal{P} := (\mathcal{P}_k^n)_{k\in\mathbb{N}_0}$ is a sequence of homogeneous monogenic polynomials which is Appell, it is easy to understand the action of the operators $\mathbb{E}$ and ∂ on $\mathcal{P}_k^n$, for each $k \geq 1$. More precisely, we have

$$\mathbb{E}\left(\mathcal{P}_k^n\right) = k\mathcal{P}_k^n$$

and combining with the hypercomplex derivative it follows that

$$\frac{1}{2}\partial\mathbb{E}\left(\mathcal{P}_k^n\right) = k^2\mathcal{P}_{k-1}^n. \tag{13}$$

This action of the hypercomplex derivation $\frac{1}{2}\partial$ in combination with the Euler operator $\mathbb{E}$ is essential for our definition of a generalized Laguerre derivative operator

$${}_LD := \frac{1}{2}\partial\mathbb{E}.$$

In this context we noticed also an interesting relationship of both operators in connection with a generalized Leibniz rule, slightly different from that considered by other authors for the expression of a product of two paravector valued functions under the action of $\bar{\partial}$. In [22], the authors deduced a generalized Leibniz rule for quaternionic-valued functions, with respect to the operator $\bar{\partial}$. Cumbersome, but straightforward calculations lead to an analogous formula for paravector-valued functions in arbitrary dimensions and with respect to the operator ∂. Since ∂ plays the role of the hypercomplex derivative operator in the monogenic function theory, the following results show a deeper analogue in higher dimensions to the well known Leibniz rule for holomorphic functions.

Theorem 3. *(generalized Leibniz rule) Let u, v be paravector-valued functions continuously differentiable in some open set of $\mathbb{R}^{n+1}$. Then*

$$\partial(u\,v) = (\partial u)v - \overline{u}\left(\overline{\partial}v\right) + 2\,\mathrm{Sc}(u\partial)v.$$

In particular, if $u = x$

$$\partial(x\,v) = (n+1)v - \overline{x}\left(\overline{\partial}v\right) + 2\mathbb{E}v.$$

We note that if v is a monogenic function, the generalized Leibniz rule reads

$$\partial(x\,v) = (n+1)v + 2\mathbb{E}v \tag{14}$$

and, for the particular choice $v = \mathcal{P}_k^n$, we obtain, taking into account the homogeneity of $\mathcal{P}_k^n$,

$$\partial(x\,\mathcal{P}_k^n) = (n+1+2k)\mathcal{P}_k^n.$$

7 Monogenic Laguerre polynomials

Consider the polynomials

$$\mathcal{Q}_k^n := \frac{\mathcal{P}_k^n}{k!},\ k = 0, 1, \ldots.$$

Recalling (13) we conclude that $\mathcal{Q} := (\mathcal{Q}_k^n)_{k\geq 0}$ is a ${}_LD$-Appell sequence, since

$${}_LD(\mathcal{Q}_k^n) = \frac{1}{2}\partial\mathbb{E}(\mathcal{Q}_k^n) = k\mathcal{Q}_{k-1}^n. \tag{15}$$

Moreover

$$({}_LD)^r(\mathcal{Q}_k^n) = \frac{k!}{(k-r)!}\mathcal{Q}_{k-r}^n,\ r \leq k. \tag{16}$$

Next, we use (12) in order to apply the exponential operator $\mathrm{Exp}_n({}_LD)$ to the sequence $\mathcal{Q}$ and obtain, for each $k \geq 1$,

$$\mathcal{L}_k^n := \mathrm{Exp}_n({}_LD)\,(\mathcal{Q}_k^n) = \sum_{r=0}^{\infty}\frac{1}{r!}\mathcal{P}_r^n({}_LD)\,(\mathcal{Q}_k^n). \tag{17}$$

The binomial-type formula (9) permits to state, for each $r \geq 0$ and $k \geq 1$, that

$$\begin{aligned}\mathcal{P}_r^n({}_LD)\,(\mathcal{Q}_k^n) &= \mathcal{P}_r^n(\tfrac{1}{2}\partial\mathbb{E})\,(\mathcal{Q}_k^n) = \mathcal{P}_r^n(\tfrac{1}{2}(\partial_0 - \partial_{\underline{x}})\mathbb{E})\,(\mathcal{Q}_k^n)\\ &= \sum_{s=0}^{r}\binom{r}{s}c_s(n)\left(\frac{1}{2}\partial_0\mathbb{E}\right)^{r-s}\left(-\frac{1}{2}\partial_{\underline{x}}\mathbb{E}\right)^{s}(\mathcal{Q}_k^n).\end{aligned}$$

Since $\mathcal{Q}_k^n$ is monogenic, $-\partial_{\underline{x}}\mathcal{Q}_k^n = \partial_0\mathcal{Q}_k^n = \frac{1}{2}\partial\mathcal{Q}_k^n$ and hence, from (16) it follows

$$\begin{aligned}\mathcal{P}_r^n({}_LD)\,(\mathcal{Q}_k^n) &= \sum_{s=0}^{r}\binom{r}{s}c_s(n)\frac{1}{2^r}\left({}_LD\right)^r(\mathcal{Q}_k^n)\\ &= \sum_{s=0}^{r}\binom{r}{s}c_s(n)\frac{1}{2^r}\frac{k!}{(k-r)!}\mathcal{Q}_{k-r}^n.\end{aligned}$$

Substituting now this expression in (17), we obtain the monogenic polynomials

$$\begin{aligned}\mathcal{L}_k^n(x) &= \sum_{r=0}^{k} \frac{1}{r!} \sum_{s=0}^{r} \binom{r}{s} c_s(n) \frac{1}{2^r} \frac{k!}{(k-r)!} \mathcal{Q}_{k-r}^n(x) \\ &= \sum_{r=0}^{k} \binom{k}{r} \frac{1}{2^r} \gamma_r(n) \mathcal{Q}_{k-r}^n(x), \quad k = 0, 1, \ldots, \end{aligned} \tag{18}$$

where

$$\gamma_r(n) = \sum_{s=0}^{r} \binom{r}{s} c_s(n) = \sum_{m=0}^{\left[\frac{r+1}{2}\right]} \binom{r+1}{2m} c_{2m}(n).$$

We notice that the sequence $(\gamma_r(n))_{r\geq 0}$ is obtained as the binomial transform of the sequence $(c_s(n))_{s\geq 0}$ (see, e.g., [23]), with inversion

$$c_r(n) = \sum_{s=0}^{r} \binom{r}{s} (-1)^{r-s} \gamma_s(n).$$

The constant polynomial $\mathcal{L}_0^n(x) \equiv 1$ is included in a natural way in the expression (18), since $\gamma_0(n) = 1$ independently of the dimension n. As a direct consequence of the fact that $(\mathcal{Q}_k^n)_{k\geq 0}$ is a ${}_LD$-Appell sequence, we have the following

Theorem 4. *The sequence of monogenic polynomials $(\mathcal{L}_k^n)_{k\geq 0}$ is a ${}_LD$-Appell sequence.*

Following the classical case, we substitute x by $-x$ in (18), and we define the *monogenic Laguerre polynomials* in $\mathbb{R}^{n+1}$ by

$$L_k^n(x) := \mathcal{L}_k^n(-x) = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \frac{1}{2^r} \gamma_r(n) \frac{\mathcal{P}_{k-r}^n(x)}{(k-r)!}, \quad k = 0, 1, \ldots. \tag{19}$$

The first monogenic Laguerre polynomials are given by

$$\begin{aligned}
L_0^n(x) &= L_0^n(x_0 + \underline{x}) = 1 \\
L_1^n(x) &= L_1^n(x_0 + \underline{x}) = -\mathcal{P}_1^n(x_0 + \underline{x}) + \tfrac{1}{2}\left(1 + \tfrac{1}{n}\right) = -\left(x_0 + \tfrac{1}{n}\underline{x}\right) + \tfrac{1}{2}\left(1 + \tfrac{1}{n}\right) \\
L_2^n(x) &= L_2^n(x_0 + \underline{x}) = \tfrac{1}{2!}\mathcal{P}_2^n(x_0 + \underline{x}) + \left(1 + \tfrac{1}{n}\right)\mathcal{P}_1^n(x_0 + \underline{x}) + \tfrac{1}{4}\left(1 + \tfrac{3}{n}\right) \\
&= \tfrac{1}{2!}\left(x_0^2 + \tfrac{2}{n}x_0\underline{x} + \tfrac{1}{n}\underline{x}^2\right) - \left(1 + \tfrac{1}{n}\right)\left(x_0 + \tfrac{1}{n}\underline{x}\right) + \tfrac{1}{4}\left(1 + \tfrac{3}{n}\right)
\end{aligned}$$

Special cases:

1. Complex case ($n = 1$)

We recall that for $n = 1$, the polynomials $\mathcal{P}_k^1$ are isomorphic to the complex powers z^k $(k = 0, 1, 2, \ldots)$. On the other hand, we have that $c_s(1) = 1$, for arbitrary s. Therefore,

$$\gamma_r(1) = \sum_{s=0}^{r} \binom{r}{s} = 2^r$$

and we obtain,

$$L_k^1(x) = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \frac{\mathcal{P}_{k-r}^1(x)}{(k-r)!} = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \frac{z^{k-r}}{(k-r)!}, \quad k = 0, 1, \ldots,$$

i.e. L_k^1 are isomorphic to the holomorphic Laguerre polynomials of one complex variable.

2. Real case ($n = 0$)

In this case, $\mathcal{P}_k^0(x_0) = x_0^k$ and $c_s(0) \equiv 1$, for all values of s, and we have again $\gamma_r(0) = 2^r$. Then, (19) leads to

$$L_k^0(x) = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \frac{x_0^{k-r}}{(k-r)!}, \; k = 0, 1, \ldots,$$

which coincides with the well-known Laguerre polynomials defined on the real line.

Following the same operational approach we can construct generalized Laguerre polynomials in $\mathbb{R}^{n+1}$. In [24] the differential operator $\frac{\partial}{\partial x} x \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial x}$ was considered associated to the real case. The hypercomplex counterpart is given by

$${}_LD^{(\alpha)} :={}_L D + \alpha \frac{1}{2} \partial, \; \alpha \in \mathbb{R}.$$

This operator is a lowering operator associated with the sequence $\mathcal{Q}^{(\alpha)} := (Q_k^{n,\alpha})_{k \geq 0}$, where

$$Q_k^{n,\alpha} := \frac{\mathcal{P}_k^n}{\Gamma(k+\alpha+1)}, \; k = 0, 1, \ldots.$$

Indeed,

$${}_LD^{(\alpha)} Q_k^{n,\alpha} = k\, Q_{k-1}^{n,\alpha}, \; k = 1, 2, \ldots.$$

Therefore, considering the exponential operator $\operatorname{Exp}_n({}_LD^{(\alpha)})$ applied to the sequence $\mathcal{Q}^{(\alpha)}$ and repeating the above procedure, we obtain for each $k \geq 1$,

$$\begin{aligned}
\mathcal{L}_k^{n,\alpha} := \operatorname{Exp}_n({}_LD^{(\alpha)}) Q_k^{n,\alpha} &= \sum_{r=0}^{\infty} \frac{1}{r!} \mathcal{P}_r^n({}_LD^{(\alpha)}) Q_k^{n,\alpha} \\
&= \sum_{r=0}^{k} \frac{1}{r!} \frac{1}{2^r} \gamma_r(n) \frac{k!}{(k-r)!} Q_{k-r}^{n,\alpha} \\
&= \sum_{r=0}^{k} \binom{k}{r} \frac{1}{2^r} \, \gamma_r(n) \, \frac{\mathcal{P}_{k-r}^n}{\Gamma(k+\alpha-r+1)}.
\end{aligned}$$

The constant polynomials $\mathcal{L}_0^{n,\alpha}(x) \equiv 1$, for any $\alpha \in \mathbb{R}$ and $x \in \mathcal{A}$ are included in a natural way in the previous expression. Using the fact that $\mathcal{Q}^{(\alpha)}$ is a ${}_LD^{(\alpha)}$-Appell set, we obtain

Theorem 5. *The sequence of monogenic polynomials $(\mathcal{L}_k^{n,\alpha})_{k \geq 0}$ is a ${}_LD^{(\alpha)}$-Appell set.*

Considering now the normalizing factor $\frac{\Gamma(k+\alpha+1)}{k!}$ and substituting x by $-x$ in the expression of $\mathcal{L}_k^{n,\alpha}$, i.e., considering the polynomials

$$L_k^{n,\alpha}(x) := \frac{\Gamma(k+\alpha+1)}{k!} \mathcal{L}_k^{n,\alpha}(-x),$$

we can define the *monogenic generalized Laguerre polynomials* of degree k ($k = 0, 1, 2, \ldots$),

$$L_k^{n,\alpha}(x) = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \frac{1}{2^r\, k!} \, \gamma_r(n) \, \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha-r+1)} \mathcal{P}_{k-r}^n(x). \tag{20}$$

The ordinary cases of the generalized Laguerre polynomials defined on the real line and on the complex plane can also be obtained from (20) by considering $n = 0$ and $n = 1$, respectively.

Notice that in the case $\alpha = 0$, we have $L_k^{n,0} \equiv L_k^n$ in a similar way as for the ordinary Laguerre and generalized Laguerre polynomials.

Remark 2. *The polynomials $L_k^{n,\alpha}$ can be directly obtained by applying the exponential operator $\operatorname{Exp}_n(-{}_LD^{(\alpha)})$ to the sequence $(Q_k^{n,\alpha}(-x))_{k \geq 0}$.*

In the origin, we have

$$L_k^{n,\alpha}(0) = \frac{1}{2^k\, k!}\gamma_k(n)\,\frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha+1)},$$

due to the homogeneity of the polynomials $\mathcal{P}_k^n$ and the fact that $\mathcal{P}_0^n(x) \equiv 1$.

A straightforward computation shows that the monogenic generalized Laguerre polynomials (20) satisfy

$$_L D^{(\alpha)} L_k^{n,\alpha}(x) = -(k+\alpha)L_{k-1}^{n,\alpha}(x). \tag{21}$$

Moreover,

$$\frac{1}{2}\partial L_k^{n,\alpha}(x) = -L_{k-1}^{n,\alpha+1}(x). \tag{22}$$

This relation implies that

$$\left(\frac{1}{2}\partial\right)^j L_k^{n,\alpha}(x) = (-1)^j\, L_{k-j}^{n,\alpha+j}(x),\;\; j = 0, 1, \ldots, k$$

and, in particular, due to the fact that $L_0^{(\alpha)}(x) \equiv 1$ it follows

$$\left(\frac{1}{2}\partial\right)^k L_k^{n,\alpha}(x) = (-1)^k\,.$$

Theorem 6. *The monogenic generalized Laguerre polynomials satisfy the equality*

$$\left(\frac{1}{2}\partial\right)^2 (x\, L_k^{n,\alpha}(x)) = -\frac{1}{2}(n+1-2\alpha)L_{k-1}^{n,\alpha+1}(x) - (k+\alpha)L_{k-1}^{n,\alpha}(x).$$

Proof. Using formula (14), we obtain

$$\left(\frac{1}{2}\partial\right)^2 (x\, v) = \frac{1}{2}(n+1)\left(\frac{1}{2}\partial\right) v + \,_L D\, v,$$

which implies that any monogenic function v satisfies the second-order differential equation

$$(x\, v)'' = \frac{1}{2}(n+1-2\alpha)v' + \,_L D^{(\alpha)} v,$$

with the notation $(.)' := \frac{1}{2}\partial(.)$. Choosing now $v = L_k^{n,\alpha}$ and using properties (21)-(22), we obtain the desired equality. □

8 Monogenic Laguerre-type exponentials

Considering once more the $_L D^{(\alpha)}$-Appell sequence $\mathcal{Q}^{(\alpha)} := (Q_k^{n,\alpha})$, we obtain from (10) the Laguerre-type exponential

$$_{L D^{(\alpha)}}\mathrm{Exp}_{\mathcal{Q}^{(\alpha)}}(x) := \sum_{k=0}^{\infty} \frac{Q_k^{n,\alpha}(x)}{k!} = \sum_{k=0}^{\infty} \frac{\mathcal{P}_k^n(x)}{k!\Gamma(k+\alpha+1)},\;\; \alpha \in \mathbb{R}. \tag{23}$$

If $\alpha = r$ is an integer and we substitute x by $-x$, we have

$$\mathcal{C}_r(x) := \sum_{k=0}^{\infty} (-1)^k \frac{\mathcal{P}_k^n(x)}{k!(k+r)!}.$$

which, for $n = 0$ (real case), coincides with the Bessel-Tricomi function (see, e.g., [25]),

$$\mathcal{C}_r(x_0) = \sum_{k=0}^{\infty} \frac{(-x_0)^k}{k!(k+r)!} = x_0^{-r/2} J_r(2\sqrt{x_0}),$$

and shows its relation to the Bessel function J_r of order r.

In the case $\alpha = 0$, the monogenic Laguerre-type exponential (23) is given by

$$ {}_L\mathrm{Exp}_n(x) := {}_{LD}\mathrm{Exp}_{\mathcal{Q}}(x) = \sum_{k=0}^{\infty} \frac{\mathcal{P}_k^n(x)}{(k!)^2}, \tag{24}$$

and can be generalized to an arbitrary m-th Laguerre-type exponential (or mL-exponential) as

$$ {}_{mL}\mathrm{Exp}_n(x) := {}_{mLD}\mathrm{Exp}_{\mathcal{Q}}(x) = \sum_{k=0}^{\infty} \frac{\mathcal{P}_k^n(x)}{(k!)^{m+1}}, \tag{25}$$

which is an eigenfunction of the operator ${}_{mL}D := \frac{1}{2}\partial\mathbb{E}^m$. Notice that for the case $m = 0$, the $0L$-exponential coincides with the exponential function defined in Section 5 and if x is a real number, then the function ${}_{mL}\mathrm{Exp}_0$ gives the generalized Laguerre-type exponential presented by Dattoli and Ricci in [12].

Considering the cases $n = 0$ (real case) and $m = 1$, ${}_L\mathrm{Exp}_0(x_0)$ can be written in terms of modified Bessel functions of the first kind,

$$ {}_L\mathrm{Exp}_0(x_0) = I_0(2\sqrt{x_0}). $$

When $x = \underline{x}$, an explicit representation of (24) can also be obtained.

Theorem 7. *The ${}_{LD}\mathrm{Exp}$-function (24) can be written in terms of hypergeometric functions as*

$$ {}_L\mathrm{Exp}_n(\underline{x}) = {}_0\mathrm{F}_3\left(;\frac{1}{2}, 1, \frac{n}{2}; -\frac{|\underline{x}|^2}{16}\right) + \omega(x)\frac{|\underline{x}|}{n}\, {}_0\mathrm{F}_3\left(;1, \frac{3}{2}, \frac{n}{2}+1; -\frac{|\underline{x}|^2}{16}\right). \tag{26}$$

Proof. We start by considering the expression of ${}_L\mathrm{Exp}_n(x_ie_i)$, for $i = 1, 2, \ldots, n$, i.e.

$$\begin{aligned} {}_L\mathrm{Exp}_n(x_ie_i) &= \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \sum_{s=0}^{k} T_s^k(n)\,(x_ie_i)^{k-s}\,(-x_ie_i)^s \\ &= \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \sum_{s=0}^{k} T_s^k(n)\,(-1)^s x_i^k e_i^k \end{aligned}$$

and applying relation (8) to obtain

$$\begin{aligned} {}_L\mathrm{Exp}_n(x_ie_i) &= \sum_{k=0}^{\infty} \frac{1}{(k!)^2} c_k(n) x_i^k e_i^k \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{\left((2m)!\right)^2} c_{2m}(n) x_i^{2m} + e_i \sum_{m=0}^{\infty} \frac{(-1)^m}{\left((2m+1)!\right)^2} c_{2m+1}(n) x_i^{2m+1}. \end{aligned}$$

After some calculation we end up with

$$ {}_L\mathrm{Exp}_n(x_ie_i) = \sum_{m=0}^{\infty} a_m(-1)^m \frac{x_i^{2m}}{m!} + e_ix_i \sum_{m=0}^{\infty} b_m(-1)^m \frac{x_i^{2m}}{m!}, \tag{27}$$

where

$$ a_m = \frac{1}{2^{4m}\left(\frac{1}{2}\right)_m (1)_m \left(\frac{n}{2}\right)_m} \quad \text{and} \quad b_m = \frac{1}{n2^{4m}(1)_m \left(\frac{3}{2}\right)_m \left(\frac{n}{2}+1\right)_m}. $$

Therefore,

$$ {}_L\mathrm{Exp}_n(x_ie_i) = {}_0\mathrm{F}_3\left(;\frac{1}{2}, 1, \frac{n}{2}; -\frac{x_i^2}{16}\right) + e_i\frac{x_i}{n}\, {}_0\mathrm{F}_3\left(;1, \frac{3}{2}, \frac{n}{2}+1; -\frac{x_i^2}{16}\right). \tag{28}$$

The expression for ${}_L\mathrm{Exp}_n(\underline{x})$ follows almost in the same way. In fact, since

$$ \underline{x} = x_1e_1 + x_2e_2 + \cdots + x_ne_n = \frac{x_1e_1 + x_2e_2 + \cdots + x_ne_n}{|\underline{x}|}|\underline{x}| = \omega(x)|\underline{x}| $$

and $\omega(x)^2 = -1$, then (27) and (28) lead to

$$ {}_L\mathrm{Exp}_n(\underline{x}) = {}_0\mathrm{F}_3\Big(;\frac{1}{2}, 1, \frac{n}{2}; -\frac{|\underline{x}|^2}{16}\Big) + \omega(x)\frac{|\underline{x}|}{n}\, {}_0\mathrm{F}_3\Big(; 1, \frac{3}{2}, \frac{n}{2}+1; -\frac{|\underline{x}|^2}{16}\Big). \qquad \square $$

Special cases:

1. For $n = 1$, (26) gives the complex function

$$ {}_L\mathrm{Exp}_1(e_1x_1) = \mathrm{Ber}_0(2\sqrt{|x_1|}) + e_1\,\mathrm{Bei}_0(2\sqrt{|x_1|}). $$

2. If $\underline{x} = x_1e_1 + x_2e_2$, ${}_L\mathrm{Exp}_2$ is the reduced quaternion valued function,

$$ {}_L\mathrm{Exp}_2(\underline{x}) = I_0(\sqrt{2|\underline{x}|})J_0(\sqrt{2|\underline{x}|}) + \omega(x)I_1(\sqrt{2|\underline{x}|})J_1(\sqrt{2|\underline{x}|}). $$

3. For the particular case $n = 3$, (26) leads to

$$ {}_L\mathrm{Exp}_n(\underline{x}) = \frac{1}{\sqrt{2|\underline{x}|}}\Bigg(\mathrm{Bei}_1(2\sqrt{2}) - \mathrm{Ber}_1(2\sqrt{2}) + \omega(x)\Big(\frac{1+|\underline{x}|}{|\underline{x}|}\mathrm{Bei}_1(2\sqrt{2}) + \sqrt{\frac{2}{|\underline{x}|}}\,\mathrm{Ber}_0(2\sqrt{2}) - \frac{|\underline{x}|-1}{|\underline{x}|}\mathrm{Ber}_1(2\sqrt{2})\Big)\Bigg). \tag{29} $$

Here, as usual, $\mathrm{Ber}_\nu(x)$ and $\mathrm{Bei}_\nu(x)$ denote the Kelvin functions, i.e., the real and imaginary parts, respectively, of the ν^{th} order Bessel function of the first kind $J_\nu(x^{3\frac{\pi}{4}i})$, where x is a real number.

Finally, we remark that the absolute convergence of the defined exponential functions ${}_{LD^{(\alpha)}}\mathrm{Exp}(x)$ (23) and ${}_{mL}\mathrm{Exp}_n(x)$ $(m = 0, 1, \dots)$ (25) is ensured because, for each $k \geq 0$, we have

$$ |\mathcal{P}_k^n(x)| \leq \sum_{s=0}^{k} T_s^k(n)\ |x|^{k-s}\ |\bar{x}|^s = |x|^k, $$

taking into account that $T_s^k(n)$ are positive and $\sum_0^k T_s^k(n) \equiv 1$, for all $n \in \mathbb{N}$.

9 Conclusions

Concluding, it is worth noticing that the question of general Clifford Algebra valued Laguerre type polynomials has been studied before, for example in [26]. By considering functions with values in the Clifford Algebra $\mathcal{C}\ell_{0,n}$ over the complex numbers, the authors use an interesting algebraic approach for the construction of Clifford-Laguerre polynomials which guarantees orthogonality with respect to some weight function. However, the Laguerre polynomials constructed in [26] are not monogenic, whereas our main concern was to obtain Laguerre polynomials which continue to be holomorphic (monogenic). The used exponential operator allowed to construct a natural generalization of the complex variable to the multidimensional case via Clifford Algebras by an analytic approach. Moreover, the definition of monogenic Laguerre-type exponentials in Section 8 shows the relevant role of Special Functions in Clifford analysis.

Forthcoming work will be the systematical use of the operational approach as a general tool for studying monogenic hypercomplex polynomials, their properties and applications.

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