CHARACTERISTIC FUNCTIONS AND AVERAGES

Assis Azevedo[1]
Department of Mathematics and Applications, University of Minho
Campus de Gualtar, 4710-057 Braga, Portugal
assis@math.uminho.pt

Abstract

Let Ω be a set and $\Omega_1,\ldots,\Omega_{m-1}$ subsets of Ω, being m an integer greater than one. For a given function $f=(f_1,\ldots f_m):\Omega\to\mathbb{R}^m$, we prove the existence of a unique function $\alpha=(\alpha_1,\ldots,\alpha_m):\Omega\to\mathbb{R}^m$ such that

$$\begin{cases} \alpha_i & = & \alpha_{i+1} \quad \text{on } \Omega_i \\ \alpha_1+\cdots+\alpha_i & = & f_1+\cdots+f_i \quad \text{on } \Omega\setminus\Omega_i, \text{ for all } i<m \\ \alpha_1+\cdots+\alpha_m & = & f_1+\cdots+f_m, \end{cases}$$

called the average function of $f:\Omega\to\mathbb{R}^m$ relatively to $(\Omega,\Omega_1,\ldots,\Omega_{m-1})$.

When Ω is a topological space and f is a continuous function, we find necessary and sufficient conditions for the continuity of the average function of f.

We write α_i as a linear combination of characteristic functions of the (coincidence) sets $\cap_{j=r}^{s}\Omega_j$, $1\le r\le s\le m-1$, belonging the coefficients to $\mathbb{Q}[f_1,\ldots,f_m]$.

1. Introduction

The motivation of this paper had its origin in a joint work with J. F. Rodrigues and L. Santos (see [1]). The problem under consideration there, was a variational inequality modelling the problem of equilibrium of N membranes, attached to rigid supports on the boundary, each one under the action of given forces $(f_1,\ldots,f_N)$ and constrained by the other membranes (see Remark 3.2).

Let Ω be a set, $m>1$, an integer, Ω_i, $1\le i\le m-1$ subsets of Ω. It is easy to characterize the sets

$$\begin{aligned} \mathcal{A}^m &= \Big\{\alpha\in\mathcal{F}(\Omega)^m:\ \Omega_i\subseteq\{x\in\Omega:\ \alpha_i(x)=\alpha_{i+1}(x)\},\ \text{for } 1\le i\le m-1\Big\}, \\ \mathcal{B}^m &= \Big\{\alpha\in\mathcal{F}(\Omega)^m:\ \Omega_i=\{x\in\Omega:\ \alpha_i(x)=\alpha_{i+1}(x)\},\ \ \text{for } 1\le i\le m-1\Big\}, \end{aligned}$$

where $\mathcal{F}(\Omega)$ is the set of real-valued functions with domain Ω.

For instance, $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathcal{A}^m$ if and only if there exists $f_1,\ldots,f_m$ such that $\alpha_1=f_1$ and $\alpha_i=\alpha_{i-1}+f_ih_{i-1}$ where h_i is a function that vanishes on Ω_i, for example the characteristic function of $\Omega\setminus\Omega_i$ or, in the case we are in a

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metric space, the distance function to Ω_i. In both cases it is easy to know when α belongs to $\mathcal{B}^m$ and when it is continuous. But in none of these cases we have $\alpha_1 + \cdots + \alpha_m = f_1 + \cdots + f_m$.

The family $(\Omega_i)_{i=1,\ldots,m-1}$ allows us to define m partitions of Ω that are crucial to the construction of the average functions. In what follows we consider $\Omega_0 = \Omega_m = \emptyset$. We also construct partitions to $\Omega \setminus \Omega_i$, for $i = 1, \ldots, m-1$.

Definition 1.1. *For $1 \leq r \leq s \leq m$ let*

$$\Omega_{r,s} = \left(\bigcap_{r \leq k \leq s-1} \Omega_k \right) \setminus (\Omega_{r-1} \cup \Omega_s) \qquad \text{if } 1 \leq r \leq s \leq m. \tag{1}$$

Notice that, with the usual convention about intersections, $\Omega_{r,r} = \Omega \setminus (\Omega_{r-1} \cup \Omega_r)$.

Lemma 1.2. *For each $i = 1, \ldots, m$,*

a) $\left(\Omega_{r,s}\right)_{1 \leq r \leq i \leq s \leq m}$ *is a partition of* Ω.

b) $\left(\Omega_{r,i}\right)_{1 \leq r \leq i}$, $\left(\Omega_{i+1,s}\right)_{i < s \leq m}$ *and* $\left(\Omega_{r,i} \cap \Omega_{i+1,s}\right)_{1 \leq r \leq i < s \leq m}$ *are partitions of* $\Omega \setminus \Omega_i$.

Proof. Is $x \notin \Omega_{i-1} \cup \Omega_i$ then $x \in \Omega_{i,i}$. If $x \in \Omega_j$, for $j \in \{i-1, i\}$ then $x \in \Omega_{r,s}$ where r is the smaller integer such that $x \in \bigcap \Omega_{k=r}^{j}$ and s is the biggest integer such that $x \in \bigcap \Omega_{k=j}^{s-1}$. And of course, if $(r,s) \neq (r',s')$ then $\Omega_{r,s} \cap \Omega_{r',s'} = \emptyset$. The other proofs are identical. $\square$

The first alinea of the this lemma can be illustrated in the following figure, for $m = 4$ and $i = 2$: in the left, we consider Ω and a general situation for Ω_1, Ω_2 and Ω_3 and in the right, we have the sets $\Omega_{1,2}$, $\Omega_{1,3}$, $\Omega_{1,4}$, $\Omega_{2,3}$ and $\Omega_{2,4}$ that define a decomposition of $\Omega_1 \cup \Omega_2$ (recall that $\Omega_{2,2}$ is $\Omega \setminus (\Omega_1 \cup \Omega_2)$):

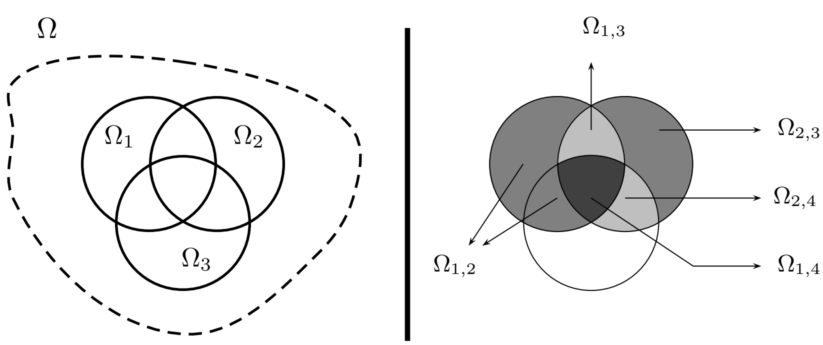

FIGURE 1. Decomposition of $\Omega_1 \cup \Omega_2$, if $m = 4$ and $i = 2$.

2. Average function

In what follows, given a set $A \subseteq \Omega$, χ_A denotes the characteristic function of A, *i.e.*, $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise, and ∂A the boundary of A. If $f_1, \dots, f_k \in \mathcal{F}(\Omega)$ we denote their average $(\frac{f_1+\cdots+f_k}{k})$ by $[f_1, \dots, f_k]$. The average of elements of $\mathcal{F}(\Omega)^m$ are defined component by component.

Definition 2.1. *With the notations above, we define the* average function of $\mathcal{F}(\Omega)$ relatively to $(\Omega, \Omega_1, \dots, \Omega_{m-1})$ *as* $\psi = (\psi_1, \dots, \psi_m) : \mathcal{F}^m(\Omega) \to \mathcal{F}^m(\Omega)$ *where*

$$(2)\ \forall i \leq m \ \ \forall f = (f_1, \dots, f_m) \in \mathcal{F}(\Omega)^n \qquad \psi_i(f) \quad = \quad \sum_{r \leq i \leq s} [f_r, \cdots, f_s]\, \chi_{\Omega_{r,s}}.$$

Note that, by Lemma 1.2, for $1 \leq i \leq m$, and using the previous notations

$$(3) \qquad \forall r \leq i \leq s, \quad \psi_i(f) = [f_r, \cdots, f_s], \text{ in } \Omega_{r,s}.$$

and in particular

$$(4) \qquad \psi_i(f) = f_i, \text{ in } \Omega \setminus (\Omega_{i-1} \cup \Omega_i).$$

If, for example, for some $k_0 \leq m-1$, Ω_k is the empty set for all $k \leq k_0$ and is equal to Ω, otherwise, then

$$\psi_i(f) \quad = \quad \begin{cases} f_i & \text{if } i \leq k_0 \\ [f_{k_0+1}, \dots, f_m] & \text{if } i > k_0. \end{cases}$$

In particular, if $k_0 = 0$, $\psi_i(f) = [f_1, \dots, f_m]$ and, if $k_0 = m-1$, ψ is the identity function.

The average function have some nice (and trivial) properties. In particular it is linear and preserves averages.

Proposition 2.2. *If* $f, g, f^1, \dots, f^k \in \mathcal{F}(\Omega)^m$ *and* $h \in \mathcal{F}(\Omega)$ *then*

a) $\psi(f+g) = \psi(f) + \psi(g)$;

b) $\psi(hf) = h\psi(f)$;

c) if $f \leq g$ *then* $\psi(f) \leq \psi(g)$;

d) $\psi([f^1, \dots, f^k]) = \big[\psi(f^1), \dots, \psi(f^k)\big]$;

e) if $x \in \Omega$ *and* $f_1(x) \leq \cdots \leq f_m(x)$ *then* $\psi_1(f)(x) \leq \cdots \leq \psi_m(f)(x)$.

Proof. The first four alineas are immediate consequences of the definition of the average function. For the last alinea, suppose that $f_1(x) \leq \cdots \leq f_m(x)$ and let $1 \leq i < m$. If $x \in \Omega_i$ then $\psi_i(f)(x) = \psi_{i+1}(f)(x)$. If $x \notin \Omega_i$ then, by Lemma (1.2), $x \in \Omega_{r,i} \cap \Omega_{i+1,s}$ for some $r \leq i < s$ and, in this case $\psi_{i+1}(f)(x) = [f_{i+1}, \dots, f_s](x) \geq f_{i+1}(x) \geq f_i(x) \geq [f_i, \dots, f_r](x) = \psi_i(f)(x)$. □

Remark 2.3. *In fact the function ψ should be denoted by $\psi^{\Omega,\Omega_1,\ldots,\Omega_{m-1}}$. We will use this notation only in the beginning of Section 3 and in the proof of the next result, where the use of an induction argument on m may induce confusion.*

As an immediate consequence of the definitions, we have, for $1 \le j \le i < m$

$$\psi_j^{\Omega,\Omega_1,\ldots,\Omega_{m-1}}(f_1,\ldots,f_m) = \psi_j^{\Omega,\Omega_1,\ldots,\Omega_{i-1}}(f_1,\ldots,f_i) \quad \text{in } \Omega \setminus \Omega_i. \tag{5}$$

Theorem 2.4. *Given $m \ge 2$ and $\Omega_1,\ldots,\Omega_{m-1}$ subsets of Ω, the average function $\psi = (\psi_1,\ldots,\psi_m)$ is the unique function satisfying the following conditions:*

a) if $1 \le i < m$, $\quad \psi_i(f_1,\ldots,f_m) = \psi_{i+1}(f_1,\ldots,f_m)$ in Ω_i;

b) if $1 \le i \le m$, $$\sum_{j=1}^{i} \psi_j(f_1,\ldots,f_m) = \sum_{j=1}^{i} f_j \qquad \text{in } \Omega\setminus\Omega_i \text{ (recall that, } \Omega_m = \emptyset).$$

Proof. a) Note that $\psi_{i+1}(f_1,\ldots,f_m) - \psi_i(f_1,\ldots,f_m)$ is equal to

$$\sum_{s=i+1}^{m} [f_r,\cdots,f_s]\,\chi_{\Omega_{i+1,s}} - \sum_{r=1}^{i}[f_r,\cdots,f_s]\,\chi_{\Omega_{r,i}}$$

which vanish in Ω_i as $\Omega_{i+1,s} \cap \Omega_i = \Omega_{r,i} \cap \Omega_i = \emptyset$.

b) We will use an induction argument on m. If $i = 1$ and $m \ge 2$ the result follows from (4). If $m = 2$ (on the left) or $m \ge 3$ and $i = 2$ (on the right) we have

$$\begin{cases} \psi_1(f_1,f_2) = f_1\,\chi_{\Omega\setminus\Omega_1} + \frac{f_1+f_2}{2}\,\chi_{\Omega_1} \\ \psi_2(f_1,f_2) = f_2\,\chi_{\Omega\setminus\Omega_1} + \frac{f_1+f_2}{2}\,\chi_{\Omega_1} \end{cases} \quad \begin{cases} \psi_1(f_1,\ldots,f_m) = f_1\,\chi_{\Omega\setminus\Omega_1} + \frac{f_1+f_2}{2}\,\chi_{\Omega_1\setminus\Omega_2} \\ \psi_2(f_1,\ldots,f_m) = f_2\,\chi_{\Omega\setminus\Omega_1} + \frac{f_1+f_2}{2}\,\chi_{\Omega_1\setminus\Omega_2} \end{cases}$$

For the induction step consider $m \ge 3$. If $2 \le i < m$ then the results follows by the induction hypothesis using the equality (5). For the case $i = m$ we prove the result on each set $\Omega_{r,m}$ ($r \le m$) that form a partition of $\Omega \setminus \Omega_m$:

- in $\Omega_{m,m} = \Omega \setminus \Omega_{m-1}$ we have, using (5), with $i = m-1$, and the induction hypothesis,

$$\begin{aligned} \sum_{j=1}^{m} \psi_j(f_1,\ldots,f_m) &= \sum_{j=1}^{m-1} f_j + \psi_m(f_1,\ldots,f_m) \\ &= \sum_{j=1}^{m} f_j, \quad \text{by (4)} \end{aligned}$$

- in $\Omega_{1,m} = \Omega_1 \cap \cdots \cap \Omega_{m-1}$, $\psi_1(f_1,\ldots,f_m) = \cdots = \psi_m(f_1,\ldots,f_m) = [f_1,\ldots,f_m]$ by alinea a) and (3), and the result follows.
- in $\Omega_{r,m} = \Omega_r \cap \cdots \cap \Omega_{m-1} \setminus \Omega_{r-1}$ (for $1 < r < m$) we have

$$\sum_{j=1}^{r-1} \psi_j(f_1,\ldots,f_m) = \sum_{j=1}^{r-1} f_j$$

using the case already proved. Then

$$\begin{aligned}\sum_{j=1}^{m}\psi_j(f_1,\ldots,f_m) &= \sum_{j=1}^{r-1} f_j + \sum_{j=r}^{m}\psi_j(f_1,\ldots,f_m)\\ &= \sum_{j=1}^{r-1} f_j + (m-r+1)\psi_m(f_1,\ldots,f_m) \text{ by alinea a)}\\ &= \sum_{j=1}^{m} f_j.\end{aligned}$$

as, by (3), with $i=s=m$, $\psi_m(f_1,\ldots,f_m)=[f_r,\ldots,f_m]$.

To prove the uniqueness, suppose that $\varphi=(\varphi_1,\ldots,\varphi_m)\in\mathcal{F}(\Omega)^m$ satisfies conditions a) and b) and let us prove that $\varphi_i(f_1,\ldots,f_m)=[f_r,\ldots,f_s]$ in $\Omega_{r,s}$, for $1\le r\le i\le s\le m$.

In fact, in $\Omega_{r,s}$, $\varphi_r(f_1,\ldots,f_m)=\cdots=\varphi_s(f_1,\ldots,f_m)$ and then

$$\begin{aligned}\varphi_i(f_1,\ldots,f_m) &= \frac{1}{s-r+1}\sum_{j=r}^{s}\varphi_j(f_1,\ldots,f_m)\\ &= \frac{1}{s-r+1}\left(\sum_{j=1}^{s}\varphi_j(f_1,\ldots,f_m)-\sum_{j=1}^{r-1}\varphi_j(f_1,\ldots,f_m)\right)\\ &= \frac{1}{s-r+1}\left(\sum_{j=1}^{s} f_j-\sum_{j=1}^{r-1} f_j\right) \qquad \text{by condition b)}\\ &= \frac{1}{s-r+1}\sum_{j=r}^{s} f_j=[f_r,\ldots,f_s]\end{aligned}$$

which completes the proof. □

We are now in conditions to studied when the average function of f belongs to $\mathcal{B}^m$ and, if Ω is a topological space and f is continuous, when this function is continuous.

Theorem 2.5. *With the previous notations:*

a) $\psi\circ\psi=\psi$*;*

b) $\mathcal{A}^m$ *is the image of* ψ*;*

c) $\mathcal{B}^m$ *is the set of all* $f=(f_1,\ldots,f_m)\in\mathcal{A}^m$ *such that, if*

$$1\le r\le i<s\le m,\ x\in\Omega_{r,i}\cap\Omega_{i+1,s} \implies [f_r,\ldots,f_i](x)\ne[f_{i+1},\ldots,f_s](x).$$

d) if Ω *is a topological space and* $f=(f_1,\ldots,f_m)$ *is a continuous functions of* $\mathcal{A}^m$ *then* $\psi(f)$ *is continuous if and only if*

$$1\le r,r'\le i<s,s'\le m,\ x\in\partial\Omega_{r,s}\cap\partial\Omega_{r',s'} \implies [f_r,\ldots,f_s](x)=[f_{r'},\ldots,f_{s'}](x).$$

Proof.

a) Let $j \in \{1,\dots,m\}$. Then in $\Omega_{r,s}$, for $r \le j \le s$,

$$\begin{aligned}(\psi\circ\psi)_j(f_1,\dots,f_m) &= \psi_j\big(\psi_1(f_1,\dots,f_m),\dots,\psi_m(f_1,\dots,f_m)\big)\\ &= \big[\psi_r(f_1,\dots,f_m),\dots,\psi_s(f_1,\dots,f_m)\big] \quad \text{by (2)}\\ &= \big[[f_r,\dots,f_s],\dots,[f_r,\dots,f_s]\big] \quad \text{by (2)}\\ &= [f_r,\dots,f_s] \;=\; \psi_j(f_1,\dots,f_m).\end{aligned}$$

b) By Theorem 2.4, the image of ψ is contained in $\mathcal{A}^m$. For the other inclusion, just note that if $(f_1,\dots,f_m)\in\mathcal{A}^m$, $1\le r\le i\le s\le m$, then $f_r=\cdots=f_s$ in $\Omega_{r,s}$ and by consequence $\psi_i(f_1,\dots,f_m)=[f_r,\dots,f_s]=f_i$.

c) Use the definition of ψ and Lemma 1.2, b).

d) Fix $i=1,\dots,m$, and $f_1,\dots,f_m\in\mathcal{F}(\Omega)$. We have a family of continuous functions $f_{r,s}:\Omega\to\mathbb{R}$ (with $1\le r\le i\le s\le m$), defined by $f_{r,s}(x)=[f_r,\dots,f_s]$, a partition of Ω, $(\Omega_{r,s})_{1\le r\le i\le s\le m}$ and $\psi_1(f_1,\dots,f_m):\Omega\to\mathbb{R}$ that is equal to $f_{r,s}$ in $\Omega_{r,s}$. In these conditions, it is well known that $\psi_1(f_1,\dots,f_m)$ is continuous if and only if $f_{r,s}=f_{r',s'}$ on $\partial\Omega_{r,s}\cap\partial\Omega_{r',s'}$ for all $(r,s),(r',s')$. □

Remark 2.6. *If we use average functions with weight we obtain similar results. For instance if $\varepsilon_1,\dots,\varepsilon_m>0$ and $\varepsilon_1+\cdots+\varepsilon_m=1$ then there exists a unique function $\Phi=(\Phi_1,\dots,\Phi_m):\mathcal{F}^m(\Omega)\to\mathcal{F}^m(\Omega)$ such that for $(f_1,\dots,f_m)\in\mathcal{F}^m(\Omega)$*

a) if $1\le i<m$, $\quad \varepsilon_i\Phi_i(f_1,\dots,f_m)=\varepsilon_{i+1}\Phi_{i+1}(f_1,\dots,f_m)$ *in Ω_i;*

b) if $1\le i\le m$, $\quad \displaystyle\sum_{j=1}^{i}\varepsilon_i\Phi_j(f_1,\dots,f_m)=\sum_{j=1}^{i}\varepsilon_i f_j$ *in $\Omega\setminus\Omega_i$.*

The idea is to consider $\Phi_i(f_1,\dots,f_m)=\frac{1}{\varepsilon_i}\psi(\varepsilon_1 f_1,\dots,\varepsilon_m f_m)$.

3. The average function using characteristic functions

Now we wish to define ψ in terms of the characteristic functions of the coincidence sets $\displaystyle\bigcap_{j=k}^{n}\Omega_j$, with $1\le k,n\le m$, with the convention that this intersection is equal to Ω if $k>n$. To simplify, we will denote those characteristic functions by $\chi_{k,n}$.

Using the equality $\chi_{A\setminus(B\cup C)}=\chi_A-\chi_{A\cap B}-\chi_{A\cap C}+\chi_{A\cap B\cap C}$, for $A,B,C\subseteq\Omega$, we obtain, for $i\le m$ and $f_1,\dots,f_m\in\mathcal{F}(\Omega)$,

$$(6)\ \psi_i(f_1,\dots,f_m) \;=\; \sum_{r\le i\le s}[f_r,\cdots,f_s]\,(\chi_{r,s-1}-\chi_{r-1,s-1}-\chi_{r,s}+\chi_{r-1,s})\,,$$

and the following result.

Theorem 3.1. *If $m \in \mathbb{N}$, $1 \leq i \leq m$ and $f_1, \dots, f_m \in \mathcal{F}$ then*

$$\psi_i(f_1, \dots, f_m) \quad = \quad f_i + \sum_{1 \leq r < s \leq m,\ r \leq i \leq s} b(r, s, i)\ \chi_{r,s-1} \tag{7}$$

where

$$b(r, s, i) \quad = \quad \begin{cases} [f_r, \dots, f_s] - [f_r, \dots, f_{s-1}] & \text{if } i = r \\ [f_r, \dots, f_s] - [f_{r+1}, \dots, f_s] & \text{if } i = s \\ \frac{2}{(s-r)(s-r+1)} \Big\{ [f_{r+1}, \dots, f_{s-1}] - [f_r, f_s] \Big\} & \text{if } r < i < s. \end{cases}$$

Proof. From equality (6) we obtain that $\psi_i(f_1, \dots, f_m)$ is equal to

$$\sum_{r \leq i \leq s} \{ [f_r, \cdots, f_s] - [f_{r+1}, \cdots, f_s] - [f_r, \cdots, f_{s-1}] + [f_{r+1}, \cdots, f_{s-1}] \}\ \chi_{r,s-1}$$

The only non trivial part is the one with $r < i < s$. In this case, if we multiply $[f_r, \cdots, f_s] - [f_{r+1}, \cdots, f_s] - [f_r, \cdots, f_{s-1}] + [f_{r+1}, \cdots, f_{s-1}]$ by $(s-r)(s-r+1)$ we obtain, successively

$$\begin{aligned} (s-r) \sum_{k=r}^{s} f_k - (s-r+1) \left[\sum_{k=r+1}^{s} f_k + \sum_{k=r}^{s-1} f_k \right] + \frac{(s-r)(s-r+1)}{s-r-1} \sum_{k=r+1}^{s-1} f_k \\ -(f_r + f_s) + \left[(s-r) - 2(s-r+1) + \frac{(s-r)(s-r+1)}{s-r-1} \right] \sum_{k=r+1}^{s-1} f_k \\ -(f_r + f_s) + \left[-(s-r+2) + \frac{(s-r)(s-r+1)}{s-r-1} \right] \sum_{k=r+1}^{s-1} f_k \\ -(f_r + f_s) + \frac{2}{s-r-1} \sum_{k=r+1}^{s-1} f_k \\ 2 \left([f_{r+1}, \dots, f_{s-1}] - [f_r, f_s] \right), \end{aligned}$$

which completes the proof. □

Remark 3.2. *The problem we considered in* ([1]) *was the following: given Ω a bounded open subset of $\mathbb{R}^N$, E a Banach space contained in $\mathcal{F}(\Omega; \mathbb{R})$, having some additional properties, a closed convex subset of E^N*

$$\mathbb{K} = \{ (v_1, \dots, v_N) \in E^N : v_1 \geq \cdots \geq v_N \},$$

$f_1, \dots, f_N \in E'$ and a given (not necessarily linear) operator $A : E \longrightarrow E'$, we wanted to prove existence of $(u_1, \dots, u_N) \in \mathbb{K}$ satisfying

$$\sum_{j=1}^{N} \langle A u_j, v_j - u_j \rangle \geq \sum_{j=1}^{N} \langle f_j, v_j - u_j \rangle \tag{8}$$

and to study some properties of this solution. With additional assumption on the f_j, $j = 1, \dots, N$, our aim was to prove that the inequality (8) could be rewritten as a *system of equations, using the coincidence sets* $\Omega_i = \{x \in \Omega : u_i(x) = u_{i+1}(x)\}$ *and their characteristic functions. This characterization of the inequality by an equality had a decisive importance in the proof of the stability of these coincidence sets under small variations of the given functions* f_j, $j = 1, \ldots, N$.

References

[1] Azevedo, A. and Rodrigues, J. F. and Santos, L., *The N-membranes problem for quasilinear degenerate systems*, Interfaces and Free Boundaries, **7** (2005), 319-317.