Moore-Penrose invertibility in involutory rings: the case $aa^\dagger = bb^\dagger$

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Abstract

In this paper, we consider Moore-Penrose invertibility in rings with a general involution. Given two von Neumann regular elements a, b in a general ring with an arbitrary involution, we aim to give necessary and sufficient conditions to $aa^\dagger = bb^\dagger$. As a special case, EP elements are considered.

1 Introduction

Let R be a (associative) ring with unity 1. We will denote, for a given $a \in R$,

$$a\{1\} := \{x \in R : axa = a\}$$

the set of von Neumann inverses (or inner inverses, or 1-inverses) of a. A particular 1-inverse of a will be written as a^-, and a is regular if $a\{1\} \neq \emptyset$. As usual, R is a regular ring if all elements of R are regular. A *reflexive inverse* a^+ of a is a 1-inverse of a that is a solution of the ring equation $xax = x$. Note that if $a^-, a^= \in a\{1\}$ then $a^-aa^=$ is a reflexive inverse of a.

An involution $*$ in R is an anti-isomorphism of degree 2 in R, that is to say, $(x^*)^* = x$, $(x+y)^* = x^* + y^*$ and $(xy)^* = y^*x^*$, for all $x, y \in R$.

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We say $a \in R$ is *Moore-Penrose invertible* (with respect to $*$) if the equations

$$axa = a,\ xax = x,\ (ax)^* = ax,\ (xa)^* = xa$$

have a common solution. If such a solution exists, then it is unique, and denoted by $a^\dagger$.

$R^\dagger$ will stand for the subset of all Moore-Penrose invertible elements of R.

Throughout this paper, R is a ring with unity and an arbitrary involution $*$ (unless otherwise stated).

This paper is motivated by the recently published [1] and [6]. There, the authors addressed to the characterization of the equality $aa^D = bb^D$, for complex matrices in the former, and for elements in general rings in the latter. A similar problem can be considered for Moore-Penrose inverses, which we aim to address in this paper. We are interested in the study of this problem for general (associative with unity) rings, and therefore no dimentional analysis nor special decompositions can be used (cf [1]).

Before we present our main results, let us focus on the characterization of Moore-Penrose invertibility of regular elements.

As a starting point, we may give an alternative characterization of the existence of the Moore-Penrose inverse of a regular element by using units in R. This will lead to a simpler condition when compared to [8, Theorem 1].

Lemma 1.1. *Let $a \in R$ be a regular element and $a^- \in a\{1\}$. Then,*

1. *$r = a^*a + 1 - a^-a$ is a unit if and only if $s = aa^*aa^- + 1 - aa^-$ is a unit, in which case*

$$r^{-1} = a^-s^{-1}a + 1 - a^*s^{-1}a$$

and

$$s^{-1} = ar^{-1}a^- + 1 - ar^{-1}a^*aa^-.$$

2. *$g = aa^* + 1 - aa^-$ is a unit if and only if $h = a^-aa^*a + 1 - a^-a$ is a unit, in which case*

$$g^{-1} = ah^{-1}a^- + 1 - ah^{-1}a^*$$

and

$$h^{-1} = a^-g^{-1}a + 1 - a^-aa^*g^{-1}a.$$

Proof. (1). $aa^*aa^- + 1 - aa^- = 1 - a(-a^*aa^- + a^-)$ is a unit if and only if $1 - (-a^*aa^- + a^-)a = a^*a + 1 - a^-a$ is a unit. The expression for r^{-1} uses the fact that aa^- and s commute.
(2) is analogous to (1). $\square$

Theorem 1.2. *Let $a \in R$ be a regular element. Then the following statements are equivalent:*

1. *$a^\dagger$ exists.*

2. *$g = aa^* + 1 - aa^-$ is a unit for one and hence all choices of $a^- \in a\{1\}$.*

3. *$r = a^*a + 1 - a^-a$ is a unit for one and hence all choices of $a^- \in a\{1\}$. Moreover,*

$$\begin{aligned} a^\dagger &= a^* \left(r^{-1}\left(a^- - a^*\right)\right)^* a^* + a^* \\ &= a^* \left(\left(a^- - a^*\right) g^{-1}\right)^* a^* + a^* \end{aligned}$$

Proof. The proof is straightforward using [8, Theorem 1], [9, Theorem 2] and Lemma 1.1. □

The results presented above were given in a ring context. These can be trivially extended to an additive category with involution, as considered, for example, in [11].

The definition of the Gelfand-Naimark property for rings was introduced in [5, Definition 4]:

Definition 1.3. *The ring R has the* Gelfand-Naimark property *(GN-property, for short) if $1 + xx^*$ is a unit, for all $x \in R$.*

Note that the ring of square complex matrices $n \times n$ with transconjugate as the involution has the GN-property. Namely, for any complex matrix X, since XX^* is positive semi-definite it has non-negative real eigenvalues. That is to say, its spectrum is a subset of $\mathbb{R}_0^+$. Since $\lambda - 1$ is an eigenvalue of XX^* if and only if λ is an eigenvalue of $I + XX^*$, then there are no zero eigenvalues of $I + XX^*$. Hence, $I + XX^*$ is invertible.

Still, the involution considered plays a crucial role on the GN-property. As an example, the same ring of complex matrices and the transposition as the involution fails to have the GN-property. Take, in this case, $X = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$.

An element $a \in R$ is $*$-*cancellable* provided $a^*ab = a^*ac$ implies $ab = ac$ and $baa^* = caa^*$ implies $ba = ca$, for $b, c \in R$.

The involution $*$ is said to be *proper* if $xx^* = 0$ implies x must vanish, for any choice of x. We note in passing that the involution is proper exactly when all elements are $*$-cancellable. Indeed, if $a^*ab = a^*ac$ then $a^*a(b-c) = 0$ which implies $(a(b-c))^*(a(b-c)) = (b-c)^*a^*a(b-c) = 0$ and thus $ab - ac = a(b-c) = 0$. Conversely. for all $x \in R$, $x^*x = 0$ implies $x^*x1 = x^*x0$, which in turn forces $x = 0$ by the $*$-cancellability of x.

A ring is said to be $*$-regular if it is a regular ring and the involution $*$ is proper.

We end this introductory section providing an alternative proof of [5, Theorem 2].

Theorem 1.4. *If R has the GN-property then $a \in R$ is regular if and only if it is Moore-Penrose invertible. Moreover, every regular ring with the GN-property is $*$-regular.*

Proof. Let $a \in R$ and $a^-, a^=$ be von Neumann inverses of a. Note that $e = aa^-$ and $f = a^=a$ are idempotents. Setting $u = 1 + (e - e^*)(e^* - e)$, then $eu = ee^*e = ue$, from which

$eR = ee^*R$, $Re = Re^*e$ since u is a unit. Thus, $e^\dagger$ exists. Analogously, $f^\dagger$ exists. Also $a^- e^\dagger$ is a 1-3 inverse of a and $f^\dagger a^=$ is a 1-4 inverse of a, from which

$$a^\dagger = (a^= a)^\dagger a^= a a^- (a a^-)^\dagger.$$

The second part of the result follows since on a regular ring R the involution is a proper if and only if $R^\dagger = R$, by [11, Lemma 3]. □

Combining the results presented above, all elements of the form $aa^* + 1 - aa^-$ from a regular ring with the GN-property are units. Also if R is $*$-regular then $aa^* + 1 - aa^-$ is a unit, for any choice of $a^- \in a\{1\}$. This is close to the GN-property, but not exactly the same. For instance, the field of complex numbers $\mathbb{C}$ with the identity ι as involution is ι-regular but fails to satisfy the GN-property. Indeed, $1 + xx^\iota = 1 + x^2$ is not a unit with $x = i$.

2 Main results

In this paper we are interested on studying the problem $aa^\dagger = bb^\dagger$ in a ring with an arbitrary involution $*$. When it is clear, and for the sake of simplicity, whenever we address to the Moore-Penrose inverse or to symmetry it is taken with respect to the fixed (arbitrary, unless otherwise stated) involution in the ring. We investigate necessary and sufficient conditions for $aa^\dagger = bb^\dagger$, using the same reasoning as in [1] and [6]. Let us first present a lemma which will be useful in the upcoming results.

Lemma 2.1. *Let $a, b \in R$ be regular elements.*

1. *There exist $a^- \in a\{1\}, b^- \in b\{1\}$ for which $(1 - bb^-)\, aa^- = 0$ if and only if*

$$(1 - bb^=)\, aa^= = 0$$

for all $a^= \in a\{1\}, b^= \in b\{1\}$.

2. *There exist $a^- \in a\{1\}, b^- \in b\{1\}$ for which $(1 - bb^-)\,(1 - a^- a) = 0$ if and only if*

$$(1 - bb^=)\,(1 - a^= a) = 0$$

for all $a^= \in a\{1\}, b^= \in b\{1\}$.

Proof. (1). If $(1 - bb^-)\, aa^- = 0$ for some $a^- \in a\{1\}, b^- \in b\{1\}$ then $bx = aa^-$ is a consistent equation. This implies, for any choice of $b^= \in b\{1\}$, and multiplying on the left hand side by $1 - bb^=$, the equality $(1 - bb^=) aa^- = 0$. Post-multiplication by $aa^=$, where $a^=$ is chosen arbitrarily in $a\{1\}$, gives the desired equality $(1 - bb^=)\, aa^= = 0$. The converse is obvious.
(2). If $(1 - bb^-)\,(1 - a^- a) = 0$, for given $a^- \in \{1\}, b^- \in \{1\}$, then

$$b\left(-b^- a^- a + b^-\right) - \left(-a^-\right) a = 1,$$

and hence $bx - ya = 1$ is a consistent ring equation. Taking arbitrary $a^{=} \in \{1\}, b^{=} \in \{1\}$, and multiplying on the left by $1 - bb^{=}$ and on the right by $1 - a^{=}a$, we have

$$(1 - bb^{=})(1 - a^{=}a) = 0.$$

The converse is obvious. □

Next, we give several equivalences to $aa^\dagger = bb^\dagger aa^\dagger$. There, we will make use of the partial orders (see [3] and [4]) $\leq$ and $\underset{*}{\leq}$ defined by

- $a \leq b$ if $aa^+ = ba^+$ and $a^+a = a^+b$ for some reflexive inverse a^+ of a;
- $a \underset{*}{\leq} b$ if $aa^* = ba^*$ and $a^*a = a^*b$.

If $a \in R^\dagger$ then $a \underset{*}{\leq} b$ implies $a \leq b$. If R is a *-regular ring then $\underset{*}{\leq} \subseteq \leq$ as subsets of $R \times R$. If e, f are symmetric idempotents in R then $e \leq f$ forces $e \underset{*}{\leq} f$. In fact, if e and f are such that $e = e^2 = e^*$ and $f = f^2 = f^*$ and if there exists a reflexive inverse e^+ of e for which $ee^+ = fe^+$ and $e^+e = e^+f$ then $e = ee^+e = fe^+e = fe^+f$. Hence $ef = fe^+f^2 = fe^+f = e$ and $fe = f^2e^+f = fe^+f = e$. This implies, if $a, b \in R^\dagger$, that $aa^\dagger \underset{*}{\leq} bb^\dagger$ *exactly* when $aa^\dagger \leq bb^\dagger$.

Proposition 2.2. *Let $a, b \in R^\dagger$ with Moore-Penrose inverses $a^\dagger$ and $b^\dagger$, respectively. The following conditions are equivalent:*

1. $aa^\dagger = bb^\dagger aa^\dagger$
2. $aa^- = bb^- aa^-$ *for some* $a^- \in a\{1\}, b^- \in b\{1\}$
3. $aa^- = bb^- aa^-$ *for all choices* $a^- \in a\{1\}, b^- \in b\{1\}$
4. $aa^* = bb^\dagger aa^* = aa^* bb^\dagger$
5. $aa^* = bb^- aa^*$ *for some* $b^- \in b\{1\}$
6. $aa^* = bb^- aa^*$ *for all choices* $b^- \in b\{1\}$
7. $a^* = a^* bb^\dagger$
8. $a = bb^\dagger a$
9. $a = bb^- a$ *for some* $b^- \in b\{1\}$
10. $a = bb^- a$ *for all* $b^- \in b\{1\}$
11. $aa^\dagger \underset{*}{\leq} bb^\dagger$
12. $a \leq bb^- a$ *for some* $b^- \in b\{1\}$

13. $a \leq bb^- a$ *for all* $b^- \in b\{1\}$

14. $aR \subseteq bb^\dagger aR$

15. $Ra^\dagger \subseteq Ra^\dagger bb^\dagger$

Proof. (1)⇔(3). Since $aa^\dagger = bb^\dagger aa^\dagger$ is equivalent to $(1-bb^\dagger)aa^\dagger = 0$, by Lemma 2.1 the result follows.
(2)⇔(3) follows directly from Lemma 2.1.
(1)⇔(4). $aa^\dagger = bb^\dagger aa^\dagger$ multiplied on the right hand side by aa^* gives the first equality. Since aa^* is symmetric, the second equality becomes trivial. Conversely, right multiplication of $aa^* = bb^\dagger aa^*$ by $(a^\dagger)^* a^\dagger$, gives (1), since $aa^*(a^\dagger)^* a^\dagger = a(a^\dagger a)^* a^\dagger = aa^\dagger$.
(3)⇒(6). Condition (3) implies, in particular, the equality $aa^\dagger = bb^- aa^\dagger$, for any choice of $b^- \in b\{1\}$. Post-multiplication by aa^* gives condition (6).
(6)⇒(5) is obvious.
(5)⇒(2). If $aa^* = bb^- aa^*$ for some $b^- \in b\{1\}$ then multiplying on the right hand side by $(a^\dagger)^* a^\dagger$ we get the equality $aa^\dagger = bb^- aa^\dagger$, for some $b^- \in b\{1\}$. (2) follows by taking $a^- = a^\dagger$.
(1)⇔(8) is obvious by multiplication of (1) on the right hand side by a and of (8) by $a^\dagger$.
(7)⇔(8), (2)⇔(9), (3)⇔(10), (1)⇔(11) are trivial.
(10)⇒(13) by the reflexivity of the partial order.
(13)⇒(12) is obvious.
(12)⇒(9). (12) implies there is, in particular, a reflexive inverse a^+ of a for which $aa^+ = bb^- aa^+$, and therefore $a = bb^- a$, for some b^-.
(1)⇒ (14) is trivial.
(14)⇒ (1). Since $aa^\dagger \in aR$ and $bb^\dagger aR = bb^\dagger aa^\dagger R$, $aa^\dagger \in bb^\dagger aa^\dagger R$ and therefore $aa^\dagger = bb^\dagger aa^\dagger x$ for some $x \in R$. By multiplication on the left hand side by $bb^\dagger$ we get $bb^\dagger aa^\dagger = bb^\dagger bb^\dagger aa^\dagger x = bb^\dagger aa^\dagger x$. Hence $aa^\dagger = bb^\dagger aa^\dagger$.
(1)⇒ (15) is trivial.
(15)⇒ (1). Note that $aa^\dagger \in Ra^\dagger$ and $Ra^\dagger bb^\dagger = Raa^\dagger bb^\dagger$. These imply $aa^\dagger = xaa^\dagger bb^\dagger$ for some $x \in R$. By multiplication on the right hand side by $bb^\dagger$ the equality $aa^\dagger = bb^\dagger aa^\dagger$ follows. □

As a remark, note that $aa^\dagger = aa^\dagger bb^\dagger \Leftrightarrow aa^\dagger = bb^\dagger aa^\dagger \Rightarrow [aa^\dagger, bb^\dagger] = 0$. Moreover, the converse on the previous implication is false.

Theorem 2.3. *Let* $a, b \in R^\dagger$ *with Moore-Penrose inverses* $a^\dagger$ *and* $b^\dagger$*, respectively. The following conditions are equivalent:*

1. $aa^\dagger = bb^\dagger$.

2. $aa^\dagger = aa^\dagger bb^\dagger$ *and* $u = aa^\dagger + 1 - bb^\dagger$ *is invertible.*

3. $aa^\dagger = aa^\dagger bb^\dagger$ *and* $v = aa^* + 1 - bb^\dagger$ *is invertible.*

4. $aa^\dagger = aa^\dagger bb^\dagger$ *and* $\exists_{b^- \in b\{1\}} : w = aa^* + 1 - bb^-$ *is invertible.*

5. $aa^\dagger = aa^\dagger bb^\dagger$ *and* $\forall_{b^= \in b\{1\}} : w = aa^* + 1 - bb^=$ *is invertible.*

6. $[aa^\dagger, bb^\dagger] = 0$ *and both* $u = aa^\dagger + 1 - bb^\dagger$ *and* $l = bb^\dagger + 1 - aa^\dagger$ *are invertible.*

7. $[aa^\dagger, bb^\dagger] = 0$ *and both* $v = aa^* + 1 - bb^\dagger$ *and* $k = bb^* + 1 - aa^\dagger$ *are invertible.*

Proof. (1) implies (2), (3), (6) and (7) is trivial.
(2)⇔(3). Note that $(aa^\dagger + 1 - bb^\dagger)(aa^* + 1 - aa^\dagger) = aa^* + 1 - bb^\dagger$, and therefore u is a unit iff v is a unit, since both $aa^* + 1 - aa^\dagger$ and u are symmetric.
(3)⇒(1). Recall that $aa^\dagger = bb^\dagger aa^\dagger$ implies $aa^* = aa^* bb^\dagger$, and hence $vaa^\dagger = aa^*$ and $vbb^\dagger = aa^*$. The invertibility of v implies that $aa^\dagger = bb^\dagger$.
(5)⇒(3) and (5)⇒(4) are trivial.
(3)⇒(5). Since $aa^* = bb^\dagger aa^*$ then we can rewrite

$$v = bb^\dagger aa^* + 1 - bb^\dagger = bb^\dagger aa^* bb^\dagger + 1 - bb^\dagger.$$

This means v is invertible if and only if $bb^\dagger aa^* bb^- + 1 - bb^-$ is a unit for any choice of b^-, using [10, Proposition 3]. That is to say, v is a unit if and only if $1 - (-bb^\dagger aa^* + 1)bb^-$ is a unit, which is equivalentto the invertibility of $1 - bb^-(-bb^\dagger aa^* + 1) = 1 + bb^\dagger aa^* - bb^- = w$ since $bb^\dagger aa^* = aa^*$, where b^- is an arbitrary 1-inverse on b.
(4)⇒(5). According to Proposition 2.2, $aa^* = bb^- aa^*$ and hence

$$w = aa^* + 1 - bb^- = bb^- aa^* + 1 - bb^- = 1 - bb^-(-aa^* + 1).$$

The invertibility of w implies the invertibility of

$$1 - (-aa^* + 1)bb^- = aa^* bb^- + 1 - bb^- = bb^- aa^* bb^- + 1 - bb^-.$$

Using [10, Proposition 3], this implies

$$bb^- aa^* bb^= + 1 - bb^= = aa^* bb^= + 1 - bb^= = 1 - (-aa^* + 1)bb^=$$

is a unit for *all* choices of $b^= \in b\{1\}$. This means that also $1 - bb^=(-aa^* + 1) = bb^= aa^* + 1 - bb^=$ is a unit. By Lemma 2.1, $bb^= aa^* = aa^*$, and the result follows.
(6)⇒(1). From $[aa^\dagger, bb^\dagger] = 0$ the equalities $bb^\dagger aa^\dagger = bb^\dagger u = bb^\dagger aa^\dagger u$ and $bb^\dagger aa^\dagger = laa^\dagger = lbb^\dagger aa^\dagger$ hold. The invertibility of u implies $bb^\dagger aa^\dagger = bb^\dagger$ and the invertibility of l implies $bb^\dagger aa^\dagger = aa^\dagger$. Hence $aa^\dagger = bb^\dagger$.
(7)⇒(3). From $[aa^\dagger, bb^\dagger] = 0$ we get $aa^\dagger k = aa^\dagger bb^* = bb^\dagger aa^\dagger k$. The invertibility of k implies $aa^\dagger = bb^\dagger aa^\dagger$. The result follows. □

3 Final remarks

We end this paper with some remarks:

1. The equivalence (1) $\Leftrightarrow$ (2) in Proposition 2.2 could be proved directly, as (1) $\Rightarrow$ (2) trivially, and if $aa^- = bb^-aa^-$ then right muliplication by $aa^\dagger$ gives $aa^\dagger = bb^-aa^\dagger$, which in turn gives, after left multiplication by $bb^\dagger$, $bb^\dagger aa^\dagger = bb^-aa^\dagger = aa^\dagger$.

2. The equality $aa^\dagger = aa^\dagger bb^\dagger$ does *not* imply $aa^\dagger = bb^\dagger$. As an example, take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = I_2$ over $\mathbb{Z}_2$, and the transposition as the involution. Then $A^\dagger = A, B^\dagger = B^{-1} = B$ and $AA^\dagger = AA^\dagger BB^\dagger$ and still $AA^\dagger \neq BB^\dagger$.

3. The invertibility of u, v or w, by itself, is *not* sufficient to $aa^\dagger = bb^\dagger$. For instance, over $\mathbb{R}$ and the transposition as involution, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^\dagger = A^*$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = B^\dagger = B^*$, then u and v are invertible. Still, $AA^\dagger \neq BB^\dagger$.

4. The invertibility of both u and l is also *not* sufficient to $aa^\dagger = bb^\dagger$. Over $\mathbb{Z}_7$, consider the transposition as the involution and take $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $A^\dagger = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $B^\dagger = B$ and both u and l are invertible. Still, $AA^\dagger \neq BB^\dagger$.

5. The co-support of a (see [5, page 374]) provides an alternative proof of (3)$\Rightarrow$(1) in Theorem 2.3. The uniqueness of the co-support, jointly with the invertibility of v and $a = bb^\dagger a$ implies $aa^\dagger = bb^\dagger$.

6. Theorem 2.3 could be used to study EP elements in R, that is, elements for which the Moore-Penrose and the group inverse exist and are equal. We may apply our theorem to derive some more equivalent conditions. Take $b = a^\dagger$.

   (a) The condition $aa^\dagger = bb^\dagger = a^\dagger a$ defines a to be EP.

   (b) Statement (2) in the theorem becomes $aa^\dagger = a(a^\dagger)^2 a = a^\dagger a^2 a^\dagger$, which implies by multiplication on the right hand side by a that $a = a(a^\dagger)^2 a^2 = a^\dagger a^2$, and so $a \in Ra^2$. Now, $a = a^2 a^\dagger u^{-1} \in a^2 R$ and therefore $a \in a^2 R \cap Ra^2$. Consequently, $a^\#$ exists ([2, Proposition 7]). Recalling $a = a^\dagger a^2$, and multiplying on the right hand side by $a^\#$ then $aa^\# = a^\dagger a$ is symmetric, which leads to $aa^\dagger = a^\dagger a$ and a is EP.

   (c) The invertibility of $v = aa^* + 1 - a^\dagger a$ in (3) jointly with $a = a^\dagger a^2$ is equivalent to a is EP.

   (d) In condition (4) and (5), it is not trivial a is EP if and only if $a = a^\dagger a^2$ and $w = aa^* + 1 - a^\dagger (a^\dagger)^-$ is a unit for one and hence all choices of $(a^\dagger)^-$.

(e) In condition (6), from $a(a^\dagger)^2a^2 = a^\dagger a^2$, multiplying on the left hand side by a, the equality $a^2 = a^2(a^\dagger)^2a^2$ holds. In this case, $u = aa^\dagger + 1 - a^\dagger a$ and $l = a^\dagger a + 1 - aa^\dagger$ are units. From $a = a^2a^\dagger u^{-1} \in a^2R$ and $a = l^{-1}a^\dagger a^2 \in Ra^2$ follows $a^\#$ exists. Multiplying both sides of $a^2 = a^2(a^\dagger)^2a^2$ by $a^\#$, we obtain $aa^\# = a(a^\dagger)^2a$ which is symmetric since $a(a^\dagger)^2a = a^\dagger a^2a^\dagger$.

(f) Condition (7) states a is EP if and only if $[aa^\dagger, a^\dagger a] = 0$ and both $k = a^\dagger(a^\dagger)^* + 1 - aa^\dagger$ and $v = aa^* + 1 - a^\dagger a$ are units. As in the previous item, the invertibility of k and v imply $a \in a^2R \cap Ra^2$, that is to say, a has a group inverse.

7. Similar considerations for EP elements could be drawn by taking, in Theorem 2.3, $b = a^*$, $a = b^\dagger$ and $a = b^*$.

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