Maximal inverse subsemigroups of the symmetric inverse semigroup on a finite-dimensional vector space

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Abstract

Yang (1999) classified the maximal inverse subsemigroups of all the ideals of the symmetric inverse semigroup $I(X)$ defined on a finite set X. Here we do the same for the semigroup $I(V)$ of all one-to-one partial linear transformations of a finite-dimensional vector space. We also show that $I(X)$ is almost never isomorphic to $I(V)$ for any set X and any vector space V, and prove that any inverse semigroup can be embedded in some $I(V)$.

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1. Introduction

Suppose V is a vector space and let $P(V)$ denote the set of all *partial linear transformations* of V: that is, all linear transformations α whose *domain*, dom α, and *range*, ran α, are subspaces of V. As for partial transformations of a set (compare [2] Vol. 1, p. 29) we define the *composition* $\alpha \circ \beta$ of $\alpha, \beta \in P(V)$ to be the linear transformation with domain $U = (\text{ran } \alpha \cap \text{dom } \beta)\alpha^{-1}$ such that, for all $u \in U$,

$$u(\alpha \circ \beta) = (u\alpha)\beta,$$

and we often write $\alpha \circ \beta$ more simply as $\alpha\beta$. Clearly, U is a subspace of V and $\alpha \circ \beta \in P(V)$ if $\alpha, \beta \in P(V)$. Also, $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$ for all $\alpha, \beta, \gamma \in P(V)$, so $(P(V), \circ)$ is a semigroup (unless stated otherwise, we use the notation and terminology of [2]). As usual, we let $\ker \alpha = \{u \in \text{dom } \alpha : u\alpha = 0\}$ and rank $\alpha = \dim \text{ran } \alpha$.

There is a natural subsemigroup of $P(V)$ – namely, $I(V)$ – which consists of all *one-to-one* partial linear transformations of V (that is, all $\alpha \in P(V)$ such that $\ker \alpha = \{0\}$); and it is easy to see that $I(V)$ is an *inverse* semigroup (that is, for each $\alpha \in I(V)$ there exists a unique $\beta \in I(V)$, namely $\beta = \alpha^{-1}$, such that $\alpha = \alpha\beta\alpha$ and $\beta = \beta\alpha\beta$). Some properties of $I(V)$ were studied in [8] in the context of 'independence algebras'. In fact, $I(V)$ can be regarded as a linear version of the well-known 'symmetric inverse semigroup' $I(X)$ defined on a set X, and we adopt this approach here. Note that we use the 'V' in $I(V)$ to denote the fact that we are considering *linear* transformations.

For example, it is well-known that the ideals of $I(X)$ are the sets

$$I_r(X) = \{\alpha \in I(X) : \text{rank } \alpha \leq r\}$$

where $0 \leq r \leq |X|$ and, for $\alpha \in I(X)$, rank $\alpha = |\text{ran } \alpha|$. Likewise, following [2] Vol. 1, p. 57, Exercise 6, it can be shown that, if $\dim V = n < \aleph_0$, then the ideals of $I(V)$ are precisely the sets

$$I_r(V) = \{\alpha \in I(V) : \text{rank } \alpha \leq r\}$$

where $0 \leq r \leq n$ (in a small way, this extends a remark in [8] p. 427). Note that $I_0(V) = \{0\}$ where 0 denotes the linear map with domain $\{0\}$ in V, and this map acts as a zero for the semigroup $I(V)$.

For arbitrary finite sets X, Yang [10] described all maximal inverse subsemigroups of each ideal of $I(X)$ in terms of the maximal subgroups of $G(X)$, the symmetric group on X, and all of these are known. Here we do the same for $I(V)$ where V is any finite-dimensional vector space over any field F, although all maximal subgroups of $GL(n, F)$, the general linear group of degree n over F, are not known at present. As stated in [11]

p. 162, “Currently, there is a fairly rich collection of examples of maximal subgroups of the general linear group. However, it is unclear ... how to find a description of all of them.”

For example, in [1] Theorem 7, Borevich states that, if $n \geq 2$ and F is any field with $|F| \geq 7$, then all maximal subgroups of $GL(n, F)$ that contain the group of diagonal matrices over F are known. On the other hand, we recall that $Z(G)$, the centre of $G = GL(n, F)$, is the set of (non-zero) scalar matrices in G and that, when $F = GF(q)$ is finite, $PGL(n, q) = GL(n, q)/Z(G)$ is the *projective general linear group* over F. It can be shown that, for $3 \leq n \leq 12$, $PGL(n, q)$ satisfies the criteria of [6] Theorem 1.2.2 and so all of its maximal subgroups are known. From this we deduce that, if $3 \leq n \leq 12$, then all subgroups of $GL(n, q)$ which are maximal and contain all the scalar matrices are known. However, simple examples show that a maximal subgroup of $GL(n, q)$ need not contain all the scalar matrices; and moreover, if one contains all the scalar matrices, then it need not contain all the diagonal matrices.

2. Preliminary notation and results

Throughout this paper, V is a vector space with finite dimension n.

As an abbreviation, we write a subset $\{e_i : i \in I\}$ of V as $\{e_i\}$, letting the subscript denote an (unspecified) index set I (this is comparable with [2] Vol. 2, p. 241 and [7] Vol. 1, p. 51). The subspace U of V generated by a linearly independent subset $\{e_i\}$ of V is denoted by $\langle e_i \rangle$, and we write $\dim U = |I|$.

Often it is necessary to construct some $\alpha \in I(V)$ by first choosing linearly independent subsets $\{e_i\}$ and $\{u_i\}$ of V, and then letting $e_i\alpha = u_i$ for each $i \in I$ and extending this action by linearity to the whole of dom $\alpha = \langle e_i \rangle$. To abbreviate matters, we simply say, given $\{e_i\}$ and $\{u_i\}$ within context, that $\alpha \in I(V)$ is defined by letting

$$\alpha = \begin{pmatrix} e_i \\ u_i \end{pmatrix}.$$

Similar notation for $I(X)$ is now standard: for example, see [9]. However, for simplicity, if X is a set and $a, b \in X$, we write a_b for the map in $I(X)$ with domain $\{a\}$ and range $\{b\}$.

Following [2] Vol. 1, p. 57, Exercise 6 (compare [8] section 3, p. 427, for endomorphisms of an independence algebra), it is easy to see that $\alpha \, \mathcal{D} \, \beta$ in $I(V)$ if and only if rank $\alpha =$ rank β (that is, $\mathcal{D} = \mathcal{J}$). Consequently, we denote each $\mathcal{D}$–class of $I(V)$ by

$$D_r = \{\alpha \in I(V) : \text{rank } \alpha = r\}$$

where $0 \leq r \leq n$ (when necessary, we use $D_r(V)$ to emphasise our use of linear maps). It is also easy to see that the idempotents of $I(V)$ are precisely the identity maps id_A whose domains A are subspaces of V. Similar facts and notation are well-known for $I(X)$: compare [9] pp. 309-310. In particular, we use $D_r(X)$ (or simply D_r within context) to denote the $\mathcal{D}$–class of $I(X)$ consisting of all elements of $I(X)$ with rank r where $0 \leq r \leq |X|$.

We begin by showing that $I(X)$ and $I(V)$ are never isomorphic if X is a set and V is a vector space over a field F where $|X| \geq 1$, $\dim V \geq 1$ and $|F| \geq 3$ (this question was not considered in [8]). For, if such an isomorphism exists, then $I_1(X)$ is isomorphic to $I_1(V)$ since these are the smallest non-zero ideals of $I(X)$ and $I(V)$, respectively. But this is impossible since every element of $I_1(X)$ is either an idempotent or a nilpotent of index 2, whereas if $k \neq 0, 1$ in F and $u \neq 0$ in V then $I_1(V)$ contains an element α with domain $\langle u \rangle$ such that $\alpha : u \mapsto ku$, and this is neither idempotent nor nilpotent.

To handle the case when $|F| = 2$, we suppose $|X| \geq 3$ and assert that $I(X)$ satisfies:

$$\text{if } \alpha \in D_2(X) \text{ where } \alpha^2 \text{ is not idempotent and } \alpha\gamma,\ \alpha^2\gamma \text{ are non-zero for some idempotent } \gamma \in D_1(X), \text{ then } \gamma\alpha = \emptyset. \qquad (*)$$

To see this, suppose $\alpha \in D_2(X)$ satisfies the initial condition and write

$$\alpha = \begin{pmatrix} a & b \\ x & y \end{pmatrix}.$$

Without loss of generality, let $\gamma = x_x$. If $x = a$ then $\alpha^2 = a_a$ (if $b \neq y$) or $\alpha^2 = \mathrm{id}_{\{a,b\}}$ (if $b = y$), contradicting our supposition that α^2 is not idempotent. On the other hand, if $x = b$ then $y \neq b$ (since α is injective) and $y \neq a$ (since α^2 is not idempotent). Hence $\alpha^2 = a_y$ and so $\alpha^2\gamma = \emptyset$, another contradiction. Therefore, $x \neq a, b$ and so $\gamma\alpha = \emptyset$. Finally, note that $(*)$ is not vacuous in $I(X)$ if $|X| \geq 3$, since the map $b \to a \to x$, where $x \neq a, b$, satisfies $(*)$.

Clearly, property $(*)$ will be preserved under an isomorphism φ from $I(X)$ onto $I(V)$. In fact, φ will map the $\mathcal{D}$–classes of $I(X)$ onto the $\mathcal{D}$–classes of $I(V)$ in an obvious way. However, $I(V)$ does not satisfy $(*)$ if $\dim V \geq 3$ and $|F| = 2$. For, in this case, if a, b are two non-zero vectors in V then they are linearly independent, and the same is true for $\{b, a + b\}$. Let

$$\alpha = \begin{pmatrix} a & b \\ b & a+b \end{pmatrix} \quad \text{and} \quad \alpha^2 = \begin{pmatrix} a & b \\ a+b & a \end{pmatrix}$$

be linear maps defined and obtained as shown, and observe that α^2 is not idempotent. Moreover, if $c = a + b$ and $\gamma \in D_1(V)$ is the idempotent with domain $\langle c \rangle$, then $\alpha\gamma$ and

$\alpha^2\gamma$ are non-zero. However, $\gamma\alpha$ is also non-zero, so property $(*)$ fails to hold in $I(V)$. Therefore, $I(X)$ and $I(V)$ are not isomorphic when $|X| \geq 3$, $\dim V \geq 3$ and $|F| = 2$. Thus we have proved most of the following result.

Theorem 1. The semigroups $I(X)$ and $I(V)$ are isomorphic only when $|X| = 1 = \dim V$ and $|F| = 2$.

Proof. For the remaining cases, we suppose $1 < |X| = n < \aleph_0$, $1 < \dim V = m < \aleph_0$ and $|F| = 2$. Clearly, the above discussion covers almost all of these situations but the argument we use here gives a more direct proof in the finite case. In fact, since V has finite dimension m over the field F, we know $V \cong F^m$ (as vector spaces) and so $|V| = 2^m$. Also, the ideals of $I(X)$ form a chain with length $n+1$; and we have a similar result for $I(V)$: its ideals form a chain with length $m+1$. Hence, if $I(X)$ and $I(V)$ are isomorphic then $n = m$ and, as mentioned above, $I_1(X)$ is isomorphic to $I_1(V)$. But the non-zero elements of $I_1(X)$ are the maps a_b where $a, b \in X$, hence $|I_1(X)| = n^2 + 1$. And similarly, since $|F| = 2$, the non-zero elements of $I_1(V)$ are the maps $u \mapsto v$ where u, v are non-zero elements in V. Hence, if $m = n$ then $|I_1(V)| = (2^n - 1)^2 + 1 = 2^{n+1}(2^{n-1} - 1) + 2$ and, since $2^{n+1} > n^2 + 1$ for every $n \geq 2$, it follows that $|I_1(V)| > |I_1(X)|$. Finally, observe that if $|X| = 1 = \dim V$ and $|F| = 2$, then $I(X)$ and $I(V)$ are simply the group of order 1 with a zero adjoined. $\square$

In view of the above, it is interesting to give a linear version of the Vagner-Preston Theorem: our proof closely follows that of [2] Vol. 1, Theorem 1.20.

Theorem 2. Any inverse semigroup can be embedded in $I(V)$ for some vector space V.

Proof. Let S be an inverse semigroup with $|S| = k$ (finite or infinite) and write $S = \{a_i\}$ with $|I| = k$. Let F be any field and let F_i be a copy of F for each $i \in I$. As in [5] p. 182, Remark (c), we let V be the vector space $\sum F_i$ over F whose basis can be identified in a natural way with $\{a_i\}$: that is, $\sum F_i$ is the set of all $(r_i)_{i\in I}$ where $r_i \in F_i$ and at most finitely many r_i are non-zero.

Let $x \in S$ and suppose ρ_x is the partial map with domain $Sx^{-1} = Sxx^{-1}$ such that $a\rho_x = ax$ $(a \in Sx^{-1})$. The range of ρ_x is $Sx^{-1}x = Sx$ and, in fact, ρ_x is one-to-one: if $a, b \in Sx^{-1}$ and $a\rho_x = b\rho_x$, then there exist $j, k \in I$ such that $a = a_jx^{-1}$ and $b = a_kx^{-1}$, so $a = a_jx^{-1} = a_jx^{-1}xx^{-1} = a_kx^{-1}xx^{-1} = a_kx^{-1} = b$. Since Sxx^{-1} is a subset of S, it is linearly independent. Hence ρ_x can be extended by linearity to an injective linear map $\rho_x : \langle a_ixx^{-1}\rangle \to V$ with range $\langle a_ix\rangle$. Since $\{a_ix\}$ and $\{a_ixx^{-1}\}$ are bases for the domain of ρ_x, we can display ρ_x as an element of $I(V)$ in two different ways.

$$\rho_x = \begin{pmatrix} a_i x^{-1} \\ a_i x^{-1} x \end{pmatrix} \quad \text{and} \quad \rho_x = \begin{pmatrix} a_i x x^{-1} \\ a_i x \end{pmatrix}.$$

We assert that $\phi : S \to I(V), x \mapsto \rho_x$, is an injective homomorphism.

Let $x, y \in S$ and suppose $x\phi = y\phi$, that is, $\rho_x = \rho_y$. Then, $\langle a_i x x^{-1} \rangle = \text{dom } \rho_x = \text{dom } \rho_y = \langle a_i y y^{-1} \rangle$ and $a\rho_x = a\rho_y$ for all $a \in \text{dom } \rho_x$. Since $x^{-1} \in Sxx^{-1} = \{a_i x x^{-1}\}$, it follows that $x^{-1}x = x^{-1}\rho_x = x^{-1}\rho_y = x^{-1}y$. On the other hand, $xx^{-1} \in \{a_i x^{-1}\} \subseteq \langle a_i x^{-1} \rangle = \langle a_i y^{-1} \rangle$ and so $xx^{-1} = \sum r_i(a_i y^{-1})$ for some scalars $r_i \in F$. Since $\{xx^{-1}\} \cup \{a_i y^{-1}\} \subseteq S$ and S is linearly independent, this implies that there exists some $i_0 \in I$ such that $r_{i_0} = 1$ and $xx^{-1} = a_{i_0} y^{-1}$. Similarly we find that $yy^{-1} = a_{i_1} x^{-1}$ for some $i_1 \in I$. Therefore, since idempotents commute in inverse semigroups, we have:

$$\begin{aligned} xx^{-1} = a_{i_0} y^{-1} &= a_{i_0} y^{-1} y y^{-1} \\ &= xx^{-1} yy^{-1} = yy^{-1} xx^{-1} = a_{i_1} x^{-1} x x^{-1} = a_{i_1} x^{-1} = yy^{-1}. \end{aligned}$$

Hence, $x = xx^{-1}x = xx^{-1}y = yy^{-1}y = y$.

To see that ϕ is a homomorphism, we let $x, y \in S$ and recall that ρ_x, ρ_y are defined in $I(V)$ by

$$\rho_x = \begin{pmatrix} a_i x^{-1} \\ a_i x^{-1} x \end{pmatrix}, \quad \rho_y = \begin{pmatrix} a_i y y^{-1} \\ a_i y \end{pmatrix}.$$

It is easy to see that $\rho_x^{-1} = \rho_{x^{-1}}$. Since $\{a_i x^{-1} x\} \cap \{a_i y y^{-1}\} = Sx^{-1}x \cap Syy^{-1} = Sx^{-1}xyy^{-1} = Sxyy^{-1} = \{a_i x y y^{-1}\}$, it follows that

$$\rho_x \rho_y = \begin{pmatrix} a_i x y y^{-1} x^{-1} \\ a_i x y y^{-1} y \end{pmatrix} = \begin{pmatrix} a_i (xy)(xy)^{-1} \\ a_i (xy) \end{pmatrix} = \rho_{xy}.$$

Hence $(xy)\phi = x\phi y\phi$. □

3. Maximal inverse subsemigroups of $I(V)$

As in [10], to describe the maximal inverse subsemigroups of $I(V)$, we begin with two Lemmas: the second of these is comparable with [3] Theorem 3.1 (we let G_n denote the group of all invertible linear transformations of V).

Lemma 1. For $1 \le r \le n-2$, $D_r \subseteq D_{r+1} D_{r+1}$.

Proof. Let $\alpha \in D_r$ and choose bases $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_r\}$ for dom α and ran α respectively, where $a_i \alpha = b_i$ for each i. Since $r \le n-2$, there exist $u_0, u_1 \in V$ such that $\{a_1, \ldots, a_r, u_0, u_1\}$ is linearly independent; and likewise $\{b_1, \ldots, b_r, v_1\}$ is linearly independent for some $v_1 \in V$. Then we have

$$\alpha = \begin{pmatrix} a_1 & \ldots & a_r \\ b_1 & \ldots & b_r \end{pmatrix} = \begin{pmatrix} a_1 & \ldots & a_r & u_0 \\ a_1 & \ldots & a_r & u_0 \end{pmatrix} \circ \begin{pmatrix} a_1 & \ldots & a_r & u_1 \\ b_1 & \ldots & b_r & v_1 \end{pmatrix}$$

where the two mappings on the right can be suitably extended by linearity to become elements of D_{r+1}. □

Lemma 2. If $\alpha \in D_{n-1}$ then $I(V) = \langle \alpha, G_n \rangle$.

Proof. Suppose $a_1, \ldots, a_{n-1}$ and $b_1, \ldots, b_{n-1}$ are bases for dom α and ran α respectively, and assume that

$$\alpha = \begin{pmatrix} a_1 & \ldots & a_{n-1} \\ b_1 & \ldots & b_{n-1} \end{pmatrix}.$$

To show that $\langle \alpha, G_n \rangle$ contains a mapping with rank $n-2$, we consider two cases.

Case 1. Suppose dom α = ran $\alpha = A$ say. Then there exists $u \notin A$ and $g \in G_n$ defined by

$$g = \begin{pmatrix} b_1 & \ldots & b_{n-2} & b_{n-1} & u \\ b_1 & \ldots & b_{n-2} & u & b_{n-1} \end{pmatrix}.$$

Since $\langle b_1, \ldots, b_{n-2} \rangle \subseteq$ dom α and α is injective, we obtain

$$\alpha g.\alpha = \begin{pmatrix} a_1 & \ldots & a_{n-2} & a_{n-1} \\ b_1 & \ldots & b_{n-2} & u \end{pmatrix} \circ \begin{pmatrix} a_1 & \ldots & a_{n-2} & a_{n-1} \\ b_1 & \ldots & b_{n-2} & b_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & \ldots & a_{n-2} \\ c_1 & \ldots & c_{n-2} \end{pmatrix}$$

for some linearly independent $c_1, \ldots, c_{n-2} \in V$.

Case 2. Suppose A = dom $\alpha \neq$ ran $\alpha = B$. Since A and B have the same finite dimension, this implies $A \not\subseteq B$. Choose $u \in A \setminus B$, let $\{c_1, \ldots, c_{n-2}, u\}$ be a basis for A and write

$$\alpha = \begin{pmatrix} c_1 & \ldots & c_{n-2} & u \\ d_1 & \ldots & d_{n-2} & v \end{pmatrix}$$

where $\{d_1, \ldots, d_{n-2}, v\}$ is a basis for B (hence $\{d_1, \ldots, d_{n-2}, v, u\}$ is a basis for V). Choose $w \notin A$ and define $g \in G_n$ by

$$g = \begin{pmatrix} d_1 & \ldots & d_{n-2} & v & u \\ c_1 & \ldots & c_{n-2} & u & w \end{pmatrix}.$$

Then

$$\alpha g = \begin{pmatrix} c_1 & \ldots & c_{n-2} & u \\ c_1 & \ldots & c_{n-2} & u \end{pmatrix}$$

and, as in Case 1, we can use this to obtain a mapping in $\langle \alpha, G_n \rangle$ with rank $n-2$.

Clearly we can repeat the above process to obtain an element of $\langle \alpha, G_n \rangle$ with rank r where $0 \leq r \leq n-1$. Then if

$$\beta = \begin{pmatrix} e_1 & \dots & e_r \\ f_1 & \dots & f_r \end{pmatrix}$$

is any element of $\langle \alpha, G_n \rangle$ with rank r, we can expand dom β and ran β to bases

$$\{e_1, \dots, e_r, e_{r+1}, \dots, e_n\} \quad \text{and} \quad \{f_1, \dots, f_r, f_{r+1}, \dots, f_n\}$$

for V. Also, if $\{u_1, \dots, u_r\}$ and $\{v_1, \dots, v_r\}$ are any linearly independent subsets of V, we can expand them to bases $\{u_1, \dots, u_r, u_{r+1}, \dots, u_n\}$ and $\{v_1, \dots, v_r, v_{r+1}, \dots, v_n\}$ for V and then define $g, h \in G_n$ by $u_i g = e_i$ and $f_i h = v_i$ for each i. Then

$$g\beta h = \begin{pmatrix} u_1 & \dots & u_r \\ v_1 & \dots & v_r \end{pmatrix} \in \langle \alpha, G_n \rangle.$$

Since this is an arbitrary element of $I(V)$ with rank r, we conclude that $I(V) = \langle \alpha, G_n \rangle$.
□

The proof of the next result follows that of [10] Theorem 2.3. For convenience, here we write $I_r(V)$ more simply as I_r.

Theorem 3. If $\dim V = n < \aleph_0$, the maximal inverse subsemigroups of $I(V)$ are precisely the following sets.

(a) $I_{n-2} \cup G_n$,

(b) $I_{n-1} \cup H$, where H is a maximal subgroup of G_n.

Proof. Clearly, $I_{n-1} \cup H$ is a maximal inverse subsemigroup of $I(V)$ in (b). Also, $I_{n-2} \cup G_n$ is an inverse semigroup. In fact, if $I_{n-2} \cup G_n \subsetneq T \subseteq I(V)$ for some inverse semigroup T then $T \cap D_{n-1} \neq \emptyset$ and so Lemma 2 implies $T = I(V)$.

Conversely, suppose S is a maximal inverse subsemigroup of $I(V)$. Then $S \cap G_n \neq \emptyset$: otherwise, $S \subseteq I_{n-1} \subsetneq I_{n-1} \cup H$ for any maximal subgroup H of G_n, contradicting the maximality of S.

Case 1. $S \cap G_n = G_n$. This means $G_n \subseteq S$ and hence $S \cap D_{n-1} = \emptyset$: otherwise, $\langle \alpha, G_n \rangle \subseteq S$ for some $\alpha \in D_{n-1}$ and then Lemma 2 implies $S = I(V)$, a contradiction as before. Consequently, $S \subseteq I_{n-2} \cup G_n$ which is an inverse subsemigroup of $I(V)$, hence $S = I_{n-2} \cup G_n$ by the maximality of S.

Case 2. $S \cap G_n = H \neq G_n$. Note that H is a group since S is inverse (that is, $\alpha \in H$ implies $\alpha^{-1} \in H$ and clearly H is a semigroup). Now $S \subseteq I_{n-1} \cup H$ which is a proper inverse subsemigroup of $I(V)$, hence $S = I_{n-1} \cup H$. Moreover, if $H \subsetneq H' \subsetneq G_n$ for some subgroup H' of G_n, we have

$$S = I_{n-1} \cup H \subsetneq I_{n-1} \cup H' \subsetneq I(V),$$

which contradicts the maximality of S. Hence, H is a maximal subgroup of G_n. □

4. Maximal inverse subsemigroups of I_r

Given a group G, two arbitrary sets J and Λ, and an arbitrary but fixed $\Lambda \times J$ matrix $P = (p_{j\lambda})$ over G^0, the group with zero adjoined, then the *Rees* $J \times \Lambda$ *matrix semigroup over* G^0 *with sandwich matrix* P is the semigroup $\mathcal{M}^0(G; J, \Lambda; P)$ of all triples (j, a, λ) with $a \in G^0$, $j \in J$ and $\lambda \in \Lambda$ and a product defined as follows: given $(j, a, \lambda), (i, b, \mu) \in \mathcal{M}^0(G; J, \Lambda; P)$, $(j, a, \lambda)(i, b, \mu) = (j, ap_{\lambda i}b, \mu)$ (compare [2] Vol. 1, p. 88: we use J instead of the more usual I to avoid any confusion with earlier notation; and we use (j, a, λ) rather than (a, j, λ) to maintain some uniformity with the notation in [10]). If $J = \Lambda$ and P is the identity $J \times J$ matrix over G^0, then we say the semigroup $\mathcal{M}^0(G; J, \Lambda; P)$ is a *Brandt semigroup* and we denote it more simply by $B(G, J)$. From [2] Theorem 3.9, the semigroup $B(G, J)$ is a completely 0-simple inverse semigroup. Also, by [2] Lemma 3.2, Green's relations on $B(G, J)$ are characterised as follows: given (j, a, λ), (i, b, μ) in $B(G, J)$, we have $(j, a, \lambda) \, \mathcal{L} \, (i, b, \mu)$ if and only if $\lambda = \mu$; and $(j, a, \lambda) \, \mathcal{R} \, (i, b, \mu)$ if and only if $j = i$.

As in [10], if S is an inverse semigroup and A is a non-empty subset of S, then we write $\langle\langle A \rangle\rangle$ for the inverse subsemigroup of S generated by A.

In [10] section 3, Yang first determines all maximal inverse subsemigroups of the Brandt semigroup $B(G, J)$ where G is a finite group and $J = \{1, \ldots, m\}$. He requires two preliminary results ([10] Lemmas 3.2 and 3.3) to do this, and both of these depend on the following result. However, Yang's version of this result covers only the finite case. Since we need to know all maximal inverse subsemigroups of $B(G, J)$ where G is an arbitrary group with identity e and J is an arbitrary index set, we include a proof of the following result for clarity and completeness (for the reason why we need such generality, see the discussion after Lemma 6 below). Here, we write B^A for the set of all maps $g : A \to B$, and represent g by $[g_a]$ (that is, A acts as an index set for the images of g in B, hence the g_a's are not necessarily distinct).

Lemma 3. For each $i \in J$, let H_{ii} denote the $\mathcal{H}$-class of $B(G, J)$ containing the element (i,e,i), put $K_i = J \setminus \{i\}$ and let $[g_j] \in G^{K_i}$. Then,

$$B(G,J) = \langle\langle H_{ii} \cup \{(i,g_j,j) : j \in K_i\}\rangle\rangle.$$

Proof. Let $(j,g,k) \in B(G,J)$. If $j = k = i$, then $(j,g,k) \in H_{ii}$. If $j = i$ and $k \neq i$, then $(j,g,k) = (i, gg_k^{-1}, i)(i, g_k, k)$ where $(i, gg_k^{-1}, i) \in H_{ii}$; and similarly, if $j \neq i$ and $k = i$, then we can write (j,g,k) as $(i,g_j,j)^{-1}(i,g_jg,i)$. If $j \neq i$ and $k \neq i$, then $(j,g,k) = (i,g_j,j)^{-1}(i,g_jgg_k^{-1},i)(i,g_k,k)$. Therefore, each (j,g,k) can be written as a finite product of elements of $H_{ii} \cup \{(i,g_j,j) : j \in K_i\}$ and their inverses, and the result follows. □

The proofs of [10] Lemmas 3.2 and 3.3 are stated for finite groups and finite index sets and, in some places, the notation and argument is not entirely clear (for example, in [10] Lemma 3.2, $I = \{1, \ldots, m\}$ and $g_1, g_2, \ldots, g_m \in G$ where $g_1 = e$, the identity of the group G, all of which appears to require G to have order at least m, but this is not intended). For this reason, and since we must deal with arbitrary groups and arbitrary index sets, we prove the following Lemmas in detail, but use the basic idea of Yang's proofs to do so.

The first Lemma determines some maximal inverse subsemigroups of $B(G,J)$.

Lemma 4. Let H be a maximal subgroup of G and let $1 \in J$. Put $H_{11}^* = \{(1,h,1) \in B(G,J) : h \in H\}$. Let $[g_j] \in G^J$ with $g_1 = e$ and let

$$B_H = \langle\langle H_{11}^* \cup \{(1,g_j,j) : j \in J\}\rangle\rangle.$$

Then B_H is isomorphic to the Brandt semigroup $B(H,J)$ and it is a maximal inverse subsemigroup of $B(G,J)$.

Proof. Let H_{ij} denote the $\mathcal{H}$–class of $B(G,J)$ containing the element (i,e,j), that is: $H_{ij} = \{(i,g,j) : g \in G\}$, and write $\{(i, g_i^{-1}hg_j, j) : h \in H\}$ as $(i, g_i^{-1}Hg_j, j)$. Consider the set $H_{ij} \cap B_H$ and let $(i,g,j) \in B_H$. Then, (i,g,j) can be written as a finite product, say $y_1 \cdots y_r$, of elements of $H_{11}^* \cup \{(1,g_j,j) : j \in J\}$ and their inverses. Clearly, $y_1 = (i, g_i^{-1}, 1)$ and $y_r = (1, g_j, j)$ and so $y_2 \cdots y_{r-1} = (1, g_igg_j^{-1}, 1)$ (for example, $(i,g,j) = y_1y_2 \cdots y_r$ and $y_1 = (i, g_i^{-1}, 1)$ imply $(1, g_ig, j) = (1, g_i, i)(i,g,j) = (1,e,1) \cdot y_2 \cdots y_r = y_2 \cdots y_r$). Now, if $y_2 = (1, g_k, k)$ for some $k \in J \setminus \{1\}$, then y_3 must equal $(k, g_k^{-1}, 1)$, so $y_2y_3 = (1,e,1)$ and we then consider $y_4 \cdots y_{r-1}$; alternatively, if $y_2 = (1, h_2, 1)$ for some $h_2 \in H$, then $y_3 \cdots y_{r-1}$ equals $(1, g', 1)$ for some $g' \in G$ and we repeat this argument by starting with y_3. It follows that $y_2 \cdots y_{r-1} \in H_{11}^*$ and there exists some h in H such that $y_2 \cdots y_{r-1} = (1,h,1)$. Thus $(i,g,j) = (i, g_i^{-1}, 1)(1,h,1)(1,g_j,j) = (i, g_i^{-1}hg_j, j) \in (i, g_i^{-1}Hg_j, j)$. Conversely, given $h \in H$, we have $(i, g_i^{-1}hg_j, j) = (i, g_i^{-1}, 1)(1, h, 1)(1, g_j, j) \in B_H$ and so $(i, g_i^{-1}hg_j, j) \in H_{ij} \cap B_H$. Hence, $H_{ij} \cap B_H = (i, g_i^{-1}Hg_j, j)$.

Now define a mapping $\theta : B(H, J) \to B_H$ as follows: $0\theta = 0$ and $(i, h, j)\theta = (i, g_i^{-1}hg_j, j)$. We assert that θ is an isomorphism from $B(H, J)$ onto B_H.

To see that θ is a morphism, let $(i, h_1, j), (r, h_2, s) \in B(H, J)$. If $j \neq r$, then

$$(i, h_1, j)\theta(r, h_2, s)\theta = (i, g_i^{-1}h_1g_j, j)(r, g_r^{-1}h_2g_s, s) = 0 = 0\theta = [(i, h_1, j)(r, h_2, s)]\theta.$$

On the other hand, if $j = r$, then

$$\begin{aligned}(i, h_1, j)\theta(r, h_2, s)\theta &= (i, g_i^{-1}h_1g_j, j)(j, g_j^{-1}h_2g_s, s) = (i, g_i^{-1}h_1g_jg_j^{-1}h_2g_s, s) \\ &= (i, g_i^{-1}h_1h_2g_s, s) = (i, h_1h_2, s)\theta \\ &= [(i, h_1, j)(j, h_2, s)]\theta = [(i, h_1, j)(r, h_2, s)]\theta.\end{aligned}$$

Clearly, given $x \in B(H, J)$, $x\theta = 0$ if and only if $x = 0$. If (i, h_1, j) and (r, h_2, s) are such that $(i, h_1, j)\theta = (r, h_2, s)\theta$, then $i = r$, $j = s$ and $g_i^{-1}h_1g_j = g_i^{-1}h_2g_j$, and so $h_1 = h_2$. Hence, $(i, h_1, j) = (r, h_2, s)$ and θ is one-to-one.

Next, let $x \in B_H \setminus \{0\}$. Since $x \in B(G, J)$, there exist $i, j \in J$ and $g \in G$ such that $x = (i, g, j)$. Thus, $x \in H_{ij} \cap B_H$ and so there is some h in H such that $x = (i, g_i^{-1}hg_j, j) = (i, h, j)\theta$. Hence, θ is an isomorphism from $B(H, J)$ onto B_H.

To show that B_H is maximal, it suffices to show that $B(H, J)$ is maximal. Let S be an inverse subsemigroup of $B(G, J)$ properly containing $B(H, J)$. Then, there exists some $(i, a, j) \in S \setminus B(H, J)$. It follows that $a \in G \setminus H$ and, since H is a maximal subgroup of G, we have $G = \langle\langle H \cup \{a\}\rangle\rangle$. Also,

$$(1, a, 1) = (1, e, i)(i, a, j)(j, e, 1) \in B(H, J).S.B(H, J) \subseteq S$$

and so $(1, a^{-1}, 1) = (1, a, 1)^{-1} \in S$. Since H is a group, every $g \in G$ can be written as a finite product, say $x_1 \cdots x_\ell$, of elements of $H \cup \{a, a^{-1}\}$. Since $B(H, J) \subseteq S$, it follows that $(1, x_t, 1) \in S$ for every $t \in \{1, \ldots, \ell\}$, and so $(1, g, 1) = (1, x_1, 1) \cdots (1, x_\ell, 1) \in S$. Hence, $H_{11} \subseteq S$. From Lemma 3, $B(G, J) = \langle\langle H_{11} \cup \{(1, g_j, j) : j \in K_1\}\rangle\rangle$ for some $[g_j] \in G^{K_1}$ with $K_1 = J \setminus \{1\}$. Let $j \in J$. Then, $(1, g_j, j) = (1, g_j, 1)(1, e, j) \in S.B(H, J) \subseteq S$. Therefore, $B(G, J) \subseteq S$ and so $B(H, J)$ is maximal. □

As in [10], a pair (J_1, J_2) is said to be a *two-partition of* J if $J = J_1 \cup J_2$ and $J_1 \cap J_2 = \emptyset$. The next Lemma determines another family of maximal inverse subsemigroups of $B(G, J)$.

Lemma 5. Let (J_1, J_2) be a two-partition of J and let $B_{J_1J_2} = B(G, J_1) \cup B(G, J_2)$. Then, $B_{J_1J_2}$ is a maximal inverse subsemigroup of $B(G, J)$.

Proof. Clearly, $B(G,J_1)\cap B(G,J_2)=\{0\}$ and $B(G,J_1), B(G,J_2)$ are inverse subsemigroups of $B(G,J)$. Also, $(i_1,g,j_1)(i_2,g',j_2)=0$ for all $(i_1,g,j_1)\in B(G,J_1)$ and $(i_2,g',j_2)\in B(G,J_2)$. Thus, $B_{J_1J_2}$ is an inverse subsemigroup of $B(G,J)$.

To see that $B_{J_1J_2}$ is maximal, let S be an inverse subsemigroup of $B(G,J)$ properly containing $B_{J_1J_2}$. Then, there exists $(i,b,j)\in S\backslash B_{J_1J_2}$. Since S is an inverse semigroup, we have $(j,b^{-1},i)=(i,b,j)^{-1}\in S$. Also, since $(i,b,j)\notin B_{J_1J_2}$, it follows that either ($i\in J_1$ and $j\in J_2$) or ($i\in J_2$ and $j\in J_1$). Hence, there exist $j_1\in J_1$, $a\in G$ and $j_2\in J_2$ such that $(j_1,a,j_2)\in S$. Let R_{j_1} denote the $\mathcal{R}$-class of $B(G,J)$ containing (j_1,a,j_2), that is, $R_{j_1}=\{(j_1,g,j): g\in G,\ j\in J\}$. Given $(j_1,g,j)\in R_{j_1}$, if $j\in J_1$, then $(j_1,g,j)\in B(G,J_1)\subseteq S$; and if $j\in J_2$, then $(j_1,g,j)=(j_1,a,j_2)(j_2,a^{-1}g,j)\in S.B(G,J_2)\subseteq S$. Therefore, $R_{j_1}\subseteq S$. From Lemma 3, if $K_{j_1}=J\setminus\{j_1\}$, then there exists $[g_j]\in G^{K_{j_1}}$ such that $B(G,J)=\langle\langle H_{j_1j_1}\cup\{(j_1,g_j,j): j\in K_{j_1}\}\rangle\rangle$. Since $H_{j_1j_1}\cup\{(j_1,g_j,j): j\in K_{j_1}\}\subseteq R_{j_1}\subseteq S$ and S is an inverse semigroup, it follows that $B(G,J)\subseteq S$, as required. □

In essence, the next result is [10] Proposition 3.4, but in a more general setting, albeit with J restricted to being an index set with at least two elements. Once again, we follow most of Yang's argument closely.

Theorem 4. Let S be a subsemigroup of the Brandt semigroup $B(G,J)$, where G is an arbitrary group with identity e and J is an index set with at least two elements. Then, S is a maximal inverse subsemigroup of $B(G,J)$ if and only if S is one of the following:

(a) $S=B(G,J_1)\cup B(G,J_2)$, where (J_1,J_2) is a two-partition of J,

(b) $S=\langle\langle H^*_{11}\cup\{(1,g_j,j): j\in K_1\}\rangle\rangle$, where $1\in J$, $K_1=J\setminus\{1\}$, $H^*_{11}=\{(1,h,1)\in B(G,J): h\in H\}$, H is a maximal subgroup of G and $[g_j]\in G^{K_1}$.

Proof. From Lemmas 4 and 5, if S is of type (a) or (b) then it is a maximal inverse subsemigroup of $B(G,J)$. Conversely, let S be a maximal inverse subsemigroup of $B(G,J)$. We assert that $S\cap H_{jj}\neq\emptyset$ for each $j\in J$. For, suppose $S\cap H_{ii}=\emptyset$ for some $i\in J$. Then, $S\cap L_i=\emptyset$ where L_i is the $\mathcal{L}$-class of $B(G,J)$ which contains the element (i,e,i). For, if $S\cap L_i\neq\emptyset$, then there exist some $j\in J$ and $a\in G$ such that $(j,a,i)\in S$, and so $(i,e,i)=(i,a^{-1},j)(j,a,i)=(j,a,i)^{-1}(j,a,i)\in S\cap H_{ii}$, a contradiction. Similarly, $S\cap R_i=\emptyset$ where R_i denotes the $\mathcal{R}$-class of $B(G,J)$ containing (i,e,i). Hence,

$$S\subseteq B(G,J\setminus\{i\})\subsetneq B(G,J\setminus\{i\})\cup H_{ii}\subsetneq B(G,J).$$

Clearly, $B(G, J \setminus \{i\}) \cup H_{ii}$ is an inverse subsemigroup of $B(G, J)$, which contradicts our assumption on S. Note that, in the above argument, we use the fact that $|J| \geq 2$.

We now consider two cases.

Case 1. Suppose $S \cap H_{11} = H_{11}$. Let $J_1 = \{j \in J : S \cap H_{1j} \neq \emptyset\}$ and $J_2 = J \setminus J_1$, and note that $J_1 \neq \emptyset$ by supposition. If $S \cap H_{1j} \neq \emptyset$, then there exists some $a \in G$ such that $(1, a, j) \in S$, and so we have $(1, g, j) = (1, ga^{-1}, 1)(1, a, j) \in H_{11}.S \subseteq S$, for every $(1, g, j) \in H_{1j}$. Hence, $S \cap H_{1j} = H_{1j}$ for each $j \in J_1$. Thus, given $(i, g, j) \in B(G, J_1)$, we have $S \cap H_{1i} = H_{1i}$ and $S \cap H_{1j} = H_{1j}$, and so $(i, g, j) = (1, g^{-1}, i)^{-1}(1, e, j) \in H_{1i}^{-1}.H_{1j} \subseteq S$. Therefore, $B(G, J_1) \subseteq S$.

Suppose there exist $j_1 \in J_1$, $a \in G$ and $j_2 \in J_2$ such that $(j_1, a, j_2) \in S$. Then, $(1, a, j_2) = (1, e, j_1)(j_1, a, j_2) \in H_{1j_1}.S \subseteq S$, since $j_1 \in J_1$ implies $S \cap H_{1j_1} \neq \emptyset$ and so $H_{1j_1} \subseteq S$ by a remark in the last paragraph. Hence $(1, a, j_2) \in S \cap H_{1j_2}$, so this set is non-empty and therefore $j_2 \in J_1$, a contradiction. Thus, since S is inverse, given $j_1 \in J_1$, $a \in G$ and $j_2 \in J_2$, both $(j_1, a, j_2) \notin S$ and $(j_2, a, j_1) \notin S$. We then have $S \subseteq B(G, J_1) \cup B(G, J_2) \subsetneq B(G, J)$. By maximality of S, it follows that $S = B(G, J_1) \cup B(G, J_2)$.

Case 2. Suppose $S \cap H_{11} \neq H_{11}$. Let $H_{11}^* = S \cap H_{11}$ and $H = \{h \in G : (1, h, 1) \in H_{11}^*\}$. We assert that H is a proper subgroup of G. Since $S \cap H_{11} \neq \emptyset$, there exists $a \in G$ such that $(1, a, 1) \in S \cap H_{11} = H_{11}^*$. Then, $(1, e, 1) = (1, a, 1)(1, a, 1)^{-1} \in H_{11}^*$ and so $e \in H$. Now suppose $h_1, h_2 \in H$. Then, $(1, h_1, 1), (1, h_2, 1) \in H_{11}^*$. Since H_{11}^* is an inverse subsemigroup of $B(G, J)$, it follows that $(1, h_1, 1)(1, h_2, 1)^{-1} = (1, h_1h_2^{-1}, 1) \in H_{11}^*$ and so $h_1h_2^{-1} \in H$. Since $S \cap H_{11} \neq H_{11}$, there exists some $g \in G$ such that $(1, g, 1) \notin S \cap H_{11} = H_{11}^*$, and so $g \notin H$ and $H \subsetneq G$.

Still in Case 2, let $J^* = \{j \in J : S \cap H_{1j} \neq \emptyset\}$ and choose $(1, g_j, j) \in S \cap H_{1j}$ for each $j \in J^*$. Clearly, $\langle\langle H_{11}^* \cup \{(1, g_j, j) : j \in J^*\}\rangle\rangle \subseteq S$. We assert that $J = J^*$. For, if $J \setminus J^* \neq \emptyset$, then $S \cap H_{1i} = \emptyset$ for each $i \in J \setminus J^*$. Now suppose $(i, a, j) \in S$ with $i \notin J^*$ and $j \in J^*$. Then, $(1, g_ja^{-1}, i) = (1, g_j, j)(j, a^{-1}, i) = (1, g_j, j)(i, a, j)^{-1} \in S$ (since S is inverse) which contradicts $S \cap H_{1i} = \emptyset$. In other words, if $(i, a, j) \in S$ then either ($i \in J^*$ and $j \in J^*$) or ($i \notin J^*$ and $j \notin J^*$). Therefore, since $1 \in J^*$ (by a remark at the start of this proof) and $H_{11} \setminus S$ is non-empty (by assumption in this case), it follows that

$$S \subsetneq B(G, J^*) \cup B(G, J \setminus J^*) \subsetneq B(G, J). \tag{1}$$

But $B(G, J^*) \cup B(G, J \setminus J^*)$ is an inverse subsemigroup of $B(G, J)$, so (1) contradicts the maximality of S. Therefore, $J = J^*$.

Next we see that $S = \langle\langle H_{11}^* \cup \{(1, g_j, j) : j \in J\}\rangle\rangle$. For convenience, we let $g_1 = e$

and for every i and j in J, we denote the subset $\{(i, g_i^{-1}hg_j, j) : h \in H\}$ of $B(G, J)$ by $(i, g_i^{-1}Hg_j, j)$. We assert that $S \cap H_{ij} = (i, g_i^{-1}Hg_j, j)$. For, if $h \in H$, we have

$$(i, g_i^{-1}hg_j, j) = (1, g_i, i)^{-1}(1, h, 1)(1, g_j, j) \in S.H_{11}^*.(S \cap H_{1j}) \subseteq S$$

and so $(i, g_i^{-1}hg_j, j) \in S \cap H_{ij}$. Conversely, let $(i, a, j) \in S \cap H_{ij}$. Then, $(1, g_iag_j^{-1}, 1) = (1, g_i, i)(i, a, j)(1, g_j, j)^{-1} \in S \cap H_{11} = H_{11}^*$ and so $g_iag_j^{-1} \in H$. Thus, $(i, a, j) = (i, g_i^{-1}hg_j, j)$ where $h = g_iag_j^{-1} \in H$, and so $S \cap H_{ij} = (i, g_i^{-1}Hg_j, j)$. Now, if $(i, a, j) \in S$, then $(i, a, j) = (i, g_i^{-1}hg_j, j) = (1, g_i, i)^{-1}(1, h, 1)(1, g_j, j)$ for some $h \in H$, and so (i, a, j) can be written as a finite product of elements of $H_{11}^* \cup \{(1, g_j, j) : j \in J\}$ and inverses of such elements. Therefore, $S \subseteq \langle\langle H_{11}^* \cup \{(1, g_j, j) : j \in J\}\rangle\rangle$, and equality follows from the containment observed earlier.

It remains to show that H is a maximal subgroup of G. Suppose this is not true. Then, there exists a proper subgroup K of G which properly contains H. Let $K_{11}^* = \{(1, k, 1) : k \in K\}$. Clearly, $H_{11}^* \subsetneq K_{11}^*$ and so S is a proper subsemigroup of the inverse semigroup $\langle\langle K_{11}^* \cup \{(1, g_j, j) : j \in J\}\rangle\rangle$, a contradiction. Therefore, H is a maximal subgroup of G, and the result follows. □

Now we turn our attention to the inverse semigroup $I_r = \{\alpha \in I(V) : \text{rank}\,\alpha \leq r\}$ for $1 \leq r \leq n$ where $n = \dim V$. From Lemma 1, the $\mathcal{D}$–class D_r generates I_r since, if $s \in \{1, \ldots, r-1\}$, then $D_s \subseteq D_r^k$ where $k = 2^{r-s}$. Since $D_r = I_r \setminus I_{r-1}$ is a $\mathcal{D}$–class of $I(V)$, the principal factor I_r/I_{r-1} is 0-simple. Moreover, I_r/I_{r-1} contains non-zero idempotents: namely, the identity maps on subspaces of V with dimension r. In fact, if η, ε are two such idempotents and $\eta\varepsilon = \varepsilon\eta = \varepsilon$, then $\text{ran}\,\varepsilon = \text{ran}\,(\varepsilon\eta) \subseteq \text{ran}\,\eta$. Since $\text{rank}\,\varepsilon = r = \text{rank}\,\eta$ and r is finite, it follows that $\text{ran}\,\varepsilon = \text{ran}\,\eta$. Therefore, $\varepsilon = \text{id}_{\text{ran}\,\varepsilon} = \text{id}_{\text{ran}\,\eta} = \eta$, and this proves that η is a primitive idempotent of I_r/I_{r-1}. Hence, I_r/I_{r-1} is completely 0-simple.

In [2] Vol. 1, pp. 95-96, the authors give a Rees matrix representation for the completely 0-simple semigroup T_r/T_{r-1}, where T_r is the set of all total transformations α on a finite set X with cardinal n such that $\text{rank}\,\alpha \leq r$ and $2 \leq r \leq n$. And, in effect, [4] Theorem 4.12 provides the Rees matrix representation for the corresponding principal factor in $T(V)$, the semigroup of all total linear transformations of a finite-dimensional vector space V. However, that result is expressed in terms of 'independence algebras', an unnecessary complication in the present situation. Therefore, we adopt a simpler approach and proceed to establish a Rees matrix representation for the principal factor I_r/I_{r-1} of $I(V)$.

Green's $\mathcal{L}$ and $\mathcal{R}$ relations on $I(X)$ are well-known for any set X: namely, $\alpha\,\mathcal{L}\,\beta$ if and only if $\text{ran}\,\alpha = \text{ran}\,\beta$; and $\alpha\,\mathcal{R}\,\beta$ if and only if $\text{dom}\,\alpha = \text{dom}\,\beta$. It is easy to show that similar statements hold for $I(V)$ and any vector space V (compare [8] p. 427, for independence algebras). Thus, we have the following result for I_r/I_{r-1}.

Lemma 6. If $1 \leq r \leq n$ and $\alpha, \beta \in I_r/I_{r-1}$ are non zero, then

(a) $\alpha \, \mathcal{L} \, \beta$ if and only if ran $\alpha =$ ran β,

(b) $\alpha \, \mathcal{R} \, \beta$ if and only if dom $\alpha =$ dom β.

Let r_V denote the family of all subspaces of V with dimension r, and note that r_V is infinite if the field F is infinite, hence our need to consider *arbitrary* Brandt semigroups. For every A in r_V, fix a basis $\{v_1^A, \ldots, v_r^A\}$ of A. From Lemma 6, I_r/I_{r-1} has $|r_V|$ non-zero $\mathcal{R}$-classes corresponding to the possible domains of dimension r and $|r_V|$ non-zero $\mathcal{L}$-classes corresponding to the possible ranges of dimension r. Let $A, B \in r_V$ and put $H_{A,B} = R_A \cap L_B$. From Lemma 6, $\alpha \in H_{A,B}$ if and only if dom $\alpha = A$ and ran $\alpha = B$, that is, if and only if

$$\alpha = \begin{pmatrix} v_1^A & \ldots & v_r^A \\ \sum_{j=1}^r x_{1j} v_j^B & \ldots & \sum_{j=1}^r x_{rj} v_j^B \end{pmatrix}$$

for some invertible $r \times r$ matrix $[x_{ij}]$ over F. In this case, we write $\alpha = (A, [x_{ij}], B)$.

Now let $\beta \in D_r$. Since ran $\alpha \cap$ dom β is a subspace of the r-dimensional vector spaces ran α and dom β (where r is finite and β is one-to-one), we have rank $(\alpha\beta) = r$ if and only if dim(ran $\alpha \cap$ dom $\beta) = r$, and this is true if and only if $B =$ ran $\alpha =$ ran $\alpha \cap$ dom $\beta =$ dom β.

Suppose rank $(\alpha\beta) = r$. Then $B =$ dom β and for some subspace D of V and some invertible $r \times r$ matrix $[y_{ij}]$ over F,

$$\beta = \begin{pmatrix} v_1^B & \ldots & v_r^B \\ \sum_{j=1}^r y_{1j} v_j^D & \ldots & \sum_{j=1}^r y_{rj} v_j^D \end{pmatrix} = (B, [y_{ij}], D).$$

Also, in this case, since ran $(\alpha\beta) \subseteq$ ran β and r is finite, we have ran $(\alpha\beta) =$ ran $\beta = D$. Analogously, we conclude that dom $(\alpha\beta) =$ dom $\alpha = A$. If $i \in \{1, \ldots, r\}$, then

$$v_i^A(\alpha\beta) = (\sum_{j=1}^r x_{ij} v_j^B)\beta = \sum_{j=1}^r x_{ij} (\sum_{k=1}^r y_{jk} v_k^D) = \sum_{k=1}^r (\sum_{j=1}^r x_{ij} y_{jk}) v_k^D.$$

Therefore, we have $\alpha\beta = (A, [z_{ij}], D)$ where $z_{ij} = \sum_{k=1}^r x_{ik} y_{kj}$ and so

$$\alpha\beta = (A, [x_{ij}][y_{ij}], D) = (A, [x_{ij}], B) \circ (\text{dom } \beta, [y_{ij}], D).$$

If rank $(\alpha\beta) < r$, then $B \neq$ dom β. Also, $\alpha\beta \notin I_r \setminus I_{r-1}$, and hence $\alpha\beta = 0$ in I_r/I_{r-1}.

Denote the group of all invertible $r \times r$ matrices over F by $G_r(F)$, and let $G = G_r(F)$ and $J = r_V$. Define the mapping $\theta : I_r/I_{r-1} \to B(G, J)$ by $0\theta = 0$ and $\alpha\theta = (A, [x_{ij}], B)$ where $v_i^A\alpha = \sum_{j=1}^r x_{ij}v_j^B$, for every $i \in \{1, \dots, r\}$. We have shown that θ is a morphism from I_r/I_{r-1} into $B(G, J)$. Clearly, θ is one-to-one: given $\alpha, \beta \in I_r \setminus I_{r-1}$ such that $(\text{dom } \alpha, [x_{ij}], \text{ran } \alpha) = \alpha\theta = \beta\theta = (\text{dom } \beta, [y_{ij}], \text{ran } \beta)$, we have

$$v_i^{\text{dom } \alpha}\alpha = \sum_{j=1}^r x_{ij}v_j^{\text{ran } \alpha} = \sum_{j=1}^r y_{ij}v_j^{\text{ran } \beta} = v_i^{\text{dom } \beta}\beta.$$

Since $\text{dom } \alpha = \text{dom } \beta$, it follows that $\alpha = \beta$. To see that θ is onto, let $[x_{ij}]$ be an invertible $r \times r$ matrix over F and $A, B \in r_V$. Now define $\alpha \in P(V)$ by

$$\alpha = \begin{pmatrix} v_1^A & \dots & v_r^A \\ \sum_{j=1}^r x_{1j}v_j^B & \dots & \sum_{j=1}^r x_{rj}v_j^B \end{pmatrix}.$$

Then, $\text{dom } \alpha = \langle v_1^A, \dots, v_r^A \rangle = A$ and, since $[x_{ij}]$ is invertible, α is one-to-one and $\text{ran } \alpha = \langle v_1^B, \dots, v_r^B \rangle = B$. Therefore, $\alpha \in I_r \setminus I_{r-1}$ and $\alpha\theta = (A, [x_{ij}], B)$. Hence θ is an isomorphism from I_r/I_{r-1} onto $B(G, J)$.

Finally, we turn to the maximal inverse subsemigroups S of I_r where $r < n$ (the case $r = n$ is handled by Theorem 3). If $I_{r-1} \setminus S \neq \emptyset$ then $S \subsetneq S \cup I_{r-1}$, which is an inverse subsemigroup of I_r. By the maximality of S, this implies $S \cup I_{r-1} = I_r$. Hence $D_r \subseteq S$ and, by Lemma 1, this implies $I_r \subseteq S$ (see the remark after the proof of Theorem 4 above). Since this contradicts our supposition, we deduce that $I_{r-1} \subseteq S$ and thus $S = I_{r-1} \cup (S \cap D_r)$. Clearly, $(S \cap D_r) \cup \{0\}$ is an inverse subsemigroup of I_r/I_{r-1}, and we assert it is a maximal one. For, if T is an inverse subsemigroup of I_r/I_{r-1} which properly contains it, then $I_{r-1} \cup (T \setminus \{0\})$ is an inverse subsemigroup of I_r (under composition) which properly contains S, a contradiction.

If $\dim V = n$ and $r = n$ then I_r/I_{r-1} is simply the group G_n of units in $I(V)$ with 0 adjoined. Therefore, $|J| = 1$ in the representation of G_n^0 as a Brandt semigroup $B(G, J)$, and so Theorem 4 is not applicable in this case. However, that result describes all maximal inverse subsemigroups of the Brandt semigroup $B(G, J)$ where G is an arbitrary group and $|J| \geq 2$; hence, via the isomorphism established above, we know all maximal inverse subsemigroups of I_r/I_{r-1} where $r < n$. Consequently, we have the following result.

Theorem 5. Let $1 \leq r < n = \dim V$ and let r_V denote the family of all subspaces of V with dimension r. If S is a subsemigroup of the inverse semigroup I_r, then S is a maximal inverse subsemigroup of I_r if and only if S is one of the following:

(a) $S = I_{r-1} \cup B_{r_{V_1}} \cup B_{r_{V_2}}$, where (r_{V_1}, r_{V_2}) is a two-partition of r_V and $B_{r_{V_i}} = \{\alpha \in I_r :$ dom α, ran $\alpha \in r_{V_i}\}$ for $i = 1, 2$,

(b) $S = I_{r-1} \cup \langle\langle H^*_{AA} \cup \{\alpha_B : B \in r'_V\}\rangle\rangle$, where $A \in r_V$, $r'_V = r_V \setminus \{A\}$, $H^*_{AA} = \{\alpha = [h_{ij}] \in I_r :$ dom $\alpha = A =$ ran α and $[h_{ij}] \in H\}$, H is a maximal subgroup of the group $G_r(F)$ of all $r \times r$ invertible matrices over F and $[\alpha_B] \in I_r^{r'_V}$ where dom $\alpha_B = A$ and ran $\alpha_B = B$ for every $B \in r'_V$.

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