Sign pattern matrices that admit M–, N–, P– or inverse M–matrices *†

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Abstract

In this paper we identify the sign pattern matrices that occur among the N–matrices, the P–matrices and the M–matrices. We also address to the class of inverse M–matrices and the related admissibility of sign pattern matrices problem.

1 Introduction

In qualitative and combinatorial matrix theory, a methodology based on the use of combinatorial information such as the signs of the elements of a matrix is very often useful in the study of some properties of matrices. A matrix whose entries are chosen from the set $\{+, -, 0\}$ is called a *sign pattern matrix*. A *zero pattern* is a sign pattern matrix whose entries are all

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equal to 0. Given an $n \times m$ real matrix $A = (a_{ij})$, we denote by $sign(A)$ the sign pattern matrix obtained from A by replacing each one of its positive entries by $+$ and each one of its negative entries by $-$. For an $n \times m$ sign pattern matrix P, we define the *sign pattern class* $\mathcal{C}(P)$ by

$$\mathcal{C}(P) = \left\{ A \in \mathbb{R}^{n \times n} \ : \ sign(A) = P \right\}.$$

A *permutation pattern* is simply a sign pattern matrix with exactly one entry in each row and column equal to $+$, and the remaining entries equal to 0. A product of the form $S^T P S$, where S is a square permutation pattern and P is a sign pattern matrix of the same order as S, is called a permutation similarity. A square sign pattern matrix whose entries lying outside the main diagonal are equal to zero is called a *diagonal pattern*, and a product of the form DPD, where D is a diagonal pattern with no zero entries in the main diagonal and P is a sign pattern matrix of the same order as D, is called a diagonal similarity. Note that $S^T P S$ and DPD are again sign pattern matrices.

A sign pattern matrix P is said to *require* a certain property $\mathcal{P}$ referring to real matrices if all real matrices in $\mathcal{C}(P)$ have the property $\mathcal{P}$, and is said to *allow* that property $\mathcal{P}$ if some real matrix in $\mathcal{C}(P)$ has the property $\mathcal{P}$. In the literature, one can find, in the last few years, an increasing interest in problems that arise from the basic question of whether a certain sign pattern matrix requires (or allows) a certain property (see, for instance, [2], [4], [5]).

In this paper, we shall consider certain classes of real matrices, namely the class of N–matrices, the class of P–matrices, the class of M–matrices and, finally, the class of inverse M–matrices. Our aim is to determine which sign pattern matrices are admissible for each one of these classes of real matrices. In other words, we shall focus on the question 'which sign pattern matrices allow the property of belonging to the class of N–matrices (respectively, P–matrices, M–matrices, inverse M–matrices)?'.

An $n \times n$ real matrix A is called an *N–matrix* if all its principal minors are negative while A is said to be a *P–matrix* if all its principal minors are positive. The class of P–matrices generalizes many important classes of matrices, such as the M–matrices and inverse M–matrices. Denote by Z_n the set of all square real matrices of order n whose off-diagonal entries are non-positive. A matrix A is called an *M–matrix* if $A \in Z_n$ and A is positive stable. Throughout this paper, M–matrices are not allowed to be singular. A nonsingular matrix A is said to be an *inverse M-matrix* if A^{-1} is an M–matrix. As in the case for P–matrices, there are many different equivalent conditions for a matrix to be an M–matrix. We shall use the following: a matrix $A \in Z_n$ is an M–matrix if and only if A is a P–matrix or, equivalently, A is nonsingular and $A^{-1} \geq 0$ ([1], Chapter 2; [3]).

We say that P is a *$\mathcal{Z}_n$–sign pattern matrix* if $P = sign(A)$ for some $A \in Z_n$.

We write $A > 0$ if A is entrywise positive.

Recall that an $n \times n$ matrix A is *reducible* if for some permutation matrix S,

$$SAS^T = \left[\begin{array}{cc} B & C \\ 0 & D \end{array} \right],$$

where B and D are square matrices, or if $n = 1$ and $A = 0$. Otherwise, A is irreducible. We define *reducible* and *irreducible sign pattern matrices* in a natural way: an $n \times n$ sign pattern matrix P is reducible if all matrices $A \in \mathcal{C}(P)$ are reducible, and irreducible otherwise.

An $n \times n$ sign pattern matrix $P = (p_{ij})$ is said to be *transitive* if for any set of distinct vertices $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$, the following condition holds:

$$p_{i_1 i_2}, p_{i_2 i_3}, \ldots, p_{i_{k-2} i_{k-1}}, p_{i_{k-1} i_k} \neq 0 \Rightarrow p_{i_1 i_k} \neq 0.$$

We denote by $J_{n \times m}$ the $n \times m$ matrix whose components are all equal to 1. Observe that $J_{k \times m} J_{m \times n} = m J_{k \times n}$ for all positive integers k, m, n.

For an $n \times n$ matrix A, the submatrix of A lying in rows α and columns β, $\alpha, \beta \subseteq \{1, ..., n\}$, is denoted by $A[\alpha|\beta]$, and the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$. Then, a real $n \times n$ matrix A is an N–matrix if $\det A[\alpha] < 0$, for all $\alpha \subseteq \{1, ..., n\}$, and a P–matrix if $\det A[\alpha] > 0$, for all $\alpha \subseteq \{1, ..., n\}$.

In [7], the authors show that any N–matrix is diagonally similar to an N–matrix in $\mathcal{S}_n$, where

$$\mathcal{S}_n = \left\{ A = (a_{ij}) \mid sign(a_{ij}) = (-1)^{i+j+1}, \text{ for all } i, j \in \{1, ..., n\} \right\}.$$

This class has as its canonical representative $A_n = (a_{ij})$ with $a_{ij} = (-1)^{i+j+1}$ for all $i, j \in \{1, \dots, n\}$. We say that P is an $\mathcal{S}_n$–sign pattern matrix if $P = sign(A_n)$.

2 Sign pattern matrices that admit N–matrices

In the next result, which, as we will see, can be derived from the fact mentioned above, we characterize all of the admissible sign pattern matrices for the class of N–matrices.

Theorem 2.1. *Let P be a sign pattern matrix. There exists a matrix A in the class $\mathcal{C}(P)$ such that A is an N–matrix if and only if P is diagonally similar to an $\mathcal{S}_n$–sign pattern matrix.*

Proof. Suppose that P is an $n \times n$ sign pattern matrix such that there exists an N–matrix A in $\mathcal{C}(P)$. We know that $sign(A) = P$ and that A is diagonally similar to an N–matrix $B \in \mathcal{S}_n$. Therefore, we can assert that there exists a diagonal matrix D such that $B = DAD^{-1}$. Let $E = sign(D)$. It is obvious that $sign(B) = EPE$. Observe that $Q = EPE$ is the $\mathcal{S}_n$–sign pattern matrix and that it is diagonally similar to P.

Conversely, assume that P is diagonally similar to the $\mathcal{S}_n$–sign pattern matrix Q and consider the following matrix B

$$B = \begin{bmatrix} -1 & a & -a^2 & \dots & (-1)^n a^{n-1} \\ a & -1 & a & \dots & (-1)^{n-1} a^{n-2} \\ -a^2 & a & -1 & \dots & (-1)^{n-2} a^{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^n a^{n-1} & (-1)^{n-1} a^{n-2} & (-1)^{n-2} a^{n-3} & \dots & -1 \end{bmatrix}$$

in $\mathcal{S}_n$, where $a > 1$. It is not difficult to prove, by applying Proposition 3.2 of [7], that B is in fact an N–matrix. Note that $sign(B) = Q$. Let E be a diagonal pattern such that $P = EQE$ and let D be the diagonal matrix whose diagonal entries are equal to 1 or -1 and $sign(D) = E$. Since $sign(DBD^{-1}) = P$ and since the class of N–matrices is invariant under diagonal similarity, we can conclude that $A = DBD^{-1}$ is an N–matrix that belongs to the class $\mathcal{C}(P)$. □

Taking into account this result and the consequent characterization of the admissible sign pattern matrices for the class of N–matrices, the natural question that arises now is whether we are able to give a conclusive answer to similar problems referring to classes of matrices defined by means of the signs of principal minors and of the signs of the entries of the matrices.

3 Sign pattern matrices that admit P– and M–matrices

In this section we derive a characterization of the admissible sign pattern matrices for the class of P–matrices and also for its subclass of M–matrices.

Theorem 3.1. *Let $P = (p_{ij})$ be an $n \times n$ sign pattern matrix. There exists a P–matrix A in $\mathcal{C}(P)$ if and only if $p_{ii} = +$ for all $i \in \{1, \ldots, n\}$.*

Proof. Let $P = (p_{ij})$ be an $n \times n$ sign pattern matrix such that there exists a P–matrix A in $\mathcal{C}(P)$. We know that $sign(A) = P$ and that all the principal minors of A are positive. In particular, $\det A[\{i\}] > 0$ for all $i \in \{1, \ldots, n\}$, which allows us to conclude that $p_{ii} = +$.

Conversely, assume that $P = (p_{ij})$ is a square sign pattern matrix of order n with $p_{ii} = +$, $i = 1, \ldots, n$. For each $\varepsilon > 0$ consider the $n \times n$ matrix $A_\varepsilon = (a_{ij})$ given by

$$a_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \quad \text{and} \quad p_{ij} = 0 \\ \varepsilon & \text{if} \quad i \neq j \quad \text{and} \quad p_{ij} = + \\ -\varepsilon & \text{if} \quad i \neq j \quad \text{and} \quad p_{ij} = - \end{cases}.$$

It is obvious that A_ε belongs to the sign pattern class $\mathcal{C}(P)$. Let $\alpha \subseteq \{1, \ldots, n\}$. Note that

$$\det A_\varepsilon[\alpha] = 1 + p_\alpha(\varepsilon),$$

where $p_\alpha(\varepsilon)$ is the null polynomial or a polynomial on ε with null constant term. Since $p_\alpha(\varepsilon)$ tends to zero as ε tends to zero, for sufficiently small values of ε, A_ε is a P–matrix belonging to the class $\mathcal{C}(P)$. □

From the preceding result and the previously mentioned characterization of M–matrices, one can easily describe the admissible sign pattern matrices for the class of M–matrices:

Theorem 3.2. *Let $P = (p_{ij})$ be an $n \times n$ sign pattern matrix. Then, there is an M–matrix A in $\mathcal{C}(P)$ if and only if P is a $\mathcal{Z}_n$–sign pattern matrix and $p_{ii} = +$ for all $i \in \{1, \ldots, n\}$.*

4 Sign pattern matrices that occur among inverse M–matrices

In this section, necessary conditions for the admissibility for inverse M–matrices are presented and some partial results related to the characterization of the admissible sign pattern matrices are achieved for this particular class of matrices.

We shall now focus on the class of inverse M–matrices. Note that if a nonsingular matrix A has nonnegative entries, $A^{-1} \in Z_n$ and A^{-1} has positive main diagonal entries, then A is an inverse M–matrix. Obviously, if a sign pattern matrix allows the property of belonging to the class of inverse M–matrices, it must be nonnegative. The following lemma will be useful for the study of the admissible sign pattern matrices for this class of matrices.

Lemma 4.1. *Given $0 < a < 1$, the following $n \times n$ matrix*

$$A = \begin{bmatrix} 1 & a & a & \ldots & a & a \\ a & 1 & a & \ldots & a & a \\ a & a & 1 & \ldots & a & a \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a & a & a & \ldots & 1 & a \\ a & a & a & \ldots & a & 1 \end{bmatrix}$$

is an inverse M–matrix.

Proof. Observe that $A = (1-a)I_n + aJ_{n\times n}$. Setting $B = cI_n + dJ_{n\times n}$, it is a simple computation to show that $AB = I_n$, and hence that $B = A^{-1}$, if and only if $c = (1-a)^{-1}$ and $d = \dfrac{-ac}{(1-a+an)}$. □

Given a real number a, denote by $C_{n,a}$ the $n \times n$ matrix whose main diagonal entries are equal to 1 and the remaining entries are all equal to a.

Simple calculations yield that $C_{n,a}J_{n\times m} = (1+(n-1)a)J_{n\times m}$, $J_{n\times m}C_{m,a} = (1+(m-1)a)J_{n\times m}$, and hence, $C_{n,a}^{-1}J_{n\times m} = (1+(n-1)a)^{-1}J_{n\times m}$ and $J_{n\times m}C_{m,a}^{-1} = (1+(m-1)a)^{-1}J_{n\times m}$ for all $a \in \mathbb{R}$ and $n, m \in \mathbb{N}$.

It is well known that an irreducible matrix $A \in Z_n$ is an M–matrix if and only if $A^{-1} > 0$ (see [1]). The next result follows from this fact.

Theorem 4.2. *Let $P = (p_{ij})$ be an $n\times n$ nonnegative irreducible sign pattern matrix. Then, there is an inverse M–matrix A in $\mathcal{C}(P)$ if and only if P is positive.*

Proof. Let us assume that there exists an inverse M–matrix A in $\mathcal{C}(P)$. Since P is irreducible, A is an irreducible matrix and, consequently, so is A^{-1}. By the result mentioned above, $A > 0$ and, therefore, P is positive.

Conversely, assume P is positive. Consider $0 < a < 1$ and the inverse M–matrix $A = C_{n,a}$. This matrix belongs to $\mathcal{C}(P)$. □

Using the definition of a transitive sign pattern matrix, we can restate a result due to Lewin and Neumann.

Theorem 4.3 ([6])**.** *If A is an inverse M–matrix, sign(A) is transitive.*

This particular theorem will be very useful in the upcoming results addressing the characterization of the admissible reducible sign-pattern matrices for the class of inverse M–matrices.

It is well known that any matrix is permutation similar to a block triangular matrix with irreducible diagonal blocks, the Frobenius normal form. Therefore, we can reduce the general case to the case of block upper triangular sign pattern matrices

$$P = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1k} \\ 0 & P_{22} & \dots & P_{2k} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & P_{kk} \end{bmatrix}$$

whose diagonal blocks $P_{11}, P_{22}, \dots, P_{kk}$ are irreducible sign pattern matrices. These diagonal blocks are the so called *irreducible components* of P.

Corollary 4.4. *Let $P = (p_{ij})$ be an $n \times n$ sign pattern matrix of the form*

$$P = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1k} \\ 0 & P_{22} & \dots & P_{2k} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & P_{kk} \end{bmatrix}$$

where $P_{11}, P_{22}, \dots, P_{kk}$ are irreducible sign pattern matrices of sizes $q_1 \times q_1, q_2 \times q_2, \dots, q_k \times q_k$, respectively. If there exists an inverse M–matrix A in $\mathcal{C}(P)$, then P_{11}, P_{22}, …, P_{kk} are positive, and each one of the blocks P_{ij}, $i \neq j$, is either positive or the zero pattern.

Proof. Let

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ 0 & A_{22} & \dots & A_{2k} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & A_{kk} \end{bmatrix}$$

be an inverse M–matrix in $\mathcal{C}(P)$. Since $A_{11}, A_{22}, \dots, A_{kk}$ are also inverse M–matrices, it follows from Theorem 4.2 that $P_{11}, P_{22}, \dots, P_{kk}$ are positive. By applying Lewin and Neumann's result, we know that P is transitive. Suppose P_{ij} $(i \neq j)$ is not the zero pattern. It is clear, then, that it has at least one positive component p_{rs}. Note that the first row of P_{ij} corresponds to the $(q_1 + \dots + q_{i-1} + 1)$–th row of P (take $q_0 = 0$), while the last row of P_{ij} corresponds to the $(q_1 + \dots + q_{i-1} + q_i)$–th row of P. Moreover, the first column of P_{ij} corresponds to the $(q_1 + \dots + q_{j-1} + 1)$–th column of P and the last column of P_{ij} corresponds to the $(q_1 + \dots + q_{j-1} + q_j)$–th row of P. Hence, $r \in \{q_1 + \dots + q_{i-1} + 1, \dots, q_1 + \dots + q_{i-1} + q_i\}$ and $s \in \{q_1 + \dots + q_{j-1} + 1, \dots, q_1 + \dots + q_{j-1} + q_j\}$. Let p_{tu} be any component of P_{ij} different from p_{rs}. Clearly, $t \in \{q_1 + \dots + q_{i-1} + 1, \dots, q_1 + \dots + q_{i-1} + q_i\}$, $u \in \{q_1 + \dots + q_{j-1} + 1, \dots, q_1 + \dots + q_{j-1} + q_j\}$ and $t \neq r$ or $u \neq s$. Observe that p_{tr} is an element of P_{ii} and p_{su} is a component of P_{jj}. We have $p_{tr} = p_{su} = +$ since P_{ii} and P_{jj} are positive. Therefore, $p_{tr}, p_{rs}, p_{su} \neq 0$ and, by the transitivity of P, $p_{tu} \neq 0$. Then, P_{ij} is positive. □

The converse of the previous result is also true for $k = 2$.

Lemma 4.5. *Any $n \times n$ sign pattern matrix P of the form*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix},$$

where P_{11} and P_{22} are positive sign pattern matrices of size $q \times q$ and $(n-q) \times (n-q)$, respectively, and P_{12} is either positive or the zero pattern, is admissible for the class of inverse M–matrices.

Proof. If $P_{12} = 0$, it is easy to prove, using Lemma 4.1 that

$$A = \begin{bmatrix} C_{q,a} & 0 \\ 0 & C_{n-q,a} \end{bmatrix},$$

with $0 < a < 1$, is an inverse M–matrix. Note that $A \in \mathcal{C}(P)$. If P_{12} is positive, let A be the following matrix

$$A = \begin{bmatrix} C_{q,a} & aJ_{q\times(n-q)} \\ 0 & C_{n-q,a} \end{bmatrix}.$$

Given that

$$A^{-1} = \begin{bmatrix} C_{q,a}^{-1} & -aC_{q,a}^{-1}J_{q\times(n-q)}C_{n-q,a}^{-1} \\ 0 & C_{n-q,a}^{-1} \end{bmatrix},$$

we only have to show that $-aC_{q,a}^{-1}J_{q\times(n-q)}C_{n-q,a}^{-1} \leq 0$. Recall that

$$C_{q,a}^{-1}J_{q\times(n-q)} = (1+(q-1)a)^{-1}J_{q\times(n-q)}$$

and

$$J_{q\times(n-q)}C_{n-q,a}^{-1} = (1+(n-q-1)a)^{-1}J_{q\times(n-q)}.$$

Hence

$$-aC_{q,a}^{-1}J_{q\times(n-q)}C_{n-q,a}^{-1} = -\frac{a}{(1+(q-1)a)(1+(n-q-1)a)}J_{q\times(n-q)}$$

and, therefore, A is an inverse M–matrix in $\mathcal{C}(P)$. □

Next we show that when P has more than two irreducible components and there are no zero patterns except for those below the principal block diagonal, P is also admissible for the class of inverse M–matrices.

Proposition 4.6. *Let P be an $n \times n$ sign pattern matrix of the form*

$$P = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1k} \\ 0 & P_{22} & \dots & P_{2k} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & P_{kk} \end{bmatrix}$$

where $P_{11}, P_{22}, \dots, P_{kk}$ are sign patterns matrices of sizes $q_1 \times q_1, q_2 \times q_2, \dots,$ $q_k \times q_k$, respectively. If P_{ij} is positive for all $j \geq i$, then there exists an inverse M–matrix A in $\mathcal{C}(P)$.

Proof. For $0 < a < 1$, let

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ 0 & A_{22} & \dots & A_{2k} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & A_{kk} \end{bmatrix}$$

be the matrix defined by $A_{ii} = C_{q_i,a}$ $(i = 1, \dots, k)$ and $A_{ij} = aJ_{q_i \times q_j}$ $(j > i)$. We claim that $A^{-1} = (B_{ij})$ with $B_{ii} = A_{ii}^{-1}$ and, for $j > i$,

$$B_{ij} = \frac{-a(1-a)^{j-i-1}}{(q_i a + (1-a))\,(q_{i+1} a + (1-a)) \dots (q_j a + (1-a))} J_{q_i \times q_j}.$$

From Lemma 4.1 it follows that each B_{ii} has negative off-diagonal entries and positive diagonal entries. If each B_{ij}, $j > i$, is as described above, it is obvious that A is an inverse M–matrix. The proof follows by induction on the number k of diagonal blocks. The case $k = 2$ is studied in the proof of the previous lemma. Suppose, now, that, given $k > 2$ the result is valid for $k - 1$. Apply this to submatrices $A[\{1, \dots, n - q_k\}]$ and $A[\{q_1 + 1, \dots, n\}]$. We only have to show, then, that B_{1k} is of the referred form. Note that

$$B_{1k} = -B_{11} A_{1k} A_{kk}^{-1} - B_{12} A_{2k} A_{kk}^{-1} - B_{13} A_{3k} A_{kk}^{-1} - \dots - B_{1k-1} A_{k-1k} A_{kk}^{-1}.$$

Recall $B_{11} = A_{11}^{-1}$ and

$$B_{1j} = \frac{-a(1-a)^{j-2}}{(q_1 a + (1-a))\,(q_2 a + (1-a)) \dots (q_j a + (1-a))} J_{q_1 \times q_j},$$

for all $j \in \{2, \ldots, k-1\}$. Hence, B_{1k} is given by the expression

$$\begin{aligned}
- \quad & A_{11}^{-1} A_{1k} A_{kk}^{-1} - \frac{-a}{(q_1 a + (1-a))(q_2 a + (1-a))} J_{q_1 \times q_2} \frac{a}{(q_k a + (1-a))} J_{q_2 \times q_k} \\
- \quad & \frac{-a(1-a)}{(q_1 a + (1-a))(q_2 a + (1-a))(q_3 a + (1-a))} J_{q_1 \times q_3} \frac{a}{(q_k a + (1-a))} J_{q_3 \times q_k} \\
- \quad & \ldots - \frac{-a(1-a)^{k-3}}{(q_1 a + (1-a))(q_2 a + (1-a)) \ldots (q_{k-1} a + (1-a))} J_{q_1 \times q_{k-1}} \times \\
\times \quad & \frac{a}{(q_k a + (1-a))} J_{q_{k-1} \times q_k}.
\end{aligned}$$

By taking $M = (q_1 a + (1-a))(q_2 a + (1-a)) \ldots (q_{k-1} a + (1-a))(q_k a + (1-a))$, we get

$$\begin{aligned}
B_{1k} \quad = \quad & (-a(q_2 a + (1-a)) \ldots (q_{k-1} a + (1-a)) + \\
& +a^2 q_2 (q_3 a + (1-a)) \ldots (q_{k-1} a + (1-a)) + \\
& +a^2 (1-a) q_3 (q_4 a + (1-a)) \ldots (q_{k-1} a + (1-a)) + \ldots + \\
& +a^2 (1-a)^{k-3} q_{k-1}) M^{-1} J_{q_1 \times q_k}.
\end{aligned}$$

Let N be the real scalar which in the equality above is right hand side multiplied by $M^{-1} J_{q_1 \times q_k}$. Observe that

$$\begin{aligned}
N \quad = \quad & -a^2 q_2 (q_3 a + (1-a)) \ldots (q_{k-1} a + (1-a)) - \\
& -a(1-a)(q_3 a + (1-a)) \ldots (q_{k-1} a + (1-a)) + \\
& +a^2 q_2 (q_3 a + (1-a)) \ldots (q_{k-1} a + (1-a)) + \\
& +a^2 (1-a) q_3 (q_4 a + (1-a)) \ldots (q_{k-1} a + (1-a)) + \ldots + \\
& +a^2 (1-a)^{k-3} q_{k-1} \\
= \quad & -a(1-a)(q_3 a + (1-a)) \ldots (q_{k-1} a + (1-a)) + \\
& +a^2 (1-a) q_3 (q_4 a + (1-a)) \ldots (q_{k-1} a + (1-a)) + \ldots + \\
& +a^2 (1-a)^{k-3} q_{k-1} \\
= \quad & -a^2 (1-a) q_3 (q_4 a + (1-a)) \ldots (q_{k-1} a + (1-a)) - \\
& -a(1-a)^2 (q_4 a + (1-a)) \ldots (q_{k-1} a + (1-a)) + \\
& +a^2 (1-a) q_3 (q_4 a + (1-a)) \ldots (q_{k-1} a + (1-a)) + \ldots + \\
& +a^2 (1-a)^{k-3} q_{k-1}
\end{aligned}$$

$$
\begin{aligned}
&= -a(1-a)^2\left(q_4a+(1-a)\right)\ldots\left(q_{k-1}a+(1-a)\right)+\\
&\quad +a^2(1-a)^2q_4\left(q_5a+(1-a)\right)\ldots\left(q_{k-1}a+(1-a)\right)+\ldots+\\
&\quad +a^2(1-a)^{k-3}q_{k-1}\\
&= \ldots\\
&= -a(1-a)^{k-2},
\end{aligned}
$$

which concludes the proof. □

We end this section with the following remark.

Remark: The irreducible sign pattern matrices that occur among the inverse M–matrices are described in Theorem 4.2. Since every matrix is permutation similar to a block triangular matrix with irreducible diagonal blocks, the general question is whether sign pattern matrices of the form

$$
P = \begin{bmatrix} P_{11} & P_{12} & \ldots & P_{1k} \\ 0 & P_{22} & \ldots & P_{2k} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & P_{kk} \end{bmatrix},
$$

whose diagonal blocks $P_{11}, P_{22}, \ldots, P_{kk}$ are the irreducible components of P, are admissible for the class of inverse M–matrices. Corollary 4.4 states that $P_{ii} > 0$, for all choices of i, and $P_{ij} = 0$ or $P_{ij} > 0$, for all $j > i$, is a necessary condition for the admissibility of such sign pattern matrices P. We strongly believe that this is also a sufficient condition. When $k = 2$ or when all $P_{ij} > 0$, the result follows from Lemma 4.5 and from Proposition 4.6. The question that remains with no answer is whether there exists an inverse M–matrix in the class $\mathcal{C}(P)$ where P is a sign pattern matrix as described above, with $P_{ii} > 0$, for all choices of i, $P_{ij} = 0$ or $P_{ij} > 0$, for all $j > i$, and at least one of these latter blocks is a zero pattern. Observe that if P_{ij} is a zero pattern, the transitivity of P may imply that some other blocks, above the principal block diagonal, are also zero patterns. It is, however, apparently hard to describe all the possibilities and, consequently, to achieve a conclusive answer. Nevertheless, it is clear that those patterns that can be expressed as direct sums of patterns where all of the blocks above the diagonal are positive work as well.

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