SIMPLICIAL RESOLUTIONS AND GANEA FIBRATIONS

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Abstract. In this work, we compare the two approximations of a path-connected space X, by the Ganea spaces $G_n(X)$ and by the realizations $\|\Lambda_\bullet X\|_n$ of the truncated simplicial resolutions emerging from the loop-suspension cotriple $\Sigma\Omega$. For a simply connected space X, we construct maps $\|\Lambda_\bullet X\|_{n-1} \to G_n(X) \to \|\Lambda_\bullet X\|_n$ over X, up to homotopy. In the case $n = 2$, we prove the existence of a map $G_2(X) \to \|\Lambda_\bullet X\|_1$ over X (up to homotopy) and conjecture that this map exists for any n.

We use the category **Top** of well pointed compactly generated spaces having the homotopy type of CW-complexes. We denote by Ω and Σ the classical loop space and (reduced) suspension constructions on **Top**.

Let $X \in$ **Top**. First we recall the construction of the Ganea fibrations $G_n(X) \to X$ where $G_n(X)$ has the same homotopy type as the n-th stage, $B_n\Omega X$, of the construction of the classifying space of ΩX:

(1) the first Ganea fibration, $p_1 \colon G_1(X) \to X$, is the associated fibration to the evaluation map $\mathrm{ev}_X \colon \Sigma\Omega X \to X$;
(2) given the nth-fibration $p_n \colon G_n(X) \to X$, let $F_n(X)$ be its homotopy fiber and let $G_n(X) \cup \mathcal{C}(F_n(X))$ be the mapping cone of the inclusion $F_n(X) \to G_n(X)$. We define now a map $p'_{n+1} \colon G_n(X) \cup \mathcal{C}(F_n(X)) \to X$ as p_n on $G_n(X)$ and that sends the (reduced) cone $\mathcal{C}(F_n(X))$ on the base point. The $(n+1)$-st-fibration of Ganea, $p_{n+1} \colon G_{n+1}(X) \to X$, is the fibration associated to p'_{n+1}.
(3) Denote by $G_\infty(X)$ the direct limit of the canonical maps $G_n(X) \to G_{n+1}(X)$ and by $p_\infty \colon G_\infty(X) \to X$ the map induced by the p_n's.

From a classical theorem of Ganea [3], one knows that the fiber of p_n has the homotopy type of an $(n+1)$-fold reduced join of ΩX with itself. Therefore the maps p_n are higher and higher connected when the integer n grows. As a consequence, if X is path-connected, the map $p_\infty \colon G_\infty(X) \to X$ is a homotopy equivalence and the total spaces $G_n(X)$ constitute approximations of the space X.

The previous construction starts with the couple of adjoint functors Ω and Σ. From them, we can construct a *simplicial space* $\Lambda_\bullet X$, defined by $\Lambda_n X = (\Sigma\Omega)^{n+1}X$ and augmented by $d_0 = \mathrm{ev}_X \colon \Sigma\Omega X \to X$. Forgetting the degeneracies, we have a *facial space* (also called restricted simplicial space in [2, 3.13]). Denote by $\|\Lambda_\bullet X\|$ the realization of this facial space (see [7] or Section 1). An adaptation of the proof of Stover (see [8, Proposition 3.5]) shows that the augmentation d_0 induces a map $\|\Lambda_\bullet X\| \to X$ which is a homotopy equivalence. If we consider the successive stages of the realization of the facial space $\Lambda_\bullet X$, we get maps $\|\Lambda_\bullet X\|_n \to X$ which constitute a second sequence of approximations of the space X. In this work, we study the relationship between these two sequences of approximations and prove the following results.

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Theorem 1. *Let $X \in \mathbf{Top}$ be a simply connected space. Then there is a homotopy commutative diagram*

$$\begin{array}{ccccc} \|\Lambda_\bullet X\|_{n-1} & \longrightarrow & G_n(X) & \longrightarrow & \|\Lambda_\bullet X\|_n \\ & \searrow & \downarrow{\scriptstyle p_n} & \swarrow & \\ & & X & & \end{array}$$

The hypothesis of simply connectivity is used only for the map $G_n(X) \to \|\Lambda_\bullet X\|_n$, see Theorem 3 and Theorem 5. In the case $n = 2$, the situation is better.

Theorem 2. *Let $X \in \mathbf{Top}$. Then there are homotopy commutative triangles*

$$\begin{array}{ccc} \|\Lambda_\bullet X\|_1 & \rightleftarrows & G_2(X) \\ & \searrow \quad \swarrow{\scriptstyle p_2} & \\ & X & \end{array}$$

We conjecture the existence of maps $\|\Lambda_\bullet X\|_{n-1} \rightleftarrows G_n(X)$ over X up to homotopy, for any n.

This work may also be seen as a comparison of two constructions: an iterative fiber-cofiber process and the realization of progressive truncatures of a facial resolution. More generally, for any cotriple, we present an adapted fiber-cofiber construction (see Definition 9) and ask if the results obtained in the case of $\Sigma\Omega$ can be extended to this setting.

Finally, we observe that a variation on a theorem of Libman is essential in our argumentation, see Theorem 4. A proof of this result, inspired by the methods developed by R. Vogt (see [9]), is presented in an Appendix.

This program is carried out in Sections 1-8 below, whose headings are self-explanatory:

Contents

1. Facial spaces

A *facial object* in a category $\mathbf{C}$ is a sequence of objects $X_0, X_1, X_2, \ldots$ together with morphisms $d_i : X_n \to X_{n-1}$, $0 \le i \le n$, satisfying the *facial identities*

$d_i d_j = d_{j-1} d_i$ $(i < j)$.

$$X_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} X_1 \underset{d_2}{\overset{d_0}{\underset{d_1}{\leftleftarrows}}}\!\!\leftarrow X_2 \qquad \cdots \qquad X_{n-1} \underset{d_n}{\overset{d_0}{\leftleftarrows}}\!\!\vdots\;\; X_n \leftleftarrows\!\!\vdots\; \cdots$$

The morphisms d_i are called *face operators*. We shall use notation like $X_\bullet$ to denote facial objects. With the obvious morphisms the facial objects in $\mathbf{C}$ form a category which we denote by $d\mathbf{C}$. An *augmentation* of a facial object $X_\bullet$ in a category $\mathbf{C}$ is a morphism $d_0 : X_0 \to X$ with $d_0 \circ d_0 = d_0 \circ d_1$. The facial object $X_\bullet$ together with the augmentation d_0 is called a *facial resolution of* X and is denoted by $X_\bullet \stackrel{d_0}{\to} X$.

1.1. **Realization(s) of a facial space.** As usual, Δ^n denotes the standard n-simplex of $\mathbb{R}^{n+1}$ and the inclusions of faces are denoted by $\delta^i : \Delta^n \to \Delta^{n+1}$. We consider the point $(0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ as the base-point of the standard n-simplex Δ^n. If X and Y are in **Top**, we denote by $X \rtimes Y$ the half smashed product $X \rtimes Y = X \times Y / * \times Y$.

A *facial space* is a facial object in **Top**. The *realization* of a facial space $X_\bullet$ is the direct limit

$$\|X_\bullet\|_\infty = \varinjlim \|X_\bullet\|_n$$

where the spaces $\|X_\bullet\|_n$ are inductively defined as follows. Set $\|X_\bullet\|_0 = X_0$. Suppose we have defined $\|X_\bullet\|_{n-1}$ and a map $\chi_{n-1} : X_{n-1} \rtimes \Delta^{n-1} \to \|X_\bullet\|_{n-1}$ (χ_0 is the obvious homeomorphism). Then $\|X_\bullet\|_n$ and χ_n are defined by the pushout diagram

$$\begin{array}{ccc} X_n \rtimes \partial\Delta^n & \xrightarrow{\varphi_n} & \|X_\bullet\|_{n-1} \\ \downarrow & & \downarrow \\ X_n \rtimes \Delta^n & \xrightarrow[\chi_n]{} & \|X_\bullet\|_n \end{array}$$

where φ_n is defined by the following requirements, for any $i \in \{0, 1, \ldots, n\}$,

$$\varphi_n \circ (X_n \rtimes \delta^i) = \chi_{n-1} \circ (d_i \rtimes \Delta^{n-1}) : X_n \rtimes \Delta^{n-1} \to \|X_\bullet\|_{n-1}.$$

It is clear that φ_1 is a well-defined continuous map. For φ_n with $n \geq 2$, this is assured by the facial identities $d_i d_j = d_{j-1} d_i$ $(i < j)$.

We also consider another realization of the facial space $X_\bullet$. The *free realization* of $X_\bullet$ is the direct limit

$$|X_\bullet|_\infty = \varinjlim |X_\bullet|_n$$

where the spaces $|X_\bullet|_n$ are inductively defined as follows. Set $|X_\bullet|_0 = X_0$. Suppose we have defined $|X_\bullet|_{n-1}$ and a map $\bar{\chi}_{n-1} : X_{n-1} \times \Delta^{n-1} \to |X_\bullet|_{n-1}$ ($\bar{\chi}_0$ is the obvious homeomorphism). Then $|X_\bullet|_n$ and $\bar{\chi}_n$ are defined by the pushout diagram

$$\begin{array}{ccc} X_n \times \partial\Delta^n & \xrightarrow{\bar{\varphi}_n} & |X_\bullet|_{n-1} \\ \downarrow & & \downarrow \\ X_n \times \Delta^n & \xrightarrow[\bar{\chi}_n]{} & |X_\bullet|_n \end{array}$$

where $\bar{\varphi}_n$ is defined by the following requirements, for any $i \in \{0, 1, \ldots, n\}$,

$$\bar{\varphi}_n \circ (X_n \times \delta^i) = \bar{\chi}_{n-1} \circ (d_i \times \Delta^{n-1}) : X_n \times \Delta^{n-1} \to |X_\bullet|_{n-1}.$$

Again the facial identities $d_i d_j = d_{j-1} d_i$ $(i < j)$ assure that $\bar{\varphi}_n$ is a well-defined continuous map. Since $\bar{\chi}_{n-1}$ is base-point preserving, so is $\bar{\varphi}_n$ and hence $\bar{\chi}_n$.

We sometimes consider facial spaces with upper indexes $X^\bullet$. In such a case, the realizations up to n are denoted by $||X^\bullet||^n$ and $|X^\bullet|^n$.

Let $X_\bullet \stackrel{d_0}{\to} X$ be a facial resolution of a space X. We define a sequence of maps $\|X_\bullet\|_n \to X$ as follows. The map $\|X_\bullet\|_0 \to X$ is the augmentation. Suppose we have defined $\|X_\bullet\|_{n-1} \to X$ such that the following diagram is commutative:

$$\begin{array}{ccc} X_{n-1} \rtimes \Delta^{n-1} & \xrightarrow{\chi_{n-1}} & \|X_\bullet\|_{n-1} \\ {\scriptstyle \mathrm{pr}}\big\downarrow & & \big\downarrow \\ X_{n-1} & \xrightarrow[(d_0)^n]{} & X, \end{array}$$

where $(d_0)^n$ denotes the n-fold composition of the face operator d_0. Consider the diagram

$$\begin{array}{ccc} X_n \rtimes \Delta^{n-1} & \xrightarrow{d_i \rtimes \Delta^{n-1}} & X_{n-1} \rtimes \Delta^{n-1} \\ {\scriptstyle X_n \rtimes \delta^i}\big\downarrow & & \big\downarrow{\scriptstyle \chi_{n-1}} \\ X_n \rtimes \partial\Delta^n & \xrightarrow{\varphi_n} & \|X_\bullet\|_{n-1} \\ {\scriptstyle \mathrm{pr}}\big\downarrow & & \big\downarrow \\ X_n & \xrightarrow[(d_0)^{n+1}]{} & X. \end{array}$$

The upper square is commutative for all i and so is the outer diagram. It follows that the lower square is commutative. We may therefore define $\|X_\bullet\|_n \to X$ to be the unique map which extends $\|X_\bullet\|_{n-1} \to X$ and which, pre-composed by χ_n, is the composite $X_n \rtimes \Delta^n \xrightarrow{\mathrm{pr}} X_n \xrightarrow{(d_0)^{n+1}} X$. Similarly, we define a sequence of maps $|X_\bullet|_n \to X$. We refer to the maps $\|X_\bullet\|_n \to X$ and $|X_\bullet|_n \to X$ as the *canonical maps* induced by the facial resolution $X_\bullet \to X$. The next statement relates these two realizations; its proof is straightforward.

Proposition 1. *Let $X_\bullet$ be a facial space. Then for each $n \in \mathbb{N}$, the canonical map $|X_\bullet|_n \to X$ factors through the canonical map $\|X_\bullet\|_n \to X$*

1.2. **Facial resolutions with contraction.** A *contraction* of a facial resolution $X_\bullet \stackrel{d_0}{\to} X$ consists of a sequence of morphisms $s : X_{n-1} \to X_n \quad (X_{-1} = X)$ such that $d_0 \circ s = \mathrm{id}$ and $d_i \circ s = s \circ d_{i-1}$ for $i \geq 1$.

Proposition 2. *Let $X_\bullet \stackrel{d_0}{\to} X$ be a facial resolution which admits a contraction $s : X_{n-1} \to X_n \quad (X_{-1} = X)$. For any $n \geq 0$, $|X_\bullet|_n$ can be identified with the quotient space $X_n \times \Delta^n / \sim$ where the relation is given by*

$$(x, t_0, ..., t_k, ..., t_n) \sim (sd_k x, 0, t_0, ..., \hat{t}_k, ..., t_n), \quad \textit{if } t_k = 0.$$

As usual, the expression $\hat{t}_k$ means that t_k is omitted. Under this identification the canonical map $|X_\bullet|_n \to X$ is given by $[x, t_0, ..., t_k, ..., t_n] \mapsto (d_0)^{n+1}(x)$ and the inclusion $|X_\bullet|_n \rightarrowtail |X_\bullet|_{n+1}$ is given by $[x, t_0, ..., t_k, ..., t_n] \mapsto [sx, 0, t_0, ..., t_k, ..., t_n]$.

Proof. We first note that the simplicial identities together with the contraction properties guarantee that the relation is unambiguously defined if various parameters are zero and also that the two maps

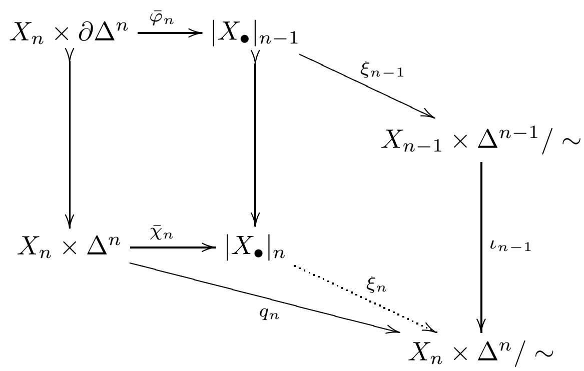
$$\begin{array}{ccc} X_n \times \Delta^n / \sim & \to & X_{n+1} \times \Delta^{n+1} / \sim \\ [x, t_0, ..., t_k, ..., t_n] & \mapsto & [sx, 0, t_0, ..., t_k, ..., t_n] \end{array}$$

and

$$\begin{array}{ccc} X_n \times \Delta^n/\sim & \to & X \\ [x, t_0, ..., t_k, ..., t_n] & \mapsto & (d_0)^{n+1}(x) \end{array}$$

that we will denote by ι_n and ε_n respectively are well-defined.

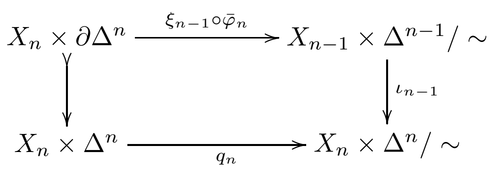
Beginning with $\xi_0 = \mathrm{id}$, we next construct a sequence of homeomorphisms $\xi_n : |X_\bullet|_n \to X_n \times \Delta^n/\sim$ inductively by using the universal property of pushouts in the diagram



where q_n is the identification map. If $t_k = 0$, the construction up to $n-1$ implies

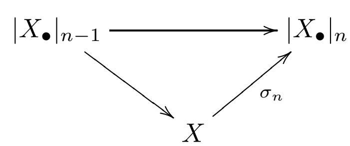
$$\xi_{n-1} \circ \bar{\varphi}_n(x, t_0, ..., t_n) = q_{n-1} \circ (d_k \times \Delta^{n-1}) = [d_k x, t_0, ...\hat{t}_k, ..., t_n].$$

Therefore, we see that the diagram



is commutative and, by checking the universal property, that it is a pushout. Thus ξ_n exists and is a homeomorphism. Through this sequence of homeomorphisms, ι_n corresponds to the inclusion $|X_\bullet|_n \rightarrowtail |X_\bullet|_{n+1}$ and ε_n to the canonical map $|X_\bullet|_n \to X$. $\square$

Proposition 3. *Let $X_\bullet \xrightarrow{d_0} X$ be a facial resolution which admits a natural contraction $s : X_{n-1} \to X_n \quad (X_{-1} = X)$. For any $n \geq 0$, the canonical map $|X_\bullet|_n \to X$ admits a (natural) section $\sigma_n : X \to |X_\bullet|_n$ and the inclusion $|X_\bullet|_{n-1} \rightarrowtail |X_\bullet|_n$ is naturally homotopic to σ_n pre-composed by the canonical map:*



In particular, if the facial resolution $X_\bullet \to *$ admits a natural contraction then the inclusions $|X_\bullet|_{n-1} \rightarrowtail |X_\bullet|_n$ are naturally homotopically trivial.

Proof. Through the identification established in Proposition 2, the section $\sigma_n : X \to |X_\bullet|_n$ is given by

$$\sigma_n(x) = [(s)^{n+1}(x), 0, ..., 0, 1].$$

Using the fact that

$$sd_n sd_{n-1} \cdots sd_2 sd_1 s = (s)^{n+1}(d_0)^n,$$

we calculate that the (well-defined) map $H : |X_\bullet|_{n-1} \times I \to |X_\bullet|_{n-1}$ given by

$$H([x, t_0, ..., t_{n-1}], u) = [sx, u, (1-u)t_0, ..., (1-u)t_{n-1}]$$

is a homotopy between the inclusion and σ_n pre-composed by the canonical map $|X_\bullet|_{n-1} \to X$. □

2. First part of Theorem 1: the map $\|\Lambda_\bullet X\|_{n-1} \to G_n(X)$

Let $X \in \mathbf{Top}$. We consider the facial resolution $\Lambda_\bullet(X) \to X$ where $\Lambda_n(X) = (\Sigma\Omega)^{n+1}X$, the face operators $d_i : (\Sigma\Omega)^{n+1}X \to (\Sigma\Omega)^n X$ are defined by $d_i = (\Sigma\Omega)^i(\mathrm{ev}_{(\Sigma\Omega)^{n-i}X})$, and the augmentation is $d_0 = \mathrm{ev}_X : \Sigma\Omega X \to X$.

Theorem 3. *Let $X \in \mathbf{Top}$. For each $n \in \mathbb{N}$, the canonical map $\|\Lambda_\bullet X\|_{n-1} \to X$ factors through the Ganea fibration $G_n(X) \to X$.*

The proof uses the next result.

Lemma 4. *Given a pushout*

$$\begin{array}{ccc} \Sigma A \rtimes \partial\Delta^n & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ \Sigma A \rtimes \Delta^n & \longrightarrow & Y' \end{array}$$

where the left-hand vertical arrow is a cofibration, then there exists a cofiber sequence $\Sigma A \wedge \partial\Delta^n \longrightarrow Y \xrightarrow{f} Y'$.

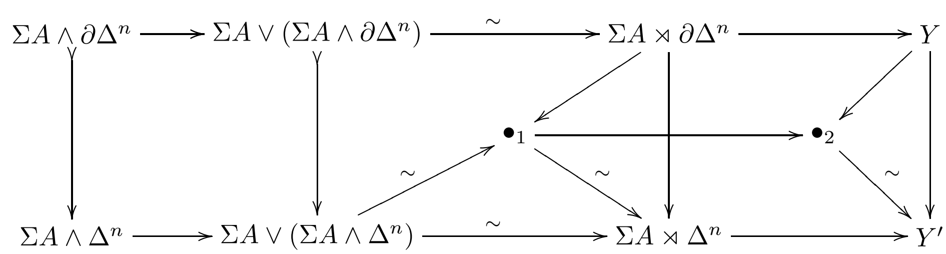
Proof. With the Puppe trick, we construct a commutative diagram

$$\begin{array}{ccc} \Sigma A \vee (\Sigma A \wedge \partial\Delta^n) & \xleftarrow{\sim} & (\Sigma A \rtimes \partial\Delta^n) \\ \downarrow & & \downarrow \\ \Sigma A \vee (\Sigma A \wedge \Delta^n) & \xleftarrow{\sim} & (\Sigma A \rtimes \Delta^n) \end{array}$$

from which we obtain a commutative diagram

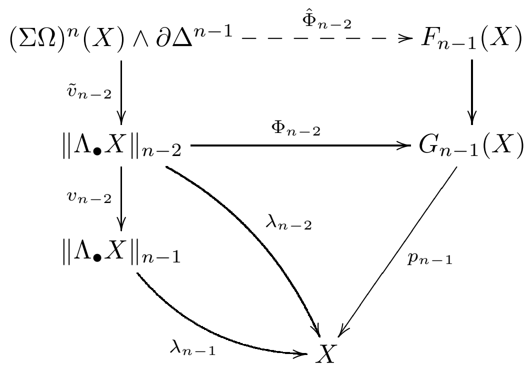
$$\begin{array}{ccc} \Sigma A \vee (\Sigma A \wedge \partial\Delta^n) & \xrightarrow{\sim} & (\Sigma A \rtimes \partial\Delta^n) \\ \downarrow & & \downarrow \\ \Sigma A \vee (\Sigma A \wedge \Delta^n) & \xrightarrow{\sim} & (\Sigma A \rtimes \Delta^n) \end{array}$$

because the left-hand vertical arrow is a cofibration. We form now



where $\bullet_1$ and $\bullet_2$ are built by pushout and the left-hand square is a pushout. The map $\bullet_2 \to Y'$ is a weak equivalence because it is induced between pushouts by the weak equivalence $\bullet_1 \to \Sigma A \rtimes \Delta^n$. □

Proof of Theorem 3. We suppose that $\Phi_{n-2}\colon \|\Lambda_\bullet X\|_{n-2} \to G_{n-1}(X)$ has been constructed over X and observe that the existence of Φ_0 is immediate. We consider the following commutative diagram



where the left-hand column is a cofibration sequence by Lemma 4. From the equalities

$$\begin{aligned} p_{n-1} \circ \Phi_{n-2} \circ \tilde{v}_{n-2} &= \lambda_{n-2} \circ \tilde{v}_{n-2} \\ &= \lambda_{n-1} \circ v_{n-2} \circ \tilde{v}_{n-2} \simeq *, \end{aligned}$$

we deduce a map $\hat{\Phi}_{n-2}\colon (\Sigma\Omega)^n(X) \wedge \partial\Delta^{n-1} \to F_{n-1}(X)$ making the diagram homotopy commutative. From the definition of $G_n(X)$ as a cofiber, this gives a map $\Phi_{n-1}\colon \|\Lambda_\bullet X\|_{n-1} \to G_n(X)$ over X. □

Instead of the explicit construction above, we can also observe that the cone length of $\|\Lambda_\bullet X\|_{n-1}$ is less than or equal to n and deduce Theorem 3 from basic results on Lusternik-Schnirelmann category, see [1].

3. The facial space $\mathcal{G}_\bullet(X)$

For a space X we denote by $P'X$ the Moore path space and by $\Omega'X$ the Moore loop space. Path multiplication turns $\Omega'X$ into a topological monoid. Given a space X, we define the facial space $\mathcal{G}_\bullet(X)$ by $\mathcal{G}_n(X) = (\Omega'X)^n$ with the face operators $d_i : (\Omega'X)^n \to (\Omega'X)^{n-1}$ given by

$$d_i(\alpha_1, ..., \alpha_n) = \begin{cases} (\alpha_2, ..., \alpha_n) & i = 0 \\ (\alpha_1, ..., \alpha_{i-1}, \alpha_i\alpha_{i+1}, ..., \alpha_n) & 0 < i < n \\ (\alpha_1, ..., \alpha_{n-1}) & i = n. \end{cases}$$

The purpose of this section is to compare the free realization of $\mathcal{G}_\bullet(X)$ to the construction of the classifying space of $\Omega'X$.

We work with the following construction of $B\Omega'X$. The classifying space $B\Omega'X$ is the orbit space of the contractible $\Omega'X$-space $E\Omega'X$ which is obtained as the direct limit of a sequence of $\Omega'X$-equivariant cofibrations $E_n\Omega'X \rightarrowtail E_{n+1}\Omega'X$. The spaces $E_n\Omega'X$ are inductively defined by $E_0\Omega'X = \Omega'X$, $E_{n+1}\Omega'X = E_n\Omega'X \cup_\theta (\Omega'X \times CE_n\Omega'X)$ where θ is the action $\Omega'X \times E_n\Omega'X \to E_n\Omega'X$ and C denotes the free (non-reduced) cone construction. The orbit spaces of the $\Omega'X$-spaces $E_n\Omega'X$ are denoted by $B_n\Omega'X$. For each $n \in \mathbb{N}$ this construction yields a cofibration $B_n\Omega'X \rightarrowtail B\Omega'X$. It is well known that for simply connected spaces this cofibration is equivalent to the nth Ganea map $G_n(X) \to X$.

Proposition 5. *For each $n \in \mathbb{N}$ there is a natural commutative diagram*

$$\begin{array}{ccc} B_n\Omega' X & \longrightarrow & |\mathcal{G}_\bullet(X)|_n \\ \downarrow & & \downarrow \\ B\Omega' X & \longrightarrow & |\mathcal{G}_\bullet(X)|_\infty \end{array}$$

in which the bottom horizontal map is a homotopy equivalence.

Proof. We obtain the diagram of the statement from a diagram of $\Omega' X$-spaces by passing to orbit spaces. Consider the facial $\Omega' X$-space $P_\bullet(X)$ in which $P_n(X)$ is the free $\Omega' X$-space $\Omega' X \times (\Omega' X)^n$ and the face operators $d_i : (\Omega' X)^{n+1} \to (\Omega' X)^n$ (which are equivariant) are given by

$$d_i(\alpha_0, ..., \alpha_n) = \left\{ \begin{array}{ll} (\alpha_0, ..., \alpha_{i-1}, \alpha_i\alpha_{i+1}, ..., \alpha_n) & 0 \le i < n \\ (\alpha_0, ..., \alpha_{n-1}) & i = n. \end{array} \right.$$

The maps $s\colon P_{n-1}(X) \to P_n(X)$ given by $s(\alpha_0, \ldots, \alpha_{n-1}) = (*, \alpha_0, \ldots, \alpha_{n-1})$ constitute a natural contraction of the facial resolution $P_\bullet(X) \to *$. By Proposition 3, the maps $|P_\bullet(X)|_{n-1} \to |P_\bullet(X)|_n$ are hence naturally homotopically trivial.

The construction of the realization of $P_\bullet(X)$ yields $\Omega' X$-spaces. We construct a natural commutative diagram of equivariant maps

$$\begin{array}{ccccccccc} E_0\Omega' X & \rightarrowtail & E_1\Omega' X & \rightarrowtail & \cdots & \rightarrowtail & E_n\Omega' X & \rightarrowtail & \cdots \\ \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_1} & & & & \downarrow{\scriptstyle g_n} & & \\ |P_\bullet(X)|_0 & \rightarrowtail & |P_\bullet(X)|_1 & \rightarrowtail & \cdots & \rightarrowtail & |P_\bullet(X)|_n & \rightarrowtail & \cdots \end{array}$$

inductively as follows: The map g_0 is the identity $\Omega' X \stackrel{=}{\to} \Omega' X$. Suppose that g_n is defined. Since the map $|P_\bullet(X)|_n \rightarrowtail |P_\bullet(X)|_{n+1}$ is naturally homotopically trivial, it factors naturally through the cone $C|P_\bullet(X)|_n$. Extend this factorization equivariantly to obtain the following commutative diagram of $\Omega' X$-spaces:

$$\begin{array}{ccc} \Omega' X \times |P_\bullet(X)|_n & \longrightarrow & |P_\bullet(X)|_n \\ \downarrow & & \downarrow \\ \Omega' X \times C|P_\bullet(X)|_n & \longrightarrow & |P_\bullet(X)|_{n+1}. \end{array}$$

Define g_{n+1} to be the composite

$$\begin{array}{ll} & E_n\Omega' X \cup_{\Omega' X \times E_n\Omega' X} (\Omega' X \times CE_n\Omega' X) \\ \to & |P_\bullet(X)|_n \cup_{\Omega' X \times |P_\bullet(X)|_n} (\Omega' X \times C|P_\bullet(X)|_n) \\ \to & |P_\bullet(X)|_{n+1}. \end{array}$$

It is clear that g_{n+1} is natural. In the direct limit we obtain a natural equivariant map $g : E\Omega' X \to |P_\bullet(X)|_\infty$. This map is a homotopy equivalence. Indeed, $E\Omega' X$ is contractible and, since each inclusion $|P_\bullet(X)|_n \rightarrowtail |P_\bullet(X)|_{n+1}$ is homotopically trivial, $|P_\bullet(X)|_\infty$ is contractible, too. For each $n \in \mathbb{N}$ we therefore obtain the following natural commutative diagram of $\Omega' X$-spaces:

$$\begin{array}{ccc} E_n\Omega' X & \longrightarrow & |P_\bullet(X)|_n \\ \downarrow & & \downarrow \\ E\Omega' X & \stackrel{\sim}{\longrightarrow} & |P_\bullet(X)|_\infty. \end{array}$$

Passing to the orbit spaces, we obtain the diagram of the statement. It follows for instance from [4, 1.16] that the map $B\Omega' X \to |\mathcal{G}_\bullet(X)|_\infty$ is a homotopy equivalence. □

Remark. Note that the upper horizontal map in the diagram of Proposition 5 is not a homotopy equivalence in general. Indeed, for $X=*$, $B_1\Omega' X$ is contractible but $|\mathcal{G}_\bullet(X)|_1 \simeq S^1$. It can, however, be shown that there also exists a diagram as in Proposition 5 with the horizontal maps reversed.

4. The facial resolution $\Omega'\Lambda_\bullet X \to \Omega' X$ admits a contraction

Consider the natural map $\gamma_X\colon \Omega' X \to \Omega'\Sigma\Omega X$, $\gamma_X(\omega,t) = (\nu(\omega,t),t)$ where $\nu(\omega,t) : \mathbb{R}^+ \to \Sigma\Omega X$ is given by

$$\nu(\omega,t)(u) = \begin{cases} \left[\omega_t, \frac{u}{t}\right], & u<t, \\ \left[c_*, 0\right], & u \geq t. \end{cases}$$

Here, c_* is the constant path $u \mapsto *$ and $\omega_t\colon I \to X$ is the loop defined by $\omega_t(s) = \omega(ts)$.

Lemma 6. *The map γ_X is continuous.*

Proof. It suffices to show that the map $\nu^\flat : \Omega' X \times \mathbb{R}^+ \to \Sigma\Omega X$, $(\omega,t,u) \mapsto \nu(\omega,t)(u)$ is continuous. Consider the subspace $W = \{\omega \in X^{\mathbb{R}^+} : \omega(0) = *\}$ of $X^{\mathbb{R}^+}$ and the continuous map $\rho : W \times \mathbb{R}^+ \to X^{\mathbb{R}^+}$ given by

$$\rho(\omega,t)(u) = \begin{cases} \omega(u), & u \leq t, \\ \omega(t), & u \geq t. \end{cases}$$

Note that if $(\omega,t) \in P'X$ then $\rho(\omega,t) = \omega$. Consider the continuous map

$$\phi : W \times \mathbb{R}^+ \times [0,\frac{\pi}{2}] \to \Sigma P' X$$

defined by

$$\phi(\omega,r,\theta) = \begin{cases} \left[\rho(\omega, r\cos\theta), r\cos\theta, \tan\theta\right], & \theta \leq \frac{\pi}{4}, \\ \left[c_*, 0, 0\right], & \theta \geq \frac{\pi}{4}. \end{cases}$$

When $r=0$, we have $\phi(\omega,r,\theta) = [c_*,0,0]$ for any θ. Therefore ϕ factors through the identification map

$$W \times \mathbb{R}^+ \times [0,\frac{\pi}{2}] \to W \times \mathbb{R}^+ \times \mathbb{R}^+, (\omega,r,\theta) \mapsto (\omega, r\cos\theta, r\sin\theta)$$

and induces a continuous map $\psi : W \times \mathbb{R}^+ \times \mathbb{R}^+ \to \Sigma P' X$. Explicitly,

$$\psi(\omega,t,u) = \begin{cases} \left[\rho(\omega,t), t, \frac{u}{t}\right], & u<t, \\ \left[c_*, 0, 0\right], & u \geq t. \end{cases}$$

Consider the continuous map $\xi : P'X \to PX$ defined by $\xi(\omega,t)(s) = \omega(ts)$. Note that $\xi(\omega,t) = \omega_t$ if $(\omega,t) \in \Omega' X$ and, in particular, that $\xi(c_*,0) = c_*$. The restriction of $\Sigma\xi \circ \psi$ to $\Omega' X \times \mathbb{R}^+$ factors through the subspace $\Sigma\Omega X$ of $\Sigma P X$ and the continuous map

$$\Omega' X \times \mathbb{R}^+ \to \Sigma\Omega X, (\omega,t,u) \mapsto (\Sigma\xi \circ \psi)(\omega,t,u)$$

is exactly $\nu^\flat$. □

Proposition 7. *The maps $s = \gamma_{(\Sigma\Omega)^n X} : \Omega'(\Sigma\Omega)^n X \to \Omega'(\Sigma\Omega)^{n+1} X$ define a contraction of the facial resolution $\Omega'\Lambda_\bullet X \to \Omega' X$.*

Proof. We have $(\Omega'(\mathrm{ev}_X) \circ \gamma_X)(\omega,t) = \Omega'(\mathrm{ev}_X)(\nu(\omega,t),t) = (\beta(\omega,t),t)$ where

$$\beta(\omega,t)(u) = \begin{cases} \omega_t(\frac{u}{t}) = \omega(u), & u<t, \\ * = \omega(u), & u \geq t. \end{cases}$$

Hence $(\Omega'(\mathrm{ev}_X) \circ \gamma_X) = \mathrm{id}_{\Omega' X}$.

In the same way one has $(\Omega'(\mathrm{ev}_{(\Sigma\Omega)^n X}) \circ \gamma_{(\Sigma\Omega)^n X}) = \mathrm{id}_{(\Sigma\Omega)^n X}$. This shows the relation $d_0 \circ s = \mathrm{id}$. It remains to check that $d_j \circ s = s \circ d_{j-1}$, for $j \geq 1$. For

$(\omega,t)\in\Omega'(\Sigma\Omega)^nX$ we have $(d_j\circ s)(\omega,t)=(\Omega'(\Sigma\Omega)^j(\mathrm{ev}_{(\Sigma\Omega)^{n-j}X})\circ\gamma_{(\Sigma\Omega)^nX})(\omega,t)=(\sigma(\omega,t),t)$ where

$$\sigma(\omega,t)(u)=\begin{cases}(\Sigma\Omega)^j(\mathrm{ev}_{(\Sigma\Omega)^{n-j}X})\left[\omega_t,\frac{u}{t}\right]=\left[(\Sigma\Omega)^{j-1}(\mathrm{ev}_{(\Sigma\Omega)^{n-j}X})\circ\omega_t,\frac{u}{t}\right], & u<t,\\ (\Sigma\Omega)^j(\mathrm{ev}_{(\Sigma\Omega)^{n-j}X})\left[c_*,0\right]=\left[c_*,0\right], & u\geq t.\end{cases}$$

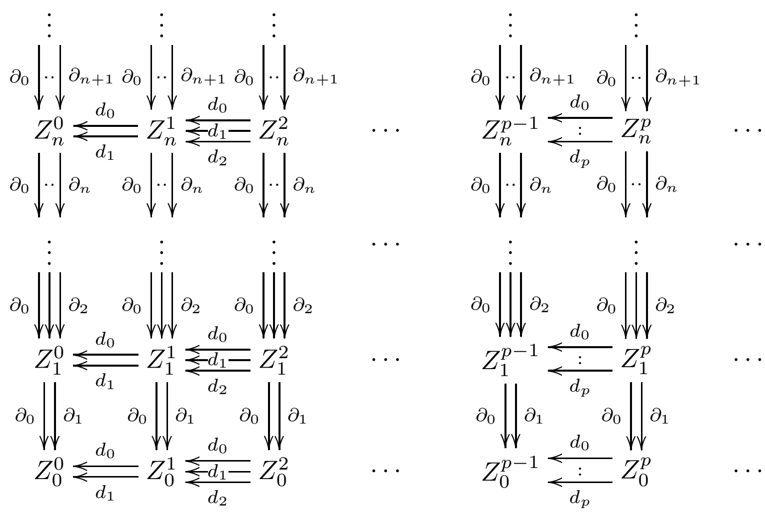
On the other hand, $(s\circ d_{j-1})(\omega,t)=(\gamma_{(\Sigma\Omega)^{n-1}X}\circ\Omega'(\Sigma\Omega)^{j-1}(\mathrm{ev}_{(\Sigma\Omega)^{n-j}X}))(\omega,t)=(\tau(\omega,t),t)$ where

$$\tau(\omega,t)(u)=\begin{cases}\left[((\Sigma\Omega)^{j-1}(\mathrm{ev}_{(\Sigma\Omega)^{n-j}X})\circ\omega)_t,\frac{u}{t}\right], & u<t,\\ \left[c_*,0\right], & u\geq t.\end{cases}$$

This shows that $d_j\circ s=s\circ d_{j-1}$ ($j\geq 1$). □

5. Second part of Theorem 1: the map $G_n(X)\to\|\Lambda_\bullet X\|_n$

A *bifacial space* is a facial object in the category d**Top** of facial spaces. We will use notations like $Z_\bullet^\bullet$ to denote bifacial spaces and refer to the upper index as the column index and to the lower index as the row index. In this way, a bifacial space can be represented by a diagram of the following type:



As in this diagram we shall reserve the notation ∂_i for the face operators of a column facial space and the notation d_i for the face operators of a row facial space. For any k, $|Z_\bullet^k|_m$ (resp. $|Z_k^\bullet|^m$) is the realization up to m of the kth column (resp. kth row) and $|Z_\bullet^\bullet|_m$ (resp. $|Z_\bullet^\bullet|^m$) is the facial space obtained by realizing each column (resp. each row) up to m.

The construction of the map $G_n(X)\to\|\Lambda_\bullet X\|_n$ relies heavily on the following result which is analogous to a theorem of A. Libman [5]. As A. Libman has pointed out to the authors, this result can be derived from [5] (private communication). For the convenience of the reader, we include, in an appendix, an independent proof of the particular case we need.

Theorem 4. *Consider a facial space $Z_\bullet^{-1}$ and a facial resolution $Z_\bullet^\bullet\xrightarrow{d_0}Z_\bullet^{-1}$ such that each row $Z_k^\bullet\xrightarrow{d_0}Z_k^{-1}$ admits a contraction. Then, for any n, there exists a not necessarily base-point preserving continuous map $|Z_\bullet^{-1}|_n\to||Z_\bullet^\bullet|^n|_n$ which is a section up to free homotopy of the canonical map $||Z_\bullet^\bullet|_n|^n\to|Z_\bullet^{-1}|_n$.*

The second part of Theorem 1 can be stated as follows.

Theorem 5. *Let $X \in$ **Top** be a simply connected space. For each $n \in \mathbb{N}$ the nth Ganea map $G_n(X) \to X$ factors up to (pointed) homotopy through the canonical map $\|\Lambda_\bullet X\|_n \to X$.*

Proof. Consider the column facial space $Z_\bullet^{-1} = \mathcal{G}_\bullet(X)$ and the facial resolution $Z_\bullet^{-1} \leftarrow Z_\bullet^\bullet$ where $Z_i^j = \mathcal{G}_i(\Lambda_j X)$. Each row facial resolution

$$Z_i^{-1} = \mathcal{G}_i(X) \leftarrow Z_i^\bullet = \mathcal{G}_i(\Lambda_\bullet X)$$

admits a contraction. Since $\mathcal{G}_0(\Lambda_\bullet X) = *$, this is clear for $i = 0$. For $i > 0$, $\mathcal{G}_i(\Lambda_\bullet X) = (\Omega'\Lambda_\bullet X)^i$. Indeed, since, by Proposition 7, the facial resolution $\Omega' X \leftarrow \Omega'\Lambda_\bullet X$ admits a contraction, its ith power also admits a contraction.

For $n \in \mathbb{N}$ consider the commutative diagram

$$\begin{array}{ccccc} B_n\Omega' X & \longrightarrow & |\mathcal{G}_\bullet(X)|_n & \longleftarrow & ||\mathcal{G}_\bullet(\Lambda_\bullet X)|_n|^n \\ \downarrow & & \downarrow & & \downarrow \\ B\Omega' X & \longrightarrow & |\mathcal{G}_\bullet(X)|_\infty & \longleftarrow & ||\mathcal{G}_\bullet(\Lambda_\bullet X)|_\infty|^n \end{array}$$

in which the left-hand square is the natural square of Proposition 5. Recall that the lower left horizontal map is a homotopy equivalence. Since X is simply connected, X is naturally weakly equivalent to $B\Omega' X$ and hence to $|\mathcal{G}_\bullet(X)|_\infty$. It follows that the map $||\mathcal{G}_\bullet(\Lambda_\bullet X)|_\infty|^n \to |\mathcal{G}_\bullet(X)|_\infty$ is weakly equivalent to the map $|\Lambda_\bullet X|_n \to X$. Since this last map factors through the map $\|\Lambda_\bullet X\|_n \to X$ and since, by Theorem 4, the upper right horizontal map of the diagram above admits a free homotopy section, we obtain a diagram

$$\begin{array}{ccc} B_n\Omega' X & \longrightarrow & \|\Lambda_\bullet X\|_n \\ \downarrow & & \downarrow \\ B\Omega' X & \xrightarrow{f} & X \end{array}$$

which is commutative up to free homotopy and in which f is a (pointed) homotopy equivalence. Since the left hand vertical map is equivalent to the Ganea map $G_n(X) \to X$, there exists a diagram

$$\begin{array}{ccc} G_n(X) & \longrightarrow & \|\Lambda_\bullet X\|_n \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & X \end{array}$$

which is commutative up to free homotopy and in which g is a (pointed) homotopy equivalence. This implies that the Ganea map $G_n(X) \to X$ factors up to free homotopy through the canonical map $\|\Lambda_\bullet X\|_n \to X$. Since X is simply connected and $\|\Lambda_\bullet X\|_n$ is connected, the Ganea map $G_n(X) \to X$ also factors up to pointed homotopy through the canonical map $\|\Lambda_\bullet X\|_n \to X$. □

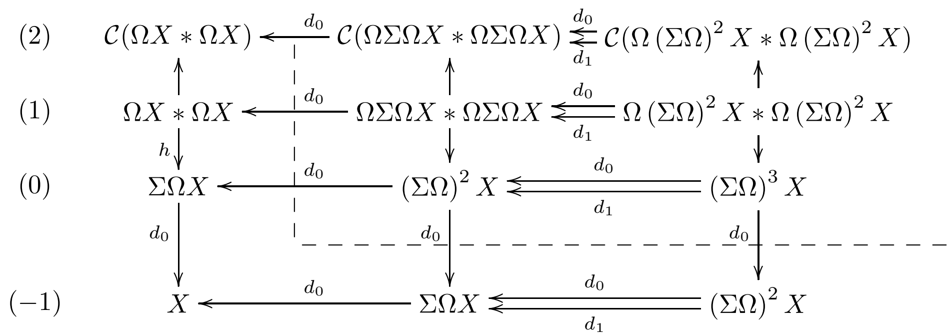
6. Proof of Theorem 2

Proof. Recall the homotopy fiber sequence

$$\Omega X * \Omega X \xrightarrow{h} \Sigma\Omega X \xrightarrow{d_0} X$$

where h is the Hopf map. This sequence is natural in X and the space $G_2(X)$ is equivalent to the pushout of $\mathcal{C}(\Omega X * \Omega X) \longleftarrow \Omega X * \Omega X \longrightarrow \Sigma\Omega X$, where $\mathcal{C}(Y)$

denotes the (reduced) cone over a space Y. We use the following diagram



We observe that

- the image of Line (-1) by Ω has a contraction in the obvious sense;
- Line (0) is the image of Line (-1) by $\Sigma\Omega$ therefore Line (0) admits a contraction;
- the face operators of Line (1) are the maps $\Omega d_i * \Omega d_i$ with the face operators d_i of Line (-1), thus Line (1) admits a contraction;
- Line (2) admits a contraction induced by the previous one.

From the expression of the Hopf map $h\colon \Omega X * \Omega X \to \Sigma\Omega X$, $h([\alpha, t, \beta]) = [\alpha^{-1}\beta, t]$, we observe that the map $H\colon (\Omega X * \Omega X) \times [0,1] \to X$, defined by $H([\alpha, t, \beta], s) = \alpha^{-1}\beta(st)$, induces a natural extension of $d_0 \circ h$ to $\mathcal{C}(\Omega X * \Omega X)$. Therefore, we can complete the diagram by maps from Line (2) to Line (-1) which are compatible with face operators.
Denote by $\tilde{G}$ the homotopy colimit of the framed part of the diagram. We have a commutative square:

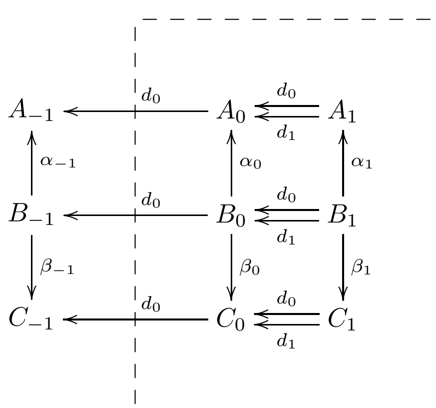
$$\begin{array}{ccc} G_2(X) & \longleftarrow & \tilde{G} \\ \downarrow & & \downarrow \\ X & \longleftarrow & \|\Lambda_\bullet X\|_1 \end{array}$$

Lemma 8 provides a homotopy section of the map $\tilde{G} \to G_2(X)$. Thus we obtain a map

$$G_2(X) \to \|\Lambda_\bullet X\|_1$$

up to homotopy over X. □

Lemma 8. *We consider the following diagram in* **Top**, *satisfying* $d_0 \circ d_0 = d_0 \circ d_1$ *and the obvious commutativity conditions.*



Let $\tilde{G}$ be the homotopy colimit of the framed part and G_{-1} be the homotopy colimit of the first column. We denote by $\tilde{d}\colon \tilde{G} \to G_{-1}$ the map induced by d_0. If the lines of the previous diagram admit contractions in the obvious sense, then the map $\tilde{d}$ has a (pointed) homotopy section.

Proof. This is a special case of a dual of a result of Libman in [5]. It is not covered by the proof of the last section but this situation is simple and we furnish an ad-hoc argument for it.

First we construct maps $f\colon A_{-1} \to \|A_\bullet\|_1$, $g\colon B_{-1} \to \|B_\bullet\|_1$ and $k\colon C_{-1} \to \|C_\bullet\|_1$ such that $\|\alpha_\bullet\|_1 \circ g \simeq f \circ \alpha_{-1}$ and $k \circ \beta_{-1} \simeq \|\beta_\bullet\|_1 \circ g$. With the same techniques as in Proposition 2, it is clear that $\|A_\bullet\|_1$ is homeomorphic to the quotient $A \rtimes \Delta^1$ by the relation $(a, t_0, t_1) \sim (sd_i a, 0, 1)$ if $t_i = 0$. So, we define f, g and k by

$$f(a) = [s_A s_A(a), 0, 1], g(b) = [s_B s_B(b), 0, 1] \text{ and } k(c) = [s_C s_C(c), 0, 1].$$

A computation gives:

$$\begin{aligned} \|\alpha_\bullet\|_1 \circ g(b) &= [\alpha_1 s_B s_B(b), 0, 1] \\ &= [s_A d_0 \alpha_1 s_B s_B(b), 0, 1] \\ &= [s_A \alpha_0 d_0 s_B s_B(b), 0, 1] \\ &= [s_A \alpha_0 s_B(b), 0, 1] \\ f \circ \alpha_1(b) &= [s_A s_A \alpha_{-1}(b), 0, 1] \\ &= [s_A s_A d_0 \alpha_0 s_B(b), 0, 1] \\ &= [s_A d_1 s_A \alpha_0 s_B(b), 0, 1] \\ &= [s_A \alpha_0 s_B(b), 1, 0], \end{aligned}$$

the last equality coming from our construction of $\|A_\bullet\|_1$. These two points, $\|\alpha_\bullet\|_1 \circ g(b)$ and $f \circ \alpha_1(b)$, are canonically joined by a path that reduces to a point if $b = *$. The same argument gives the similar result for k. We observe now that these homotopies give a map between the two mapping cylinders which is a section up to pointed homotopy. □

7. Open questions

The main open question after these results concerns the existence of maps over X up to homotopy, $G_n(X) \to \|\Lambda_\bullet X\|_{n-1}$ for any n. This question is related to the Lusternik-Schnirelman category (LS-category in short) $\operatorname{cat} X$ of a topological space X. Recall that $\operatorname{cat} X \leq n$ if and only if the Ganea fibration $G_n(X) \to X$ admits a section. The truncated resolutions bring a new homotopy invariant $\ell_{\Sigma\Omega}(X)$ defined in a similar way as follows:

$$\ell_{\Sigma\Omega}(X) \leq n \text{ if the map } \|\Lambda_\bullet X\|_{n-1} \to X \text{ admits a homotopical section.}$$

From Theorem 1 and Theorem 2, we know that this new invariant coincides with the LS-category for spaces of LS-category less than or equal to 2 and satisfies

$$\operatorname{cat} X \leq \ell_{\Sigma\Omega}(X) \leq 1 + \operatorname{cat} X.$$

Grants to the result in dimension 2, $\ell_{\Sigma\Omega}(X)$ does not coincide with the cone length. We conjecture its equality with the LS-category and the existence of maps $G_n(X) \to \|\Lambda_\bullet X\|_{n-1}$ over X up to homotopy.

We now extend our study by considering a cotriple T. Recall that a cotriple (T, η, ε) on **Top** is a functor $T :$ **Top** $\to$ **Top** together with two natural transformations $\eta_X : T(X) \to X$ and $\varepsilon_X : T(X) \to T^2(X)$ such that:

$$\varepsilon_{F(X)} \circ \varepsilon_X = F(\varepsilon_X) \circ \varepsilon_X \text{ and } \eta_{T(X)} \circ \varepsilon_X = T(\eta_X) \circ \varepsilon_X = \mathrm{id}_{T(X)}.$$

It is well known that T gives a simplicial space $\Lambda_\bullet^T X$ defined by $\Lambda_n^T X = T^{n+1}(X)$. From it, we deduce a facial space and the truncated realizations $\|\Lambda_\bullet^T X\|_n$. If T satisfies $T(*) \sim *$, takes its values in suspensions and $\Omega'(\Lambda_\bullet^T X)$ admits a contraction, a careful reading of the proofs in this work shows that we get the same conclusions as in Theorem 1 and Theorem 2 with the Ganea spaces $G_n(X)$ and the realizations $\|\Lambda_\bullet^T X\|_i$.

We could also use a construction of the Ganea spaces adapted to the cotriple T as follows.

Definition 9. Let T be a cotriple and X be a space, the *nth fibration of Ganea associated to T and X* is defined inductively by:
– $p_1^T \colon G_1^T(X) \to X$ is the associated fibration to $\eta_X \colon T(X) \to X$,
– if $p_n^T \colon G_n^T(X) \to X$ is defined, we denote by $F_n^T(X)$ its homotopy fiber and build a map ${p'}_{n+1}^T \colon G_n^T(X) \cup \mathcal{C}(T(F_n^T(X)) \to X$ as p_n^T on $G_n^T(X)$ and sending the cone $\mathcal{C}(T(F_n^T(X))$ on the base point. The fibration p_{n+1}^T is the associated fibration to ${p'}_{n+1}^T$.

The results of this paper and the questions above have their analog in this setting. New approximations of spaces arise from the truncated realizations $\|\Lambda_\bullet^T X\|_i$ and from the adapted fiber-cofiber constructions. One natural problem is to look for a comparison between them. These questions can also be stated in terms of LS-category. For instance, does the Stover resolution (see [8]) of a space by wedges of spheres give the s-category defined in [6]?

8. Appendix: Proof of Theorem 4

The purpose of this appendix is to give a proof of Theorem 4. This proof is contained in the Subsection 8.2 below and uses the constructions and notation of the following subsection.

8.1. **n-facial spaces and n-rectifiable maps.** Let $n \geq 0$ be an integer. A facial space $X_\bullet$ is a *n-facial space* if, for any $k \geq n+1$, $X_k = *$. To any facial space $Y_\bullet$, we can associate an n-facial space $T_\bullet^n(Y)$ by setting $T_k^n(Y) = Y_k$ if $k \leq n$ and $T_k^n(Y) = *$ if $k \geq n+1$. Obviously, for any $k \leq n$, we have $|T_\bullet^n(Y)|_k = |Y_\bullet|_k$.

Let $Y_\bullet$ be a facial space with face operators $\partial_i : Y_k \to Y_{k-1}$. We associate to $Y_\bullet$ two n-facial spaces $I_\bullet^n(Y)$ and $J_\bullet^n(Y)$ and morphisms $\eta, \zeta, \pi, \overline{\pi}$ which induce homotopy equivalences between the realizations up to n and such that the following diagram is commutative:

$$\begin{array}{ccccc} T_\bullet^n(Y) & \xrightarrow{\eta} & I_\bullet^n(Y) & \xleftarrow{\zeta} & J_\bullet^n(Y) \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \overline{\pi}} & \\ & & T_\bullet^n(Y). & & \end{array}$$

For any integer $k \geq 1$ we denote by $\partial_{\underline{k}}$ the set $\{\partial_0, ..., \partial_k\}$ of the $(k+1)$ face operators $\partial_i : Y_k \to Y_{k-1}$ and, for any integer $l \geq k$, we set $\partial_{\underline{k}\,:\,\underline{l}} := \partial_{\underline{k}} \times \partial_{\underline{k+1}} \times ... \times \partial_{\underline{l}}$.

The n-facial space $J_\bullet^n(Y)$. For $0 \leq k \leq n$, consider the space:

$$\left(Y_k \times \Delta^0\right) \coprod \coprod_{m=1}^{n-k} \left(\partial_{\underline{k+1}\,:\,\underline{k+m}} \times Y_{k+m} \times \Delta^m\right).$$

An element of this space will be written $(\partial_{i_1},...,\partial_{i_m},y,t_0,...,t_m)$ with the convention $(\partial_{i_1},...,\partial_{i_m},y,t_0,...,t_m)=(y,1)$ if $m=0$. Set

$$J^n_k(Y):=\left(\left(Y_k\times\Delta^0\right)\coprod\coprod_{m=1}^{n-k}\left(\partial_{\underline{k+1}:\underline{k+m}}\times Y_{k+m}\times\Delta^m\right)\right)/\sim$$

where the relations are given by

$$(\partial_{i_1},...,\partial_{i_m},y,t_0,...,t_m)\sim(\partial_{i_1},...,\partial_{i_{m-1}},\partial_{i_m}y,t_0,...,t_{m-1}),\quad \text{if } t_m=0,$$

and

$$(\partial_{i_1},...,\partial_{i_p},\partial_{i_{p+1}},...\partial_{i_m},y,t_0,...,t_m)\sim(\partial_{i_1},...,\partial_{i_{p+1}-1},\partial_{i_p},...\partial_{i_m},y,t_0,...,t_m),$$

if $t_p=0$ and $i_p<i_{p+1}$.

Together with the face operators $J\partial_i:J^n_k(Y)\to J^n_{k-1}(Y)$, $0\le i\le k$, defined by

$$J\partial_i(\partial_{i_1},...,\partial_{i_m},y,t_0,...,t_m)=(\partial_i,\partial_{i_1},...,\partial_{i_m},y,0,t_0,...,t_m),$$

$J^n_\bullet(Y)$ is a n-facial space.

The n-facial space $I^n_\bullet(Y)$. For $0\le k\le n$, we consider now the space:

$$\left(Y_k\times\Delta^1\right)\coprod\coprod_{m=1}^{n-k}\left(\partial_{\underline{k+1}:\underline{k+m}}\times Y_{k+m}\times\Delta^{m+1}\right).$$

We write $(\partial_{i_1},...,\partial_{i_m},y,t_0,...,t_{m+1})$ the elements of that space with the convention $(\partial_{i_1},...,\partial_{i_m},y,t_0,...,t_{m+1})=(y,t_0,t_1)$ if $m=0$. The space $I^n_k(Y)$ is defined to be the quotient

$$I^n_k(Y):=\left(\left(Y_k\times\Delta^1\right)\coprod\coprod_{m=1}^{n-k}\left(\partial_{\underline{k+1}:\underline{k+m}}\times Y_{k+m}\times\Delta^{m+1}\right)\right)/\sim$$

with respect to the relations

$$(\partial_{i_1},...,\partial_{i_m},y,t_0,...,t_{m+1})\sim(\partial_{i_1},...,\partial_{i_{m-1}},\partial_{i_m}y,t_0,...,t_m),\quad \text{if } t_{m+1}=0,$$

and

$$(\partial_{i_1},...,\partial_{i_p},\partial_{i_{p+1}},...\partial_{i_m},y,t_0,...,t_{m+1})\sim(\partial_{i_1},...,\partial_{i_{p+1}-1},\partial_{i_p},...\partial_{i_m},y,t_0,...,t_{m+1}),$$

if $t_{p+1}=0$ and $i_p<i_{p+1}$.

Together with the face operators $I\partial_i:I^n_k(Y)\to I^n_{k-1}(Y)$, $0\le i\le k$, defined by

$$I\partial_i(\partial_{i_1},...,\partial_{i_m},y,t_0,t_1,...,t_{m+1})=(\partial_i,\partial_{i_1},...,\partial_{i_m},y,t_0,0,t_1,...,t_{m+1}),$$

$I^n_\bullet(Y)$ is a n-facial space.

The morphisms $\eta,\zeta,\pi,\overline{\pi}$. The facial maps $\eta:T^n_\bullet(Y)\to I^n_\bullet(Y)$, $\zeta:J^n_\bullet(Y)\to I^n_\bullet(Y)$, $\pi:I^n_\bullet(Y)\to T^n_\bullet(Y)$ and $\overline{\pi}:J^n_\bullet(Y)\to T^n_\bullet(Y)$ are respectively defined (for $k\le n$) by:

$$\eta_k(y)=(y,1,0),$$

$$\zeta_k(\partial_{i_1},...,\partial_{i_m},y,t_0,...,t_m)=(\partial_{i_1},...,\partial_{i_m},y,0,t_0,...,t_m),$$

$$\pi_k(\partial_{i_1},...,\partial_{i_m},y,t_0,...,t_{m+1})=\partial_{i_1}\cdots\partial_{i_m}y\quad\text{and}\quad\pi_k(y,t_0,t_1)=y,$$

$$\overline{\pi}_k=\pi_k\circ\zeta_k.$$

We have $\pi_k \circ \eta_k = \mathrm{id}$ so that the following diagram is commutative:

$$\begin{array}{ccccc} T^n_\bullet(Y) & \xrightarrow{\eta} & I^n_\bullet(Y) & \xleftarrow{\zeta} & J^n_\bullet(Y) \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \overline{\pi}} & \\ & & T^n_\bullet(Y). & & \end{array}$$

In order to see that these morphisms induce homotopy equivalences between the realizations up to n, it suffices to see that, for any k, $0 \leq k \leq n$, the maps $\eta_k, \zeta_k, \pi_k, \overline{\pi}_k$ are homotopy equivalences. Thanks to the commutativity of the diagram above we just have to check it for the maps π_k and $\overline{\pi}_k$. These two maps admit a section: we have already seen that $\pi_k \circ \eta_k = \mathrm{id}$ and, on the other hand, the map $\varphi_k : T^n_k(Y) \to J^n_k(Y)$ given by $\varphi_k(y) = (y, 1)$ (which does not commute with the face operators) satisfies $\overline{\pi}_k \circ \varphi_k = \mathrm{id}$. The conclusion follows then from the fact that the two homotopies

$$\begin{array}{rcl} H_k : I^n_k(Y) \times I & \to & I^n_k(Y) \\ ((\partial_{i_1}, ..., \partial_{i_m}, y, t_0, ..., t_{m+1}), u) & \mapsto & (\partial_{i_1}, ..., \partial_{i_m}, y, u + (1-u)t_0), \\ & & \qquad (1-u)t_1, .., (1-u)t_{m+1}) \\ \overline{H}_k : J^n_k(Y) \times I & \to & J^n_k(Y) \\ ((\partial_{i_1}, ..., \partial_{i_m}, y, t_0, ..., t_m), u) & \mapsto & (\partial_{i_1}, ..., \partial_{i_m}, y, u + (1-u)t_0, \\ & & \qquad (1-u)t_1, .., (1-u)t_m) \end{array}$$

satisfy $H_k(-,0) = \mathrm{id}$, $H_k(-,1) = \eta_k \circ \pi_k$ and $\overline{H}_k(-,0) = \mathrm{id}$, $\overline{H}_k(-,1) = \varphi_k \circ \overline{\pi}_k$.

n-rectifiable map. We write $\varphi : T^n_\bullet(Y) \dashrightarrow J^n_\bullet(Y)$ to denote the collection of maps $\varphi_k : T^n_k(Y) \to J^n_k(Y)$ given by $\varphi_k(y) = (y, 1)$. Recall that φ is not a morphism of facial spaces since it does not satisfy the usual rules of commutation with the face operators. In the same way we write $\psi : Y_\bullet \dashrightarrow Z_\bullet$ for a collection of maps $\psi_k : Y_k \dashrightarrow Z_k$ which do not satisfy the usual rules of commutation with the face operators and we say that ψ is an *n-rectifiable map* if there exists a morphism of facial spaces $\overline{\psi} : J^n_\bullet(Y) \to T^n_\bullet(Z)$ such that $\overline{\psi}_k \circ \varphi_k = \psi_k$ for any $k \leq n$. So, an n-rectifiable map $\psi : Y_\bullet \dashrightarrow Z_\bullet$ induces a map between the realizations up to n of the facial spaces $Y_\bullet$ and $Z_\bullet$.

8.2. **Proof of Theorem 4.** Let $Z^\bullet_\bullet \stackrel{d_0}{\to} Z^{-1}_\bullet$ be a facial resolution of a facial space $Z^{-1}_\bullet$ such that each row $Z^\bullet_k \stackrel{d_0}{\to} Z^{-1}_k$ admits a contraction and let $n \geq 0$. We first note that the realization of $Z^\bullet_\bullet$ up to p along the rows and up to n along the columns leads to two canonical maps:

$$||Z^\bullet_\bullet|^p|_n \to |Z^{-1}_\bullet|_n \qquad ||Z^\bullet_\bullet|_n|^p \to |Z^{-1}_\bullet|_n.$$

Induction on p and standard colimit arguments show that these two maps are equal (up to homeomorphism). Here we prove that $||Z^\bullet_\bullet|^p|_n \to |Z^{-1}_\bullet|_n$ admits a homotopy section.

For any k, we denote by s_k the contraction of the kth row

$$Z^{-1}_k \xleftarrow{d_0} Z^0_k \underset{d_1}{\overset{d_0}{\leftleftarrows}} Z^1_k \underset{d_2}{\overset{d_0}{\Lleftarrow}}{\scriptstyle d_1}\; X^2_k \qquad \cdots \qquad Z^{n-1}_k \underset{d_n}{\overset{d_0}{\leftleftarrows}}{\scriptstyle \vdots}\; Z^n_k$$

and, in order to simplify the notation we will write L_k for the realization up to n of this facial space. That is, $L_k = |Z^\bullet_k|^n$. Recall, from Proposition 2, that the existence of the contraction permits the following description of L_k:

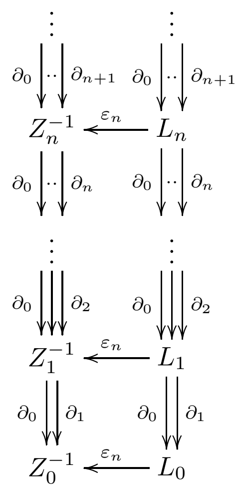
$$L_k = Z_k^n \times \Delta^n / \sim$$

where the relation is given by

$$(z, t_0, ..., t_i, ..., t_n) \sim (s_k d_i z, 0, t_0, ..., \hat{t}_i, ..., t_n) \quad \text{if } t_i = 0.$$

With respect to this description, the canonical map $L_k \to Z_k^{-1}$ is given by $[z, t_0, ..., t_i, ..., t_n] \mapsto d_0^{n+1} z$ and is denoted by ε_n (without reference to k).

Realizing all the lines, we obtain a facial map:



The face operators $\partial_i : L_k \to L_{k-1}$ are given by $\partial_i[z, t_0, ..., t_n] = [\partial_i z, t_0, ..., t_n]$. Our aim is thus to see that the map obtained after realization (and always denoted by ε_n)

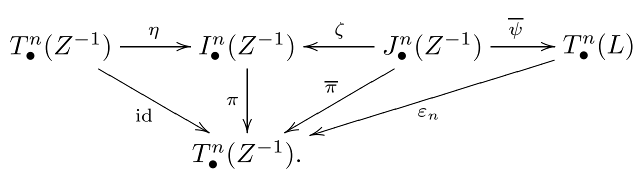
$$|Z_\bullet^{-1}|_n \xleftarrow{\varepsilon_n} |L_\bullet|_n$$

admits a section up to homotopy.

For each k, the map $\varepsilon_n : L_k \to Z_k^{-1}$ admits a (strict) section given by $z \mapsto [s_k^{n+1} z, 0, 0, ..., 0, 1]$ which we denote by ψ_k. The collection ψ of these maps does not define a facial map since the contraction s_k are not required to commute with the face operators ∂_i of the columns. The key is that $\psi : Z_\bullet^{-1} \dashrightarrow L_\bullet$ is an n-rectifiable map. We can indeed consider, for each $k \leq n$, the (well-defined) map $\overline{\psi}_k : J_k^n(Z^{-1}) \to L_k$ given by:

$$\overline{\psi}_k(\partial_{i_1}, ..., \partial_{i_m}, z, t_0, ..., t_m) = [s_k^{n+1-m} \partial_{i_1} s_{k+1} \partial_{i_2} s_{k+2} ... \partial_{i_m} s_{k+m} z, 0, ..., 0, t_0, ..., t_m].$$

Straightforward calculation shows that the maps $\overline{\psi}_k$ commute with the face operators ∂_i so that the collection $\overline{\psi}$ is a facial map. This morphism also satisfies $\overline{\psi}_k \circ \varphi_k = \psi_k$ for any $k \leq n$ (which implies that ψ is an n-rectifiable map) and $\varepsilon_n \overline{\psi} = \overline{\pi}$. We have hence the following commutative diagram:



Since the morphisms η, ζ, π and $\overline{\pi}$ induce homotopy equivalence between the realizations up to n, we get the following situation after realization:

$$\begin{array}{ccccccc}
|T^n_\bullet(Z^{-1})|_n & \xrightarrow{\sim} & |I^n_\bullet(Z^{-1})|_n & \xleftarrow{\sim} & |J^n_\bullet(Z^{-1})|_n & \xrightarrow{\overline{\psi}} & |T^n_\bullet(L)|_n \\
& {\searrow}^{\mathrm{id}} & \downarrow{\sim} & {\swarrow}^{\sim} & & {\swarrow}_{\varepsilon_n} & \\
& & |T^n_\bullet(Z^{-1})|_n . & & & &
\end{array}$$

Since $|T^n_\bullet(Z^{-1})|_n = |Z^{-1}_\bullet|_n$ and $|T^n_\bullet(L)|_n = |L_\bullet|_n$, we obtain that the map $|L_\bullet|_n \to |Z^{-1}_\bullet|_n$ admits a homotopy section. □

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