The link between regularity and strong-pi-regularity

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Abstract

It is shown that if all powers of a ring element a are regular, then a will be strongly-pi-regular exactly when a suitable word in the powers of a and their inner inverses is a unit.

1 Introduction

An element m in a ring R is *regular* if there exists m^-, referred to an inner inverse, such that $mm^-m = m$. The set of all inner inverses of m will be denoted by $m\{1\}$. We say m is *strongly-pi-regular* if it has a *Drazin inverse* m^d that satisfies $xmx = x$ and $mx = xm$, as well as $m^kxm = m^k$ for some k [2]. The smallest such k, say $k = s$, is called the *index* of m and denoted by $i(m)$. When $i(m) \leq 1$ we say that m has a group inverse, which will be denoted by $m^\#$. In particular m will be a unit if and only if $i(m) = 0$. The index $i(m)$ can also be characterized as the smallest k for which there exist x and y such that $a^{k+1}x = a^k = ya^{k+1}$. Given ring elements x and y, we say they are *orthogonal*, denoted by $x \perp y$, if $xy = yx = 0$.

It is known that if m is strongly-pi-regular, then $m^{i(m)}$ is regular and in fact belongs to a multiplicative group, ensuring that $(m^{i(m)})^\#$ exists. We propose to solve the converse problem, namely that of characterizing strong-pi-regularity in terms of the regularity of suitable powers of m together with the existence of a word, in powers of m and their inner inverses, that is a unit.

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2 The regular stack

Suppose we are given an element m in R and assume that m and all its powers are regular. For each power we pick a fixed inner inverse. That is, we assume a fixed list

$$\left\{m^-, (m^2)^-, \ldots, (m^k)^-, \ldots\right\}.$$

We define the fixed idempotents $E_k = m^k(m^k)^-$ for $k = 1, 2, \ldots$ and set $e = E_1 = mm^-$. It is easily seen that

$$(1)\ em = m \quad (2)\ eE_k = E_k = E_k^2 \quad (3)\ E_k m E_k = mE_k \quad (4)\ E_k E_{k+1} = E_{k+1}.$$

We now consider the map $\phi : R \to R$ defined by $\phi(x) = mxe + 1 - exe$ and construct the sequence of elements $m_k = \phi(E_k) = x_k + y_k$ for $k = 1, 2, \ldots$, where $x_k = mE_k e$ and $y_k = 1 - eE_k e$.

It should be observed that $\phi(1) = \phi(e)$ is a unit precisely when m has a group inverse [7], and that $\phi(a)$ is a unit exactly when am has a group inverse [3].

In addition we see that

$$x_k y_k = mE_k e - mE_k eE_k e = 0,$$

$$y_k x_k = mE_k e - eE_k emE_k e = mE_k e - E_k mE_k e = 0,$$

and therefore we have an orthogonal splitting $m_k = x_k + y_k$.

We now claim that the elements m_k are in fact regular and may recursively be generated.

Lemma 2.1. *If $m_k = \phi\left(m^k(m^k)^-\right)$ then there exists an inner inverse m_{k-1}^- such that*

$$m_k = m_{k-1}^2 m_{k-1}^- + 1 - m_{k-1} m_{k-1}^-,$$

with $m_0 = m$.

Proof. For $i \geq 1$ we have $m_i = x_i + y_i$, in which **both** components are regular. Indeed, $y_i = 1 - m^i(m^i)^- mm^-$ is idempotent, and x_i has $m^i(m^{i+1})^- mm^-$ as an inner inverse; calling this x_i^- we have that $x_i x_i^- = m^{i+1}(m^{i+1})^- mm^-$, and that $y_i x_i^- = 0$ since $eE_i em^i = mm^- m^i(m^i)^- mm^- m^i = m^i$.

We can, therefore, take $m_{k-1}^- = x_{k-1}^- + y_{k-1}$, which in turn gives

$$\begin{aligned} m_k &= m^{k+1}(m^k)^- mm^- + 1 - m^k(m^k)^- mm^- \\ &= x_{k-1}x_{k-1}x_{k-1}^- + y_{k-1} + 1 - x_{k-1}x_{k-1}^- - y_{k-1} \\ &= (x_{k-1} + y_{k-1})(x_{k-1}x_{k-1}^- + y_{k-1}) + 1 - (x_{k-1}x_{k-1}^- + y_{k-1}) \\ &= m_{k-1}^2 m_{k-1}^- + 1 - m_{k-1}m_{k-1}^-, \end{aligned}$$

as desired. □

Using the previous lemma, we can now express m_k alternatively as

$$m_k = m^{k+1}(m^k)^- mm^- + 1 - m^k(m^k)^- mm^-.$$

3 Index Results

Let us now use the above regular stack to obtain suitable index results. Suppose that m is strongly-pi-regular, and consider the associated sequences

$$\begin{aligned} u_k &= m^{k+1}(m^k)^- + 1 - m^k(m^k)^-, \\ w_k &= m^- m^{k+1}(m^k)^- m + 1 - m^- m^k(m^k)^- m \\ \text{and} \quad v_k &= (m^k)^- m^{k+1} + 1 - (m^k)^- m^k. \end{aligned}$$

We shall first need the following fact:

Lemma 3.1 ([1]). *If $1 + ab$ has a Drazin inverse, then $1 + ba$ has a Drazin inverse and*

$$i(1 + ab) = i(1 + ba).$$

Proof. Suppose $1+ab$ has a Drazin inverse and has index $i(1+ab) = k$. Then $(1+ab)^{k+1}x = (1 + ab)^k = y(1 + ab)^{k+1}$, for some x and y in R. This means that $(1 + ba)^{k+1}(1 - bxa) = (1+ba)^k = (1-bya)(1+ba)^{k+1}$ and thus $i(1+ba) \leq i(1+ab)$. Again by interchanging a and b, we obtain equality. □

By applying this lemma we may conclude that $i(m_k) = i(u_k) = i(w_k) = i(v_k)$.

We now recall the following lemma:

Lemma 3.2 ([5]). *Given m strongly-pi-regular,*

$$i(m^2 m^- + 1 - mm^-) = i(m) - 1.$$

As a consequence we may deduce that $i(m_k) = t$ if and only if $i(m_{k+1}) = t - 1$.

We shall also need the following result, which can be deduced from the proof of [2, Theorem 4].

Lemma 3.3. *If $a^{k+1}x = a^k = ya^{k+1}$, then $a^d = a^k x^{k+1} = y^{k+1}a^k$ and $aa^d = a^k x^k = y^k a^k$.*

Proof. Repeatedly pre-multiplying the first equality by a and post-multiplying by x, shows that $a^{k+r}x^r = a^k$ for all $r = 1, 2, \ldots$, and in particular, when $r = k$, $a^{2k}x^k = a^k$. By symmetry we also get $a^k = y^k a^{2k}$. The latter two equalities ensure that a^k has a group inverse of the form $(a^k)^\# = y^k a^k x^k = y^k a^{2k} x^{2k} = a^k x^{2k} = y^{2k} a^k$. This implies that $a^d = a^{k-1}(a^k)^\# = a^{k-1}a^k x^{2k} = \left(a^{k+(k-1)}x^{k-1}\right) x^{k+1} = a^k x^{k+1}$, and by symmetry $a^d = y^{k+1}a^k$.

Lastly we also see that $aa^d = a^{k+1}x^{k+1} = \left(a^{k+1}x\right) x^k = a^k x^k$ and by symmetry $aa^d = y^k a^k$. □

Combining these results, we now may state the following theorem:

Theorem 3.4. *The following conditions are equivalent:*

1. *$i(m) = s$.*
2. *s is the smallest integer such that $m^s + 1 - m^s(m^s)^-$ is a unit.*

3. s is the smallest integer such that $m^{2s}(m^s)^- + 1 - m^s(m^s)^-$ is a unit.

4. s is the smallest integer such that m_s is a unit.

5. s is the smallest integer such that u_s is a unit.

6. m_ℓ is strongly-pi-regular and $i(m_\ell) = s - \ell$, for one and hence all $0 \leq \ell \leq s$.

7. u_ℓ is strongly-pi-regular and $i(u_\ell) = s - \ell$, for one and hence all $0 \leq \ell \leq s$.

When the conditions are satisfied,

$$\begin{aligned} m^d &= u_s^{-1} m^s v_s^{-s} = m^s v_s^{-s-1} \\ &= u_s^{-s} m^s v_s^{-1} = u_s^{-s-1} m^s \\ &= m^{s-1} u_s^{-(s+1)} m^{s+2} v_s^{-(s+1)}. \end{aligned}$$

Proof. The equivalences between (1), (2) and (3) are known (see [6]).
Since $i(m_\ell) = t \Leftrightarrow i(m_{\ell+1}) = t - 1$, we may, by using this argument recursively, conclude that $i(m) = s$ is equivalent to $i(m_\ell) = s - \ell$.
(6) is equivalent to (7) (respectively (4) is equivalent to (5)), by applying Lemma 3.1 with $b = mm^-$ and $a = m^{\ell+1}(m^\ell)^- - m^\ell(m^\ell)^-$ (respectively $a = m^{s+1}(m^s)^- - m^s(m^s)^-$).
It is obvious that (6) is sufficient to (4) and that (7) is sufficient to (5).
Finally, we now prove that (5) implies (1). As u_s is a unit and $u_s m^s = m^{s+1}$, we have $m^s = u_s^{-1} m^{s+1}$. Likewise, u_s being a unit implies that $v_s = (m^s)^- m^{s+1} + 1 - (m^s)^- m^s$ is a unit, which in turns yields $m^s = m^{s+1} v_s^{-1}$. Therefore, $m^s \in m^{s+1}R \cap Rm^{s+1}$ and $m^d = m^{s-1} u_s^{-(s+1)} m^{s+2} v_s^{-(s+1)}$. □

We may in fact compute the Drazin inverses of the three associated sequences $\{u_k\}, \{v_k\}$ and $\{w_k\}$. It suffices to compute the former.

Theorem 3.5. *If $i(m) = s$ then, for all $0 \leq \ell \leq s$,*

$$u_\ell^d = m^d m^\ell (m^\ell)^- + 1 - m^\ell (m^\ell)^-.$$

Proof. Set $X = m^\ell$ and $A = m(m^\ell)^-$ so that $u_\ell = XA + (1 - E_\ell)$. From the last theorem we recall that $i(u_\ell) = i(m) - \ell$. Now observe that u_ℓ is a sum of two orthogonal elements, and since u_ℓ is strongly-pi-regular so are each of the two orthogonal summands. In particular, $m^{\ell+1}(m^\ell)^-$ is strongly-pi-regular and we obtain the expression

$$\begin{aligned} (u_\ell)^d &= (mE_\ell)^d + 1 - E_\ell \qquad (1) \\ &= (XA)^d + 1 - E_\ell, \qquad (2) \end{aligned}$$

where $E_\ell = m^\ell \left(m^\ell\right)^-$.

Next, we turn to the computation of $(XA)^d = (mE_\ell)^d$. We claim that $(XA)^{k+1} y = (XA)^k$, where $y = m^d m^\ell (m^\ell)^-$. Indeed, it follows by induction that $(XA)^i = m^{i+\ell}(m^\ell)^-$, and hence we have

$$\begin{aligned}(XA)^{k+1}y &= m^{k+\ell+1}(m^\ell)^- m^\ell m^d (m^\ell)^- \\ &= m^\ell m^{k+1} m^d (m^\ell)^- \\ &= m^{k+\ell}(m^\ell)^- = (XA)^k.\end{aligned}$$

We now apply Lemma 3.3 to obtain $(XA)^d = (XA)^k y^{k+1}$.
Again, by induction, $y^i = (m^d)^i m^\ell (m^\ell)^-$, and hence $y^{k+1} = (m^d)^{k+1} m^\ell (m^\ell)^-$, which gives

$$\begin{aligned}(XA)^d = (XA)^k y^{k+1} &= m^{\ell+k}(m^\ell)^-(m^d)^{k+1} m^\ell (m^\ell)^- \\ &= m^{\ell+k}(m^\ell)^- m^\ell (m^d)^{k+1}(m^\ell)^- \\ &= m^\ell m^k (m^d)^{k+1}(m^\ell)^- \\ &= m^d m^\ell (m^\ell)^-\end{aligned}$$

and

$$\begin{aligned}(XA)^d XA &= (mE_\ell)^d mE_\ell \\ &= m^d m^\ell (m^\ell)^- m^{\ell+1}(m^\ell)^- \\ &= m^d m^{\ell+1}(m^\ell)^-.\end{aligned}$$

Lastly, substituting the expression for $(XA)^d$ in equation (2), we arrive at

$$\begin{aligned}(u_\ell)^d &= m^d E_\ell + 1 - E_\ell \\ &= m^d m^\ell (m^\ell)^- + 1 - m^\ell (m^\ell)^-\end{aligned}$$

which is the desired expression. □

We close with some pertinent remarks.

Remarks

1. If m_k is a unit for one choice of $(m^k)^-$ then it is a unit for all such choices.
   Indeed, the fact that m_k is a unit implies that $i(m) = s$, which implies, from the proof, that $m_s = m^{s+1}(m^s)^= mm^- + 1 - m^s(m^s)^= mm^-$ is also a unit.

2. If u_s is a unit for one choice of $(m^s)^-$ then it is a unit for all such choices.

3. In a ring, a^2 may be regular without a being regular. For example let $a = 4$ in $\mathbb{Z}_8$.

4. In a ring it can happen that an element a is regular without a^2 being regular. Indeed, let $A = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$ be over $\mathbb{Z}_4$. Then A has an inner inverse $A^- = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ while $A^2 = 2A$ does not, since $(2A)X(2A) = 0 \neq 2A$.

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References

[1] N. Castro-González, C. Mendes Araújo, P. Patrício, A note on generalized inverses of a sum in rings, accepted in *Bull. Aust. Math. Soc.*, `doi:10.1017/S0004972710000080`.

[2] M. P. Drazin, Pseudo-inverses in associative rings and semigroups, *Amer. Math. Monthly*, 65 (1958) 506–514.

[3] F. J. Hall, R. E. Hartwig, Algebraic properties of governing matrices used in Cesaro-Neumann iterations, *Rev. Roumaine Math. Pures Appl.* 26 (1981), no. 7, 959–978.

[4] R. E. Hartwig, More on the Souriau-Frame algorithm and the Drazin inverse, *SIAM J. Appl. Math.* 31 (1976), no. 1, 42–46.

[5] P. Patrício, A. Veloso da Costa, On the Drazin index of regular elements, *Cent. Eur. J. Math.* 7 (2009), no. 2, 200–205.

[6] R. Puystjens, M. C. Gouveia, Drazin invertibility for matrices over an arbitrary ring, *Linear Algebra Appl.* 385 (2004), 105–116.

[7] R. Puystjens, R. E. Hartwig, The group inverse of a companion matrix, *Linear and Multilinear Algebra*, 43 (1997), no. 1-3, 137–150.