Submitted to the Bulletin of the Australian Mathematical Society
doi:10.1017/S ...

GENERALIZED INVERSES OF A SUM IN RINGS

N. CASTRO-GONZÁLEZ✉ , C. MENDES-ARAÚJO and PEDRO PATRICIO

Abstract

We study properties of the Drazin index of regular elements in a ring with a unity 1. We give expressions for generalized inverses of $1 - ba$ in terms of generalized inverses of $1 - ab$. In our development we prove that the Drazin index of $1 - ba$ is equal to the Drazin index of $1 - ab$.

2000 *Mathematics subject classification.* primary 15A09; secondary 16U99.
Keywords and phrases: Regular element, reflexive inverse, Drazin index, Drazin inverse, EP elements.

1. Introduction

Let $\mathcal{R}$ be a ring with a unity 1. An element a is said to be regular if there is an element x such that $axa = a$. If it exists, then it is called an inner inverse of a (von Neumann inverse). We will denote by $a\{1\} = \{x \in \mathcal{R} \mid axa = a\}$ the set of all inner inverses of a and we will write a^- to designate a member of $a\{1\}$. A reflexive inverse a^+ of a is an inner and outer inverse of a, that is, $a^+ \in a\{1\}$ and $a^+aa^+ = a^+$.

An element a is said to be Drazin invertible provided there is a common solution for the equations

$$xax = x, \quad ax = xa, \quad a^kxa = a^k \text{ for some } k \geq 0.$$

If a common solution exists, then it is unique and it will be denoted by a^D (see [2]). The smallest integer k for which the above equations hold is called the Drazin index of a, denoted by $\text{ind}(a)$.

First researcher was partially supported by Project MTM2007-67232, "Ministerio de Educación y Ciencia" of Spain.
Second and third researchers were supported by the Portuguese Foundation for Science and Technology-FCT through the research program POCTI.

The Drazin index can be characterized in terms of right and left ideals generated by a power of a as follows [7]: $\text{ind}(a) = k$ if and only if k is the smallest non-negative integer for which $a^k\mathcal{R} = a^{k+1}\mathcal{R}$ and $\mathcal{R}a^k = \mathcal{R}a^{k+1}$, or equivalently, $a^k \in a^{k+1}\mathcal{R} \cap \mathcal{R}a^{k+1}$.

If a is Drazin invertible with $\text{ind}(a) = 1$, then a is regular. In the former case the Drazin inverse of a is known as the group inverse of a, denoted by $a^\sharp$. It is well known that the smallest k for which $(a^k)^\sharp$ exists equals $\text{ind}(a) = k$, and $a^D = (a^k)^\sharp a^{k-1} = a^{k-1}(a^k)^\sharp$.

If there exists an element $a^\pi \in \mathcal{R}$ such that a^π is idempotent, $aa^\pi = a^\pi a$, aa^π is nilpotent, and $a + a^\pi$ is nonsingular, then it is called a spectral idempotent of a; such element is unique (if it exists). We know that a is Drazin invertible if and only the spectral idempotent of a exists. In this case we have $a^D = (a + a^\pi)^{-1}(1 - a^\pi)$ and $a^\pi = 1 - aa^D$. Characterizations of ring elements with related spectral idempotents are given in [4], [5].

Let $\mathcal{R}$ be a ring with an involution $x \to x^*$ such that $(x^*)^* = x$, $(x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, for all $x, y \in \mathcal{R}$. We say that a is Moore-Penrose invertible if the equations

$$bab = b, \quad aba = a, \quad (ab)^* = ab, \quad (ba)^* = ba$$

have a common solution; such solution is unique if it exists (see [2], [6]), and it will be denoted by $a^\dagger$.

We say that an element a is EP if a is Moore-Penrose invertible and $aa^\dagger = a^\dagger a$. An element a is generalized EP if there exists $k \in \mathbb{N}$ such that a^k is EP.

Barnes [1] proved that the ascents (descents) of $I - RS$ and $I - SR$ are equal for bounded operators on Banach spaces $R \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, X)$. Consequently, the Drazin indices of $I - RS$ and $I - SR$ are equal. In this paper we deal with the Drazin index of $1 - ab$ and $1 - ba$ in rings, and therefore neither functional calculi and operator theory can be used. Moreover, we provide a formula for the reflexive inverse, the group inverse and the Drazin inverse of $1 - ba$ in terms of the corresponding generalized inverse of $1 - ab$.

In our development, we extend the following characterization of the Drazin index given by Puystjens and Hartwig [10]: Given a regular element $a \in \mathcal{R}$, then

$$\text{ind}(a) \leq 1 \Leftrightarrow \text{ind}(a + 1 - aa^-) = 0, \text{ for one and hence all choices of } a^- \in a\{1\}.$$

2. Auxiliary results

In this section we give some auxiliary lemmas. We start with an elementary known result.

LEMMA 2.1. *Let $a, b \in \mathcal{R}$. Then $1 - ab$ is invertible if and only if $1 - ba$ is invertible.*

LEMMA 2.2. *Let a be a regular element. Then, given a natural n,*

$$(a + 1 - aa^-)^n = (a^2a^- + 1 - aa^-)^n + \sum_{i=1}^{n} a^i(1 - aa^-). \tag{2.1}$$

PROOF. The proof is by induction on n. Denote $z = a + 1 - aa^-$ and $x = a^2a^- + 1 - aa^-$. It is clear that $z = x + a(1 - aa^-)$. Assuming (2.1) to hold for k, we will prove it for $k + 1$.

We note that $zx = x^2 + a(1 - aa^-)$ and $za = a^2$. Now, by the induction step

$$\begin{aligned} z^{k+1} &= z\left(x^k + \sum_{i=1}^{k} a^i(1 - aa^-)\right) \\ &= x^{k+1} + a(1 - aa^-) + \sum_{i=1}^{k} a^{i+1}(1 - aa^-) \\ &= x^{k+1} + \sum_{i=1}^{k+1} a^i(1 - aa^-). \end{aligned}$$

□

LEMMA 2.3. *Let $a, b \in \mathcal{R}$. Then, given a natural n,*

$$(1 - ba)^n = 1 - bra \quad \textit{and} \quad (1 - ab)^n = 1 - rab,$$

where $r = \sum_{j=0}^{n-1}(1 - ab)^j$.

PROOF. It can be easily proved by induction on n. □

In [5] the authors give the following characterization of EP elements in a ring.

LEMMA 2.4. *Let $\mathcal{R}$ be a ring with an involution $x \to x^*$. For $a \in \mathcal{R}$ the following conditions are equivalent:*

(i) *a is EP.*
(ii) *a is Drazin and Moore-Penrose invertible and $a^D = a^\dagger$.*
(iii) *a is group invertible and $a^\pi = (a^*)^\pi$.*

3. Main results

The following theorem is an answer to a question raised by Patricio and Veloso in [8] about the equivalence between $\mathrm{ind}(a^2a^- + 1 - aa^-) = k$ and $\mathrm{ind}(a + 1 - aa^-) = k$, and provides a new characterization of the Drazin index.

THEOREM 3.1. *Let a be a regular non-invertible element. The following conditions are equivalent:*

(i) $\mathrm{ind}(a) = k + 1$.
(ii) $\mathrm{ind}(a^2a^- + 1 - aa^-) = k$, *for one and hence all choices of* $a^- \in a\{1\}$.
(iii) $\mathrm{ind}(a + 1 - aa^-) = k$, *for one and hence all choices of* $a^- \in a\{1\}$.

PROOF. The equivalence (i)⇔(ii) is proved in [8, Theorem 2.1]. We proceed to show that (ii)⇒(iii). Denote $x = a^2a^- + 1 - aa^-$ and $z = a + 1 - aa^-$. Assume $\mathrm{ind}(x) = k$, or equivalently, $\mathrm{ind}(a) = k + 1$. Then $x^k = x^{k+1}\mathcal{R}$ and $a^{k+1} = a^{k+2}w$ for some $w \in \mathcal{R}$. By (2.1),

$$\begin{aligned}
z^k\mathcal{R} &= \left(1 + \sum_{i=1}^{k} a^i(1 - aa^-)\right) x^k\mathcal{R} \\
&= \left(1 + \sum_{i=1}^{k} a^i(1 - aa^-)\right) x^{k+1}\mathcal{R} \\
&= \left(z^{k+1} - \sum_{i=1}^{k+1} a^i(1 - aa^-) + \sum_{i=1}^{k} a^i(1 - aa^-)\right)\mathcal{R} \\
&= \left(z^{k+1} - a^{k+1}(1 - aa^-)\right)\mathcal{R} = (z^{k+1} - a^{k+2}w(1 - aa^-))\mathcal{R} \\
&= z^{k+1}(1 - aw(1 - aa^-))\mathcal{R} \subseteq z^{k+1}\mathcal{R}.
\end{aligned}$$

This gives $z^k\mathcal{R} = z^{k+1}\mathcal{R}$. On the other hand, since $\mathrm{ind}(x) = k$ we also have $x^k = ux^{k+1}$ for some $u \in \mathcal{R}$. By (2.1),

$$\begin{aligned}
\mathcal{R}z^k &= \mathcal{R}\left(x^k + \sum_{i=1}^{k} a^i(1 - aa^-)\right) \\
&= \mathcal{R}\left(ux^{k+1} + \sum_{i=1}^{k} a^i(1 - aa^-)\right) \\
&= \mathcal{R}\left(u - u\sum_{i=1}^{k+1} a^i(1 - aa^-) + \sum_{i=1}^{k} a^i(1 - aa^-)\right) z^{k+1} \subseteq \mathcal{R}z^{k+1}.
\end{aligned}$$

From this we conclude that $\mathcal{R}z^k = \mathcal{R}z^{k+1}$. Consequently, $\mathrm{ind}(z) \le k$.

By symmetrical arguments, we can show that $\operatorname{ind}(z) = k$ implies that $\operatorname{ind}(x) \leq k$. Further, suppose $\operatorname{ind}(z) < k$, having $\operatorname{ind}(x) = k$, then we would get that $\operatorname{ind}(x) \leq k - 1$, and we would arrive to a contradiction. Therefore $\operatorname{ind}(z) = k$. □

We can state the symmetrical of Theorem 3.1.

Corollary 3.2. *Let a be a regular non-invertible element. The following conditions are equivalent:*

(i) $\operatorname{ind}(a) = k + 1$.
(ii) $\operatorname{ind}(a^- a^2 + 1 - a^- a) = k$, *for one and hence all choices of* $a^- \in a\{1\}$.
(iii) $\operatorname{ind}(a + 1 - a^- a) = k$, *for one and hence all choices of* $a^- \in a\{1\}$.

The following corollary is an extension of the analogous result for the Drazin index of a complex partitioned matrix over $\mathbb{C}$ [3, Theorem 7.7.5].

Corollary 3.3. *Let $\mathcal{R}$ be any ring with unity. If* $M = \begin{pmatrix} A & B \\ C & CA^{-1}B \end{pmatrix} \in \mathcal{R}_{n\times n}$, *where* $A \in \mathcal{R}_{r\times r}$ *is invertible, then* $\operatorname{ind}(M) = \operatorname{ind}(A + BCA^{-1}) + 1$.

Proof. We have $M^- = \begin{pmatrix} A^{-1} & 0 \\ -CA^{-1} & I \end{pmatrix}$ is an inner inverse of M and

$$M + I - MM^- = \begin{pmatrix} A + BCA^{-1} & 0 \\ C - CA^{-1}(I - BCA^{-1}) & I \end{pmatrix}.$$

Using the following known result for block triangular matrices,

$$\max\{\operatorname{ind}(I), \operatorname{ind}(A + BCA^{-1})\} \leq \operatorname{ind}(M + I - MM^-) \leq \operatorname{ind}(A + BCA^{-1}) + \operatorname{ind}(I),$$

we conclude that $\operatorname{ind}(M + I - MM^-) = \operatorname{ind}(A + BCA^{-1})$. Now, that $\operatorname{ind}(M) = \operatorname{ind}(A + BCA^{-1}) + 1$ follows from Theorem 3.1. □

It is well known that $1 - ba$ is regular if and only if $1 - ab$ is regular. Moreover, if $(1 - ab)^-$ is an inner inverse of $1 - ab$ then $(1 - ba)^- = 1 + b(1 - ab)^- a$ is an inner inverse of $1 - ba$. In the sequel, we will extend the same reasoning to other generalized inverses, namely reflexive, group and Drazin inverse.

THEOREM 3.4. *Let $a, b \in \mathcal{R}$. If $(1-ab)^+$ is a reflexive inverse of $1-ab$, then a reflexive inverse of $1-ba$ is given by*

$$(1-ba)^+ = 1 + b\left((1-ab)^+ - pq\right)a,$$

where $p = 1-(1-ab)^+(1-ab)$ and $q = 1-(1-ab)(1-ab)^+$.

PROOF. Let $x = 1 + b\left((1-ab)^+ - pq\right)a$. Then

$$(1-ba)x = 1 - bqa.$$

Further,

$$(1-ba)x(1-ba) = 1 - ba - bqa(1-ba)a = 1 - ba$$

and

$$\begin{aligned} x(1-ba)x &= x - xbqa \\ &= x - bqa - b\left((1-ab)^+ - pq\right)abqa \\ &= x, \end{aligned}$$

where we have simplified writing $ab = 1-(1-ab)$ and using relations $(1-ab)(1-ab)^+(1-ab) = (1-ab)$ and $(1-ab)^+(1-ab)(1-ab)^+ = (1-ab)^+$. □

THEOREM 3.5. *Let $a, b \in \mathcal{R}$. If $1-ab$ is group invertible, then $1-ba$ is group invertible and*

$$(1-ba)^\sharp = 1 + b\left((1-ab)^\sharp - (1-ab)^\pi\right)a,$$

where $(1-ab)^\pi = 1-(1-ab)^\sharp(1-ab)$.

PROOF. Let $x = 1+b\left((1-ab)^\sharp - (1-ab)^\pi\right)a$. First, we note that $(1-ab)^\sharp$ is a reflexive inverse that commutes with $1-ab$. In view of the preceding theorem we have that x is reflexive inverse of $1-ba$. Next, we will prove that x commutes with $1-ba$. We have

$$x(1-ba) = 1 - ba + b(1-ab)^\sharp(1-ab)a = 1 - b(1-ab)^\pi a$$

and, similarly, $(1-ba)x = 1-b(1-ab)^\pi a$ which gives $x(1-ba) = (1-ba)x$. Therefore x verifies the three equations involved in the definition of group inverse.

□

Theorem 3.6. *Let $a, b \in \mathcal{R}$. If $1-ab$ is Drazin invertible with* $\operatorname{ind}(1-ab) = k$*, then $1-ba$ is Drazin invertible with* $\operatorname{ind}(1-ba) = k$ *and*

$$(1-ba)^D = 1 + b\left((1-ab)^D - (1-ab)^\pi r\right)a,$$

where $r = \sum_{j=0}^{k-1}(1-ab)^j$.

Proof. Assume $\operatorname{ind}(1-ab) = k \geq 2$. Then $(1-ab)^k$ is group invertible and Theorem 3.1 leads to $\operatorname{ind}(1-(1-(1-ab)^k)(1-ab)^k((1-ab)^k)^\sharp) = 0$. By Lemma 2.3 we have

$$1-(1-ab)^k = rab \quad \text{and} \quad 1-(1-ba)^k = bra, \tag{3.1}$$

where $r = \sum_{j=0}^{k-1}(1-ab)^j$. According to the above relations, $1-rab(1-ab)^k((1-ab)^k)^\sharp$ is invertible and by Lemma 2.1 we have that $1-b(1-ab)(1-ab)^D ra$ is invertible. Further,

$$\begin{aligned}(1-b(1-ab)(1-ab)^D ra)(1-ba)^k &= (1-ba)^k - b(1-ab)(1-ab)^D ra(1-ba)^k \\ &= (1-ba)^k - b(1-ab)^k ra \\ &= (1-bra)(1-ba)^k = (1-ba)^{2k}.\end{aligned}$$

From this it follows that $(1-ba)^k = (1-b(1-ab)(1-ab)^D ra)^{-1}(1-ba)^{2k} \in \mathcal{R}(1-ba)^{k+1}$. On the other hand,

$$\begin{aligned}(1-ba)^k(1-b(1-ab)(1-ab)^D ra) &= (1-ba)^k - (1-ba)^k b(1-ab)(1-ab)^D ra \\ &= (1-ba)^k - b(1-ab)^k ra = (1-ba)^{2k}\end{aligned}$$

and hence $(1-ba)^k = (1-ba)^{2k}(1-b(1-ab)(1-ab)^D ra)^{-1} \in (1-ba)^{k+1}\mathcal{R}$.

Therefore $(1-ba)^k \in \mathcal{R}(1-ba)^{k+1} \cap (1-ba)^{k+1}\mathcal{R}$, which implies $\operatorname{ind}(1-ba) \leq k$.

Further, analysis similar to that of the last part of the proof of Theorem 3.1 shows that $\operatorname{ind}(1-ab) = k$. Now, $(1-ba)^D = ((1-ba)^k)^\sharp(1-ba)^{k-1}$. In view of (3.1) and applying Theorem 3, it follows

$$\begin{aligned}((1-ba)^k)^\sharp = (1-bra)^\sharp &= 1 + b\left((1-rab)^\sharp - (1-rab)^\pi\right)ra \\ &= 1 + b\left(\left((1-ab)^k\right)^\sharp - \left((1-ab)^k\right)^\pi\right)ra \\ &= 1 + b\left(\left((1-ab)^D\right)^k - (1-ab)^\pi\right)ra.\end{aligned}$$

Hence,

$$\begin{aligned}(1-ba)^D &= \Big(1+b\Big(\big((1-ab)^D\big)^k-(1-ab)^\pi\Big)ra\Big)(1-ba)^{k-1}\\ &= (1-ba)^{k-1}+b\Big(\big((1-ab)^D\big)^k-(1-ab)^\pi\Big)(1-ab)^{k-1}ra\\ &= 1-br'a+b\Big((1-ab)^D r-(1-ab)^\pi(1-ab)^{k-1}\Big)a\\ &= 1+b\Big((1-ab)^D-(1-ab)^\pi r'-(1-ab)^\pi(1-ab)^{k-1}\Big)a\\ &= 1+b\Big((1-ab)^D-(1-ab)^\pi r\Big)a,\end{aligned}$$

where $r' = \sum_{j=0}^{k-2}(1-ab)^j$, completing the proof. □

Let $\mathcal{R}_{n\times n}$ the ring of $n\times n$ matrices over $\mathcal{R}$. Any matrix $A\in\mathcal{R}_{r\times n}$ ($B\in\mathcal{R}_{n\times r}$) with $r<n$ may be enlarged to square $n\times n$ matrix A' (B') by adding zeros. Then we can compute a generalized inverse of $I-BA=I-B'A'$ using preceding results in the ring $\mathcal{R}_{n\times n}$. Finally, we can rewrite the corresponding expression for the generalized inverse of $I-B'A'$ in terms of A and B, getting that formulas similar to that in the preceding theorems hold for rectangular matrices A and B.

Example 3.7. We consider the following matrices with entries in the univariate polynomial ring in x over $\mathbb{Z}_8$, the ring of integers modulo 8:

$$A=\begin{pmatrix}x & 2 & 1\end{pmatrix} \quad\text{and}\quad B=\begin{pmatrix}7x\\ 2\\ x^2+3\end{pmatrix}.$$

Then

$$I-BA=\begin{pmatrix}x^2+1 & 2x & x\\ 6x & 5 & 6\\ 7x^3+5x & 6x^2+2 & 7x^2+6\end{pmatrix} \quad\text{and}\quad 1-AB=2.$$

The zero degree polynomial equal to 2 is nilpotent of index 3 and, so, $\mathrm{ind}(1-AB)=3$ and $(1-AB)^D=0$. Applying Theorem 3 we get

$$\begin{aligned}(I-BA)^D &= I+\begin{pmatrix}7x\\ 2\\ x^2+3\end{pmatrix}(0-1(1+2+2^2))\begin{pmatrix}x & 2 & 1\end{pmatrix}\\ &= \begin{pmatrix}7x^2+1 & 6x & 7x\\ 2x & 5 & 2\\ x^3+3x & 2x^2+6 & x^2+4\end{pmatrix}.\end{aligned}$$

We know that in general $1-ab$ is EP may not imply that $1-ba$ is EP. In the following result we give a necessary and sufficient condition for such implication to hold.

COROLLARY 3.8. *Let $\mathcal{R}$ be a ring with an involution $x \to x^*$. If $1-ab$ is EP, then $1-ba$ is EP if and only if $a^*(1-ab)^\pi b^* = b(1-ab)^\pi a$. In this case,*

$$(1-ba)^\dagger = 1 + b\left((1-ab)^\dagger - (1-(1-ab)(1-ab)^\dagger)\right)a.$$

PROOF. Since $1-ab$ is EP, by Lemma 2.4 we have that $1-ab$ is group invertible and Moore-Penrose invertible and $(1-ab)^\sharp = (1-ab)^\dagger$. Now, from Theorem 3 it follows that $1-ba$ is also group invertible and $(1-ba)^\sharp = 1 + b((1-ab)^\sharp - (1-ab)^\pi)a$, and consequently, $(1-ba)^\pi = b(1-ab)^\pi a$. Thus, by Lemma 2.4, $1-ba$ is EP if and only if $((1-ba)^*)^\pi = (1-ba)^\pi$, that is,

$$(b(1-ab)^\pi a)^* = b(1-ab)^\pi a.$$

Hence, using that $((1-ab)^*)^\pi = (1-ab)^\pi$, the result follows. □

COROLLARY 3.9. *Let $\mathcal{R}$ be a ring with an involution $x \to x^*$. If $1-ab$ is generalized EP, then $1-ba$ is generalized EP if and only if $(ra)^*(1-ab)^\pi b^* = b(1-ab)^\pi ra$, where $r = \sum_{j=0}^{k-1}(1-ab)^j$ and $k = \mathrm{ind}(1-ab)$.*

PROOF. Since $1-ab$ is generalized EP then there exists the smallest integer $k \in \mathbb{N}$ such that $(1-ab)^k$ is EP. From Lemma 2.4 we can deduce that $\mathrm{ind}(1-ab) = k$. Now, by Lemma 2.3 we have $(1-ab)^k = 1-rab$, where r is defined as in the statement of this corollary. By preceding corollary, $(1-ba)^k = 1-bra$ is EP if and only if $(b(1-ab)^\pi ra)^* = b(1-ab)^\pi ra$, completing the proof. □

In this example we show that the existence of the Moore-Penrose of $1-ab$ does not imply the existence of the Moore-Penrose of $1-ba$.

EXAMPLE 3.10. Consider the following matrices over the field $\mathbb{C}$ of complex numbers, with the involution defined by $A^\star = A^T$:

$$A = \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix} \text{ and } \quad B = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

Then

$$I - AB = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad I - BA = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix},$$

and, further,

$$(I - AB)^{\star}(I - AB) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad (I - BA)^{\star}(I - BA) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since rank $(I - AB) = 1$ and rank $(I - AB)^{\star}(I - AB) =$ rank $(I - AB)(I - AB)^{\star} = 1$ we conclude, applying [9, Theorem 1], that $I - AB$ is Moore-Penrose invertible. On the other hand, since rank $(I - BA) = 1$ and rank $(I - BA)^{\star}(I - BA) = 0$ we conclude that $I - BA$ is not Moore-Penrose invertible.

References

[1] B.A. Barnes,*Common operator properties of the linear operators RS and SR*, Proc. Am. Math. Soc. 126 (1998), 1055-1061.

[2] A. Ben-Israel and T. N. E. Greville, Generalized Inverses. Theory and Applications (Second Edition), Springer-Verlag, New York, 2003.

[3] S. L. Campbell, C. D. Meyer Jr., *Generalized Inverse of Linear Transformations*, Pitman, London, (1979); Dover, New York, (1991).

[4] N. Castro-González, J. Y. Vélez-Cerrada, *Elements in rings and Banach algebras with related spectral idempotents*, J. Aust. Math. Soc., **80** (2006), 383–396.

[5] J. J. Koliha, P. Patricio, *Elements of rings with equal spectral idempotents*, J. Aust. Math. Soc., **72** (2002), 137–152.

[6] R. E. Hartwig, *Block generalized inverses*, Arch. Rational Mech. Anal., 61, (1976), 197–251.

[7] R. E. Hartwig, J. Shoaf, *Group inverse of bidiagonal and triangular Toeplitz matrices*, J. Austral. Math. Soc. Ser. A, 24 (1977), 10–34.

[8] P. Patricio, A. Veloso da Costa, *On the Drazin index of regular elements*, Cent. Eur. J. Math. 7(2) (2009), 200–208.

[9] M. H. Pearl, *Generalized inverses of matrices with entries taken from an arbitrary field*, Linear Algebra and Its Applications, 1 (1968), 571–587.

[10] R. Puystjens, R. E. Hartwig, *The group of a companion matrix*, Linear and Multilinear Algebra, 43 (1997), 137–150.

N. Castro-González, Facultad de Informática, Universidad Politécnica de Madrid, 28660 Boadilla del Monte, Madrid, Spain
e-mail: nieves@fi.upm.es

C. Mendes-Araújo, Centro de Matemática, Universidade do Minho, 4710-057 Braga, Portugal
e-mail: clmendes@math.uminho.pt

Pedro Patricio, Centro de Matemática, Universidade do Minho, 4710-057 Braga, Portugal
e-mail: pedro@math.uminho.pt