

ON THE CAUCHY PROBLEM FOR A COUPLED SYSTEM OF KDV EQUATIONS: CRITICAL CASE

M. PANTHEE AND M. SCIALOM

ABSTRACT. We investigate some well-posedness issues for the initial value problem associated to the system

$$\begin{cases} u_t + \partial_x^3 u + \partial_x(u^2 v^3) = 0, \\ v_t + \partial_x^3 v + \partial_x(u^3 v^2) = 0, \end{cases}$$

for given data in low order Sobolev spaces $H^s(\mathbb{R}) \times H^s(\mathbb{R})$. We prove local and global well-posedness results utilizing the sharp smoothing estimates associated to the linear problem combined with the contraction mapping principle. For data with small Sobolev norm we obtain global solution whenever $s \geq 0$ by using global smoothing estimates. In particular, for data satisfying $\delta < \|(u_0, v_0)\|_{L^2 \times L^2} < \|(S, S)\|_{L^2 \times L^2}$, where S is solitary wave solution, we get global solution whenever $s > 3/4$. To prove this last result, we apply the splitting argument introduced by Bourgain [5] and further simplified by Fonseca, Linares and Ponce [6, 7].

1. INTRODUCTION

Let us consider the initial value problem (IVP)

$$\begin{cases} u_t + \partial_x^3 u + \partial_x(u^p v^{p+1}) = 0, \\ v_t + \partial_x^3 v + \partial_x(u^{p+1} v^p) = 0, & x, t \in \mathbb{R}, p \in \mathbb{Z}^+ \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \end{cases} \quad (1.1)$$

where $u = u(x, t)$, $v = v(x, t)$ are real valued functions.

2000 *Mathematics Subject Classification.* 35Q35, 35Q53.

Key words and phrases. Korteweg-de Vries equation, Cauchy problem, local & global well-posedness.

M. P. was partially supported by FCT, Portugal through POCI 2010/FEDER and the grant SFRH/BPD/22018/2005 and FAPESP Brazil and M. S. was partially supported by FAPESP Brazil.

This system contains a pair of Korteweg-de Vries (KdV) equations coupled through nonlinear terms and is a special case of a general class of nonlinear evolution equations considered in [1]. The following quantities are conserved by the flow of (1.1):

$$\int_{\mathbb{R}} u \, dx, \quad \int_{\mathbb{R}} v \, dx, \quad (1.2)$$

$$\frac{1}{2} \int_{\mathbb{R}} (u^2 + v^2) \, dx \quad (1.3)$$

and

$$\frac{1}{2} \int_{\mathbb{R}} \left(u_x^2 + v_x^2 - \frac{2}{p+1} u^{p+1} v^{p+1} \right) dx. \quad (1.4)$$

This model has been extensively studied in recent years. Alarcon, Angulo and Montenegro [2] considered the IVP (1.1) and proved local and global well-posedness results for given data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 1$. To get global results they used the above conserved quantities satisfied by the flow of (1.1) along with some size restriction on the given data depending on the values of p . They also studied the existence and nonlinear stability of the solitary wave solution to this model from the point of view of the abstract theory of Grillakis, Shatah and Strauss [9]. In [2] the solitary wave solution to the system (1.1) were shown to be orbitally stable for $p < 2$ and unstable for $p > 2$. To obtain the instability result they followed a method established by Bona, Souganidis and Strauss [4] in the KdV context.

Some particular cases of the IVP (1.1) have also been a matter of interest in recent literature. When $p = 1$, the system (1.1) reduces to a coupled system of modified KdV (mKdV) equations. In this case, Montenegro [17] obtained local well-posedness for data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 1/4$ and global well-posedness for $s \geq 1$, which is in accordance with the single mKdV equation. Recently, in [18] this result was improved by showing that the local solution can be extended to any time interval $[0, T]$ whenever $s > 4/9$. To obtain this global result, the frequency splitting technique introduced by Bourgain [5] and further simplified by Fonseca, Linares and Ponce [6] has been used.

When $p = 2$, the IVP (1.1) turns out to be a coupled system of critical KdV equations, i.e.,

$$\begin{cases} u_t + \partial_x^3 u + \partial_x(u^2 v^3) = 0, \\ v_t + \partial_x^3 v + \partial_x(u^3 v^2) = 0, & x, t \in \mathbb{R}, p \in \mathbb{Z}^+ \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x). \end{cases} \quad (1.5)$$

We say the system (1.5) (i.e., (1.1) with $p = 2$) is critical because we have global solutions in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ for all data when $p < 2$ and global solutions only for small data (i.e., data small in $H^1 \times H^1$ -norm) when $p > 2$. Also, the solitary wave solutions are orbitally stable for $p < 2$ and unstable for $p > 2$. This feature for $p = 2$ resembles that of the critical generalized KdV equation

$$u_t + u_{xxx} + (u^k)_x = 0,$$

with $k = 5$. So, naming the system (1.5) critical seems well justified.

In the case of the critical KdV equation, the size restriction on the initial data needed to obtain global solutions in $H^1(\mathbb{R})$ is $\|u_0\|_{L^2} < \|S\|_{L^2}$, where S is the solitary wave solution to the critical KdV equation. Merle in [16] proved that there exists $u_0 \in H^1$, with $\|u_0\|_{L^2} > \|S\|_{L^2}$, such that the corresponding solution to the critical KdV equation blows-up in finite time. We do not know whether we can have a result of blow-up solution in the case of system (1.5) with initial data $u_0 \neq v_0$.

Recently, exploiting the criticality of the system (1.5), Hakkaev and Kirchev in [10] studied the stability of the solitary wave solution. The authors in [10] used the ideas and techniques introduced by Angulo, Bona, Linares and Scialom in [3] to obtain analogous results to that for the critical KdV equation.

In this work we are interested in addressing some questions related to the well-posedness of the IVP (1.5) for given data in low regularity Sobolev spaces $H^s(\mathbb{R}) \times H^s(\mathbb{R})$. We will improve the well-posedness results obtained by Alarcon, Angulo and Montenegro in [2]. Our notion of well-posedness includes existence, uniqueness, persistence property and continuous dependence of the solution upon the data. If the local solution can be extended

to any time interval, we say the IVP (1.5) is globally well-posed. If any one condition in the definition of well-posedness fails we say that the IVP is ill-posed.

Our results for the system (1.5) are in the same spirit to that of the critical KdV equation obtained by Kenig, Ponce and Vega in [14]. They proved that the IVP associated to the critical KdV equation is globally well-posed for small data in $H^s(\mathbb{R})$, $s \geq 0$. To obtain this result, they used the sharp version of the smoothing effects of Kato type (see [11]) satisfied by the group associated to the linear problem, combined with the contraction mapping principle. Before stating the main results, let us define notation that will be used throughout this work.

Notation : We use \hat{f} to denote the Fourier transform of f and is defined as,

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$$

of f . The L^2 -based Sobolev space of order s will be denoted by H^s with norm

$$\|f\|_{H^s} = \left(\int_{\mathbb{R}} (1 + \xi^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

We denote by $X^s = H^s(\mathbb{R}) \times H^s(\mathbb{R})$ and $X = L^2(\mathbb{R}) \times L^2(\mathbb{R})$. The Riesz potential of order $-s$ is denoted by $D_x^s = (-\partial_x^2)^{s/2}$. For $f : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$ we define the mixed $L_x^p L_T^q$ -norm by

$$\|f\|_{L_x^p L_T^q} = \left(\int_{\mathbb{R}} \left(\int_0^T |f(x, t)|^q dt \right)^{p/q} dx \right)^{1/p},$$

with usual modifications when $p = \infty$. We replace T by t if $[0, T]$ is the whole real line \mathbb{R} . Also we define $\|(f, g)\|_{L_x^p L_T^q} = \|f\|_{L_x^p L_T^q} + \|g\|_{L_x^p L_T^q}$. We use the letter c to denote various constants whose exact values are immaterial and which may vary from one line to the next.

Now, we are in position to state the main results of this work. Our first result is concerned with L^2 -well-posedness,

Theorem 1.1. *Let $(u_0, v_0) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$. Then there exists $\delta > 0$ such that for $\|(u_0, v_0)\|_{L^2 \times L^2} < \delta$, the IVP (1.5) admits a unique solution (u, v) satisfying*

$$(u, v) \in C(\mathbb{R} : L^2(\mathbb{R}) \times L^2(\mathbb{R})), \tag{1.6}$$

$$\|\partial_x u\|_{L_x^\infty L_t^2} < \infty, \quad \|\partial_x v\|_{L_x^\infty L_t^2} < \infty, \quad (1.7)$$

$$\|u\|_{L_x^5 L_t^{10}} < \infty, \quad \|v\|_{L_x^5 L_t^{10}} < \infty. \quad (1.8)$$

Moreover, the map $(\tilde{u}_0, \tilde{v}_0) \mapsto (\tilde{u}, \tilde{v})$ from $\{(u_0, v_0) \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) : \|(u_0, v_0)\|_{L^2 \times L^2} < \delta\}$ into the class defined by (1.6) to (1.8) is Lipschitz.

The second result deals with the H^s -well-posedness, where $s > 0$.

Theorem 1.2. *Let $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > 0$. There exists $\delta > 0$ such that for $\|(u_0, v_0)\|_{H^s \times H^s} < \delta$, the IVP (1.5) admits a unique solution (u, v) satisfying*

$$(u, v) \in C(\mathbb{R} : H^s(\mathbb{R}) \times H^s(\mathbb{R})), \quad (1.9)$$

$$\|\partial_x u\|_{L_x^\infty L_t^2} < \infty, \quad \|\partial_x v\|_{L_x^\infty L_t^2} < \infty, \quad (1.10)$$

$$\|D_x^s \partial_x u\|_{L_x^\infty L_t^2} < \infty, \quad \|D_x^s \partial_x v\|_{L_x^\infty L_t^2} < \infty, \quad (1.11)$$

$$\|u\|_{L_x^5 L_t^{10}} < \infty, \quad \|v\|_{L_x^5 L_t^{10}} < \infty, \quad (1.12)$$

$$\|D_x^s u\|_{L_x^5 L_t^{10}} < \infty, \quad \|D_x^s v\|_{L_x^5 L_t^{10}} < \infty. \quad (1.13)$$

Moreover, the map $(\tilde{u}_0, \tilde{v}_0) \mapsto (\tilde{u}, \tilde{v})$ from $\{(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R}) : \|(u_0, v_0)\|_{H^s \times H^s} < \delta\}$ into the class defined by (1.9) to (1.13) is Lipschitz.

The previous theorems give global solutions to the IVP (1.5) for data with small $H^s \times H^s$ norm. The following theorem asserts well-posedness for the IVP (1.5) for arbitrary data in $H^s \times H^s$, but in this case we only have local solutions whose proven existence time depends on the data.

Theorem 1.3. *Let $(u_0, v_0) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$. There exist $T = T(u_0, v_0) > 0$ and a unique strong solution (u, v) to the IVP (1.5) satisfying*

$$(u, v) \in C([-T, T] : L^2(\mathbb{R}) \times L^2(\mathbb{R})), \quad (1.14)$$

$$\|\partial_x u\|_{L_x^\infty L_T^2} < \infty, \quad \|\partial_x v\|_{L_x^\infty L_T^2} < \infty, \quad (1.15)$$

$$\|u\|_{L_x^5 L_T^{10}} < \infty, \quad \|v\|_{L_x^5 L_T^{10}} < \infty. \quad (1.16)$$

Moreover, for any $T' \in (0, T)$, there exists a neighborhood \mathcal{V} of (u_0, v_0) in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ such that the map $(\tilde{u}_0, \tilde{v}_0) \mapsto (\tilde{u}, \tilde{v})$ from \mathcal{V} into the class defined by (1.14) to (1.16) with T' in place of T is Lipschitz.

If $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > 0$, then the previous result extends to the class

$$(u, v) \in C([-T, T] : H^s(\mathbb{R}) \times H^s(\mathbb{R})),$$

$$\|D_x^s \partial_x u\|_{L_x^\infty L_T^2} < \infty, \quad \|D_x^s \partial_x v\|_{L_x^\infty L_T^2} < \infty,$$

in the above time interval $[-T, T]$.

For $s > 0$ existence time for solutions can be shown to depend only on the $H^s \times H^s$ norm of the data. More precisely, we have the following result.

Theorem 1.4. *Suppose $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > 0$. Then there exist $T = T(\|u_0\|_{s,2}, \|v_0\|_{s,2}) > 0$ and a unique solution (u, v) to the IVP (1.5) satisfying*

$$(u, v) \in C([-T, T] : H^s(\mathbb{R}) \times H^s(\mathbb{R})), \quad (1.17)$$

$$\|\partial_x u\|_{L_x^\infty L_T^2} < \infty, \quad \|\partial_x v\|_{L_x^\infty L_T^2} < \infty, \quad (1.18)$$

$$\|u\|_{L_x^5 L_T^{10}} < \infty, \quad \|v\|_{L_x^5 L_T^{10}} < \infty, \quad (1.19)$$

$$\|D_x^s u\|_{L_x^5 L_T^{10}} + \|D_t^{s/3} u\|_{L_x^5 L_T^{10}} < \infty, \quad \|D_x^s v\|_{L_x^5 L_T^{10}} + \|D_t^{s/3} v\|_{L_x^5 L_T^{10}} < \infty, \quad (1.20)$$

$$\|D_x^s \partial_x u\|_{L_x^\infty L_T^2} + \|D_t^{s/3} \partial_x u\|_{L_x^\infty L_T^2} < \infty, \quad \|D_x^s \partial_x v\|_{L_x^\infty L_T^2} + \|D_t^{s/3} \partial_x v\|_{L_x^\infty L_T^2} < \infty. \quad (1.21)$$

Moreover, for any $T' \in (0, T)$, there exists a neighborhood \mathcal{V} of (u_0, v_0) in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ such that the map $(\tilde{u}_0, \tilde{v}_0) \mapsto (\tilde{u}, \tilde{v})$ from \mathcal{V} into the class defined by (1.17) to (1.21) with T' in place of T is Lipschitz.

Our next interest is to extend the local solution obtained in Theorem 1.4. Note that, for given data $(u_0, v_0) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ with $\|(u_0, v_0)\|_{L^2 \times L^2} < \delta$, we have from Theorem

1.1, a global solution to the IVP (1.5). In [20], Weinstein proved the following Gagliardo-Nirenberg type inequality for $u \in H^1(\mathbb{R})$,

$$\frac{1}{3} \int u^6 \leq \frac{1}{(\int S^2)^2} \left(\int u^2 \right)^2 \int u_x^2, \quad (1.22)$$

where S is the solitary wave solution for (1.5).

Now, using (1.22), the conserved quantities mentioned earlier and the fact that

$$\int u^3 v^3 \leq \frac{1}{2} \left(\int u^6 + v^6 \right),$$

one can obtain an *a priori* estimate for $\|(u, v)\|_{H^1 \times H^1}$ provided

$$\|(u_0, v_0)\|_{L^2 \times L^2} < \|(S, S)\|_{L^2 \times L^2}. \quad (1.23)$$

This *a priori* estimate yields global well-posedness for the IVP (1.5) for initial data in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ satisfying (1.23). This situation is similar to the one we discussed above for a single critical KdV equation. Hence, a natural goal is to obtain global solutions for data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > 0$ satisfying $\delta < \|(u_0, v_0)\|_{L^2 \times L^2} < \|(S, S)\|_{L^2 \times L^2}$. A partial result in this direction is our next theorem.

Theorem 1.5. *Let $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, where $s > \frac{3}{4}$. Suppose that $\|(u_0, v_0)\|_{L^2 \times L^2} < \|(S, S)\|_{L^2 \times L^2}$. Then the unique solution to the IVP (1.5) given by Theorem 1.4 can be extended to any interval of time $[0, T]$.*

Using Duhamel's principle, we prove these theorems by considering the associated integral equation associated to the IVP (1.5), i.e,

$$\begin{cases} u(t) = U(t)u_0 - \int_0^t U(t-t') \partial_x(u^2 v^3)(t') dt' \\ v(t) = U(t)v_0 - \int_0^t U(t-t') \partial_x(u^3 v^2)(t') dt'. \end{cases} \quad (1.24)$$

So, our interest will be to solve (1.24). We use the contraction mapping principle in appropriate metric spaces to prove Theorems 1.1 - 1.4. While, to prove the global well-posedness result of Theorem 1.5, we use the frequency splitting argument introduced by Bourgain in [5].

This paper is organized as follows. In Section 2 we record some preliminary estimates associated to the linear problem and other relevant results. In Section 3 we give a proof of the local well-posedness results and global well-posedness results for small data. Finally, a proof of the global well-posedness result for data not so small will be given in Section 4.

2. PRELIMINARY ESTIMATES

In this section we give some linear estimates associated to the IVP (1.5). These estimates are not new and can be found in the literature. Consequently, we just sketch the idea of the proof and mention the references where they can be found. Let $U(t)$ be the group generated by the operator ∂_x^3 . First, let us state the smoothing effects.

Lemma 2.1. *If $u_0 \in L^2(\mathbb{R})$, then*

$$\|\partial_x U(t)u_0\|_{L_x^\infty L_t^2} \leq c\|u_0\|_{L^2}. \quad (2.1)$$

If $g \in L_x^1 L_t^2$, then for any $T > 0$

$$\sup_{[-T, T]} \|\partial_x \int_{-\infty}^{\infty} U(t-t')g(\cdot, t') dt'\|_{L_x^2} \leq c\|g\|_{L_x^1 L_t^2}. \quad (2.2)$$

Proof. For the proof of the homogeneous smoothing effect (2.1), see Section 4 in [13] (see also [14]). Inequality (2.2) follows from the dual version of the smoothing effect (2.1). \square

Following is the double smoothing effect that obtains for solutions to the non-homogeneous linear problem

$$\begin{cases} u_t + \partial_x^3 u = f, \\ u(x, 0) = 0. \end{cases} \quad (2.3)$$

Lemma 2.2. *If $f \in L_x^1 L_t^2$ then*

$$\|\partial_x^2 \int_0^t U(t-t')f(\cdot, t') dt'\|_{L_x^\infty L_t^2} \leq c\|f\|_{L_x^1 L_t^2}. \quad (2.4)$$

Proof. See [12, 14]. \square

Now we give the maximal function estimates.

Lemma 2.3. *If $u_0 \in H^{1/4}$, then*

$$\|U(t)u_0\|_{L_x^4 L_T^\infty} \leq c \|D_x^{1/4} u_0\|_{L^2}. \quad (2.5)$$

If $u_0 \in H^s$, $s > 3/4$ and $0 < T < 1$ then

$$\|U(t)u_0\|_{L_x^2 L_T^\infty} \leq c \|u_0\|_{H^s}. \quad (2.6)$$

Proof. The proof of the estimates (2.5) and (2.6) can be found in [12] and [15]. \square

Some more estimates.

Lemma 2.4. *If $u_0 \in L^2(\mathbb{R})$, then*

$$\|U(t)u_0\|_{L_x^5 L_t^{10}} \leq c \|u_0\|_{L^2}. \quad (2.7)$$

If $g \in L_x^{5/4} L_t^{10/9}$

$$\left\| \int_0^t U(t-t')g(\cdot, t') dt' \right\|_{L_x^5 L_t^{10}} \leq c \|g\|_{L_x^{5/4} L_t^{10/9}}. \quad (2.8)$$

Proof. The estimates of this theorem can be found in [14]. The estimate (2.7) follows by interpolating (2.1) and (2.6). The estimate (2.8) follows by using interpolation in BMO spaces, see [14]. \square

The proof of the following result, that is the chain rule for fractional derivatives, can be found in [14].

Lemma 2.5. *Let $\alpha \in (0, 1)$. Let $p, p_1, p_2, q, q_2 \in (1, \infty)$, $q_1 \in (1, \infty]$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then*

$$\|D_x^\alpha F(f)\|_{L_x^p L_T^q} \leq c \|F'(f)\|_{L_x^{p_1} L_T^{q_1}} \|D_x^\alpha f\|_{L_x^{p_2} L_T^{q_2}}. \quad (2.9)$$

Following is the Leibniz's rule for fractional derivatives whose proof is also given in [14].

Lemma 2.6. *Let $\alpha \in (0, 1)$, $\alpha_1, \alpha_2 \in [0, \alpha]$, $\alpha_1 + \alpha_2 = \alpha$. Let $p, p_1, p_2, q, q_1, q_2 \in (1, \infty)$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then*

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L_x^p L_T^q} \leq c \|D_x^{\alpha_1} f\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\alpha_2} g\|_{L_x^{p_2} L_T^{q_2}} \quad (2.10)$$

Moreover, for $\alpha_1 = 0$ the value $q_1 = \infty$ is allowed.

The next lemma is a Sobolev type inequality, known as Gagliardo-Nirenberg inequality, whose proof is given in [8].

Lemma 2.7. *Let q, r be any numbers satisfying $1 \leq q, r \leq \infty$ and let j, m be any integers satisfying $0 \leq j < m$. If $f \in C_0^m(\mathbb{R}^n)$, then*

$$\|D^j f\|_{L^p} \leq c \|D^m f\|_{L^r}^\theta \|f\|_{L^q}^{1-\theta}, \quad (2.11)$$

where

$$\frac{1}{p} = \frac{j}{n} + \theta \left(\frac{1}{r} - \frac{m}{n} \right) + (1 - \theta) \frac{1}{q},$$

for all θ in the interval $\frac{j}{m} \leq \theta \leq 1$, and c depends only on n, m, j, q, r, θ .

Let us record the following result which plays crucial role in our argument to prove Theorem 1.3.

Lemma 2.8. *Let $u_0 \in L^2(\mathbb{R})$. Then for any $\epsilon > 0$, there exist $T = T(u_0; \epsilon) > 0$ and $\delta = \delta(u_0; \epsilon)$ such that if $\|u_0 - \tilde{u}_0\|_{L^2} < \delta$, then*

$$\|\partial_x U(t) \tilde{u}_0\|_{L_x^\infty L_T^2} < \epsilon \quad (2.12)$$

and

$$\|U(t) \tilde{u}_0\|_{L_x^5 L_T^{10}} < \epsilon. \quad (2.13)$$

Proof. We give details to obtain the estimate (2.12), the proof of (2.13) is similar. Using the linear estimate (2.1), if one takes $\delta < \epsilon/2c$, to show (2.12) it is enough to prove that

$$\|\partial_x U(t) u_0\|_{L_x^\infty L_T^2} < \frac{\epsilon}{2}. \quad (2.14)$$

Let us take $w_0 \in \mathcal{S}(\mathbb{R})$ such that $\|u_0 - w_0\|_{L^2} < \epsilon/4c$. Now, using the estimate (2.1), Sobolev inequality and group property we get,

$$\begin{aligned} \|\partial_x U(t)u_0\|_{L_x^\infty L_T^2} &\leq \|\partial_x U(t)(u_0 - w_0)\|_{L_x^\infty L_T^2} + \|\partial_x U(t)w_0\|_{L_x^\infty L_T^2} \\ &\leq c\|u_0 - w_0\|_{L_x^2} + cT^{1/2}\|U(t)\partial_x w_0\|_{L_x^\infty L_T^\infty} \\ &\leq \frac{\epsilon}{4} + cT^{1/2}\|w_0\|_{2,2}. \end{aligned} \quad (2.15)$$

Now choose T small enough such that $cT^{1/2}\|w_0\|_{2,2} < \epsilon/4$ to obtain (2.12). \square

In sequel, we record an inequality which is crucial in the proof of Theorems 1.4 and 1.5 and can be found in [14]. We have,

$$\|f\|_{L_x^5 L_T^{10}} \leq cT^{1/p}\|f\|_{L_x^5 L_T^q} \leq cT^{s/3}\|D_t^{s/3}f\|_{L_x^5 L_T^{10}}, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{10}, \quad q = q(s) \in (10, \infty), \quad (2.16)$$

where the first inequality follows from the Hölder's inequality and the second follows from the Gagliardo-Nirenberg inequality (2.11). Note that, we can get inequality (2.16) even for functions that are defined only in the interval $[-T, T]$. In this case one needs to extend them to the real line with zero values outside this interval to be able to define Fourier transform and hence fractional derivative in the time variable.

Before leaving this section we give some identities that will be useful in the proof of Theorem 1.4.

Lemma 2.9. *The following identities hold*

$$D_t^{s/3}U(t)u_0 = cD_x^s U(t)u_0 \quad (2.17)$$

and

$$D_t^{s/3} \int_{-\infty}^{\infty} U(t-t')g(t') dt' = cD_x^s \int_{-\infty}^{\infty} U(t-t')g(t') dt'. \quad (2.18)$$

Proof. The proof of the identities (2.17) and (2.18) follows easily by using a simple observation

$$\begin{aligned} U(t)u_0 &= c \int_{-\infty}^{\infty} e^{i(t\xi^3+x\xi)} \hat{u}_0(\xi) d\xi \\ &= c \int_{-\infty}^{\infty} e^{it\eta} e^{ix\eta^{1/3}} \frac{\hat{u}_0(\eta^{1/3})}{\eta^{2/3}} d\eta. \end{aligned} \tag{2.19}$$

□

3. PROOF OF THE LOCAL RESULTS AND GLOBAL RESULTS FOR SMALL DATA

Proof of Theorem 1.1: We will prove this theorem following the argument in [14]. Let us define a metric space,

$$\mathcal{X} = \{(u, v) \in C(\mathbb{R} : X(\mathbb{R})) : \|(u, v)\| < \infty\},$$

where

$$\|(u, v)\| = \max\{\|u\|, \|v\|\},$$

with

$$\|f\| = \|f\|_{L_t^\infty L_x^2} + \|\partial_x f\|_{L_x^\infty L_t^2} + \|f\|_{L_x^5 L_t^{10}}. \tag{3.1}$$

Let $\mathcal{X}_a = \{(u, v) \in \mathcal{X} : \|(u, v)\| < a\}$ be a ball in \mathcal{X} .

Now, we define the following application,

$$\begin{cases} \Phi_{u_0}[u, v](t) := U(t)u_0 - \int_0^t U(t-t')\partial_x(u^2v^3)(t') dt' \\ \Psi_{v_0}[u, v](t) := U(t)v_0 - \int_0^t U(t-t')\partial_x(u^3v^2)(t') dt'. \end{cases} \tag{3.2}$$

We show that, for some $a > 0$ and $\delta > 0$, the application $\Phi \times \Psi$ maps \mathcal{X}_a into \mathcal{X}_a and is a contraction.

Exploiting the symmetry of the system, we will only estimate the first component Φ . The estimates for the second component Ψ are similar.

Using the linear estimate (2.2) and Hölder's inequality we obtain,

$$\begin{aligned}
\|\Phi\|_{L_x^2} &\leq \|U(t)u_0\|_{L_x^2} + \left\| \int_0^t U(t-t')\partial_x(u^2v^3)(t') dt' \right\|_{L_x^2} \\
&\leq c\|u_0\|_{L_x^2} + c\|u^2v^3\|_{L_x^1L_t^2} \\
&\leq c\|u_0\|_{L_x^2} + c\|u\|_{L_x^5L_t^{10}}^2 \|v\|_{L_x^5L_t^{10}}^3 \\
&\leq c\|(u_0, v_0)\|_{L_x^2 \times L_x^2} + c\|(u, v)\|^5.
\end{aligned} \tag{3.3}$$

Therefore,

$$\|\Phi\|_{L_t^\infty L_x^2} \leq c\|(u_0, v_0)\|_{L_x^2 \times L_x^2} + c\|(u, v)\|^5. \tag{3.4}$$

Similarly, using (2.1), (2.4) and Hölder's inequality, we get

$$\begin{aligned}
\|\partial_x \Phi\|_{L_x^\infty L_t^2} &\leq \|\partial_x U(t)u_0\|_{L_x^\infty L_t^2} + \left\| \partial_x \int_0^t U(t-t')\partial_x(u^2v^3)(t') dt' \right\|_{L_x^\infty L_t^2} \\
&\leq c\|u_0\|_{L_x^2} + c\|u^2v^3\|_{L_x^1L_t^2} \\
&\leq c\|(u_0, v_0)\|_{L_x^2 \times L_x^2} + c\|(u, v)\|^5.
\end{aligned} \tag{3.5}$$

Finally, the use of (2.7), (2.8) and Hölder's inequality yields,

$$\begin{aligned}
\|\Phi\|_{L_x^5L_t^{10}} &\leq \|U(t)u_0\|_{L_x^5L_t^{10}} + \left\| \int_0^t U(t-t')\partial_x(u^2v^3)(t') dt' \right\|_{L_x^5L_t^{10}} \\
&\leq c\|u_0\|_{L_x^2} + c\|\partial_x(u^2v^3)\|_{L_x^{5/4}L_t^{10/9}} \\
&\leq c\|u_0\|_{L_x^2} + c\|u^2v^2\partial_x v\|_{L_x^{5/4}L_t^{10/9}} + c\|uv^3\partial_x u\|_{L_x^{5/4}L_t^{10/9}} \\
&\leq c\|u_0\|_{L_x^2} + c\|u\|_{L_x^5L_t^{10}}^2 \|v\|_{L_x^5L_t^{10}}^2 \|\partial_x v\|_{L_x^\infty L_t^2} + c\|u\|_{L_x^5L_t^{10}} \|v\|_{L_x^5L_t^{10}}^3 \|\partial_x u\|_{L_x^\infty L_t^2} \\
&\leq c\|(u_0, v_0)\|_{L_x^2 \times L_x^2} + c\|(u, v)\|^5.
\end{aligned} \tag{3.6}$$

From (3.4), (3.5) and (3.6) we obtain,

$$\|\Phi\| \leq c\|(u_0, v_0)\|_{L^2 \times L^2} + c\|(u, v)\|^5. \tag{3.7}$$

In an analogous manner we can get,

$$\|\Psi\| \leq c\|(u_0, v_0)\|_{L_x^2 \times L_x^2} + c\|(u, v)\|^5. \tag{3.8}$$

Hence, for $(u, v) \in \mathcal{X}_a$,

$$\|(\Phi, \Psi)\| \leq c\|(u_0, v_0)\|_{L_x^2 \times L_x^2} + c\|(u, v)\|^5 \leq c\delta + ca^5. \quad (3.9)$$

Let us choose δ such that $c(10c\delta)^4 \leq 1/2$ and $a \in (2c\delta, 3c\delta)$. With these choices we get from (3.9),

$$\|(\Phi, \Psi)\| \leq \frac{a}{2} + \frac{a}{2}.$$

Therefore, $\Phi \times \Psi$ maps \mathcal{X}_a into \mathcal{X}_a .

Now, we move to show that $\Phi \times \Psi$ is a contraction. For this, let $(u, v), (u_1, v_1) \in \mathcal{X}_a$.

Using the argument employed to obtain (3.4), we get,

$$\begin{aligned} & \|\partial_x(\Phi[u, v] - \Phi[u_1, v_1])\|_{L_x^\infty L_t^2} = \\ & = \left\| \partial_x \int_0^t U(t-t') \partial_x (u^2 v^3 - u_1^2 v_1^3)(t') dt' \right\|_{L_x^\infty L_t^2} \\ & \leq c \|u^2 v^3 - u_1^2 v_1^3\|_{L_x^1 L_t^2} \\ & \leq c \|v^3 u(u - u_1)\|_{L_x^1 L_t^2} + c \|v^3 u_1(u - u_1)\|_{L_x^1 L_t^2} + c \|u_1^2 v^2(v - v_1)\|_{L_x^1 L_t^2} \\ & \quad + c \|u_1^2 v u_1(v - v_1)\|_{L_x^1 L_t^2} + c \|u_1^2 v_1^2(v - v_1)\|_{L_x^1 L_t^2} \quad (3.10) \\ & \leq c \|(u, v)\|^4 \|u - u_1\| + c \|(u_1, v_1)\|^2 \|(u, v)\|^2 \|v - v_1\| \\ & \quad + c \|(u, v)\|^3 \|(u_1, v_1)\| \|u - u_1\| + c \|(u_1, v_1)\|^3 \|(u, v)\| \|v - v_1\| \\ & \quad + c \|(u_1, v_1)\|^4 \|v - v_1\| \\ & \leq 5ca^4 \|(u - u_1, v - v_1)\|. \end{aligned}$$

Similarly, with the argument used in (3.6), one gets

$$\begin{aligned}
& \|\Phi[u, v] - \Phi[u_1, v_1]\|_{L_x^5 L_t^{10}} = \\
& = \left\| \int_0^t U(t-t') \partial_x (u^2 v^3 - u_1^2 v_1^3)(t') dt' \right\|_{L_t^5 L_x^{10}} \\
& \leq c \|\partial_x (u^2 v^3 - u_1^2 v_1^3)\|_{L_x^{5/4} L_t^{10/9}} \\
& \leq c \{ \|uv^2 \partial_x v(u - u_1)\|_{L_x^{5/4} L_t^{10/9}} + \|u_1 v^2 \partial_x v(u - u_1)\|_{L_x^{5/4} L_t^{10/9}} + \|u_1^2 v \partial_x v(v - v_1)\|_{L_x^{5/4} L_t^{10/9}} \\
& \quad + \|u_1^2 v_1 \partial_x v(v - v_1)\|_{L_x^{5/4} L_t^{10/9}} + \|u_1^2 v_1^2 \partial_x (v - v_1)\|_{L_x^{5/4} L_t^{10/9}} + \|v^3 \partial_x u(u - u_1)\|_{L_x^{5/4} L_t^{10/9}} \\
& \quad + \|u_1 v^2 \partial_x u(v - v_1)\|_{L_x^{5/4} L_t^{10/9}} + \|u_1 v v_1 \partial_x u(v - v_1)\|_{L_x^{5/4} L_t^{10/9}} \\
& \quad + \|u_1 \partial_x u v_1^2 (v - v_1)\|_{L_x^{5/4} L_t^{10/9}} + \|u_1 v_1^3 \partial_x (u - u_1)\|_{L_x^{5/4} L_t^{10/9}} \} \\
& =: c(A_1 + \dots + A_{10}).
\end{aligned} \tag{3.11}$$

One can get estimates for A_1, \dots, A_{10} using Hölder's inequality. For the sake of clarity let us present estimates for A_1, A_3 and A_7 , the rest are analogous.

$$A_1 \leq c \|u\|_{L_x^5 L_t^{10}} \|v^2\|_{L_x^{5/2} L_t^5} \|\partial_x v\|_{L_x^\infty L_t^2} \|u - u_1\|_{L_x^5 L_t^{10}} \leq c \|(u, v)\|^4 \|u - u_1\|. \tag{3.12}$$

$$A_3 \leq c \|u_1^2\|_{L_x^{5/2} L_t^5} \|v\|_{L_x^5 L_t^{10}} \|\partial_x v\|_{L_x^\infty L_t^2} \|v - v_1\|_{L_x^5 L_t^{10}} \leq c \|(u_1, v_1)\|^2 \|(u, v)\|^2 \|v - v_1\|. \tag{3.13}$$

$$A_7 \leq c \|u_1\|_{L_x^5 L_t^{10}} \|v^2\|_{L_x^{5/2} L_t^5} \|\partial_x u\|_{L_x^\infty L_t^2} \|v - v_1\|_{L_x^5 L_t^{10}} \leq c \|(u_1, v_1)\| \|(u, v)\|^3 \|v - v_1\|. \tag{3.14}$$

Now, inserting estimates for A_1, \dots, A_{10} in (3.11) we get

$$\|\Phi[u, v] - \Phi[u_1, v_1]\|_{L_x^5 L_t^{10}} \leq 10ca^4 \|(u - u_1, v - v_1)\|. \tag{3.15}$$

Also, using the arguments employed to get estimates (3.5) and (3.10), it is easy to obtain

$$\|\Phi[u, v] - \Phi[u_1, v_1]\|_{L_t^\infty L_x^2} \leq 5ca^4 \|(u - u_1, v - v_1)\|. \tag{3.16}$$

With an analogous argument we can obtain the similar estimates for Ψ too.

Combining all these estimates and the choice $c(10c\delta)^4 \leq 1/2$ we get

$$\|(\Phi[u, v] - \Phi[u_1, v_1], \Psi[u, v] - \Psi[u_1, v_1])\| \leq 1/2\|(u - u_1, v - v_1)\|. \quad (3.17)$$

Hence $(\Phi, \Psi) : \mathcal{X}_a \rightarrow \mathcal{X}_a$ is a contraction. The rest of the proof follows a standard argument. \square

Now, we provide proof for the second result.

Proof of Theorem 1.2. As in the proof of the previous theorem, let us consider a ball

$$\mathcal{X}_a^s = \{(u, v) \in C(\mathbb{R} : X^s(\mathbb{R})) : \|(u, v)\|_s < a\},$$

in the complete metric space

$$\mathcal{X}^s = \{(u, v) \in C(\mathbb{R} : X^s(\mathbb{R})) : \|(u, v)\|_s < \infty\},$$

where

$$\|(u, v)\|_s = \max\{\|u\|_s, \|v\|_s\},$$

with

$$\|f\|_s = \|D_x^s f\|_{L_t^\infty L_x^2} + \|\partial_x f\|_{L_x^\infty L_t^2} + \|D_x^s \partial_x f\|_{L_x^\infty L_t^2} + \|f\|_{L_x^5 L_t^{10}} + \|D_x^s f\|_{L_x^5 L_t^{10}}. \quad (3.18)$$

Now our aim is to show that, for some $a > 0$ and $\delta > 0$, the application $\Phi \times \Psi$ defined by (3.2) maps \mathcal{X}_a^s into \mathcal{X}_a^s and is a contraction.

Here also, using symmetry of the system we will only estimate the first component Φ .

The estimates for the norms $\|\partial_x \Phi\|_{L_x^\infty L_t^2}$ and $\|\Phi\|_{L_x^5 L_t^{10}}$ are already obtained in (3.5) and (3.6) respectively. Now, using the linear estimates established in Section 2 along with

Leibniz rule and chain rule for fractional derivatives we obtain

$$\begin{aligned}
\|D_x^s \Phi\|_{L_x^2} &\leq \|D_x^s U(t)u_0\|_{L_x^2} + \|D_x^s \int_0^t U(t-t')\partial_x(u^2v^3)(t') dt'\|_{L_x^2} \\
&\leq c\|u_0\|_{H^s} + c\|D_x^s(u^2v^3)\|_{L_x^1L_t^2} \\
&\leq c\|u_0\|_{H^s} + c\|D_x^s(u^2v^3) - D_x^s(u^2)v^3 - u^2D_x^s(v^3)\|_{L_x^1L_t^2} \\
&\quad + c\|D_x^s(u^2)v^3\|_{L_x^1L_t^2} + c\|u^2D_x^s(v^3)\|_{L_x^1L_t^2} \\
&\leq c\|u_0\|_{H^s} + c\|D_x^s(u^2)\|_{L_x^{5/2}L_t^5}\|v^3\|_{L_x^{5/3}L_t^{10/3}} + c\|u^2\|_{L_x^{5/2}L_t^5}\|D_x^s(v^3)\|_{L_x^{5/3}L_t^{10/3}} \\
&\leq c\|u_0\|_{H^s} + c\|u\|_{L_x^5L_t^{10}}\|D_x^s u\|_{L_x^5L_t^{10}}\|v\|_{L_x^5L_t^{10}}^3 + c\|u\|_{L_x^5L_t^{10}}^2\|D_x^s v\|_{L_x^5L_t^{10}}\|v\|_{L_x^5L_t^{10}}^2 \\
&\leq c\|(u_0, v_0)\|_{H^s \times H^s} + c\|(u, v)\|_s^5.
\end{aligned} \tag{3.19}$$

Therefore

$$\|D_x^s \Phi\|_{L_t^\infty L_x^2} \leq c\|(u_0, v_0)\|_{H^s \times H^s} + c\|(u, v)\|_s^5. \tag{3.20}$$

Similarly,

$$\begin{aligned}
\|D_x^s \partial_x \Phi\|_{L_x^\infty L_t^2} &\leq \|D_x^s \partial_x U(t)u_0\|_{L_x^\infty L_t^2} + \|D_x^s \partial_x \int_0^t U(t-t')\partial_x(u^2v^3)(t') dt'\|_{L_x^\infty L_t^2} \\
&\leq c\|D_x^s u_0\|_{L_x^2} + c\|\partial_x^2 \int_0^t U(t-t')D_x^s(u^2v^3)(t') dt'\|_{L_x^\infty L_t^2} \\
&\leq c\|u_0\|_{H^s} + c\|D_x^s(u^2v^3)\|_{L_x^1L_t^2} \\
&\leq c\|(u_0, v_0)\|_{H^s \times H^s} + c\|(u, v)\|_s^5,
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
\|D_x^s \Phi\|_{L_x^5 L_t^{10}} &\leq \|D_x^s U(t)u_0\|_{L_x^5 L_t^{10}} + \|D_x^s \int_0^t U(t-t')\partial_x(u^2v^3)(t') dt'\|_{L_x^5 L_t^{10}} \\
&\leq c\|D_x^s u_0\|_{L_x^2} + c\|D_x^s \partial_x(u^2v^3)\|_{L_x^{5/4}L_t^{10/9}} \\
&\leq c\|u_0\|_{H^s} + c\|D_x^s(u^2\partial_x v^3)\|_{L_x^{5/4}L_t^{10/9}} + c\|D_x^s(\partial_x(u^2)v^3)\partial_x u\|_{L_x^{5/4}L_t^{10/9}} \\
&\leq c\|(u_0, v_0)\|_{H^s \times H^s} + c\|(u, v)\|_s^5.
\end{aligned} \tag{3.22}$$

Therefore, combining (3.5), (3.6) and (3.20) – (3.22) we obtain,

$$\|\Phi\|_s \leq c\|(u_0, v_0)\|_{H^s \times H^s} + c\|(u, v)\|_s^5. \quad (3.23)$$

In an analogous manner it is easy to get,

$$\|\Psi\|_s \leq c\|(u_0, v_0)\|_{H^s \times H^s} + c\|(u, v)\|_s^5. \quad (3.24)$$

Hence, for $(u, v) \in \mathcal{X}_a^s$,

$$\|(\Phi, \Psi)\|_s \leq c\|(u_0, v_0)\|_{H^s \times H^s} + c\|(u, v)\|_s^5 \leq c\delta + ca^5. \quad (3.25)$$

Let us choose δ such that $c(10c\delta)^4 \leq 1/2$ and $a \in (2c\delta, 3c\delta)$. With these choices we get from (3.25),

$$\|(\Phi, \Psi)\|_s \leq \frac{a}{2} + \frac{a}{2}.$$

Therefore, $\Phi \times \Psi$ maps \mathcal{X}_a^s into \mathcal{X}_a^s .

With the similar argument, one can prove that $\Phi \times \Psi$ is a contraction. The rest of the proof follows standard argument. \square

Remark 3.1. *In the Theorems 1.1 and 1.2 we obtained the global solution to the IVP (1.5) for given data with $H^s \times H^s$ norm less than δ . It would be interesting to find the exact value of δ . As a motivation for this, there is a similar result for the single critical KdV equation in the work of Angulo, Bona, Linares and Scialom in [3].*

Proof of Theorem 1.3. First, let us prove the theorem for given data in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. Now define a complete metric space \mathcal{X}^T , in which we are going to find solution to the IVP (1.5), by

$$\mathcal{X}^T = \{(u, v) \in C([-T, T] : X(\mathbb{R})) : \|(u, v)\| < \infty\},$$

where

$$\|(u, v)\| = \max\{\|u\|, \|v\|\},$$

with

$$\|u\| = \|u - U(t)u_0\|_{L_T^\infty L_x^2} + \|\partial_x u\|_{L_x^\infty L_T^2} + \|u\|_{L_x^5 L_T^{10}} \quad (3.26)$$

and similar for v . Also, define a ball

$$\mathcal{X}_a^T = \{(u, v) \in \mathcal{X}^T : \|(u, v)\| < a\}.$$

We show that for some $a > 0$, $\Phi \times \Psi$ defined by (3.2) maps \mathcal{X}_a^T into \mathcal{X}_a^T and is a contraction.

Now, using linear estimates and Lemma 2.8 we get,

$$\begin{aligned} \|\Phi - U(t)u_0\|_{L_T^\infty L_x^2} &\leq \left\| \int_0^t U(t-t') \partial_x(u^2 v^3)(t') dt' \right\|_{L_T^\infty L_x^2} \\ &\leq c \|u^2 v^3\|_{L_x^1 L_T^2} \leq c \|u\|_{L_x^5 L_T^{10}}^2 \|v\|_{L_x^5 L_T^{10}}^3 \\ &\leq c \|(u, v)\|^5, \end{aligned} \quad (3.27)$$

$$\begin{aligned} \|\partial_x \Phi\|_{L_x^\infty L_T^2} &\leq \|\partial_x U(t)u_0\|_{L_x^\infty L_T^2} + \left\| \partial_x \int_0^t U(t-t') \partial_x(u^2 v^3)(t') dt' \right\|_{L_x^\infty L_T^2} \\ &\leq c\epsilon + c \|u^2 v^3\|_{L_x^1 L_T^2} \\ &\leq c\epsilon + c \|(u, v)\|^5 \end{aligned} \quad (3.28)$$

and similarly,

$$\|\Phi\|_{L_x^5 L_T^{10}} \leq c\epsilon + c \|(u, v)\|^5. \quad (3.29)$$

In an analogous manner we can obtain similar estimates for Ψ too.

Combining all these estimates we obtain for $(u, v) \in \mathcal{X}_a^T$,

$$\|(\Phi, \Psi)\| \leq c\epsilon + c \|(u, v)\|^5 \leq c\epsilon + ca^5. \quad (3.30)$$

If we choose ϵ and a in such a way that $c\epsilon + ca^4 < a$, we get $(\Phi, \Psi) \in \mathcal{X}_a^T$.

For $(u, v), (u_1, v_1) \in \mathcal{X}_a^T$, a similar argument leads to

$$\|(\Phi[u, v] - \Phi[u_1, v_1], \Psi[u, v] - \Psi[u_1, v_1])\| \leq 10ca^4 \|(u - u_1, v - v_1)\|. \quad (3.31)$$

Thus, for $10ca^4 < 1/2$, $\Phi \times \Psi$ is a contraction on \mathcal{X}_a^T .

Note that, if we choose $\epsilon > 0$ such that $c(10c\epsilon)^4 < 1/2$ and $a \in (2c\epsilon, 3c\epsilon)$ then the both conditions $c\epsilon + ca^4 < a$ and $10ca^4 < 1/2$ are satisfied. A standard argument completes

the rest of the proof in this case. The general case is similar, since as in the Proof of Theorem 1.2, the estimates in the involved norms appear linearly after interpolation. \square

Proof of Theorem 1.4. Following the procedure employed in the proof of the previous theorems, let us consider a ball

$$\mathcal{X}_a^T = \{(u, v) \in C([-T, T] : X^s(\mathbb{R})) : \|(u, v)\|_s < a\},$$

in a complete metric space, in which we are going to find solution to the IVP (1.5)

$$\mathcal{X}^T = \{(u, v) \in C([-T, T] : X^s(\mathbb{R})) : \|(u, v)\|_s < \infty\},$$

where

$$\|(u, v)\|_s = \max\{\|u\|_s, \|v\|_s\},$$

with

$$\begin{aligned} \|f\|_s &= \|D_s^s f\|_{L_T^\infty L_x^2} + \|\partial_x f\|_{L_x^\infty L_T^2} + \|D_x^s \partial_x f\|_{L_x^\infty L_T^2} + \|f\|_{L_x^5 L_T^{10}} + \|D_x^s f\|_{L_x^5 L_T^{10}} \\ &\quad + \|D_t^{s/3} f\|_{L_x^5 L_T^{10}} + \|D_t^{s/3} \partial_x f\|_{L_x^\infty L_T^2}. \end{aligned} \quad (3.32)$$

Our aim is to show that, for some $a > 0$ and $T > 0$, the application $\Phi \times \Psi$ defined by (3.2) maps \mathcal{X}_a^T into \mathcal{X}_a^T and is a contraction.

As earlier, we will estimate only the first component Φ . Using the linear estimates established in section 2, Leibniz rule and chain rule for fractional derivatives along with the estimate (2.16) we obtain

$$\begin{aligned} \|D_x^s \Phi\|_{L_x^2} &\leq c\|u_0\|_{H^s} + c\|u\|_{L_x^5 L_T^{10}} \|D_x^s u\|_{L_x^5 L_T^{10}} \|v\|_{L_x^5 L_T^{10}}^3 + c\|u\|_{L_x^5 L_T^{10}}^2 \|D_x^s v\|_{L_x^5 L_T^{10}} \|v\|_{L_x^5 L_T^{10}}^2 \\ &\leq c\|u_0\|_{H^s} + cT^{s/3} \|D_t^{s/3} u\|_{L_x^5 L_T^{10}} \|D_x^s u\|_{L_x^5 L_T^{10}} T^s \|D_t^{s/3} v\|_{L_x^5 L_T^{10}}^3 \\ &\quad + cT^{2s/3} \|D_t^{s/3} u\|_{L_x^5 L_T^{10}}^2 \|D_x^s v\|_{L_x^5 L_T^{10}} T^{2s/3} \|D_t^{s/3} v\|_{L_x^5 L_T^{10}}^2 \\ &\leq c\|u_0\|_{H^s} + cT^{4s/3} \|u\|_s^2 \|v\|_s^3 + cT^{4s/3} \|u\|_s^2 \|v\|_s^3 \\ &\leq c\|(u_0, v_0)\|_{H^s \times H^s} + cT^{4s/3} \|(u, v)\|_s^5. \end{aligned}$$

Therefore,

$$\|D_x^s \Phi\|_{L_T^\infty L_x^2} \leq c\|(u_0, v_0)\|_{H^s \times H^s} + cT^{4s/3} \|(u, v)\|_s^5. \quad (3.33)$$

Similarly,

$$\begin{aligned} \|\partial_x \Phi\|_{L_x^\infty L_T^2} &\leq c\|u_0\|_{H^s} + c\|u\|_{L_x^5 L_T^{10}}^2 \|v\|_{L_x^5 L_T^{10}}^3 \\ &\leq c\|(u_0, v_0)\|_{H^s \times H^s} + cT^{4s/3} \|(u, v)\|_s^5, \end{aligned} \quad (3.34)$$

and

$$\|D_x^s \partial_x \Phi\|_{L_x^\infty L_T^2} \leq c\|(u_0, v_0)\|_{H^s \times H^s} + cT^{4s/3} \|(u, v)\|_s^5. \quad (3.35)$$

Finally,

$$\begin{aligned} \|\Phi\|_{L_x^5 L_T^{10}} &\leq c\|u_0\|_{H^s} + c\|u\|_{L_x^5 L_T^{10}}^2 \|v\|_{L_x^5 L_T^{10}}^2 \|\partial_x v\|_{L_x^\infty L_T^2} + c\|u\|_{L_x^5 L_T^{10}} \|v\|_{L_x^5 L_T^{10}}^3 \|\partial_x u\|_{L_x^\infty L_T^2} \\ &\leq c\|(u_0, v_0)\|_{H^s \times H^s} + cT^{4s/3} \|(u, v)\|_s^5, \end{aligned} \quad (3.36)$$

and

$$\|D_x^s \Phi\|_{L_x^5 L_T^{10}} \leq c\|(u_0, v_0)\|_{H^s \times H^s} + cT^{4s/3} \|(u, v)\|_s^5. \quad (3.37)$$

Now, using (2.17), (2.18) and the argument employed to get estimates for other norms we can get,

$$\|D_t^{s/3} \Phi\|_{L_x^5 L_T^{10}} \leq c\|(u_0, v_0)\|_{H^s \times H^s} + cT^{4s/3} \|(u, v)\|_s^5 \quad (3.38)$$

and

$$\|D_t^{s/3} \partial_x \Phi\|_{L_x^\infty L_T^2} \leq c\|(u_0, v_0)\|_{H^s \times H^s} + cT^{4s/3} \|(u, v)\|_s^5. \quad (3.39)$$

From (3.33) to (3.39) we obtain,

$$\|\Phi\|_s \leq c\|(u_0, v_0)\|_{H^s \times H^s} + cT^{4s/3} \|(u, v)\|_s^5. \quad (3.40)$$

In an analogous manner one can easily get,

$$\|\Psi\|_s \leq c\|(u_0, v_0)\|_{H^s \times H^s} + cT^{4s/3} \|(u, v)\|_s^5. \quad (3.41)$$

Hence, for $(u, v) \in \mathcal{X}_a^T$,

$$\|(\Phi, \Psi)\|_s \leq c\|(u_0, v_0)\|_{H^s \times H^s} + cT^{4s/3}\|(u, v)\|_s^5 \leq c\delta + ca^5. \quad (3.42)$$

Let us choose $a = 2c\|(u_0, v_0)\|_{H^s \times H^s}$ and T such that $cT^{4s/3}a^4 < 1/2$. With these choices we get from (3.42),

$$\|(\Phi, \Psi)\|_s \leq \frac{a}{2} + \frac{a}{2}.$$

Therefore, $\Phi \times \Psi$ maps \mathcal{X}_a^T into \mathcal{X}_a^T .

With the similar argument, one can prove that $\Phi \times \Psi$ is a contraction. The rest of the proof follows standard argument. \square

Remark 3.2. *From the choice of a and T in the proof of Theorem 1.4 it is clear that the local existence time is given by*

$$T \leq c\|(u_0, v_0)\|_{H^s \times H^s}^{-3/s}. \quad (3.43)$$

4. PROOF OF THE GLOBAL RESULT WITH DATA NOT SO SMALL

As mentioned in the introduction, we use the frequency splitting argument of Bourgain. In particular, we closely follow the scheme in [7]. We decompose the given data $(u_0, v_0) \in X^s$, $s < 1$ to low and high frequency terms as,

$$\begin{cases} u_0(x) = (\chi_{\{|\xi| \leq N\}} \widehat{u}_0(\xi))^\vee(x) + (\chi_{\{|\xi| > N\}} \widehat{u}_0(\xi))^\vee(x) := \phi_1(x) + \phi_2(x) \\ v_0(x) = (\chi_{\{|\xi| \leq N\}} \widehat{v}_0(\xi))^\vee(x) + (\chi_{\{|\xi| > N\}} \widehat{v}_0(\xi))^\vee(x) := \psi_1(x) + \psi_2(x), \end{cases} \quad (4.1)$$

where $N \gg 1$ arbitrary but fixed for now, whose exact value will be selected later.

Then we have, $(\phi_1, \psi_1) \in X^\beta$, $0 < \beta \leq 1$ and $(\phi_2, \psi_2) \in X^\rho$, $0 < \rho \leq s < 1$ with

$$\|(\phi_1, \psi_1)\|_{X^\beta} \lesssim N^{\beta-s} \lesssim N^{1-s}, \quad \|(\phi_1, \psi_1)\|_X < \|(S, S)\|_X, \quad (4.2)$$

and

$$\|(\phi_2, \psi_2)\|_{X^\rho} \lesssim N^{\rho-s}, \quad 0 < \rho \leq s < 1. \quad (4.3)$$

We evolve (ϕ_1, ψ_1) according to the IVP

$$\begin{cases} u_{1t} + u_{1xxx} + (u_1^2 v_1^3)_x = 0 \\ v_{1t} + v_{2xxx} + (u_1^3 v_1^2)_x = 0 \\ u_1(x, 0) = \phi_1(x), \quad v_1(x, 0) = \psi_1(x), \end{cases} \quad (4.4)$$

which is the same as the IVP (1.5). We evolve (ϕ_2, ψ_2) according to the difference equation

$$\begin{cases} u_{2t} + u_{2xxx} + ((u_1 + u_2)^2 (v_1 + v_2)^3)_x - (u_1^2 v_1^3)_x = 0 \\ v_{2t} + v_{2xxx} + ((u_1 + u_2)^3 (v_1 + v_2^2))_x - (u_1^3 v_1^2)_x = 0 \\ u_2(x, 0) = \phi_2(x), \quad v_2(x, 0) = \psi_2(x), \end{cases} \quad (4.5)$$

with coefficients depending on the solution (u_1, v_1) to the IVP (4.4). It is clear that $u = u_1 + u_2$ and $v = v_1 + v_2$ solve the IVP (1.5). For simplicity, let us write (4.5) as

$$\begin{cases} u_{2t} + u_{2xxx} + \partial_x F = 0 \\ v_{2t} + v_{2xxx} + \partial_x G = 0 \\ u_2(x, 0) = \phi_2(x), \quad v_2(x, 0) = \psi_2(x), \end{cases} \quad (4.6)$$

where

$$\begin{aligned} F = & 3u_1^2 v_1^2 v_2 + 3u_1^2 v_1 v_2^2 + u_1^2 v_2^3 + 2u_1 u_2 v_1^3 + 6u_1 u_2 v_1^2 v_2 + 6u_1 u_2 v_1 v_2^2 \\ & + 2u_1 u_2 v_2^3 + u_2^2 v_1^3 + 3u_2^2 v_1^2 v_2 + 3u_2^2 v_1 v_2^2 + u_2^2 v_2^3, \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} G = & 2u_1^3 v_1 v_2 + u_1^3 v_2^2 + 3u_1^2 u_2 v_1^2 + 6u_1^2 u_2 v_1 v_2 + 3u_1^2 u_2 v_2^2 + 3u_1 u_2^2 v_1^2 \\ & + 6u_1 u_2^2 v_1 v_2 + 3u_1 u_2^2 v_2^2 + u_2^3 v_1^2 + 2u_2^3 v_1 v_2 + u_2^3 v_2^2. \end{aligned} \quad (4.8)$$

Note that from Theorem 1.4 we have the existence result for the IVP (4.4). To get the existence result for the IVP (4.6) we need the following theorem.

Theorem 4.1. *Suppose the initial data (ϕ_1, ψ_1) of the IVP (4.4) satisfy*

$$\begin{cases} \|(\phi_1, \psi_1)\|_X \leq c \\ \|(\phi_1, \psi_1)\|_{X^1} \leq cN^{1-s}. \end{cases} \quad (4.9)$$

Then for the existence time $T \sim c\|(\phi_1, \psi_1)\|_{X^1}^{-3} \sim cN^{-3(1-s)}$ obtained in Theorem 1.4

(i) The solution (u_1, v_1) to the IVP (4.4) satisfies,

$$\sup_t \|(u_1(t), v_1(t))\|_{X^1} = \sup_t [\|u_1(t)\|_{H^1} + \|v_1(t)\|_{H^1}] \leq cN^{1-s}. \quad (4.10)$$

(ii) Moreover, for any $\beta \in (0, 1)$, the solution (u_1, v_1) to the IVP (4.4) satisfies,

$$\|(u_1, v_1)\|_\beta \sim N^{(1-s)\beta}, \quad (4.11)$$

where $\|(u_1, v_1)\|_\beta = \max\{\|u_1\|_\beta, \|v_1\|_\beta\}$ and $\|f\|_\beta$ as in (3.32).

Proof. The proof of (4.10) follows by using the conservation laws combined with the Gagliardo-Nirenberg inequality. The estimate (4.11) can be obtained by using the hypothesis (4.9) and the local well-posedness result. \square

The following theorem provides the local existence result for the IVP (4.6) whose existence time coincides with that of (u_1, v_1) .

Theorem 4.2. *Let $(\phi_2, \psi_2) \in X^s$, $s > 0$ and (u_1, v_1) be the unique solution given by Theorem 1.4 satisfying the conditions of Theorem 4.1. Then there exists a unique solution (u_2, v_2) to the IVP (4.6) in the same interval of existence of (u_1, v_1) , $[0, T]$ such that,*

$$(u_2, v_2) \in C([-T, T] : H^s(\mathbb{R}) \times H^s(\mathbb{R})), \quad (4.12)$$

$$\|\partial_x u_2\|_{L_x^\infty L_T^2} < \infty, \quad \|\partial_x v_2\|_{L_x^\infty L_T^2} < \infty, \quad (4.13)$$

$$\|u_2\|_{L_x^5 L_T^{10}} < \infty, \quad \|v_2\|_{L_x^5 L_T^{10}} < \infty, \quad (4.14)$$

$$\|D_x^s u_2\|_{L_x^5 L_T^{10}} + \|D_t^{s/3} u_2\|_{L_x^5 L_T^{10}} < \infty, \quad \|D_x^s v_2\|_{L_x^5 L_T^{10}} + \|D_t^{s/3} v_2\|_{L_x^5 L_T^{10}} < \infty, \quad (4.15)$$

$$\|D_x^s \partial_x u_2\|_{L_x^\infty L_T^2} + \|D_t^{s/3} \partial_x u_2\|_{L_x^\infty L_T^2} < \infty, \quad \|D_x^s \partial_x v_2\|_{L_x^\infty L_T^2} + \|D_t^{s/3} \partial_x v_2\|_{L_x^\infty L_T^2} < \infty. \quad (4.16)$$

Proof. The proof of this theorem follows the same argument used to prove Theorem 1.4. As earlier, we consider the equivalent integral equation associated to the IVP (4.6),

$$\begin{cases} u_2(t) = U(t)\phi_2 - \int_0^t U(t-t')\partial_x F(t') dt' \\ v_2(t) = U(t)\psi_2 - \int_0^t U(t-t')\partial_x G(t') dt', \end{cases} \quad (4.17)$$

where F and G are defined in (4.7) and (4.8) respectively.

Let us define a ball

$$\mathcal{X}_a^T = \{(u_2, v_2) \in C([0, T] : X^s(\mathbb{R})) : \|(u_2, v_2)\|_s < a\},$$

in a complete metric space

$$\mathcal{X}^T = \{(u_2, v_2) \in C([0, T] : X^s(\mathbb{R})) : \|(u_2, v_2)\|_s < \infty\},$$

where

$$\|(u_2, v_2)\|_s = \max\{\|u_2\|_s, \|v_2\|_s\},$$

with $\|f\|_s$ as in (3.32).

Finally, we define,

$$\begin{cases} \Phi_{\phi_2}[u_2, v_2] = U(t)\phi_2 - \int_0^t U(t-t')\partial_x F(t') dt' \\ \Psi_{\psi_2}[u_2, v_2] = U(t)\psi_2 - \int_0^t U(t-t')\partial_x G(t') dt'. \end{cases} \quad (4.18)$$

and show that, for some $a > 0$ and $T > 0$, the application $\Phi \times \Psi$ maps \mathcal{X}_a^T into \mathcal{X}_a^T and is a contraction. As in the previous cases we can estimate each component separately.

Using the estimate (2.2) we obtain

$$\|D_x^s \Phi\|_{L_x^2} \leq \|\phi\|_s + c \|\partial_x \int_0^T U(t-t') D_x^s F(t')\|_{L_x^2} \leq \|D_x^s F\|_{L_x^1 L_T^2}. \quad (4.19)$$

Now, considering the term $3u_1^2 v_1^2 v_2$ in F and using Hölder's inequality, Leibniz rule and chain rule for fractional derivatives along with the estimate (2.16) we get

$$\begin{aligned}
\|D_x^s(3u_1^2v_1^2v_2)\|_{L_x^1L_T^2} &\leq c\|D_x^s(u_1^2v_1^2)v_2\|_{L_x^1L_T^2} + c\|u_1^2v_1^2\|_{L_x^{\frac{5}{4}}L_T^{\frac{5}{2}}}\|D_x^sv_2\|_{L_x^5L_T^{10}} \\
&\leq c\|D_x^s(u_1^2v_1^2)\|_{L_x^{\frac{5}{4}}L_T^{\frac{5}{2}}}\|v_2\|_{L_x^5L_T^{10}} + c\|u_1v_1\|_{L_x^{\frac{5}{2}}L_T^5}^2\|D_x^sv_2\|_{L_x^5L_T^{10}} \\
&\leq c\left[\|D_x^s(u_1^2)v_1^2\|_{L_x^{\frac{5}{4}}L_T^{\frac{5}{2}}} + \|u_1^2\|_{L_x^{\frac{5}{2}}L_T^5}\|D_x^sv_1^2\|_{L_x^{\frac{5}{2}}L_T^5}\right]\|v_2\|_{L_x^5L_T^{10}} \\
&\quad + \|u_1\|_{L_x^5L_T^{10}}^2\|v_1\|_{L_x^5L_T^{10}}^2\|D_x^sv_2\|_{L_x^5L_T^{10}} \\
&\leq c\left[\|D_x^s(u_1^2)\|_{L_x^{\frac{5}{2}}L_T^5}\|v_1^2\|_{L_x^{\frac{5}{2}}L_T^5} + \|u_1\|_{L_x^5L_T^{10}}^2\|D_x^sv_1\|_{L_x^5L_T^{10}}\|v_1\|_{L_x^5L_T^{10}}\right]\|v_2\|_{L_x^5L_T^{10}} \\
&\quad + \|u_1\|_{L_x^5L_T^{10}}^2\|v_1\|_{L_x^5L_T^{10}}^2\|D_x^sv_2\|_{L_x^5L_T^{10}} \\
&\leq c\left[\|D_x^su_1\|_{L_x^5L_T^{10}}\|u_1\|_{L_x^5L_T^{10}}\|v_1\|_{L_x^5L_T^{10}}^2 + \|u_1\|_{L_x^5L_T^{10}}^2\|D_x^sv_1\|_{L_x^5L_T^{10}}\|v_1\|_{L_x^5L_T^{10}}\right]\|v_2\|_{L_x^5L_T^{10}} \\
&\quad + \|u_1\|_{L_x^5L_T^{10}}^2\|v_1\|_{L_x^5L_T^{10}}^2\|D_x^sv_2\|_{L_x^5L_T^{10}} \\
&\leq cT^{4s/3}\left[\|D_x^su_1\|_{L_x^5L_T^{10}}\|D_t^{s/3}u_1\|_{L_x^5L_T^{10}}\|D_t^{s/3}v_1\|_{L_x^5L_T^{10}}^2\right. \\
&\quad \left. + \|D_t^{s/3}u_1\|_{L_x^5L_T^{10}}^2\|D_x^sv_1\|_{L_x^5L_T^{10}}\|D_t^{s/3}v_1\|_{L_x^5L_T^{10}}\right]\|D_t^{s/3}v_2\|_{L_x^5L_T^{10}} \\
&\quad + cT^{4s/3}\|D_t^{s/3}u_1\|_{L_x^5L_T^{10}}^2\|D_t^{s/3}v_1\|_{L_x^5L_T^{10}}^2\|D_x^sv_2\|_{L_x^5L_T^{10}} \\
&\leq cT^{4s/3}\|(u_1, v_1)\|_s^4\|(u_2, v_2)\|_s.
\end{aligned}$$

We can obtain similar estimates for the other terms in F too and get from (4.19)

$$\begin{aligned}
\|D_x^s\Phi\|_{L_x^2} &\leq c\|\phi_2\|_s + cT^{4s/3}\left\{\|(u_1, v_1)\|_s^4 + \|(u_1, v_1)\|_s^3\|(u_2, v_2)\|_s + \|(u_1, v_1)\|_s^2\|(u_2, v_2)\|_s^2\right. \\
&\quad \left. + \|(u_1, v_1)\|_s\|(u_2, v_2)\|_s^3 + \|(u_2, v_2)\|_s^4\right\}\|(u_2, v_2)\|_s \\
&\leq c\|\phi_2\|_s + cT^{4s/3}\left\{\|(u_1, v_1)\|_1^{4s} + \|(u_1, v_1)\|_1^{3s}\|(u_2, v_2)\|_s + \|(u_1, v_1)\|_1^{2s}\|(u_2, v_2)\|_s^2\right. \\
&\quad \left. + \|(u_1, v_1)\|_1^s\|(u_2, v_2)\|_s^3 + \|(u_2, v_2)\|_s^4\right\}\|(u_2, v_2)\|_s.
\end{aligned} \tag{4.20}$$

With the analogous argument, we can get estimates similar to (4.20) for the other norms involved in the definition of $\|\cdot\|_s$ and obtain

$$\|\Phi\|_s \leq \|\phi_2\|_{H^s} + cT^{4s/3}f(\|(u_1, v_1)\|_1, \|(u_2, v_2)\|_s)\|(u_2, v_2)\|_s, \tag{4.21}$$

where f is an appropriate polynomial in its arguments.

Also, one can derive similar estimates for $\|\Psi\|_s$ and combine those with (4.21) to have

$$\|(\Phi, \Psi)\|_s \leq \|(\phi_2, \psi_2)\|_{H^s \times H^s} + cT^{4s/3} f(\|(u_1, v_1)\|_1, \|(u_2, v_2)\|_s) \|(u_2, v_2)\|_s. \quad (4.22)$$

Now, choosing $a = \max\{\|(\phi_1, \psi_1)\|_{X^1}, \|(\phi_2, \psi_2)\|_{X^s}\}$ (which is $\|(\phi_1, \psi_1)\|_{X^1}$) and choosing T as in Theorem 4.1 we get $cT^{4s/3}a^4 < 1/2$. Since, $f(\|(u_1, v_1)\|_1, \|(u_2, v_2)\|_s) \leq ca^4$, $\Phi \times \Psi$ maps the ball \mathcal{X}_a^T into \mathcal{X}_a^T .

We can prove that the mapping $\Phi \times \Psi$ is a contraction with a similar argument. This concludes the proof of the theorem. \square

The following proposition provides the L^2 norm estimates for the solution (u_2, v_2) .

Proposition 4.3. *Define $\|(u_2, v_2)\|_0 = \max\{\|u_2\|_0, \|v_2\|_0\}$ where,*

$$\|f\|_0 = \|f\|_{L_T^\infty L_x^2} + \|\partial_x f\|_{L_x^\infty L_T^2} + \|f\|_{L_x^5 L_T^{10}}.$$

Let (u_1, v_1) and (u_2, v_2) be solutions to the IVPs (4.4) and (4.6) with $(\phi_1, \psi_1) \in X^1$ and $(\phi_2, \psi_2) \in X^s$ respectively satisfying $\|(\phi_1, \psi_1)\|_{X^1} \sim N^{1-s}$ and $\|(\phi_2, \psi_2)\|_X \sim N^{-s}$, $0 < s < 1$. Then

$$\|(u_2, v_2)\|_0 \sim N^{-s}. \quad (4.23)$$

Proof. The proof follows by using the equivalent integral equation (4.17), the linear estimates and the choice of T in Theorem 4.2. So, we omit the details. \square

The following Proposition gives the estimates for the X^1 and X norms of the inhomogeneous part of the evolution of the high frequency part.

Proposition 4.4. *Let F and G be given by (4.7) and (4.8) with (u_1, v_1) and (u_2, v_2) solutions to the IVPs (4.4) and (4.6) respectively. Define,*

$$(z_1(t), z_2(t)) = \left(- \int_0^t U(t-t') \partial_x F(t') dt', - \int_0^t U(t-t') \partial_x G(t') dt' \right). \quad (4.24)$$

Let (ϕ_1, ψ_1) and (ϕ_2, ψ_2) satisfy the hypothesis of Proposition 4.3. If $0 < s < 1$, then

$$\sup_{t \in [0, T]} \|(z_1(t), z_2(t))\|_{X^1} \leq cN^{1-2s} \quad (4.25)$$

and

$$\|(z_1(t), z_2(t))\|_X \leq cN^{-s}. \quad (4.26)$$

Proof. We apply the estimate (2.2) to get

$$\|\partial_x z_1\|_{L^2} = \|\partial_x \int_0^t U(t-t') \partial_x F(t') dt'\|_{L^2} \leq c \|\partial_x F\|_{L_x^1 L_T^2}. \quad (4.27)$$

As in the proof of the Theorem 4.2, we consider the term $u_1^2 v_1^2 v_2$ and use argument as in [7] together with the choice of T and estimates in Theorem 4.1 and Proposition 4.3 to get

$$\begin{aligned} \|\partial_x (u_1^2 v_1^2 v_2)\|_{L_x^1 L_T^2} &\leq \|u_1^2 v_1^2 v_{2x}\|_{L_x^1 L_T^2} + \|u_1^2 v_1 v_{1x} v_2\|_{L_x^1 L_T^2} + \|u_1 u_{1x} v_1^2 v_2\|_{L_x^1 L_T^2} \\ &\leq c \|u_1\|_{L_x^4 L_T^\infty}^2 \|v_1\|_{L_x^4 L_T^\infty}^2 \|v_{2x}\|_{L_x^\infty L_T^2} + c \|u_1\|_{L_x^5 L_T^{10}}^2 \|v_1\|_{L_x^5 L_T^{10}} \|v_{1x}\|_{L_x^5 L_T^{10}} \|v_2\|_{L_x^5 L_T^{10}} \\ &\quad + c \|u_1\|_{L_x^5 L_T^{10}} \|u_{1x}\|_{L_x^5 L_T^{10}} \|v_1\|_{L_x^5 L_T^{10}}^2 \|v_2\|_{L_x^5 L_T^{10}} \\ &\leq c \|u_1\|_{L_x^4 L_T^\infty}^2 \|v_1\|_{L_x^4 L_T^\infty}^2 \|v_{2x}\|_{L_x^\infty L_T^2} + cT \|D_t^{1/3} u_1\|_{L_x^5 L_T^{10}}^2 \|D_t^{1/3} v_1\|_{L_x^5 L_T^{10}} \|v_{1x}\|_{L_x^5 L_T^{10}} \|v_2\|_{L_x^5 L_T^{10}} \\ &\quad + cT \|D_t^{1/3} u_1\|_{L_x^5 L_T^{10}} \|u_{1x}\|_{L_x^5 L_T^{10}} \|D_t^{1/3} v_1\|_{L_x^5 L_T^{10}}^2 \|v_2\|_{L_x^5 L_T^{10}} \\ &\leq c \|(u_1, v_1)\|_{\frac{4}{1}}^4 \|(u_2, v_2)\|_0 + cT \|(u_1, v_1)\|_1^4 \|(u_2, v_2)\|_0 \\ &\leq c \|(u_1, v_1)\|_1 \|(u_2, v_2)\|_0 + cT \|(u_1, v_1)\|_1^4 \|(u_2, v_2)\|_0 \\ &\leq cN^{1-s} N^{-s} + cN^{-3(1-s)} N^{4(1-s)} N^{-s} \\ &\leq cN^{1-2s}. \end{aligned} \quad (4.28)$$

We can obtain similar estimates for the other terms in F too.

Using an analogous argument we can get,

$$\|z_1\|_{L_x^2} \leq cN^{-s}.$$

Finally, we can also obtain similar estimates for z_2 and that concludes the proof. \square

Now, we are in position to provide proof of the global well-posedness result.

Proof of Theorem 1.5. Let $(u_0, v_0) \in X^s(\mathbb{R})$, $0 < s < 1$ such that $\|(u_0, v_0)\|_X < \|(S, S)\|_X$. Also, consider $N \gg 1$ be arbitrary but fixed to be determined later. Let us decompose the initial data as in (4.1) to

$$\begin{cases} u_0(x) = \phi_1(x) + \phi_2(x), \\ v_0(x) = \psi_1(x) + \psi_2(x). \end{cases} \quad (4.29)$$

Then we have,

$$\begin{cases} \|(\phi_1, \psi_1)\|_{X^\beta} \leq cN^{\beta(1-s)}, & 0 < \beta \leq 1 \\ \|(\phi_1, \psi_1)\|_X < \|(S, S)\|_X. \end{cases} \quad (4.30)$$

$$\|(\phi_2, \psi_2)\|_{X^\rho} \leq cN^{\rho-s}, \quad 0 < \rho \leq s < 1. \quad (4.31)$$

Consider the IVP (4.4) with initial data $(\phi_1, \psi_1) \in X^1$. From Theorem 1.4 there exists T satisfying

$$T \leq c\|(\phi_1, \psi_1)\|_{X^1}^{-3} \sim N^{-3(1-s)}, \quad (4.32)$$

such that the IVP (4.4) has a unique solution (u_1, v_1) in the interval $[0, T]$. Moreover, from (4.10) we have

$$\sup_{t \in [0, T]} \|(u_1(t), v_1(t))\|_{X^1} \leq cN^{1-s}. \quad (4.33)$$

Now, we consider the IVP (4.6) with initial data (ϕ_2, ψ_2) . In Theorem 4.2 we found that the IVP (4.6) has a unique solution (u_2, v_2) defined in the same interval of existence of the solution (u_1, v_1) , $[0, T]$ and is given by (1.24), i.e.

$$\begin{cases} u_2(t) = U(t)\phi_2 + z_1(t) \\ v_2(t) = U(t)\psi_2 + z_2(t). \end{cases} \quad (4.34)$$

where $z_1(t)$ and $z_2(t)$ are given by (4.24).

As mentioned earlier, $u = u_1 + u_2$ and $v = v_1 + v_2$ solve the IVP (1.5) in the time interval $[0, T]$.

Given $\tilde{T} > 0$ arbitrary, we are interested in extending the solution (u, v) of the IVP (1.5) to the interval $[0, \tilde{T}]$. For this, we iterate the above process in each interval of size T unless covering the whole interval. Now, at the time $t = T$ we have,

$$\begin{cases} u(T) = u_1(T) + U(T)\phi_2 + z_1(T) \\ v(T) = v_1(T) + U(T)\psi_2 + z_2(T). \end{cases} \quad (4.35)$$

Now we decompose $(u(T), v(T))$ as,

$$\begin{cases} u(T) = \tilde{u}_1(T) + \tilde{u}_2(T) \\ v(T) = \tilde{v}_1(T) + \tilde{v}_2(T), \end{cases} \quad (4.36)$$

where,

$$\begin{cases} \tilde{u}_1(T) = u_1(T) + z_1(T), & \tilde{u}_2(T) = U(T)\phi_2 \\ \tilde{v}_1(T) = v_1(T) + z_2(T), & \tilde{v}_2(T) = U(T)\psi_2, \end{cases} \quad (4.37)$$

and evolve $(\tilde{u}_1(T), \tilde{v}_1(T))$ and $(\tilde{u}_2(T), \tilde{v}_2(T))$ according to the IVPs (4.4) and (4.6) respectively. Using previous procedure, to get solution to the IVP (1.5) in $[T, 2T]$ we must guarantee that $(\tilde{u}_1(T), \tilde{v}_1(T))$ and $(\tilde{u}_2(T), \tilde{v}_2(T))$ satisfy the respective conditions (4.30) and (4.31).

Since $U(t)$ is unitary in H^ρ , $(\tilde{u}_2(T), \tilde{v}_2(T))$ satisfies the same growth condition as that of (ϕ_2, ψ_2) , i.e, $(\tilde{u}_2(T), \tilde{v}_2(T)) \in X^\rho$ and

$$\|(\tilde{u}_2(T), \tilde{v}_2(T))\|_{X^\rho} = \|(\phi_2, \psi_2)\|_{X^\rho} \leq cN^{\rho-s}, \quad \rho \leq s.$$

Now, let us check how is the growth of the X^1 -norm and the X -norm of $(\tilde{u}_1(T), \tilde{v}_1(T))$. Using the estimate (4.33) and Proposition 4.4 we get

$$\begin{aligned} \|(\tilde{u}_1(T), \tilde{v}_1(T))\|_{X^1} &\leq \|(u_1(T), v_1(T))\|_{X^1} + \|(z_1(T), z_2(T))\|_{X^1} \\ &\leq cN^{1-s} + cN^{1-2s}. \end{aligned} \quad (4.38)$$

On the other hand, using (4.35) and the conservation law (1.3) we obtain,

$$\begin{aligned} \|(\tilde{u}_1(T), \tilde{v}_1(T))\|_X &\leq \|(u(T), v(T)) - (U(T)\phi_2, U(T)\psi_2)\|_X \\ &\leq \|(u(T), v(T))\|_X + \|(\phi_2, \psi_2)\|_X \\ &\leq \|(u_0, v_0)\|_X + N^{-s}. \end{aligned} \quad (4.39)$$

Therefore, choosing N sufficiently large enough, we obtain $\|(\tilde{u}_1(T), \tilde{v}_1(T))\|_X < \|(S, S)\|_X$. Hence, the conditions (4.30) and (4.31) are satisfied for the first iteration.

To cover the interval $[0, \tilde{T}]$ we must iterate the above process \tilde{T}/T times. As seen earlier, in each iteration, there will be a contribution of $\|(z_1, z_2)\|_{X^1}$ and $\|(z_1, z_2)\|_X$. From (4.38) we see that the total contribution of $\|(z_1, z_2)\|_{X^1}$ to cover $[0, \tilde{T}]$ is, $(\tilde{T}/T)N^{1-2s}$.

Thus the X^1 -norm of (z_1, z_2) will grow uniformly as N^{1-s} on the interval $[0, \tilde{T}]$ if we have,

$$\frac{\tilde{T}}{T}N^{1-2s} < cN^{1-s}. \quad (4.40)$$

Now, using $T \sim N^{-3(1-s)}$ from (4.32) we see that (4.40) is equivalent to,

$$\tilde{T}N^{3-4s} < c. \quad (4.41)$$

Therefore, to guarantee (4.40) we must choose $N = N(\tilde{T})$ large, satisfying

$$N(\tilde{T}) = \tilde{T}^{\frac{1}{4s-3}},$$

with $4s - 3 > 0$, i.e. $s > 3/4$.

Let us show, with this choice the X -norm is also controlled by $\|(S, S)\|_X$. We know from (4.39) that, in each step there is a contribution of N^{-s} . Therefore, the total contribution to cover the interval $[0, \tilde{T}]$ is $(\tilde{T}/T)N^{-s}$. Now, with the choice of N we get,

$$\frac{\tilde{T}}{T}N^{-s} \leq c\tilde{T}N^{3(1-s)}N^{-s} \leq c,$$

and we are done as in [7].

Hence, we conclude that the IVP (1.5) has global solution whenever $s > 3/4$. \square

REFERENCES

- [1] M. Ablowitz, D. Kaup, A. Newell and H. Segur, *Nonlinear evolution equations of physical significance*, Phys. Rev. Lett., **31**(2) (1973) 125–127.
- [2] E. Alarcon, J. Angulo and J. F. Montenegro, *Stability and instability of solitary waves for a nonlinear dispersive system*, Nonlinear Analysis, **36** (1999) 1015–1035.
- [3] J. Angulo, J. Bona, F. Linares and M. Scialom, *Scaling, stability and singularities for nonlinear dispersive wave equations: the critical case*, Nonlinearity **15** (2002) 759–786.
- [4] J. L. Bona, P. Souganidis and W. Strauss, *Stability and instability of solitary waves of Korteweg-de Vries type equation*, Proc. Roy. Soc. London Ser A, **411** (1987) 395–412.
- [5] J. Bourgain, *Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity*, Internat. Math. Res. Notices, **5** (1998) 253–283.
- [6] G. Fonseca, F. Linares and G. Ponce, *Global well-posedness for the modified Korteweg-de Vries equation*, Communications in PDE, **24** (1999) 683–705.
- [7] G. Fonseca, F. Linares and G. Ponce, *Global existence of the critical generalized KdV equation*, Proc. Amer. Math. Soc. **131** (2003) 1847–1855.
- [8] A. Friedman, *Partial Differential Equations*, Holt, Rinehart and Winston, Inc. (1969).
- [9] M. Grillakis, J. Shatah and W. Strauss, *Stability theory of solitary waves in the presence of symmetry I*, J. Funct. Anal., **74** (1987) 160–197.
- [10] S. Hakkaev and K. Kirchev, *Stability of solitary waves for a nonlinear dispersive system in critical case*, Preprint (2003).
- [11] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Advances in Mathematics Supplementary Studies, Studies in Appl. Math., **8** (1983) 93–128.
- [12] C. E. Kenig, G. Ponce and L. Vega, *Well-posedness of the initial value problem for the Korteweg-de Vries equation*, J. Amer. Math. Soc., **4** (2) (1991) 323–347.
- [13] C. E. Kenig, G. Ponce and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana University Math. J., **40** (1) (1991) 33–69.
- [14] C. E. Kenig, G. Ponce and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math., **46** (1993) 527–620.

- [15] C. E. Kenig and A. Ruiz, *A strong type $(2,2)$ estimate for a maximal operator associated to the Schrödinger equation*, Trans. Amer. Math. Soc., **230** (1983) 239–246.
- [16] F. Merle, *Existence of blow-up solutions in the energy space for the critical generalized KdV equation*, J. Amer. Math. Soc., **14** (2001) 555–578.
- [17] J. F. Montenegro, *Sistemas de equações não-lineares. Estudo local, global e estabilidade de ondas solitárias*, Ph. D. Thesis, IMPA, Rio de Janeiro (1995).
- [18] M. Panthee, *Properties of solutions to some nonlinear dispersive models*, Ph.D. Thesis, IMPA, Rio de Janeiro (2004).
- [19] Stein E. and Weiss G., *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, (1971).
- [20] M. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math., **39** (1986) 51–67.

CENTER FOR MATHEMATICAL ANALYSIS, GEOMETRY AND DYNAMICAL SYSTEMS, IST, AV. ROVISCO PAIS 1, 1049-001, LISBON, PORTUGAL.

E-mail address: mpanthee@math.ist.utl.pt

IMECC-UNICAMP, 13083-970, CAMPINAS, SÃO PAULO, BRAZIL

E-mail address: scialom@ime.unicamp.br